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SOLUTIONS TO THE NONAUTONOMOUS BISTABLE EQUATION WITH SPECIFIED MORSE INDEX. PART I: EXISTENCE

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ABSTRACT. We investigate the existence of unstable solutions of specified Morse index for the equation $\varepsilon^2 u_{xx} - f(x, u) = 0$ on a finite interval and Neumann boundary conditions.

1. INTRODUCTION

We consider the boundary value problem

$$\begin{aligned} &\varepsilon^2 u_{xx} - f(x, u) = 0, && 0 < x < 1, \\ &u_x = 0, && x = 0, 1, \end{aligned} \tag{1.1}$$

with $\varepsilon > 0$ a small parameter, $uf(x, u) > 0$ for $|u|$ large, and $f(x, \cdot)$ having at least three nondegenerate zeros (see Figure 1). In particular we assume the existence of a smooth function $\gamma(x)$ defined on $[0, 1]$ such that

$$f(x, \gamma(x)) = 0, \qquad f_u(x, \gamma(x)) < 0. \tag{1.2}$$

Under the assumptions above plus a minor technical hypothesis (see (H3) in §2) we establish:

Theorem. *Given any nonnegative integer k there is a nonempty open set $\mathscr{E}_k \subset \mathbb{R}$ such that*

(1)
$$\lim_{\bar{\varepsilon}\to 0} \operatorname{meas}(\mathscr{E}_k \cap (0, \bar{\varepsilon}))/\bar{\varepsilon} = 1.$$

(2) *For each ε in $\mathscr{E}_k$ problem (1.1) has $(k+1)$ pairs of solutions $u^+_{\varepsilon, j}$, $u^-_{\varepsilon, j}$, $j = 0, 1, \dots, k$, with the property that the eigenvalue problem*

$$\begin{aligned} &\varepsilon^2 h_{xx} - f_u(x, u_\varepsilon)h = \lambda h, && 0 < x < 1, \\ &h_x = 0, && x = 0, 1, \end{aligned} \tag{1.3}$$

where $u_\varepsilon = u^{\pm}_{\varepsilon, j}$, has exactly j positive eigenvalues.

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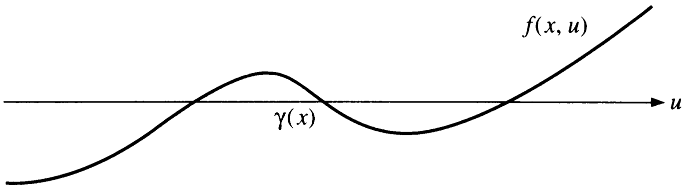

FIGURE 1

This is our main result. It states that (1.1) possesses solutions of arbitrary *Morse index* k, for ε sufficiently small. The corresponding result for f *independent of* x is classical and can be obtained by an elementary argument. We sketch it since basically it is the argument we use to settle the general case. Letting $u = \gamma + v$, we write the equation (1.1) as a first order system

$$v_x = w, \qquad w_x = g(v)/\varepsilon^2, \tag{1.4}$$

where $g(v) = f(\gamma+v) - f(\gamma) = f(\gamma+v)$. Letting $(v_\varepsilon(x, v_0), w_\varepsilon(x, v_0))$ denote the solution to (1.4) with initial conditions $v_\varepsilon(0, v_0) = v_0$, $w_\varepsilon(0, v_0) = 0$, we observe that u_ε satisfies (1.1) if and only if $w_\varepsilon(1, v_0) = 0$. If we let $\theta_\varepsilon(x, v_0)$ stand for the angle swept by the vector

$$\vec{V}_\varepsilon(\xi, v_0) = \begin{bmatrix} v_\varepsilon(\xi, v_0) \\ w_\varepsilon(\xi, v_0) \\ 0 \end{bmatrix}$$

as ξ goes from 0 to x, then $w_\varepsilon(1, v_0) = 0$ for $v_0 \neq 0$ is equivalent to

$$\theta_\varepsilon(v_0) \equiv \theta_\varepsilon(1, v_0) = k\pi, \qquad k \in \mathbb{Z}.$$

The angle $\theta_\varepsilon(x, v_0)$ can be defined also for $v_0 = 0$ so that $\theta_\varepsilon(x, v_0)$ is jointly continuous in (ε, x, v_0) for $\varepsilon > 0$. In fact by performing a classical transformation due to Prüffer [Pr] by setting $v = \rho \cos\theta$, $w = -\rho\sin\theta$ we discover that $\theta_\varepsilon(\cdot, v_0)$ satisfies

$$\theta_{\varepsilon x} = \sin^2\theta_\varepsilon - \frac{g(v_\varepsilon(x, v_0))}{\varepsilon^2 v_\varepsilon(x, v_0)}\cos^2\theta_\varepsilon, \qquad \theta_\varepsilon(0) = 0. \tag{1.5}$$

As $v_0 \to 0$ the solution to (1.5) approaches uniformly over $[0, 1]$ the solution to

$$\theta_{\varepsilon x} = \sin^2\theta_\varepsilon - \frac{1}{\varepsilon^2} g_v(x, 0)\cos^2\theta_\varepsilon, \qquad \theta_\varepsilon(0) = 0. \tag{1.6}$$

Note that (1.6) corresponds to the linearization of (1.5) about the equilibrium solution $(0, 0)$ (γ is an equilibrium solution to the equation in (1.1) in the autonomous case). Next the angle ψ_ε is introduced with a definition analogous to that of θ_ε but with the vector $\partial\vec{V}(\xi, v_0)/\partial v_0$ now taking the place of $\vec{V}(\xi, v_0)$. While $\theta_\varepsilon(1, v_0)$ characterizes the solutions to the boundary value problem, $\psi_\varepsilon(1, v_0)$ can be used to determine the number of positive eigenvalues of (1.3). In fact if $\psi_\varepsilon(v_0) \equiv \psi_\varepsilon(1, v_0)$, then the number of positive eigenvalues $n^+(v_0)$ of (1.3) about u_ε is given by the formula

$$n^+(v_0) = \text{Integer Part}(\psi_\varepsilon(v_0)/\pi + 1). \tag{1.7}$$

This formula is the basis of the whole argument. It relates the number of positive eigenvalues to the rotation of the tangent line to the solution curve.

In the autonomous case γ is an equilibrium solution and provides the center about which the solutions rotate. This fact is crucial. As was noticed above, (1.6) is the linearized equation about γ. A continuity argument provides the link between $\theta_\varepsilon(v_0)$ and the easy to compute $\psi_\varepsilon(0)$. Finally $\theta_\varepsilon(v_0)$ is related to $\psi_\varepsilon(v_0)$ via the Gauss map. In the nonautonomous case γ ($\gamma(x)$ now) is not generally a solution and this is the major obstruction one has to overcome for pursuing this approach. The following proposition does exactly that.

Proposition. *Assume f satisfies the conditions of the Theorem. Then there is an open set $\mathscr{E} \subset \mathbb{R}$ such that*

(1) $\lim_{\bar{\varepsilon}\to 0} \operatorname{meas}(\mathscr{E} \cap (0, \bar{\varepsilon}))/\bar{\varepsilon} = 1$,
(2) *For each ε in $\mathscr{E}$ problem* (1.1) *has a solution $\bar{u}_\varepsilon$,*
(3) *$\bar{u}_\varepsilon$ converges uniformly to $\gamma(x)$ on $[0, 1]$ as $\varepsilon \to 0$, ε in $\mathscr{E}$.*

This proposition is the major technical tool of the paper. The solution $\bar{u}_\varepsilon$ plays the role of the center about which the solution curves rotate. Its proof, we found out, could not be based on a perturbation argument or on a sub-supersolution argument simply because $\gamma(x)$ is increasingly (as $\varepsilon \to 0$) unstable, but instead proceeds by a careful analysis of the initial value problem. The set $\mathscr{E}$ is not equal to $(0, \bar{\varepsilon})$ in general, as we show by example, and thus the existence of $\bar{u}_\varepsilon$ is not to be expected for all small ε.

Another complication arises from the fact that (1.1) cannot be analyzed on the plane and the instability of the solutions is viewed through the rotation $\psi_\varepsilon(v_0)$ in the tangent flow of the three-dimensional dynamical system. In general $\psi_\varepsilon(v_0)$ is impossible to compute but by a homotopy which "unwinds" the curve $\mathscr{C} = \{(v_\varepsilon(1, v_0), w_\varepsilon(1, v_0)) \colon w_0 \in I\}$ for certain intervals I, we show the existence of $v_k > 0$ such that $\theta_\varepsilon(v_k) = k\pi$ and $\psi_\varepsilon(v_k) \in [(k-1)\pi, k\pi)$ for any k provided ε is small. Once a candidate for v_k has been identified, the idea is this: Let $I = [v_k, v_0]$ where v_0 is sufficiently large so that $\psi_\varepsilon(v_0) \in (-\pi/2, \pi/2)$. Smoothly attach a curve to $\mathscr{C}$ whose rotation is known and such that the new closed curve is smoothly homotopic to a simple closed curve (see Figure 4). $\psi(v_k)$ can then be computed since the angle through which the tangent turns is invariant under a smooth homotopy. This argument is given for a general system in the Winding Lemma at the end of §3.

Equation (1.1) has been used as a model for population genetics (e.g., [AW, P, K, Fi]) but is fundamental to many physical systems in which at least two isolated stable states exist. It clearly includes as a special case the important bistable nonlinearity

$$f(x, u) = u(u-1)(u-\gamma(x)), \qquad 0 < \gamma(x) < 1. \tag{1.8}$$

Stable solutions to (1.1) for the choice (1.8) have been completely characterized in [AMP]. The present paper initiates the systematic study of unstable solutions. In this connection we should mention Kurland's work [K] showing existence of oscillating solutions for (1.1) with f given by (1.8). These solutions are found by changing to action-angle variables and by averaging (1.1) considered as a three-dimensional dynamical system with x as a slowly varying dependent variable. In [K] no information concerning the index is provided, and although averaging may be applied to the linearized action-angle system, it can only provide an estimate on the index. Both the number of oscillations and the index of the solutions found in [K] grow unboundedly as $\varepsilon \to 0$, and therefore the solutions with a given Morse index that we study lie outside the set of solutions in [K]. Our technique is different from Kurland's. In fact, for f satisfying our assumptions, one does not have in general the monotonicity of the "time map" which is guaranteed for an f of the form (1.8) and is basic for the proof of existence of oscillatory solutions in [K]. Without (1.8) or another stringent requirement on f it is unclear whether one can use the elegant argument in [K] to obtain solutions. In fact we provide an example at the end of §2 showing nonexistence of solutions in a neighborhood of γ (which is equivalent to a uniform disk about the origin in the action-angle plane). Nonexistence is a direct consequence of the monotonicity of the time map failing. It is also worth noting that more than the oscillatory nature of solutions is needed to compute their index. Indeed in [AMP] solutions with many oscillations are found which have index 0, i.e., they are stable with respect to the parabolic equation. The proof of the Winding Lemma at the end of §3 gives an explicit geometrical condition which allows one to find solutions of a specified index. In a forthcoming paper [ABF] we study the shape of the unstable solutions of a given index k as $\varepsilon \to 0$.

Our motivation for studying unstable solutions comes from the realization that the *global attractor* is the relevant object to study for identifying the physically relevant solutions of the associated evolution problem

$$u_t = \varepsilon^2 u_{xx} + f(x, u), \qquad u_x = 0, \quad \text{at } x = 0, 1. \tag{1.9}$$

The global attractor is typically made up of the unstable stationary solutions and their unstable manifolds (see [H, Hel, 2, L, BF1, 2, A, BV]). Thus the long-time behavior of trajectories is determined to a large extent by the nature of the unstable stationary states.

The paper is structured as follows: in §2 we give a precise statement of the hypotheses and of the Proposition and provide its proof. In §3 we prove the Theorem.

2. The center

Let $f(x, u)$ be a smooth (C^3 will do), bounded function. Furthermore suppose:

(H1) There is smooth function γ on $[0, 1]$ such that

$$f(x, \gamma(x)) \equiv 0, \qquad p(x) \equiv f_u(x, \gamma(x)) < 0,$$

(H2) There exists $R > 0$ such that $uf(x, u) > 0$ for $|u| \geq R$.

(H3) $\gamma(x)$ satisfies at $x = 0, 1$: $\gamma' = (\gamma''/p)' = 0$.

Remarks. 1. (H3) is not very restrictive. For instance it holds if γ is constant in neighborhoods of 0 and 1.

2. One can easily see that all solutions to (1.1) must satisfy $\|u\|_\infty \leq R$. This allows us, without loss of generality, to replace (H2) by the seemingly stronger condition

(H2)$'$ There exists $\beta > 0$ such that $uf(x, u) \geq \beta|u|$ for $|u| \geq R$. We can, and will, also assume without loss of generality that $f_u(x, u)$ is globally Lipschitz in u, uniformly in x on $[0, 1]$. These extra conditions are achieved by modifying f when $|u| \geq R$ and then noting that any solutions to the modified problem are solutions to the original problem.

Proposition. *Assume that* $f \in C^3$ *satisfies* (H1)–(H3) *and is uniformly bounded on* $[0, 1] \times \mathbb{R}$. *Then there is an open set* $\mathscr{E} \subset \mathbb{R}$ *such that*

(i) $\lim_{\bar{\varepsilon} \to 0} \operatorname{meas}(\mathscr{E} \cap (0, \bar{\varepsilon}))/\bar{\varepsilon} = 1$,
(ii) *For each* ε *in* $\mathscr{E}$ *problem* (1.1) *has a solution* $\bar{u}_\varepsilon$,
(iii) $\bar{u}_\varepsilon$ *converges uniformly to* $\gamma(x)$ *on* $[0, 1]$ *as* $\varepsilon \to 0$, ε *in* $\mathscr{E}$.

Proof. Change variables in (1.1) by letting

$$u = \gamma + \varepsilon^2(w + \gamma''/p). \tag{2.1}$$

Since γ and p are given functions we need only show that solutions of the form (2.1) exist with $\varepsilon^2 w$ as small as desired. Using (H3), our new problem is

$$\begin{aligned} &\varepsilon^4 w'' - f(x, \gamma + \varepsilon^2(w + \gamma''/p)) + \varepsilon^2\gamma'' + \varepsilon^4(\gamma''/p)'' = 0, \\ &w'(0) = w'(1) = 0. \end{aligned} \tag{2.2}$$

Since $f(x, \gamma(x)) = 0$ and $p(x) = f_u(x, \gamma(x))$, Taylor's formula yields

$$f(x, \gamma + \varepsilon^2(w + \gamma''/p)) = \varepsilon^2 pw + \varepsilon^2\gamma'' + \varepsilon^2 g(\varepsilon, x, w) \tag{2.3}$$

where
(2.4)

$$g(\varepsilon, x, w) = \int_0^1 \left\{ f_u\left(x, \gamma + s\varepsilon^2\left(w + \frac{\gamma''}{p}\right)\right) - f_u(x, \gamma) \right\} ds \left(w + \frac{\gamma''}{p}\right).$$

Notice that for some $K_0 > 0$

$$|g(\varepsilon, x, w)| \le \varepsilon^2 K_0(1 + |w(x)|^2)$$

for all $w \in C([0, 1])$. Combining this with (2.3) we see that (2.2) can be written in the form

$$\varepsilon^2 w'' - pw - h(\varepsilon, x, w) = 0, \qquad w'(0) = w'(1) = 0 \tag{2.5}$$

where for some $K > 0$

$$|h(\varepsilon, x, w)| = |g(\varepsilon, x, w) - \varepsilon^2(\gamma''/p)''| \le \varepsilon^2 K(1 + |w(x)|^2). \tag{2.6}$$

Now let

$$W^\varepsilon(s, x) = \begin{pmatrix} W_{11}^\varepsilon & W_{12}^\varepsilon \\ W_{21}^\varepsilon & W_{22}^\varepsilon \end{pmatrix}(s, x)$$

be the fundamental matrix solution for the equation

$$\varepsilon^2 w'' - pw = 0$$

satisfying

$$W^\varepsilon(t, t) = I, \quad \text{for all } t.$$

At this point we need an auxiliary linear result whose proof we postpone for a while.

Lemma. *Let* $q \in C^1([0, 1])$ *be strictly positive and let* $W^\varepsilon(x, s)$ *be the fundamental matrix solution of*

$$\varepsilon^2 w'' + q^2(x) w = 0. \tag{$*$}$$

Then there is a constant C, *independent of* ε *such that for all* $x, s \in [0, 1]$

(P1) $|W_{11}^\varepsilon(x, s)|$, $|W_{22}^\varepsilon(x, s)| \le C$,
(P2) $|W_{12}^\varepsilon(x, s)|/\varepsilon$, $|W_{21}^\varepsilon(x, s)|\varepsilon \le C$.

Furthermore,

(P3) W_{11}^ε, W_{22}^ε, $\varepsilon W_{21}^\varepsilon$, *and* $W_{12}^\varepsilon/\varepsilon$ *have the form*

$$a(\varepsilon)\sin\left(-\frac{1}{\varepsilon}\int_s^x q + j(\varepsilon)\right)$$

with $j(\varepsilon) = j(x, s; \varepsilon)$ *bounded in* ε *and* $a(\varepsilon) = a(x, s; \varepsilon)$ *bounded uniformly away from zero.*

We continue our proof of the Proposition. Note that the lemma applies since p is negative (see (H1)). We observe that solutions of the equation in (2.5) with initial conditions

$$w(0) = w_0, \qquad w'(0) = 0 \tag{2.7}$$

may be written

$$\begin{aligned} w(x) &= w_0 W_{11}^\varepsilon(0, x) + \frac{1}{\varepsilon^2}\int_0^x W_{12}^\varepsilon(s, x)h(\varepsilon, s, w(s))\,ds, \\ w'(x) &= w_0 W_{21}^\varepsilon(0, x) + \frac{1}{\varepsilon^2}\int_0^x W_{22}^\varepsilon(s, x)h(\varepsilon, s, w(s))\,ds. \end{aligned} \tag{2.8}$$

Using (2.6), (P1), and (P2) we deduce from (2.8)

$$\overline{w}(x) \le (|w_0| + \varepsilon K(1 + \overline{w}^2(x)))C \tag{2.9}$$

where $\overline{w}(x) \equiv \max_{0\le y\le x}|w(y)|$. If $|w_0| \le 1$ and $\varepsilon < \tilde{\varepsilon} \equiv [CK(1+(C+1)^2)]^{-1}$ then (2.9) implies

$$|w(x)| \le C + 1 \quad \text{on } [0, 1]. \tag{2.10}$$

The second equation in (2.8) now gives

$$w'(1; w_0) = w_0 W_{21}^\varepsilon(0, 1) + \chi(\varepsilon, w_0) \tag{2.11}$$

where by (2.6), (P1), and (2.10)

$$|\chi(\varepsilon, w_0)| \le CK(1 + (C+1)^2) \equiv M. \tag{2.12}$$

Now $W_{21}^\varepsilon(0, 1)$ varies continuously in ε and so by (P3), given $\alpha > 0$ sufficiently small

$$\widetilde{\mathscr{E}} \equiv \{\varepsilon : \varepsilon W_{21}^\varepsilon(0, 1) > \alpha\} \quad \text{satisfies condition (i).}$$

Taking $w_0 = \pm 1$ in (2.11) we see that $w'(1; 1)$ and $w'(1; -1)$ have opposite signs provided $\varepsilon \in \widetilde{\mathscr{E}} \cap (0, \tilde{\varepsilon}) \cap (0, \alpha/M) \equiv \mathscr{E}$. It follows that for $\varepsilon \in \mathscr{E}$ there exists $w_0 \in (-1, 1)$ such that $w'(1; w_0) = 0$, that is, $w(\cdot; w_0)$ satisfies (2.5) and $\overline{u}_\varepsilon$ given by (2.1) is the desired solution, satisfying (iii) by (2.10). The proof of the Proposition is complete. □

Proof of Lemma. Introduce new variables

$$a(x) = a(x, s; \varepsilon) \quad \text{and} \quad \sigma(x) = \sigma(x, s; \varepsilon)$$

with $a \ge 0$ by setting

$$qw = a\cos\sigma; \qquad \varepsilon w' = a\sin\sigma. \tag{2.13}$$

Differentiating with respect to x and using $(*)$ to eliminate w we obtain

$$\begin{aligned} &\frac{q'}{q} a\cos\sigma + \frac{q}{\varepsilon} a\sin\sigma = a'\cos\sigma - a\sigma'\sin\sigma, \\ &-\frac{q}{\varepsilon} a\cos\sigma = a'\sin\sigma + a\sigma'\cos\sigma. \end{aligned} \tag{2.14}$$

From this it follows that

$$a' = \frac{q'}{q} a\cos^2\sigma, \qquad \sigma' = -\frac{q}{\varepsilon} - \frac{1}{2}\frac{q'}{q}\sin 2\sigma. \tag{2.15}$$

The first equation implies that

$$a(s)c \le a(x) \le a(s)C \quad \text{for all } x, s \in [0, 1] \tag{2.16}$$

for some positive constants c and C depending only on q. The second equation in (2.15) has a unique solution which satisfies

(2.17)

$$\sigma(x) = -\frac{1}{\varepsilon}\int_s^x q + \sigma(s) + j(x, s; \varepsilon) \quad \text{where } j(x, s; \varepsilon) = -\frac{1}{2}\int_s^x \frac{q'}{q}\sin 2\sigma.$$

Clearly, there is a constant D such that $|j(x, s; \varepsilon)| \le D$ for all $\varepsilon > 0$ and $x, s \in [0, 1]$. Now $W_{11}^\varepsilon(x, s)$ is the solution to $(*)$ satisfying $W_{11}^\varepsilon(s, s) = 1$ and $\partial W_{11}^\varepsilon(s, s)/\partial x = 0$. Furthermore $W_{21}^\varepsilon = \partial W_{11}^\varepsilon/\partial x$. From (2.13) we find

$$q(s) = a(s)\cos\sigma(s), \qquad 0 = a(s)\sin\sigma(s) \tag{2.18}$$

and therefore $\sigma(s) = 0$ and $a(s) = a(s, s; \varepsilon) = q(s)$. Therefore (2.13) gives

$$W_{11}^\varepsilon(x, s) = \frac{a(x, s; \varepsilon)}{q(x)}\cos\left(-\frac{1}{\varepsilon}\int_s^x q + j(x, s; \varepsilon)\right), \tag{2.19}$$

$$W_{21}^\varepsilon(x, s) = \frac{a(x, s; \varepsilon)}{\varepsilon}\sin\left(-\frac{1}{\varepsilon}\int_s^x q + j(x, s; \varepsilon)\right). \tag{2.20}$$

Let $\tilde{a}$, $\tilde{\sigma}$, and $\tilde{j}$ be the expressions for a, σ, and j corresponding to the solution W_{12}^ε and its derivative W_{22}^ε. Then because $W_{12}^\varepsilon(s, s) = 0$ and $W_{22}^\varepsilon(s, s) = 1$, (2.13) gives $\tilde{\sigma}(s) = \pi/2$ and $\tilde{a}(s) = \varepsilon$ so that

$$W_{12}^\varepsilon(x, s) = \frac{-\tilde{a}(x, s; \varepsilon)}{q(x)}\sin\left(-\frac{1}{\varepsilon}\int_s^x q + \tilde{j}(x, s; \varepsilon)\right), \tag{2.21}$$

$$W_{22}^\varepsilon(x, s) = \frac{-\tilde{a}(x, s; \varepsilon)}{\varepsilon}\cos\left(-\frac{1}{\varepsilon}\int_s^x q + \tilde{j}(x, s; \varepsilon)\right). \tag{2.22}$$

Properties (P1)–(P3) now follow from (2.19)–(2.22) by using (2.16) together with the values of $a(s)$ and $\tilde{a}(s)$ computed above. The proof of the lemma is complete. □

Remark. The set $\mathscr{E}$ in the Proposition cannot be replaced by an open interval of the type $(0, \varepsilon_0)$ as the following example shows.

Example. There exists a function f satisfying (H1)–(H3), a positive number Δ, and a sequence $\varepsilon_n \to 0$ such that (1.1) has no solution within Δ distance of γ for each $\varepsilon = \varepsilon_n$.

Proof of Claim. Let $\Delta > 0$ be a fixed small number. For $c > 0$ and fixed let $\gamma(x) = c[x^6 - 5x^4 + 7x^2]$ and define $f(x, u) = \gamma(x) - u$ for (x, u) such that

$|\gamma(x) - u| \le \Delta$, smooth and bounded for all (x, u), and satisfying (H2). Then it is clear that (H1)–(H3) hold. In fact $p(x) \equiv -1$ and $f_u(x, u) = -1$ for all (x, u) such that $|\gamma(x) - u| \le \Delta$. It follows that if $u(x)$ is a solution to (1.1) uniformly within Δ of $\gamma(x)$, then there is a solution $w(x)$ to (2.5) with $h(\varepsilon, x, w) = \varepsilon^2 \gamma^{(4)}(x)$. Using (2.8) we find

$$w(x) = w_0 \cos\left(\frac{x}{\varepsilon}\right) + \varepsilon \int_0^x \sin\left(\frac{x-s}{\varepsilon}\right) \gamma^{(4)}(s)\, ds.$$

Taking $\varepsilon_n = 1/n\pi$, $n = 1, 2, \dots$, we find

$$\begin{aligned} w'(1) &= (-1)^n \int_0^1 \cos(n\pi s)\gamma^{(4)}(s)\, ds \\ &= 5!c(-1)^n \int_0^1 (3s^2 - 1)\cos(n\pi s)\, ds \neq 0. \end{aligned}$$

Thus we see that no solution Δ-close to γ exists for this choice of ε_n.

3. Solutions of specified index

Let f satisfy hypotheses (H1), (H2), and (H3) of §2. As before $f(x, u)$ is taken to be a smooth function with $f_u(x, u)$ globally Lipschitz in u. Under these assumptions we have

Theorem. *Given any integer $k > 0$, there is an open set $\mathscr{E}_k \subset \mathbb{R}$ such that*

(i) $$\lim_{\bar\varepsilon \to 0} \operatorname{meas}(\mathscr{E}_k \cap (0, \bar\varepsilon))/\bar\varepsilon = 1,$$

(ii) *For each ε in $\mathscr{E}_k$, (1.1) has $(k+1)$ pairs of solutions $u^+_{\varepsilon,j}$, $u^-_{\varepsilon,j}$, $0 \le j \le k$, with the property that for each j the eigenvalue problem (1.3) has exactly j positive eigenvalues.*

Proof.

Step 1. Change variables in (1.1) by letting $u = \bar u_\varepsilon + v$ where $\bar u_\varepsilon$ is the solutions provided by the Proposition for $\varepsilon \in \mathscr{E}$. Thus we consider

$$\begin{aligned} &\varepsilon^2 v'' - g^\varepsilon(x, v) = 0, \qquad 0 < x < 1, \\ &v'(0) = v'(1) = 0, \end{aligned} \tag{3.1}$$

where $g^\varepsilon(x, v) = f(x, \bar u_\varepsilon + v) - f(x, \bar u_\varepsilon)$. Note that $v \equiv 0$ is a solution to (3.1).

It is convenient to write (3.1) as a system

$$v' = w, \qquad w' = g^\varepsilon(x, v)/\varepsilon^2 \tag{3.2}$$

with boundary conditions $w(0) = w(1) = 0$.

Let $(v_\varepsilon(x; v_0), w_\varepsilon(x; v_0))$ denote the solution to (3.2) with initial conditions $v(0) = v_0$, $w(0) = 0$. Then solutions to (1.1) correspond to numbers v_0 satisfying $w_\varepsilon(1; v_0) = 0$.

For $v_0 \neq 0$ we use $\theta(y; v_0)$ to denote the angle through which $(v_\varepsilon(x; v_0), w_\varepsilon(x; v_0))$ turns as x goes from 0 to y, clockwise being positive. Similarly, for all v_0, we use $\psi(y; v_0)$ to denote the angle through which

$$(h_\varepsilon(x; v_0), k_\varepsilon(x; v_0)) \equiv (\partial/\partial v_0)(v_\varepsilon(x; v_0), w_\varepsilon(x; v_0))$$

turns as x goes from 0 to y. Note that $(h_\varepsilon(x\,;v_0), k_\varepsilon(x\,;v_0))$ is tangent to the curve $\mathscr{C} = \{(v_\varepsilon(x\,;v_0), w_\varepsilon(x\,;v_0)): v_0 \in \mathbb{R}\}$ and satisfies the variational system

$$(3.3) \qquad h'_\varepsilon = k_\varepsilon\,, \qquad k'_\varepsilon = g^\varepsilon_v(x\,, v_\varepsilon(x\,;v_0))h_\varepsilon/\varepsilon^2$$

with initial conditions $h_\varepsilon(0) = 1$, $k_\varepsilon(0) = 0$. Changing to polar coordinates in (3.3) yields

$$\psi'(x\,;v_0) = \sin^2\psi(x\,;v_0) - (1/\varepsilon^2)g^\varepsilon_v(x\,, v_\varepsilon(x\,;v_0))\cos^2\psi(x\,;v_0)\,,$$
$$\psi(0\,;v_0) = 0.$$

In particular when $v_0 = 0$ we get

$$(3.4) \qquad \psi'(x\,;0) = \sin^2\psi(x\,;0) - (1/\varepsilon^2)g^\varepsilon_v(x\,,0)\cos^2\psi(x\,;0).$$

But $g^\varepsilon_v(x\,,0) = f_u(x\,,\overline{u}_\varepsilon(x))$ and using the Proposition we may choose $\varepsilon \in \mathscr{E}$ so small that $f_u(x\,,\overline{u}_\varepsilon(x)) < -\delta$ for fixed $\delta > 0$ sufficiently small. Comparing (3.4) with

$$\psi' = \sin^2\psi + (\delta/\varepsilon^2)\cos^2\psi$$

leads us to the conclusion that for any positive integer k there exists $\varepsilon_k > 0$ such that for $\varepsilon \in \mathscr{E}_k \equiv \mathscr{E} \cap (0\,,\varepsilon_k)$

$$(3.5) \qquad \psi(1\,;0) > k\pi.$$

From (3.2) using (H2)$'$ and the Proposition we also conclude, by possibly taking ε_k smaller, that

$$|v_\varepsilon(x\,;v_0)| > R \quad \text{for } 0 \le x \le 1 \text{ whenever } |v_0| > R.$$

This allows us to apply the Winding Lemma below to obtain numbers $\overline{v}_j < 0 < v_j$, $0 \le j \le k$, such that $w_\varepsilon(1\,;v_j) = 0$, $\theta(1\,;v_j) = j\pi$, and generically

$$\psi(1\,;v_j) \in ((j-1)\pi\,, j\pi)\,, \qquad 0 \le j \le k \quad (\text{similarly for } \overline{v}_j).$$

The significance of this last statement is manifest in the next step linking it with the index.

Step 2. Let F be a smooth function and suppose $\overline{v}$ is a solution to

$$(3.6) \qquad \begin{aligned} &v'' + F(x\,,v) = 0\,, \qquad 0 < x < 1\,;\\ &v'(0) = v'(1) = 0.\end{aligned}$$

Consider the eigenvalue problem

$$(3.7) \qquad h'' + F_v(x\,,\overline{v}(x))h = \lambda h\,, \qquad h'(0) = h'(1) = 0.$$

Then we claim that the number of positive eigenvalues of (3.7) is given by the integer part of $(\psi(1)/\pi + 1)$ where $\psi(x)$ is the solution to

$$(3.8) \qquad \begin{aligned} &\psi' = \sin^2\psi + F_v(x\,,\overline{v}(x))\cos^2\psi\,,\\ &\psi(0) = 0.\end{aligned}$$

Proof. Writing $h = r\cos\tilde{\psi}$, $h' = -r\sin\tilde{\psi}$ in (3.7) yields

$$(3.9) \qquad \begin{aligned} &\tilde{\psi}' = \sin^2\tilde{\psi} + [F_v(x\,,\overline{v}(x)) - \lambda]\cos^2\tilde{\psi}\,,\\ &\tilde{\psi}(0) = 0\,,\\ &\tilde{\psi}(1) = n\pi \quad \text{for some integer } n.\end{aligned}$$

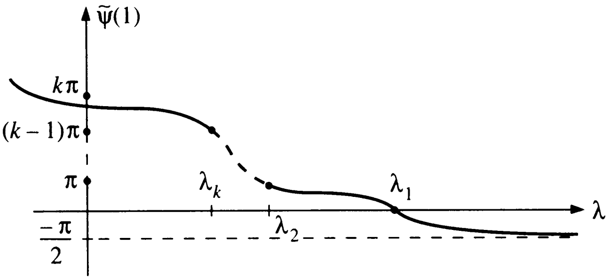

FIGURE 2

Note that the right-hand side of the differential equation in (3.9) is 1 whenever $\tilde{\psi} = \pi/2 + j\pi$ and therefore $\tilde{\psi}(x) > -\pi/2$ independently of x and λ.

If we vary λ in (3.9) we also see that $\tilde{\psi}(1)$ is a decreasing function of λ with $\tilde{\psi}(1) \to -\pi/2$ as $\lambda \to \infty$. Furthermore, eigenvalues of (3.9) are those values of λ for which $\tilde{\psi}(1) = n\pi$ for some integer n (see Figure 2).

Thus, the number of positive eigenvalues, i.e., the number of times $\tilde{\psi}(1)$ takes on a value of $n\pi$ for λ in $(0, \infty)$, is the integer part of $(\tilde{\psi}(1)/\pi + 1)$ corresponding to $\lambda = 0$; but $\tilde{\psi} = \psi$ when $\lambda = 0$. This proves the claim. This idea may be extended to the case where the diffusion coefficient is x-dependent (see [FH2 and R]).

Step 3. To complete the proof of the Theorem for $\varepsilon \in \mathscr{E}_k$ we take $u^+_{\varepsilon,j}(x) = \overline{u}_\varepsilon(x) + v_\varepsilon(x; v_j)$ (and $u^-_{\varepsilon,j}(x) = \overline{u}_\varepsilon(x) + v_\varepsilon(x; \overline{v}_j)$), and observe that the eigenvalue problem (1.3) is equivalent to the eigenvalue problem

$$\varepsilon^2 h'' - g^\varepsilon_v(x, v_\varepsilon(x; v_0))h = \lambda h, \qquad h'(0) = h'(1) = 0$$

with $v_0 = v_j$ or $\overline{v}_j$, since $g^\varepsilon_v(x, v_\varepsilon(x; v_0)) = f_u(x, u^\pm_{\varepsilon,j}(x))$ in this case. The proof of the Theorem is complete. □

In the proof of the theorem above we made use of a winding result that we now state in more general form than we actually need but may be interesting in its own right. Consider the system

$$(*) \qquad v' = a(x, v, w), \qquad w' = b(x, v, w)$$

with a and b smooth functions satisfying

$$(*.1) \qquad a(x, 0, 0) = b(x, 0, 0) = 0 \quad \text{for all } x.$$

We will denote the solution to $(*)$ passing through the point (v_0, w_0) at $x = 0$ by $(v(x; v_0, w_0), w(x; v_0, w_0))$. Assume

$(*.2)$ For some $L > 0$ solutions $(v(x; v_0, 0), w(x; v_0, 0))$ exist for $x \in [0, L]$ and all $v_0 \in \mathbb{R}$.

$(*.3)$ There are positive numbers R and R_1 such that $|v_0| \geq R$ implies

$$|v(x; v_0, 0)| \geq R_1 \text{ for } x \in [0, L].$$

Remark. Assumption $(*.3)$ is satisfied if either

$$wa(x, v, w) \geq 0 \text{ and } vb(x, v, w) > 0 \quad \text{for } |v| \geq R$$

or

$$wa(x, v, w) \leq 0 \text{ and } vb(x, v, w) < 0 \quad \text{for } |v| \geq R.$$

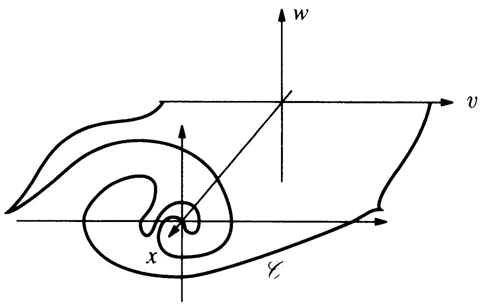

FIGURE 3

Our discussion here will concern the curve $\mathscr{C} \equiv \{(v(L; v_0, 0), w(L; v_0, 0)) : -\infty < v_0 < \infty\}$ and how it may wind around the origin $(v, w) = (0, 0)$, which is a point on $\mathscr{C}$ because of $(*.1)$. (See Figure 3.)

For brevity we write $(v(v_0), w(v_0))$ in place of $(v(L; v_0, 0), w(L; v_0, 0))$. Introducing polar coordinates (r, θ) on the (v, w)-plane, we can consider the total angle through which the solution starting at $(v_0, 0)$ rotates before it reaches $(v(v_0), w(v_0))$. Call this angle $\theta(v_0)$ and note that it is not restricted to $[0, 2\pi]$. We do not define θ at $v_0 = 0$ except through continuity of θ. We will take the clockwise direction to be positive.

Now consider the evolution of the tangent to the initial line $w = 0$. This tangent is $(\partial/\partial v_0)(v(x; v_0, 0), w(x; v_0, 0))$ and satisfies the equation obtained by linearizing $(*)$ at the solution $(v(x; v_0, 0), w(x; v_0, 0))$. For all v_0, including $v_0 = 0$, we denote by $\psi(v_0)$ the angle through which this tangent turns as x goes from 0 to L. We are interested in the relationship between $\theta(v_0)$ and $\psi(v_0)$.

The Winding Lemma. *Let $k \geq 1$ be an integer and suppose $\psi(0) > k\pi$. Then for each integer $j \in [1, k]$ there exist $\overline{v}_j < 0 < v_j$ such that*

$$\theta(\overline{v}_j) = \theta(v_j) = j\pi \quad \textit{and} \quad \psi(\overline{v}_j), \psi(v_j) \in [(j-1)\pi, j\pi].$$

Proof. For fixed $j \in [1, k]$ we show the existence of v_j. For $\overline{v}_j$ the proof is analogous. Consider $\mathscr{C}^+$, that part of $\mathscr{C}$ corresponding to $v_0 > 0$. Let $\overline{v}_0$ be the minimal value for which $v(v_0) \geq R_1$ for all $v_0 \geq \overline{v}_0$. Then by $(*.3)$ $\overline{v}_0 \in (0, R]$ and it is clear that

$$\theta(v_0) \in (-\pi/2, \pi/2) \quad \text{for all } v_0 \geq \overline{v}_0 \tag{3.10}$$

and

$$\psi(\overline{v}_0) \in [-\pi/2, \pi/2]. \tag{3.11}$$

It is also clear that there exists $\hat{v}_0 > 0$ sufficiently small such that

$$\theta(v_0) > k\pi \quad \text{for } 0 < v_0 < \hat{v}_0. \tag{3.12}$$

Let v_j be the largest value of v_0 for which $\theta(v_0) = j\pi$. Such a v_j exists by (3.10) and (3.12). To make the argument more transparent we will assume

$$\theta(\overline{v}_0) = 0, \qquad \psi(\overline{v}_0) = -\pi/2, \qquad \psi(v_j) = \pi/2 \pmod{\pi}. \tag{3.13}$$

If (3.13) does not hold, simple modifications of $\mathscr{C}^+$ give a curve satisfying (3.13). Because the modifications produce well-understood changes in θ and ψ, assuming (3.13) does not diminish the generality in our argument.

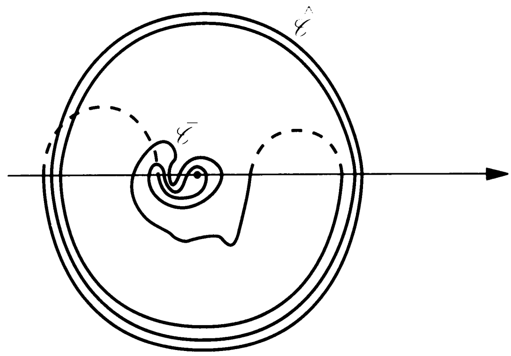

FIGURE 4

For $\overline{R}$ sufficiently large that portion of $\mathscr{C}^+$ corresponding to $v \in [v_j, \overline{v}_0]$, call it $\overline{\mathscr{C}}$, lies in the disk of radius $\overline{R}$. Now we form a closed curve by attaching another curve $\widehat{\mathscr{C}}$ to $\overline{\mathscr{C}}$ as follows (see Figure 4): From $(v(v_j), w(v_j))$ take a semicircle out to where the circle of radius $\overline{R}+1$ meets the ray $\theta = j\pi$. To the free end of this semicircle smoothly attach a spiral which lies in the annulus of inner radius $\overline{R}$ and outer radius $\overline{R}+1$ and which rotates through an angle of $-j\pi$. Now smoothly attach another semicircle from the free end of the spiral to $(v(\overline{v}_0), w(\overline{v}_0))$.

The closed curve we have formed has many self-intersections; however, it is the orthogonal projection onto the (v, w)-plane of a simple closed curve, Γ, on the surface $S = \{(z, r, \theta): z = \theta, \ r_1 < r < \overline{R}+2, \ \theta_1 < \theta < (j+1)\pi\}$. Here $r_1 > 0$ is chosen so that $\overline{\mathscr{C}}$ lies outside the disk of radius r_1, and $\theta_1 + \pi < \min\{\theta(v_0): v_j < v_0 < \overline{v}_0\}$ (see Figure 5).

We now show Γ can be smoothly deformed into a small nearly elliptical curve. Let H be a diffeomorphism from S to the plane, P, and let Γ' be the image of Γ under H. Now apply a motion by curvature flow to Γ' to deform

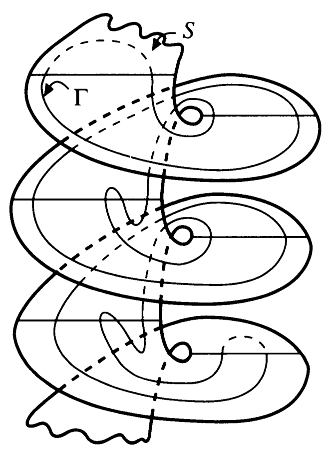

FIGURE 5

it to an asymptotically circular curve as it contracts to a point (see [G]). The application of H^{-1} to the family of curves generated from Γ' leads to a family of simple closed curves in S, becoming essentially elliptical as it contracts to a point.

Now consider the orthogonal projection onto the (u, v)-plane of this smoothly changing family of simple closed curves in S. Initially the curve is $\overline{\mathscr{C}} \cup \widehat{\mathscr{C}}$ and eventually it is essentially elliptical. Furthermore, no singularities develop in the tangent to this curve as the motion by curvature in P drives the evolution, by the results in [G] and due to the fact that S is never parallel to the normal to the (u, v)-plane. It follows that, even though $\overline{\mathscr{C}} \cup \widehat{\mathscr{C}}$ is self-intersecting, the tangent to this curve rotates through -2π as we traverse its length once, starting at $(v(\overline{v}_0), 0)$ and proceeding first along $\overline{\mathscr{C}}$. But $\widehat{\mathscr{C}}$ was constructed so that the tangent to it is rotated through $-(2+j)\pi$ as we move along it from $(v(v_j), 0)$ to $(v(\overline{v}_0), 0)$, the -2π contribution coming from the semicircles. This implies that the tangent to $\overline{\mathscr{C}}$ rotates through $-2\pi - (-(2+j)\pi) = j\pi$ as we proceed along it from $(v(\overline{v}_0), 0)$ to $(v(v_j), 0)$. Thus, under assumption (3.13), $\psi(v_j) = (j - \frac{1}{2})\pi$. Generically $\psi(v) \in ((j-1)\pi, j\pi)$. The proof of the Winding Lemma is complete. □

Remarks. 1. A similar result holds for $k < -1$.

2. Conditions (∗.2) and (∗.3) can be weakened. All we are doing here is restricting the motion of a solution starting with v_0 large.

3. There are other ways to homotopy $\overline{\mathscr{C}} \cup \widehat{\mathscr{C}}$ smoothly to a simple closed curve or to measure the rotation of the tangent as we tranverse $\overline{\mathscr{C}}$. Using the results in [G] seemed to be conceptually easy.

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