STABILITY OF SPECIAL INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

VINCENZO ANCONA AND GIORGIO OTTAVIANI

ABSTRACT. We prove that the special instanton bundles of rank $2n$ on $\mathbb{P}^{2n+1}(\mathbb{C})$ with a symplectic structure studied by Spindler and Trautmann are stable in the sense of Mumford-Takemoto. This implies that the generic special instanton bundle is stable. Moreover all instanton bundles on $\mathbb{P}^5$ are stable. We get also the stability of other related vector bundles.

INTRODUCTION

The instanton bundles of rank $2n$ on $\mathbb{P}^{2n+1}(\mathbb{C})$ were first defined by Okonek and Spindler in [OS], answering a question posed by Salamon in [Sal], where the Penrose transformation was generalized to any quaternionic manifold. In fact $\mathbb{P}^{2n+1}(\mathbb{C})$ is the twistor space over $\mathbb{P}^n(\mathbb{H})$.

Then Spindler and Trautmann gave in [ST2] a remarkable description of the class of special instanton bundles. A special instanton bundle on $\mathbb{P}^{2n+1}$ of quantum number k can be defined [ST2, Proposition 4.2] by an exact sequence

$$0 \to \mathscr{O}(-1)^k \to S^* \to E \to 0$$

where S is a Schwarzenberger bundle of rank $2n + k$ which is in turn defined by a special exact sequence (see 2.6)

$$0 \to \mathscr{O}(-1)^k \to \mathscr{O}^{2n+2k} \to S \to 0.$$

When $n = 1$ the special instanton bundles are the so-called special 't Hooft bundles (see [HN] and [BT]). In this case the stability of E is easy to check because rank $E = 2$ and moreover E is symplectic.

When $k = 1$ then $S^* = \Omega^1(1)$ and E is a nullcorrelation bundle. Also in this case E is symplectic and moreover it is homogeneous and irreducible under the action of the symplectic group. So it is stable by the theorem of Ramanan [U]. Ein gave in [E] an alternative proof of the stability of nullcorrelation bundles without using homogeneity.

When $k \geq 2$, $n \geq 2$ the stability of E is left as an open problem in [ST2]. The purpose of this paper is in fact to prove the following

Main theorem.

(i) *Every special symplectic instanton bundle on $\mathbb{P}^{2n+1}$ is stable.*

(ii) *The generic special instanton bundle on* $\mathbb{P}^{2n+1}$ *is stable.*
(iii) *Every instanton bundle on* $\mathbb{P}^5$ *is stable.*

(i) will be proved in Theorem 3.7. (ii) is an easy consequence of (i) and will be proved in Corollary 3.10. (iii) is Theorem 3.6. The proof of (iii) is based on a characterization of symplectic instanton bundles (Proposition 2.23).

Our proof of (i) is by induction on n, as in [E] for $k = 1$. But Ein's proof does not work if $k \geq 2$ because it relies on properties of the restriction of E to generic two codimensional linear subspaces, which are in fact well known only for a small family of linear subspaces (see Theorem 3.1).

So we give a cohomological criterion for the stability of a symplectic bundle (Theorem 3.5) which is similar to a criterion of Hoppe [H] and we apply it to our situation. The only "hard" tool that we use is a consequence of the results of Kobayashi, Lübke, Donaldson, Uhlenbeck, and Yau on Hermite-Einstein bundles, precisely that if E is a stable bundle then the exterior powers $\bigwedge^q E$ are semistable and direct sum of stable bundles.

In general Okonek, Spindler, and Trautmann define [OS, ST2] an instanton bundle of quantum number k on $\mathbb{P}^{2n+1}$ as a bundle of rank $2n$ satisfying the following properties:

(i) The Chern polynomial of E is $c_t(E) = (1 - t^2)^{-k}$.
(ii) E has natural cohomology in the range $-2n - 1 \leq q \leq 0$.
(iii) E has trivial splitting type.
(iv) E is simple.

We show (Proposition 2.11) that every $2n$-bundle satisfying (i) and (ii) is simple; hence, (iv) is superfluous (see Definition 2.12).

We get that also the Schwarzenberger bundles are stable (Theorem 2.2 and Corollary 2.9). Moreover the pullback of a special symplectic instanton bundle under a finite morphism $\mathbb{P}^{2n+1} \rightarrow \mathbb{P}^{2n+1}$ remains stable (Corollary 3.17).

After this paper has been written we received a preprint of Bohnhorst and Spindler [BS] where the stability of rank n Schwarzenberger bundles on $\mathbb{P}^n$ and other related bundles was proved.

1. PRELIMINARIES

For basic facts and notations about vector bundles we refer to [OSS]. We use the definition of stability of Mumford-Takemoto.

Fact 1.1. *Let E be a vector bundle. Then for $q \leq \mathrm{rk}(E)$:*

$$\mu\left(\bigwedge^q E\right) = \frac{c_1(\bigwedge^q E)}{\mathrm{rk}(\bigwedge^q E)} = \frac{\binom{\mathrm{rk}(E) - 1}{q - 1} c_1(E)}{\binom{\mathrm{rk}(E)}{q}} = q\mu(E).$$

The following useful criterion can be used to establish the stability of a vector bundle:

Theorem 1.2 (Hoppe [H, Lemma 2.6]). *Let X be projective manifold with $\mathrm{Pic}(X) = \mathbb{Z}$, and let E be a vector bundle on X. If $H^0(X, (\bigwedge^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq \mathrm{rk}(E) - 1$ then E is stable.*

The condition of the theorem is not necessary. The simplest counterexamples are the nullcorrelation bundles N on $\mathbb{P}^{2n+1}$. In fact $(\bigwedge^q N)_{\text{norm}}$ contains $\mathcal{O}$ as direct summand (see also Lemma 1.10).

Remark 1.3. Let X, E be as in Theorem 1.2 and suppose $c_1(E) = 0$. If $H^0(X, (\bigwedge^q E)_{\text{norm}}(-1)) = 0$ for $1 \leq q \leq \text{rank}(E) - 1$ then E is semistable.

We need also the following well known (compare to [OSS, Corollary to II, 1.2.8]) result.

Lemma 1.4. *Let E, F be stable vector bundles on $\mathbb{P}^m$ such that $c_1(E) = c_1(F) = 0$. Then every morphism $\phi \colon E \to F$ is either zero or an isomorphism.*

Proof. Let $\phi \colon E \to F$ be a nonzero morphism. Then $\text{Im}\,\phi$ is a nonzero subsheaf of F. If $\text{rk}(\text{Im}\,\phi) < \text{rk}(F)$ then as F is stable we have $c_1(\text{Im}\,\phi) < 0$; this implies $c_1(\text{Ker}\,\phi) > 0$, contradicting the semistability of E. Hence $\text{rk}(\text{Im}\,\phi) = \text{rk}(F)$. If $\text{rk}(E) > \text{rk}(F)$ then $\text{rk}(\text{Ker}\,\phi) > 0$, and the stability of E implies $c_1(\text{Ker}\,\phi) < 0$, whence $c_1(\text{Im}\,\phi) > 0$, which is again a contradiction. Then E, F have the same rank and ϕ is injective. This implies that $\det \phi \colon \mathcal{O} \to \mathcal{O}$ (and thus also ϕ) is an isomorphism.

We have the following important characterization (for the relative definitions see, e.g., [Lü]).

Theorem 1.5 (Kobayashi, Lübke, Donaldson, Uhlenbeck, Yau [K, Lü, D, UY]). *Let E be a vector bundle E on a projective manifold X.*

   (i) *If E is Hermite-Einstein then it is semistable and a direct sum of stable bundles (with respect to the Hermite-Einstein metric).*
   (ii) *If E is stable then it is Hermite-Einstein.*

We will use only the following

Corollary 1.6. *Let E be a stable vector bundle on a projective manifold X. Then $\bigwedge^q E$ is semistable and a direct sum of stable bundles (hence they have the same slope $\mu = c_1 / \text{rk}$).*

Proof. If suffices to note that if E is Hermite-Einstein also $\bigwedge^q E$ is [Lü]. The semistability of $\bigwedge^q E$ was known since [M2].

Fact 1.7. *Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then we have the exact sequences involving alternating and symmetric powers:*

(a) $\quad 0 \to \bigwedge^q E \to \bigwedge^q F \to \bigwedge^{q-1} F \otimes G \to \cdots \to F \otimes S^{q-1} G \to S^q G \to 0$,

(b) $\quad 0 \to S^q E \to S^{q-1} E \otimes F \to \cdots \to E \otimes \bigwedge^{q-1} F \to \bigwedge^q F \to \bigwedge^q G \to 0$.

Fact 1.8. *Let E be a vector bundle. $\forall j \geq 0$ $\bigwedge^j E \otimes E$ contains $\bigwedge^{j+1} E$ as direct summand. The natural morphisms are locally given by*

$$\bigwedge^{j+1} E \to \bigwedge^j E \otimes E, \quad \bigwedge^j E \otimes E \to \bigwedge^{j+1} E$$

$$e_1 \wedge \cdots \wedge e_{j+1} \mapsto \frac{1}{j+1} \sum_{i=1}^{j+1} (-1)^{j-i+1}(e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{j+1}) \otimes e_i,$$

$$(e_1 \wedge \cdots \wedge e_j) \otimes e_h \mapsto e_1 \wedge \cdots \wedge e_j \wedge e_h.$$

Definition 1.9. A vector bundle E is called symplectic if there exists an isomorphism $\phi\colon E \to E^*$ such that $\phi^* = -\phi$. This is equivalent to the existence of a nondegenerate form $\Gamma \in H^0(\bigwedge^2 E)$.

Lemma 1.10. *Let E be a symplectic vector bundle. For every j such that $0 \leq j \leq \mathrm{rk}(E)/2 - 1$, $\mathcal{O}$ is a direct summand of $\bigwedge^{2j} E$ and E is a direct summand of $\bigwedge^{2j+1} E$.*

Proof. We have the following stronger claim: for $j \leq \mathrm{rk}(E)/2$, $\bigwedge^{j-2} E$ is a direct summand of $\bigwedge^j E$. In fact let $\Gamma \in H^0(\bigwedge^2 E)$ be a nondegenerate form. We have a canonical morphism $\phi\colon \bigwedge^{j-2} E \to \bigwedge^j E$ locally given by $e_1 \wedge \cdots \wedge e_{j-2} \mapsto e_1 \wedge \cdots \wedge e_{j-2} \wedge \Gamma$. In [B, Chapter 9, §5, n.3] it is shown that (for E a vector space) ϕ is injective and there exists a canonical direct summand of $\mathrm{Im}\,\phi$. As everything is natural, the same construction yields the result for vector bundles. Then remember that $\bigwedge^{r-i} E \cong \bigwedge^{r+i} E$ (set $2r = \mathrm{rk}(E)$).

2. Schwarzenberger and instanton bundles

The classical Schwarzenberger bundles on $\mathbb{P}^m$ were first defined in [Sch] in the following way:

Definition 2.1. Let $r \geq m$ be an integer. The exact sequence of sheaves on $\mathbb{P}^m$

$$(1) \qquad 0 \to \mathcal{O}(-1)^{r-m+1} \xrightarrow{\phi} \mathcal{O}^{r+1} \to \mathscr{E}_m^r \to 0$$

where the map ϕ is given by the matrix

$$r-m+1 \left\{ \begin{bmatrix} x_0 & & \cdots & x_m & & \\ & \ddots & & & \ddots & \\ & & x_0 & \cdots & & x_m \end{bmatrix} \right.$$

defines an m-bundle $\mathscr{E}_m^r$ on $\mathbb{P}^m$. We call a (classical) Schwarzenberger bundle and we denote by E_m^r any bundle of the form $E_m^r \simeq g^* \mathscr{E}_m^r$ for some $g \in \mathrm{Aut}(\mathbb{P}^m)$.

These bundles were independently studied by Tango in [T].

Theorem 2.2. *A Schwarzenberger bundle E_m^r is stable.*

Proof. The proof is a simple application of the criterion of Hoppe (Theorem 1.2). We have $\mu(E_m^r) = (r-m+1)/m$. Then by $\mu(\bigwedge^q E_m^r) = q(r-m+1)/m > 0$. Then $(\bigwedge^q E_m^r)_{\mathrm{norm}} = (\bigwedge^q E_m^r)(t)$ with $t \leq -1$. So it is sufficient to prove that $H^0((\bigwedge^q E_m^r)(-1)) = 0$ for $1 \leq q \leq m-1$. Taking wedge powers of (1) as in 1.7(b) we have an exact sequence on $\mathbb{P}^m$:

$$\to \mathcal{O}(-q-1)^{\binom{r-m+q}{q}} \to \cdots \to \mathcal{O}(-2)^{\binom{r+1}{q-1}(r-m+1)}$$

$$\to \mathcal{O}(-1)^{\binom{r+1}{q}} \to \left(\bigwedge^q E_m^r\right)(-1) \to 0.$$

The thesis follows cutting into short exact sequences or by a spectral argument.

Corollary 2.3. *Let E_m^r be a Schwarzenberger bundle as in 2.1.*

$$h^0(E_m^r \otimes E_m^{r\,*}) = 1.$$
$$h^1(E_m^r \otimes E_m^{r\,*}) = m(r+2)(r-m+1) + 1 - (r+1)^2.$$
$$h^2(E_m^r \otimes E_m^{r\,*}) = 0.$$

Hence, Schwarzenberger bundles E_m^r give smooth points in the moduli space M_m^r of all stable bundles with the same rank and Chern classes. They belong to an irreducible component of dimension $m(r+2)(r-m+1)+1-(r+1)^2$.

Proof. We have the exact sequences

(2) $$0 \to E_m^{r\,*} \to \mathcal{O}^{r+1} \to \mathcal{O}(1)^{r-m+1} \to 0,$$

(3) $$0 \to E_m^{r\,*}(-1) \to \mathcal{O}(-1)^{r+1} \to \mathcal{O}^{r-m+1} \to 0,$$

(4) $$0 \to E_m^{r\,*}(-1)^{r-m+1} \to (E_m^{r\,*})^{r+1} \to E_m^r \otimes E_m^{r\,*} \to 0.$$

By (2) and (3) we obtain $H^2(E_m^{r\,*}) = H^2(E_m^{r\,*}(-1)) = H^3(E_m^{r\,*}(-1)) = 0$. Then (4) gives $h^2(E_m^r \otimes E_m^{r\,*}) = 0$ and

$$h^0(E_m^r \otimes E_m^{r\,*}) - h^1(E_m^r \otimes E_m^{r\,*})$$
$$= (r+1)[h^0(E_m^{r\,*}) - h^1(E_m^{r\,*})] - (r-m+1)[h^0(E_m^{r\,*}(-1)) - h^1(E_m^{r\,*}(-1))]$$

(by (2) and (3))

$$= (r+1)[(r+1) - (m+1)(r-m+1)] + (r-m+1)^2$$
$$= (r+1)^2 + (r-m+1)[(r-m+1) - (r+1)(m+1)]$$
$$= (r+1)^2 - (r-m+1)m(r+2).$$

E_m^r is stable, hence simple. This gives the result.

Remark 2.4. In [Sch] the moduli space $\mathcal{M}_m^r$ of Schwarzenberger bundles E_m^r is computed. It coincides with a point when $r = m$ and with $\mathrm{PGL}(m+1)/\mathrm{PGL}(2)$ when $r \geq m+1$. Then when $r \geq m+2$ $\mathcal{M}_m^r \subsetneq M_m^r$. In fact the proof of Theorem 2.2 is still valid if the matrix of the map ϕ in (1) is replaced by an arbitrary matrix with maximum rank.

Definition 2.5 [ST]. Let $k \geq 0$. The exact sequence of sheaves on $\mathbb{P}^{2n+1}$

(5) $$0 \to \mathcal{O}(-1)^k \xrightarrow{\phi} \mathcal{O}^{2n+2k} \to \mathscr{S}_n^k \to 0$$

where the morphism ϕ is given by the matrix

$$\left\{ \begin{bmatrix} x_0 & \cdots & x_n & & & y_0 & \cdots & y_n & \\ & \ddots & & \ddots & & & \ddots & & \ddots \\ & & x_0 & \cdots & x_n & & & y_0 & \cdots & y_n \end{bmatrix} \right.$$

$(x_0, \ldots, x_n, y_0, \ldots, y_n)$ homog. coord. on $\mathbb{P}^{2n+1}$,

defines a $(2n+k)$-bundles $\mathscr{S}_n^k$ on $\mathbb{P}^{2n+1}$. We call a (generalized) Schwarzenberger bundle and we denote by S_n^k any bundle of the form $S_n^k \simeq g^* \mathscr{S}_n^k$ for some $g \in \mathrm{Aut}(\mathbb{P}^{2n+1})$. Often we will use S for S_n^k.

Remark 2.6. On the line $r = \{x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0\}$ we have $\mathscr{S}_n^k|_r \simeq \mathcal{O}(1)^k \oplus \mathcal{O}^{2n}$. It follows from a semicontinuity argument that this is the generic splitting.

Definition-Theorem 2.7. *There exist injective morphisms $\mathcal{O}(-1)^k \to S_n^{k*}$ on $\mathbb{P}^{2n+1}$ such that the quotient E in the sequence*

(6) $$0 \to \mathcal{O}(-1)^k \to S_n^{k*} \to E \to 0$$

is a vector bundle of rank $2n$. *Any* E *constructed in this way is called a special instanton bundle of quantum number* k.

Proof. See [ST2].

The above definitions motivate the following general:

Theorem 2.8. *Let* a, b *be integers,* $0 < a \leq b$. *Let* T, E *be vector bundles on* $\mathbb{P}^m$ *defined by exact sequences*

(i) $$0 \to \mathcal{O}(-1)^b \to \mathcal{O}^{m+a+b-1} \to T \to 0,$$

(ii) $$0 \to \mathcal{O}(-1)^a \to T^* \to E \to 0.$$

Then T *is stable and* E *is simple.*

Proof. In order to prove that T is stable we apply again Theorem 1.2. We have $\mu(T) = b/(m + a - 1)$, so that $\mu(\bigwedge^q T) = qb/(m + a - 1) > 0$. Then $(\bigwedge^q T)_{\text{norm}} = (\bigwedge^q T)(j)$ for some $j \leq -1$. We take wedge powers of (i) as in 1.7(b) and we get

$$0 \to \mathcal{O}(-q-1)^{\binom{b+q-1}{q}} \to \cdots \to \mathcal{O}(-1)^{\binom{m+a+b-1}{q}} \to \left(\bigwedge^q T\right)(-1) \to 0.$$

From this sequence it follows that

(7) $$h^0\left(\bigwedge^q T(-1)\right) = 0 \quad \text{for } q \leq m - 1$$

and then $h^0((\bigwedge^q T)_{\text{norm}}) = 0$ for $q \leq m - 1$. Consider now that

$$\text{rk}(T) = m + a - 1,$$

$$\mu\left(\bigwedge^{m+t} T(-t-1)\right) = (m+t)\mu(T) - t - 1 = \frac{(m+t)b - (m+a-1)(t+1)}{m+a-1}$$

$$= \frac{m(b-t-1) - (a-t-1) + t(b-a)}{m+a-1}$$

$$\geq \frac{(m-1)(a-t-1)}{m+a-1} > 0 \quad \text{for } t \leq a - 2.$$

Hence

$$\left(\bigwedge^{m+t} T\right)_{\text{norm}} = \left(\bigwedge^{m+t} T\right)(j) \quad \text{for } j \leq -t-2, t \leq a - 2.$$

Thus it suffices to prove that

(8) $$H^0\left(\left(\bigwedge^{m+t} T\right)(-t-2)\right) = 0 \quad \text{for } 0 \leq t \leq a - 2.$$

We will show (8) by induction on t. Applying 1.7(a) to the dual of (ii) we get after twisting

$$0 \to \bigwedge^m T(-2) \to \left[\bigwedge^{m-1} T(-1)^a\right] \to \cdots$$

$(\bigwedge^m E^* = 0$ because $\text{rk}(E) = m - 1)$. Using (7) in this sequence we prove (8) for $t = 0$.

In the same way the analogous sequence

$$0 \to \bigwedge^{m+t} T(-t-2) \to \left[\bigwedge^{m+t-1} T(-t-1)\right]^a \to \cdots$$

gives the inductive step. This concludes the proof that T is stable. Now tensor by E the sequence dual to (ii) and obtain

$$0 \to E^* \otimes E \to T \otimes E \to E(1)^a \to 0.$$

Thus $h^0(E^* \otimes E) \le h^0(T \otimes E)$. Now tensor by T the sequence (ii) and get

$$0 \to T(-1)^a \to T \otimes T^* \to E \otimes T \to 0.$$

From (i) we have $h^1(T(-1)) = 0$ and from the first part of the proof it follows $h^0(T \otimes T^*) = 1$. Then the result follows from a simple analysis of the cohomology sequence associated to the above sequence.

Corollary 2.9. *A generalized Schwarzenberger bundle is stable. A special instanton bundle is simple.*

Remark 2.10. The fact that the special instanton bundles are simple was proved by Spindler and Trautmann [ST2] in a different way.

Proposition 2.11. *Let E be a rank $2n$ bundle on $\mathbb{P}^{2n+1}$ such that E has natural cohomology in the range $-2n - 1 \le q \le 0$ and $c_t(E) = (1 - t^2)^{-k}$. Then E is simple.*

Proof. Exactly as in Corollary 1.4 of [OS] we find using Beilinson sequence [OSS] that E is the cohomology of a monad

$$0 \to \mathcal{O}(-1)^c \xrightarrow{\alpha} \mathcal{O}^{2n+a+b} \xrightarrow{\beta} \mathcal{O}(1)^b \to 0$$

for some a, b, $c > 0$. Set $T = (\text{Ker}\,\beta)^*$, then $\text{rk}(T) = 2n + a$, $c_1(T) = b$ and the following sequence is exact:

$$0 \to \mathcal{O}(-1)^c \to T^* \to E \to 0.$$

This implies that $c = b = a = k$ and we are in the hypothesis of Theorem 2.8. This concludes the proof.

Definition 2.12. We define an *instanton bundle* of quantum number k on $\mathbb{P}^{2n+1}$ as a bundle of rank $2n$ satisfying the following properties:
  (i)   the Chern polynomial of E is $c_t(E) = (1 - t^2)^{-k}$,
  (ii) E has natural cohomology in the range $-2n - 1 \le q \le 0$,
  (iii) E has trivial splitting type.

Proposition 2.11 shows that instanton bundles are simple: hence this definition agrees with the one in [ST2]. Spindler and Trautmann have shown [ST2] that a special instanton bundle as in 2.7 is an instanton bundle and is simple.

This definition differs slightly from the one in [OS], where also the property that E is symplectic is assumed.

Definition 2.13. Let c, d, m be positive integers such that $m \le d - c$. A vector bundle S on $\mathbb{P}^m$ arising from an exact sequence

(9) $$0 \to \mathcal{O}(-1)^c \to \mathcal{O}^d \to S \to 0$$

is called a *Schwarzenberger type bundle* (STB).

A Schwarzenberger bundle as in Definition 2.1 or 2.5 is a STB.

Proposition 2.14. *Let S be a stable Schwarzenberger type bundle as in Definition 2.13. Then*

$$h^0(S \otimes S^*) = 1.$$
$$h^1(S \otimes S^*) = 1 - c^2 - d^2 + cd(m + 1).$$
$$h^2(S \otimes S^*) = 0.$$

Hence, STB cannot be stable if $c^2 + d^2 - cd(m + 1) > 1$.

Proof. Exactly as in Corollary 2.3.

Remark 2.15. In [ST2] the moduli space of generalized Schwarzenberger bundles is computed. It is irreducible and of dimension $4n^2 + 8n - 3$ (it is G/H in the notations of [ST2]). By Proposition 2.14 generalized Schwarzenberger bundles give smooth points in the moduli space of all stable bundles with the same rank and Chern classes, and they belong to an irreducible component of dimension $(k - 1)(4n^2 + 4nk - k - 1)$.

Remark 2.16. The property to be a STB is open. In fact, if S is a STB then the only nonzero $h^i(S(-j))$ in the range $0 \le i$, $j \le m$ are $h^0(S) = d$ and $h^{m-1}(S(-m)) = c$, and this property is open. Conversely if a bundle S has $h^i(S(-j))$ exactly as above, the Beilinson sequence [OSS] shows that it is a STB. Now observe that if $k \ge 3$ then $4n^2 + 8n - 3 < (k - 1)(4n^2 + 4nk - k - 1)$. It follows that for $k \ge 3$ there exist stable STB which are not generalized Schwarzenberger bundles.

Proposition 2.17. *Every instanton bundle E on $\mathbb{P}^{2n+1}$ of quantum number k appears as a quotient*

$$(10) \qquad\qquad 0 \to \mathscr{O}(-1)^k \to S^* \to E \to 0$$

where S is a stable Schwarzenberger type bundle arising from an exact sequence

$$(11) \qquad\qquad 0 \to \mathscr{O}(-1)^k \to \mathscr{O}^{2n+2k} \to S \to 0.$$

S is uniquely determined by E and by the sequence (10).

Proof. We refer to Theorem 2.8 and to the proof of Proposition 2.11 for the existence and the stability of S. We underline that the existence of S relies on the Beilinson sequence (see [OS]). Suppose now that there exist two exact sequences

$$0 \to \mathscr{O}(-1)^k \to S^* \xrightarrow{p} E \to 0, \quad 0 \to \mathscr{O}(-1)^k \to S'^* \to E \to 0$$

with S, S' stable STB. As $H^1(S(-1)) = 0$ the morphism p can be lifted to a morphism $S^* \to S'^*$ such the following diagram is commutative:

$$
\begin{array}{ccc}
S^* & \xrightarrow{\ p\ } & E \\[4pt]
\downarrow & & \downarrow \text{id}_E \\[4pt]
S'^* & \longrightarrow & E
\end{array}
$$

Now the stability of S, S' implies that $S \simeq S'$.

In [KO] it is shown that the simple bundles on $\mathbb{P}^m$ have a coarse moduli space. We can show, applying the following criterion, that this moduli space is separated at points corresponding to instanton bundles.

Proposition 2.18 (Kosarew, Okonek [KO, Proposition 6.6]). *Let F, G be non-isomorphic simple vector bundles on $\mathbb{P}^m$. If the points associated to F, G are nonseparated in the moduli space of simple bundles, then there are nontrivial morphisms $\phi \colon F \to G$ and $\psi \colon G \to F$ with $\phi \circ \psi = 0$, $\psi \circ \phi = 0$.*

Proposition 2.19. *Let F, G be a nonisomorphic instanton bundles on $\mathbb{P}^{2n+1}$ with the same quantum number k. Then the associated points in the moduli space of simple bundles are separated.*

Proof. We apply Proposition 2.18. Let us suppose that there exist nontrivial morphisms $\phi \colon F \to G$ and $\psi \colon G \to F$ with $\psi \circ \phi = 0$. We have the exact sequences (see Proposition 2.17)

$$0 \to \mathcal{O}(-1)^k \to S^* \xrightarrow{\gamma} F \to 0, \quad 0 \to \mathcal{O}(-1)^k \to S'^* \xrightarrow{\delta} G \to 0,$$

where S, S' are stable. As $H^1(S(-1)) = 0$ it is possible to lift the morphism $\phi \circ \gamma$ to a nontrivial morphism $S^* \xrightarrow{\alpha} S'^*$ in order that the following diagram commutes:

$$
\begin{array}{ccc}
S^* & \xrightarrow{\gamma} & F \\
\downarrow{\alpha} & & \downarrow{\phi} \\
S'^* & \xrightarrow{\delta} & G
\end{array}
$$

As S, S' are stable, it follows that α is an isomorphism and that $S \simeq S'$. In the same way we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-1)^k & \longrightarrow & S^* & \xrightarrow{\gamma} & F & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\alpha} & & \downarrow{\phi} & & \\
0 & \longrightarrow & \mathcal{O}(-1)^k & \longrightarrow & S^* & \xrightarrow{\delta} & G & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\beta} & & \downarrow{\psi} & & \\
0 & \longrightarrow & \mathcal{O}(-1)^k & \longrightarrow & S^* & \xrightarrow{\gamma} & F & \longrightarrow & 0
\end{array}
$$

We have $\gamma \circ \beta \circ \alpha = \psi \circ \phi \circ \gamma = 0$; then $\beta \circ \alpha$ can be lifted to a nontrivial morphism $S^* \to \mathcal{O}(-1)^k$. This is a contradiction because $h^0(S(-1)) = 0$.
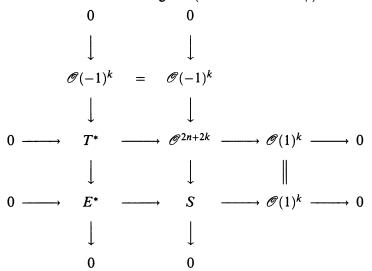
Remark 2.20. The argument of the proof of Proposition 2.19 can be easily adapted in order to show that if F, G are symplectic instanton bundles on $\mathbb{P}^{2n+1}$ with $\mathrm{Hom}(F, G) \neq 0$ then $F \simeq G$.

Proposition 2.21. *Let E be an instanton bundle. Then E^* is an instanton bundle.*

Proof. We have the diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
\mathscr{O}(-1)^k \\
\downarrow \\
\mathscr{O}^{2n+2k} \\
\end{array}
$$

$$0 \longrightarrow E^* \longrightarrow S \xrightarrow{\phi} \mathscr{O}(1)^k \longrightarrow 0$$

ϕ is surjective and induces the diagram (we set $T^* = \operatorname{Ker} \phi$)

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathscr{O}(-1)^k & = & \mathscr{O}(-1)^k & & \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow & T^* & \longrightarrow & \mathscr{O}^{2n+2k} & \longrightarrow & \mathscr{O}(1)^k & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \| & \\
0 \longrightarrow & E^* & \longrightarrow & S & \longrightarrow & \mathscr{O}(1)^k & \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}
$$

From Theorem 2.8 T^* is stable, in particular $h^0(T^*) = 0$. Hence $h^0(E^*) = 0$. From this fact and Serre duality it follows easily that E^* has natural cohomology in the range $-2n - 1 \le q \le 0$ and hence it is an instanton bundle.

Remark 2.22. With the notations of the above proof E is self-dual if and only if $S \simeq T$ (see Proposition 2.17). In the language of monads, if E is the cohomology of the monad $0 \to \mathscr{O}(-1)^k \xrightarrow{A} \mathscr{O}^{2n+2k} \xrightarrow{B} \mathscr{O}(1) \to 0$ then E^* is the cohomology of the dual monad $0 \to \mathscr{O}(-1)^k \xrightarrow{B^t} \mathscr{O}^{2n+2k} \xrightarrow{A^t} \mathscr{O}(1) \to 0$.

Proposition 2.23. *Let E be an instanton bundle. E is symplectic if and only if $h^0(\bigwedge^2 E) \ne 0$ and $h^0(\bigwedge^2 E^*) \ne 0$.*

Proof. The "only if" part follows by definition. So we may suppose that there exist nonzero morphisms $\phi \in H^0(\bigwedge^2 E)$, $\psi \in H^0(\bigwedge^2 E^*)$. They induce nonzero

morphisms $\phi' : E^* \to E$, $\psi' : E \to E^*$. From the proof of Proposition 2.19 it follows that $\phi' \circ \psi' \neq 0$. E is simple (Proposition 2.11), hence ϕ' and ψ' are symplectic isomorphisms.

3. SPECIAL SYMPLECTIC INSTANTON BUNDLES ARE STABLE

We have defined in 2.7 the special instanton bundles.

Theorem 3.1 [ST2, 5.9; ST1, 3.6.1]. *Let E be a special instanton bundle on $\mathbb{P}^{2n+1}$. Suppose moreover that E is symplectic. There exists a one-dimensional family of linear subspaces $\mathbb{P}^{2n-1}$ such that $E|_{\mathbb{P}^{2n-1}} \simeq E' \oplus \mathcal{O}^2$ where E' is a special symplectic instanton bundle on $\mathbb{P}^{2n-1}$.*

For further use we need the following:

Lemma 3.2. *Let S be a STB bundle on $\mathbb{P}^{2n+1}$ (see Definition 2.13). Let $0 \leq q \leq n$, $n \geq 2$. Then*

   (i) $H^0(\bigwedge^q S(-1)) = 0$.
   (ii) $H^1(\bigwedge^q S(-2)) = 0$.
   (iii) $H^2(\bigwedge^q S(-3)) = 0$.
   (iv) $H^0(\bigwedge^q S \otimes S(-1)) = 0$.
   (v) $H^1(\bigwedge^q S \otimes S(-2)) = 0$.

Proof. Taking wedge powers of (9) as in 1.7(b) we get the sequence

$$0 \to \mathcal{O}(-q)^{\binom{c+q-1}{q}} \to \cdots \to \mathcal{O}(-1)^{c\binom{d}{q-1}} \to \mathcal{O}^{\binom{d}{q}} \to \bigwedge^q S \to 0.$$

Tensoring this equation by $\mathcal{O}(-1)$, $\mathcal{O}(-2)$, $\mathcal{O}(-3)$ it is easy to check respectively (i), (ii), (iii). Now tensor the sequence (9) by $\bigwedge^q S(-1)$ and obtain

$$(12) \qquad 0 \to \left[\bigwedge^q S(-2)\right]^k \to \left[\bigwedge^q S(-1)\right]^{2n+2k} \to \bigwedge^q S \otimes S(-1) \to 0.$$

Now the cohomology sequence associated to (12) together with (i) and (ii) give (iv). Finally (v) follows using (ii) and (iii) in the sequence obtained by tensoring (12) by $\mathcal{O}(-1)$.

Lemma 3.3. *Let S be a STB bundle and let E be an instanton bundle as in 2.17. Then for $0 \leq q \leq n$, $n \geq 2$,*

   (i) $H^0(\bigwedge^q S \otimes E^*(-1)) = 0$.
   (ii) $H^1(\bigwedge^q S \otimes E^*(-2)) = 0$.

Proof. Tensor the sequence dual to (10) by $\bigwedge^q S(-1)$ and get

$$(13) \qquad 0 \to \bigwedge^q S \otimes E^*(-1) \to \bigwedge^q S \otimes S(-1) \to \left[\bigwedge^q S\right]^k \to 0.$$

Now (iv) of Lemma 3.2 gives (i). Tensoring (13) by $\mathcal{O}(-1)$, (i) and (v) of Lemma 3.2 give (ii).

Lemma 3.4. *Let E be an instanton bundle. For $0 \leq q \leq n$, $n \geq 2$, the following hold:*

   (i) $H^0(E) = 0$.
   (ii) $H^0(\bigwedge^q E^*(-1)) = 0$.
   (iii) $H^1(\bigwedge^q E^*(-2)) = 0$.
   (iv) $H^0(E^* \otimes \bigwedge^q E^*(-1)) = 0$.
   (v) $H^1(E^* \otimes \bigwedge^q E^*(-2)) = 0$.

If E is symplectic then $\bigwedge^q E^* \simeq \bigwedge^{2n-q} E^*$ and the vanishing above hold for
$0 \leq q \leq 2n$.

Proof. (i) follows from (10) because $h^0(S^*) = 0$ (S is stable from Proposition
2.17). Using Fact 1.8, (ii) and (iii) follow respectively from (iv) and (v). Now
consider the wedge power as in 1.7(a) of the dual of (10)

$$0 \to \bigwedge^q E^* \to \bigwedge^q S \to \left[\bigwedge^{q-1} S(1)\right]^k \to \cdots.$$

Tensoring this sequence by $E^*(-1)$ we have

$$(14) \quad 0 \to \bigwedge^q E^* \otimes E^*(-1) \to \bigwedge^q S \otimes E^*(-1) \to \left[\bigwedge^{q-1} S \otimes E^*\right]^k \to \cdots.$$

The sequence (14) and (i) of Lemma 3.3 give (iv). Tensoring (14) by $\mathscr{O}(-1)$,
(i) and (ii) of Lemma 3.3 give (v).

We are now going to prove the stability of a special symplectic instanton
bundle. It will be a consequence of the following general statement:

Theorem 3.5. *Let E be a symplectic vector bundle on a projective manifold X
with $\mathrm{Pic}(X) = \mathbb{Z}$ such that for q odd, $1 \leq q \leq \frac{1}{2}\mathrm{rk}(E)$ the following hold:*

(i) $h^0(\bigwedge^q E) = 0$.
(ii) $h^0(\bigwedge^q E \otimes E) = 1$.

Then E is stable.

Proof. (i) implies that no coherent subsheaf of odd rank destabilizes E (see,
e.g., the proof of Lemma 2.6 in [H]). Let us suppose that there is a destabilizing
subsheaf Z of E of rank $2t$. Furthermore, E is semistable by Remark 1.3,
then $c_1(Z) = 0$.

We have an exact sequence

$$(15) \qquad\qquad 0 \to Z \to E \xrightarrow{p} Q \to 0$$

where we may suppose Q torsion-free [OSS]. We have also nonzero morphisms
$f: \mathscr{O} \simeq (\bigwedge^{2t} Z)^{**} \to \bigwedge^{2t} E$ and $g: \bigwedge^{2t} E \to \bigwedge^{2t-1} E \otimes Q$ given locally by

$$e_1 \wedge \cdots \wedge e_{2t} \mapsto \frac{1}{2t} \sum_{i=1}^{2t} (-1)^i (e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{2t}) \otimes p(e_i).$$

Consider now the sequence

$$\mathscr{O} \xrightarrow{f} \bigwedge^{2t} E \xrightarrow{g} \bigwedge^{2t-1} E \otimes Q.$$
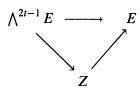
On the open set where Q is locally free this sequence is the first part of 1.7(a)
and it is exact. As $\bigwedge^{2t-1} E \otimes Q$ is torsion-free, this implies that $g \circ f = 0$
Tensoring (15) by $\bigwedge^{2t-1} E$ we get

$$(16) \qquad 0 \to Z \otimes \bigwedge^{2t-1} E \to E \otimes \bigwedge^{2t-1} E \xrightarrow{h} \bigwedge^{2t-1} E \otimes Q \to 0.$$

Fact 1.8 implies that $\bigwedge^{2t} E$ is a direct summand of $\bigwedge^{2t-1} E \otimes E$; it is easy
to check that $h|_{\bigwedge^{2t} E} = g$. From Lemma 1.10 it follows that $\bigwedge^{2t} E \simeq \mathscr{O} \oplus$

$B_t \subset E \otimes \bigwedge^{2t-1} E$, and by hypothesis (ii) $h^0(B_t) = 0$. Then $f(\mathscr{O}) \subset \mathscr{O}$ and $g(\mathscr{O}) = 0$: it follows that the direct summand $\mathscr{O} \subset E \otimes \bigwedge^{2t-1} E$ goes to zero with h in (16). Looking at the cohomology sequence associated to (16) we get that the morphism $\bigwedge^{2t-1} E \longrightarrow E$ corresponding to $\mathscr{O} \subset \bigwedge^{2t} E \subset E \otimes \bigwedge^{2t-1} E \simeq$ $Hom(\bigwedge^{2t-1} E, E)$ goes to zero with $H^0(h)$ and then factors in a diagram

$$\bigwedge^{2t-1} E \longrightarrow E$$
$$\searrow \quad \nearrow$$
$$Z$$

This is a contradiction because by hypothesis (ii) the only (up to a constant) nonzero morphism $\bigwedge^{2t-1} E \to E$ is the projection on the direct summand (see Lemma 1.10), which is surjective.

Theorem 3.6. *Let E be an instanton bundle on $\mathbb{P}^5$. Then E is stable.*

Proof. If E is symplectic the statement follows from Theorem 3.5. In any case we have $\bigwedge^2 E \simeq \bigwedge^2 E^*$ (rank $E = 4$) and $\bigwedge^3 E \simeq E^*$. If E is not symplectic from Proposition 2.23 it follows that $h^0(\bigwedge^2 E) = 0$. E^* is an instanton bundle (Proposition 2.21), then $h^0(E^*) = 0$ from (i) of Lemma 3.4. We conclude using Hoppe criterion 1.2.

Theorem 3.7. *Let E be a special symplectic instanton bundle on $\mathbb{P}^{2n+1}$. Then*

(i) $h^0(\bigwedge^{2j+1} E) = 0$ $\forall j : 0 \leq j \leq n - 1$.
(ii) $h^0(\bigwedge^{2j+1} E \otimes E) = 1$ $\forall j : 0 \leq j \leq n - 1$.
(iii) $\bigwedge^{2j+2} E = \mathscr{O} \oplus B_{j+1}$ with $h^0(B_j) = 0$ $\forall j : 0 \leq j \leq n - 1$.
(iv) E is stable.
(v) $h^0(\bigwedge^{2j} E \otimes E) = 0$ $\forall j : 0 \leq j \leq n - 1$.

Proof. We prove all these statements together by induction on n (they are trivial for $n = 1$). The cases $j = 0$ and $j = n - 1$ of (i), (ii), (iii), (v) follow easily from Lemma 1.10, Remark 2.10, and Lemma 3.4(i). Let $H \simeq \mathbb{P}^{2n-1}$ as in Theorem 3.1. We have the exact sequence

$$0 \to \bigwedge^{2j+1} E(-2) \to \left[\bigwedge^{2j+1} E(-1) \right]^2 \to \bigwedge^{2j+1} E \to \bigwedge^{2j+1} E|_H \to 0$$

By Lemma 3.4(ii) and (iii), the restriction map $H^0(\bigwedge^{2j+1} E) \to H^0(\bigwedge^{2j+1} E|_H)$ is injective. Lemma 1.10 implies $\bigwedge^{2j+1} E \simeq E \oplus C_j$ and $E|_H \simeq E' \oplus \mathscr{O}^2$, with E' special symplectic instanton bundle on H, follows from Theorem 3.1. Then

$$E' \oplus \mathscr{O}^2 \oplus C_j|_H \simeq [E \oplus C_j]|_H \simeq \bigwedge^{2j+1} E|_H \simeq \bigwedge^{2j+1} E' \oplus \left[\bigwedge^{2j} E' \right]^2 \oplus \bigwedge^{2j-1} E'.$$

By the inductive hypothesis (we may suppose $0 < j < n-1$) $\bigwedge^{2j+1} E' = E' \oplus C'_j$, $\bigwedge^{2j-1} E' = E' \oplus C'_{j-1}$, $\bigwedge^{2j} E' = \mathscr{O} \oplus B'_j$ with $h^0(B'_j) = 0$, $h^0(\bigwedge^{2j+1} E') = h^0(\bigwedge^{2j-1} E') = 0$. This implies $h^0(C_j|_H) = h^0([B'_j]^2 \oplus E' \oplus C'_j \oplus C'_{j-1}) = 0$. Thus $h^0(C_j) = 0$ and this proves (i).

Now $\bigwedge^{2j+1} E \otimes E \simeq (E \otimes E) \oplus (C_j \otimes E)$ and by 2.10 in order to prove (ii) it suffices to prove $h^0(C_j \otimes E) = \dim \mathrm{Hom}(E, C_j) = 0$. Consider a morphism $\phi \colon E \to C_j$ and its restriction

$$\phi_H : E' \oplus \mathcal{O}^2 \simeq E|_H \longrightarrow C_j|_H \simeq [B_j']^2 \oplus E' \oplus C_j' \oplus C_{j-1}'.$$

By Lemma 3.4(iv) and (v) the restriction map $H^0(E \otimes C_j) \to H^0(E \otimes C_j|_H)$ is injective, so it suffices to prove that $\phi_H = 0$.

By the inductive hypothesis (ii) and (v), we have

$$h^0\left(\bigwedge^{2j-1} E' \otimes E'\right) = h^0\left(\bigwedge^{2j+1} E' \otimes E'\right) = 1, \quad h^0\left(\bigwedge^{2j} E' \otimes E'\right) = 0.$$

Then $H^0(C_j' \otimes E') = \mathrm{Hom}(E', C_j') = 0$, $H^0(C_{j-1}' \otimes E') = \mathrm{Hom}(E', C_{j-1}') = 0$, $H^0(B_j' \otimes E') = \mathrm{Hom}(E', B_j') = 0$. Thus $\phi_H(E') \subset E'$ and $\phi_H(\mathcal{O}^2) = 0$. If $\phi_H \neq 0$ then the composition $\phi_H^* \circ \phi_H$ would be different from zero (because E' is simple). Thus $\phi^* \circ \phi \neq 0$ and, as E is simple, E would be a direct summand of C_j. In this case $\mathcal{O}_H^2$ would be a direct summand of $C_j|_H$ and this is a contradiction because $h^0(C_j|_H) = 0$. This proves (ii).

(iii) is now a trivial consequence of (ii) and of Lemma 1.10.

(iv) is a consequence of (i), (ii) and of Theorem 3.5.

Then, by Corollary 1.6, $\bigwedge^{2j} E$ is a direct sum of stable bundles with $c_1 = 0$. Lemma 1.4 implies that any morphism $E \to \bigwedge^{2j} E \simeq \mathcal{O} \oplus B_j$ is either an isomorphism on a direct summand or zero. E cannot be a direct summand of B_j because $h^0(E|_H) = 2$ and $h^0(B_j'|_H) = 1$. This proves (v).

Theorem 3.8 (Ein, Spindler, Trautmann). *There exists a coarse irreducible moduli space for special instanton bundles on $\mathbb{P}^{2n+1}$ of quantum number k. Its dimension is $2n^2 + 3n$ for $k = 1$ and $2nk + 4(n+1)^2 - 7$ for $k \geq 2$.*

Proof. For $k = 1$ we have the nullcorrelation bundles (see [E]). For $k \geq 2$ the moduli space is a quotient of $G \times F$ in the notations of [ST2, Theorem 6.3].

Remark 3.9. For k odd the moduli space is fine. Anyway, for k even there is always a universal family over $G \times F$ [ST2, §8].

Corollary 3.10. *There exists a dense open subset of the coarse moduli space M of special instanton bundles such that the corresponding bundles are stable.*

Proof. By Theorem 3.7 there exists at least one special instanton bundle which is stable. Then use that stability is an open property [M1].

Corollary 3.11. *Let E be an instanton bundle on $\mathbb{P}^{2n+1}$ of quantum number k. Then*
$$h^0(E \otimes E^*) = 1,$$
$$h^1(E \otimes E^*) - h^2(E \otimes E^*) = 1 - k^2 + 8n^2k - 4n^2 + 3nk^2 - 2n^2k^2.$$

Proof. The result follows from the sequences (10), (11), and

$$0 \longrightarrow S(-1)^k \longrightarrow S^* \otimes S \longrightarrow E \otimes S \longrightarrow 0,$$
$$0 \longrightarrow E \otimes E^* \longrightarrow E \otimes S \longrightarrow E(1)^k \longrightarrow 0.$$

Corollary 3.12. *If $n = 2$, $k \geq 8$, or $n = 3$, $k \geq 5$ or $n = 4, 5, 6$, $k \geq 4$ or $n \geq 7$, $k \geq 3$, then for every instanton bundle E on $\mathbb{P}^{2n+1}$ of quantum number k we have $H^2(E \otimes E^*) \neq 0$.*

Proof. The inequalities in the statement are exactly the positive integral solutions $(n \geq 2)$ to

$$1 - k^2 + 8n^2k - 4n^2 + 3nk^2 - 2n^2k^2 < 2nk + 4(n+1)^2 - 7$$

(see Theorem 3.8 and Corollary 3.11).

Theorem 3.13. *Let E be an instanton bundle and E^* be its dual (see Proposition 2.21). They are constructed from the sequences:*

$$0 \to \mathcal{O}(-1)^k \to S^* \to E \to 0, \quad 0 \to \mathcal{O}(-1)^k \to T^* \to E^* \to 0$$

$$(S, T \text{ are } STB).$$

Then

$$H^2(E \otimes E^*) \simeq H^2(S^* \otimes E^*) \simeq H^2(S^* \otimes T^*).$$

If E is self-dual we have in particular $H^2(E \otimes E^*) \simeq H^2(S^* \otimes S^*)$, so that $h^2(E \otimes E^*)$ depends only on S.

Proof. Tensoring the sequence defining E by E^* we get

$$0 \to E^*(-1)^k \to S^* \otimes E^* \to E \otimes E^* \to 0.$$

It is easy to check

$$H^2(E^*(-1)) = H^3(E^*(-1)) = 0.$$

Hence it follows $H^2(E \otimes E^*) \simeq H^2(S^* \otimes E^*)$. Tensoring the sequence defining E^* by S^* we get

$$0 \to S^*(-1)^k \to S^* \otimes T^* \to S^* \otimes E^* \to 0.$$

It is easy to check

$$H^2(S^*(-1)) = H^3(S^*(-1)) = 0.$$

Hence it follows $H^2(S^* \otimes E^*) \simeq H^2(S^* \otimes T^*)$.

Theorem 3.14. *Let E be an instanton bundle on $\mathbb{P}^{2n+1}$ of quantum number $k = 2$ defined from the sequence $0 \to \mathcal{O}(-1)^2 \to S^* \to E \to 0$. We have*

$$h^0(S^*(1)) = 2n + 2, \quad h^1(S^*(1)) = 0, \quad h^0(E(1)) = 2n,$$
$$h^1(E \otimes E^*) = 4n^2 + 12n - 3, \quad h^2(E \otimes E^*) = 0.$$

In particular the moduli space of stable instanton bundle with $k = 2$ is smooth of dimension $4n^2 + 12n - 3$.

Proof. We have $\text{rank}(S) = 2n + 2$, $c_1(S) = 2$, $S^*(1) \simeq \bigwedge^{2n+1} S(-1)$. As an application of Fact 1.7(b) we have

$$0 \to \mathcal{O}(-2n-2)^{2n+2} \to \cdots \to \mathcal{O}(-1)^{\binom{2n+4}{2n+1}} \to \bigwedge^{2n+1} S(-1) \to 0.$$

From this sequence it follows that $h^1(S^*(1)) = h^1(\bigwedge^{2n+1} S(-1)) = 0$. Then it is easy to check that $h^0(S^*(1)) = 2n + 2$ and $h^0(E(1)) = 2n$. Tensoring the sequence $0 \to E^* \to S \to \mathcal{O}(1)^2 \to 0$ by S^* we get $0 \to S^* \otimes E^* \to S^* \otimes S \to S^*(1)^2 \to 0$. It follows from Proposition 2.14 that $h^2(S^* \otimes S) = 0$ and from the first part of the proof that $h^1(S^*(1)) = 0$. From the last sequence we obtain $h^2(S^* \otimes E^*) = 0$. From Theorem 3.13 and Corollary 3.11 the result follows.

Remark 3.15. A further analysis (see [AO]) shows that the generic special instanton bundle E on $\mathbb{P}^5$ with $k = 3$ satisfies $h^1(E \otimes E^*) = 54$, $h^2(E \otimes E^*) = 0$, while if E is a symplectic special instanton bundle on $\mathbb{P}^5$ with $k = 3$ we have $h^1(E \otimes E^*) = 57$, $h^2(E \otimes E^*) = 3$. Let $M = M(\mathbb{P}^5; 0, 3, 0, 6)$ be the moduli space of stable 4-bundles on $\mathbb{P}^5$ with Chern classes $c_1 = 0$, $c_2 = 3$, $c_3 = 0$, $c_4 = 6$. It follows that the component of M containing special instanton bundles is reduced but singular and has dimension 54 (compare with the dimension of the moduli space of special instanton bundles which is 41 by Theorem 3.8).

Corollary 3.16. *Let* $X \xrightarrow{f} \mathbb{P}^{2n+1}$ *be a cyclic covering of* $\mathbb{P}^{2n+1}$ *with* $\mathrm{Pic}(X) = \mathbb{Z}$ *and let* E *be a special symplectic instanton bundle on* $\mathbb{P}^{2n+1}$. *Then* f^*E *is stable.*

Proof. We can apply Theorem 3.5 because f^*E is symplectic. We have $f_*\mathscr{O} \simeq \bigoplus_{i=0}^{p}\mathscr{O}(-ai)$ for some a, $p > 0$ (see, e.g., [BPV, Lemma I.17.2]). Then for q odd we have

$$h^0\left(X, \bigwedge^q f^*E\right) = h^0\left(\mathbb{P}^{2n+1}, \bigwedge^q E \otimes f_*\mathscr{O}\right) = \sum_{i=0}^{p} h^0\left(\mathbb{P}^{2n+1}, \bigwedge^q E(-ai)\right) = 0.$$

In the same way $h^0(\bigwedge^q f^*E \otimes f^*E) = 1$.

Corollary 3.17. *Let* n *be odd and* Q_n *be the* n-*dimensional smooth quadric. Let* $f : \mathbb{P}^n \to \mathbb{P}^n$ *or* $f : Q_n \to \mathbb{P}^n$ *be a finite morphism. Then the pullback* f^*E *of a special symplectic instanton bundle is stable.*

Proof. The same proof of Corollary 3.16 works because in both cases $f_*\mathscr{O} \simeq \bigoplus_{i=0}^{m}\mathscr{O}(a_i)$ with $a_i < 0$ except $a_0 = 0$.

REFERENCES

[AO]   V. Ancona and G. Ottaviani, *On moduli of instanton bundles on* $\mathbb{P}^{2n+1}$, Pacific J. Math. (to appear).

[B]   N. Bourbaki, *Éléments de mathématique. Algèbre,* Livre II, Springer-Verlag, Hermann, Paris, 1959.

[BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces,* Springer-Verlag, Berlin, Heidelberg, and New York, 1984.

[BS]   G. Bohnhorst and H. Spindler, *The stability of certain vector bundles on* $\mathbb{P}^n$, (Proc. Bayreuth Conference), Lecture Notes in Math., vol. 1507, Springer-Verlag, Berlin, Heidelberg, and New York, 1992, pp. 39–50.

[BT]   W. Böhmer and G. Trautmann, *Special instanton bundles and Poncelet curves* (Proc. Lambrecht Conf.), Lecture Notes in Math., vol. 1273, Springer-Verlag, Berlin, Heidelberg, and New York, 1987, pp. 325–336.

[D]   S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles,* Proc. London Math. Soc. (3) **50** (1985), 1–26.

[E]   L. Ein, *Some stable vector bundles on* $\mathbb{P}^4$ *and* $\mathbb{P}^5$, J. Reine Angew. Math. **337** (1982), 142–153.

[H]   H. J. Hoppe, *Generischer spaltungstyp und zweite Chernklasse stabiler Vektorraumbündel vom rang 4 auf* $\mathbb{P}^4$, Math. Z. **187** (1984), 345–360.

[HN]   A. Hirschowitz and M. S. Narasimhan, *Fibrés de 't Hooft spéciaux et applications* (Proc. Nice Conf. 1981), Enumerative Geometry and Classical Algebraic Geometry (P. LeBarz and Y. Hervier, eds.), Birkhäuser, Basel and Boston, Mass., 1982, pp. 143–163.

[K]   S. Kobayashi, *Curvature and stability of vector bundles,* Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), 158–162.

[KO] S. Kosarew and C. Okonek, *Global moduli spaces and simple holomorphic vector bundles*, Publ. Research Inst. Math. Sci. **25** (1989), 1–19.

[Lü] M. Lübke, *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math. **42** (1983), 245–257.

[M1] M. Maruyama, *Openness of a family of torsion-free sheaves*, J. Math. Kyoto Univ. **16** (1976), 627–637.

[M2] _____, *The theorem of Grauert-Mülich-Spindler*, Math. Ann. **255** (1981), 317–333.

[OSS] C. Okonek and H. Spindler, *Mathematical instanton bundles on* $\mathbb{P}^{2n+1}$, J. Reine Angew. Math. **364** (1986), 35–50.

[OS] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, Basel and Boston, Mass., 1980.

[Sal] S. M. Salamon, *Quaternionic structures and twistor spaces*, Global Riemannian Geometry (N. J. Willmore and T. J. Hitchin, eds.), Ellis Horwood, London, 1984.

[Sch] R. L. E. Schwarzenberger, *Vector bundles on the projective plane*, Proc. London Math. Soc. **11** (1961), 623–640.

[ST1] H. Spindler and G. Trautmann, *Rational normal curves and the geometry of special instanton bundles on* $\mathbb{P}^{2n+1}$, Math. Gottingensis **18** (1987).

[ST2] _____, *Special instanton bundles on* $\mathbb{P}^{2n+1}$, *their geometry and their moduli*, Math. Ann. **286** (1990), 559–592.

[T] H. Tango, *On* $(n-1)$*-dimensional projective spaces contained in the Grassmann variety* $\mathrm{Gr}(n, 1)$, J. Math. Kyoto Univ. **14** (1974), 415–460.

[U] H. Umemura, *On a theorem of Ramanan*, Nagoya Math. J. **69** (1978), 131–138.

[UY] K. Uhlenbeck and S. T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Pure Appl. Math. **39** (1986), S257–S293.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI FIRENZE, I-50134 FIRENZE, ITALY
E-mail address: vancona@vm.idg.fi.cnr.it

DIPARTIMENTO DI MATEMATICA, II UNIVERSITÀ, I-00133 ROMA, ITALY
Current address: Dipartimento di Matematica Applicata, Università de Firenze, 50139 Firenze, Italy
E-mail address: ottaviani@ingfi1.ing.unifi.it