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UNIQUENESS THEOREMS FOR PARAMETRIZED ALGEBRAIC CURVES

PETER HALL

ABSTRACT. Let $L_1, \dots, L_n$ be lines in $\mathbb{P}^2$ and let $f, g\colon \mathbb{P}^1 \to \mathbb{P}^2$ be nonconstant algebraic maps. For certain configurations of lines $L_1, \dots, L_n$, the hypothesis that, for $i = 1, \dots, n$, the inverse images $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are equal, not necessarily with the same multiplicities, implies that f is identically equal to g.

INTRODUCTION

The subject of this paper derives from Nevanlinna's "five-point uniqueness theorem" of 1926:

Theorem (Nevanlinna [7, 8]). *Let $f, g\colon \mathbb{C} \to \mathbb{CP}^1$ be nonconstant meromorphic functions and let $a_1, \dots, a_5$ be five distinct points of $\mathbb{CP}^1$ such that, for $i = 1, \dots, 5$, the inverse images $f^{-1}(a_i)$ and $g^{-1}(a_i)$ are the same set, but not necessarily with the same multiplicities. Then $f \equiv g$.*

A striking feature of this theorem is that the multiplicities are left out of the hypothesis.

Value-distribution theory had its beginning in the attempt to find analogues for entire functions of theorems on polynomials [6]. Thus, for example, product representations of an entire function were investigated as analogues of the fundamental theorem of algebra. In contrast, later results in value-distribution theory have sometimes appeared before the corresponding results in algebra. An example is the theorem due to Nevanlinna that we have just stated. Its analogue for rational functions was published in 1971:

Theorem (Adams-Straus [1, remark before Theorem 3]). *Let $f, g\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be nonconstant rational functions and let $a_1, \dots, a_4$ be four distinct points of the image $\mathbb{CP}^1$ such that, for $i = 1, \dots, 4$, the inverse images $f^{-1}(a_i)$ and $g^{-1}(a_i)$ are the same set, but not necessarily with the same multiplicities. Then $f \equiv g$.*

Pizer [9] gives an example to show that three points do not suffice and states two unsolved problems.

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One may ask for generalizations of Nevanlinna's theorem to higher dimensions. The present author [3] has published some theorems of this kind for holomorphic curves in the projective plane $\mathbb{CP}^2$. It appeared, in that investigation, that it might be more reasonable to tackle the problem for algebraic curves first.

The algebraic curves discussed in this paper are *parametrized* algebraic curves, or curves as maps, as opposed to the curves as subvarieties which are studied almost exclusively in the literature. Our results are valid over any algebraically closed field of characteristic zero. In the projective plane $\mathbb{P}^2$, consider a set of lines $L_1, \dots, L_n$. We make the somewhat strong general position requirement that there are no algebraic relations of genus 0 among $L_1, \dots, L_n$. Let $f, g: \mathbb{P}^1 \to \mathbb{P}^2$ be parametrized algebraic curves. The theorems of this paper give three different hypotheses that each imply $f \equiv g$. In Theorem 1, due to Fujimoto, we take $n = 4$; we assume that the image of f does not lie in a line and that, for each i, the divisors $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are equal. In Theorem 2 we take $n = 26$ and assume that $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are the same set, but possibly with different multiplicities. In Theorem 3 we take $n = 12$ and assume that the divisors $\min(f^{-1}(L_i), 2)$ and $\min(g^{-1}(L_i), 2)$ are equal. There is no reason to think that the values of n in Theorems 2 and 3 are the best possible. It is likely that lower values could be obtained by making a more elaborate analysis of singularities.

The proofs of Theorems 2 and 3 are arguments by contradiction involving the degree of an algebraic curve. Use is made of a construction introduced in our paper on uniqueness theorems for holomorphic curves [3]. If $f, g: \mathbb{P}^1 \to \mathbb{P}^2$ are parametrized algebraic curves, then for each z in $\mathbb{P}^1$ such that $f(z) \neq g(z)$ there is a point $f \wedge g(z)$ in the dual plane $\mathbb{P}^{2*}$ defined by the Plücker coordinates of the line through $f(z)$ and $g(z)$. This defines a map $f \wedge g: \mathbb{P}^1 \to \mathbb{P}^{2*}$ that is used to estimate the number of z in $\mathbb{P}^1$ for which $f(z)$ and $g(z)$ both lie on a given line L_i. A separate estimate is made of the multiple intersections of f and g with the lines L_i, and this is what allows us to leave the multiplicities out of the hypothesis. In each of these estimate we regard the image of a parametrized curve $c: \mathbb{P}^1 \to \mathbb{P}^2$ as a subvariety and apply the Plücker formulas. The idea is that, if the map c passes a large number of times through the point p of $\mathbb{P}^2$, then its image has a singularity of large multiplicity at p.

1. Parametrized algebraic curves

Throughout this paper we work over a fixed algebraically closed field K of characteristic 0. We write $\mathbb{P}^n$ for the projective space of dimension n over K.

If $\varphi: \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function, then, using an inhomogeneous coordinate, we may write φ as the quotient of two polynomials with no common factor. The *degree* $\deg \varphi$ is the greater of the degrees of these polynomials. If $\deg \varphi > 0$ then we define the *multiplicity* of φ at a point z_0 in the domain $\mathbb{P}^1$ to be the integer $\mu = \mu(z_0) \geq 1$ such that

$$\varphi(z) = (z - z_0)^\mu + \text{ higher order terms}$$

in some system of local coordinates about $\varphi(z_0)$. Then we have the *Riemann-*

Hurwitz formula [5]

$$\sum_{z\in\mathbb{P}^1}(\mu(z)-1)=2\deg\varphi-2. \tag{1}$$

A *parametrized algebraic curve* is an algebraic map $f: F\to V$, where F is a Riemann surface and V is an algebraic variety. We shall always take F to be the projective line $\mathbb{P}^1$ and V to be the projective plane $\mathbb{P}^2$. We emphasize that this paper studies curves as maps rather than as subvarieties.

Let $f: \mathbb{P}^1\to\mathbb{P}^2$ be a parametrized algebraic curve and let z be an inhomogeneous coordinate on $\mathbb{P}^1$. Then we may write f in the form

$$f(z)=(\varphi_0(z), \varphi_1(z), \varphi_2(z)), \tag{2}$$

where $\varphi_0, \varphi_1, \varphi_2$ are polynomials with no common factor. We call equation (2) a *reduced polynomial representation* for f. The *degree* deg f is defined to be the greatest of the degrees of $\varphi_0, \varphi_1, \varphi_2$. There exists a rational function $f_1: \mathbb{P}^1\to\mathbb{P}^1$ and a parametrized algebraic curve $f_2: \mathbb{P}^1\to\mathbb{P}^2$ such that $f=f_2\circ f_1$ and f_2 cannot be factored through a rational function of degree greater than 1. The degrees are related by

$$\deg f=\deg f_1\times\deg f_2. \tag{3}$$

If $\gamma: \mathbb{P}^1\to\mathbb{P}^2$ is a nonconstant parametrized algebraic curve that cannot be factored through a rational function of degree greater than 1, then we may regard γ as a subvariety of $\mathbb{P}^2$ and apply the classical theory of plane algebraic curves. Corresponding to each point x_0 of $\mathbb{P}^1$ there is a *place* of γ with *centre* $\gamma(x_0)$, that is, a local parametrization by formal power series [10, p. 116]. If γ is not a line, it is always possible to choose homogeneous coordinates on $\mathbb{P}^2$ so that the place of γ corresponding to x is of the form

$$\gamma(x)=(1, (x-x_0)^r, (x-x_0)^{r+s}+\text{higher order terms}). \tag{4}$$

Here $r=r(x_0)\geq 1$ is an integer called the *order* and $s=s(x_0)\geq 1$ is an integer called the *class*.

For each point x_0 of $\mathbb{P}^1$ there is a well-defined *tangent line to* γ *at* x_0. In the coordinates used for the representation (4), the tangent line consists of those points of $\mathbb{P}^2$ that have the third coordinate equal to zero. We introduce the *dual plane* $\mathbb{P}^{2*}$ [4, 5, 10], the points of which are in bijection with the lines of $\mathbb{P}^2$. The *dual curve* $\gamma^*: \mathbb{P}^1\to\mathbb{P}^{2*}$ takes each x_0 in $\mathbb{P}^1$ to the point of $\mathbb{P}^{2*}$ corresponding to the tangent line to γ at x_0.

Let $p=(p_0, p_1, p_2)$ and $q=(q_0, q_1, q_2)$ be distinct points of $\mathbb{P}^2$. The point of $\mathbb{P}^{2*}$ corresponding to the line joining p and q is written $p\wedge q$ and its *Plücker coordinates* are

$$p\wedge q=(p_1q_2-p_2q_1, p_2q_0-p_0q_2, p_0q_1-p_1q_0). \tag{5}$$

We shall need the Plücker formulas and the genus formula for rational curves with arbitrary singularities. There is an extensive treatment in Hensel and Landsberg [4]; a more modern reference is Iitaka [5]. Not all authors use the same nomenclature for these formulas.

For a curve γ of genus 0 the *first and second Plücker formulas* are

$$\deg\gamma^*+\sum(f(x)-1)=2\deg\gamma-2, \tag{6}$$

$$2\sum(r(x)-1)+\sum(s(x)-1)=3\deg\gamma-6, \tag{7}$$

where the sums are taken over all places x of γ. To state the genus formula, we assign a multiplicity to each point on γ. If p is a point of $\mathbb{P}^2$ that lies on γ, the multiplicity $m(p)$ is the sum of $r(x)$ over all the places x of γ with centre at p. To calculate the genus it is necessary in general to include the *infinitely near points* that arise in the resolution of singularities. For the present purpose these are irrelevant, and we shall use the *genus formula* for r in the form of the inequality

$$\sum m(p)(m(p)-1)\le(\deg\gamma-1)(\deg\gamma-2), \tag{8}$$

where the sum is taken over all points p of $\mathbb{P}^2$ that lie on γ. In the proofs of Theorems 2 and 3 we shall need the following lemma based on the genus formula.

Lemma. *Let γ be a rational algebraic curve in $\mathbb{P}^2$ with $\deg\gamma\ge 2$ (considered as a subvariety) and let $p_1, \dots, p_n$ be distinct points on γ. Then*

$$\sum_{i=1}^{n}(m(p_i)-1)<n^{1/2}(\deg\gamma-1), \tag{9}$$

where $m(p_i)$ is the multiplicity of γ at p_i.

Proof. By the genus formula (8) we have

$$\sum_{i=1}^{n}m(p_i)(m(p_i)-1)\le(\deg\gamma-1)(\deg\gamma-2).$$

Consider all vectors $(x_1, \dots, x_n)$ of real numbers such that

$$\sum_{i=1}^{n}x_i(x_i-1)\le(\deg\gamma-1)(\deg\gamma-2). \tag{10}$$

The greatest value of $\sum_{i=1}^{n}(x_i-1)$ occurs when $x_1=x_2=\cdots=x_n$. In this case (10) becomes

$$nx_1(x_1-1)\le(\deg\gamma-1)(\deg\gamma-2),$$

so that $n(x_1-1)^2<(\deg\gamma-1)^2$ and $x_1-1<n^{-1/2}(\deg\gamma-1)$. It follows that

$$\sum_{i=1}^{n}(x_i-1)=n(x_1-1)<n^{1/2}(\deg\gamma-1),$$

which implies the lemma.

2. The uniqueness problem

Let $f, g\colon \mathbb{P}^1\to\mathbb{P}^2$ be nonconstant parametrized algebraic curves and let $L_1, \dots, L_n$ be distinct lines in $\mathbb{P}^2$. If we assume that, for $i=1, \dots, n$, the inverse images $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are "the same," in some sense, does it follow that f is identically equal to g? We shall consider three different senses in which f and g may be "the same." First, $f^{-1}(L_i)$ may be the same set as $g^{-1}(L_i)$, ignoring questions of multiplicity. This is the hypothesis treated in Theorem 2. Secondly, $f^{-1}(L_i)$ and $g^{-1}(L_i)$ may be equal as divisors on

$\mathbb{P}^1$, which we express by saying that $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are *equal, counting multiplicities.* This is treated in Theorem 1, due to Fujimoto. Thirdly, the truncated divisors $\min(f^{-1}(L_i), 2)$ and $\min(g^{-1}(L_i), 2)$ may be equal, which we express by saying that $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are *equal, counting multiplicities up to* 2. This hypothesis is a natural one to make on account of the dimensions of the spaces in the problem. It is treated in Theorem 3.

We shall impose certain hypotheses of general position on the lines $L_1, \dots, L_n$. In the most usual sense, to say that $L_1, \dots, L_n$ are in general position means that no three of them are coincident; in other words, they satisfy no linear relation. In Theorems 2 and 3 it is necessary also to impose the condition that, for each positive integer d, no subset of $L_1, \dots, L_n$ containing $\frac{1}{2}(d^2+3d+2)$ lines satisfies a relation of degree d and genus 0. In Theorem 1 a hypothesis of this type would be necessary if the curve $f: \mathbb{P}^1 \to \mathbb{P}^2$ were allowed to have a linear image.

Theorem 1 (Fujimoto). *Let $f, g: \mathbb{P}^1 \to \mathbb{P}^2$ be parametrized algebraic curves such that the image of f does not lie in a line. Let $L_1, \dots, L_4$ be four lines in general position in $\mathbb{P}^2$ such that, for $i = 1, \dots, 4$, the inverse images $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are equal, counting multiplicities. Then $f \equiv g$.*

This theorem is not stated in Fujimoto's paper, but the proof, which is by elementary algebra, is the same as Fujimoto's proof of his Theorem II [2, p. 11]. We conclude this section by discussing two examples.

Example 1. Let L_0, L_1, L_2 be the coordinate axes in $\mathbb{P}^2$. Let $\varphi_0, \varphi_1, \varphi_2$ be nonconstant polynomials with no common zeros and let c_0, c_1, c_2 be nonzero constants. Then the curves $f, g: \mathbb{P}^1 \to \mathbb{P}^2$ defined by $f = (\varphi_0, \varphi_1, \varphi_2)$, $g = (c_0\varphi_0, c_1\varphi_1, c_2\varphi_2)$ satisfy $f^{-1}(L_i) = g^{-1}(L_i)$, with the same multiplicities, for $i = 0, 1, 2$. More generally, if ψ_0, ψ_1, ψ_2 are polynomials such that, for $i = 0, 1, 2$, the zeros of ψ_i are the same as the zeros of φ_i, but with different multiplicities, then f and $h = (\psi_0, \psi_1, \psi_2)$ satisfy $f^{-1}(L_i) = h^{-1}(L_i)$, not counting multiplicities, for $i = 0, 1, 2$.

Example 2 (Fujimoto [2, p. 13]). Take a rational function $\varphi: \mathbb{P}^1 \to \mathbb{P}^1$ and compose φ with two linear maps from $\mathbb{P}^1$ to $\mathbb{P}^2$. For example, let $f = (\varphi, 0, 1)$, $g = (0, 1, \varphi)$. For any nonzero element a of K, the line L_a in $\mathbb{P}^2$ joining $(a, 0, 1)$ and $(0, 1, a)$ satisfies $f^{-1}(L_a) = g^{-1}(L_a)$, with the same multiplicities. The lines L_a lie on a nondegenerate conic in $\mathbb{P}^{2*}$ and so they are all in linear general position. Thus, if, in Theorem 1, we assume that f is nonconstant but allow the image of f to be a line, then no number of lines in linear general position is great enough to yield the conclusion that $f \equiv g$. It suffices to take six lines $L_1, \dots, L_6$ in linear general position and assume in addition that $L_1, \dots, L_6$ do not all lie on a conic in $\mathbb{P}^{2*}$.

3. A uniqueness theorem not counting multiplicities

Theorem 2. *Let $\{L_i : i = 1, \dots, 26\}$ be a set of distinct lines in $\mathbb{P}^2$ such that for no positive integer d do $\frac{1}{2}(d^2 + 3d + 2)$ of the L_i satisfy an irreducible algebraic relation of degree d and genus 0. Let $f, g: \mathbb{P}^1 \to \mathbb{P}^2$ be nonconstant algebraic maps such that, for $i = 1, \dots, 26$, the inverse images $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are the same set, but not necessarily with the same multiplicities. Then $f \equiv g$.*

Proof. We shall begin by considering n distinct lines $L_1, \dots, L_n$ and subsequently determine the least value of n that suffices to establish the conclusion.

If the image of f is one of the lines L_i, then the image of g is also L_i, and the theorem reduces to the one-dimensional case. Assume then that the image of f is not any of the L_i, and the same for g. Let $N(f, L_i)$ be the number of times f passes through L_i, counted with multiplicities. Make the corresponding definition of $N(g, L_i)$. Then

$$\sum_{i=1}^{n}(N(f, L_i) + N(g, L_i)) = n(\deg f + \deg g). \tag{11}$$

Under the assumption that $f \not\equiv g$, we shall estimate the left-hand member of (11) in terms of a smaller multiple of $\deg f + \deg g$ and thus obtain a contradiction.

The hypothesis is that, for $i = 1, \dots, 26$, we have $f^{-1}(L_i) = g^{-1}(L_i)$, possibly with different multiplicities. We write $N_{\text{shared}}(L_i)$ for the number of points in $f^{-1}(L_i)$, each counted once, and let

$$\begin{aligned} N_{\text{multiple}}(f, L_i) &= N(f, L_i) - N_{\text{shared}}(L_i), \\ N_{\text{multiple}}(g, L_i) &= N(g, L_i) - N_{\text{shared}}(L_i). \end{aligned} \tag{12}$$

We shall estimate N_{shared} using a construction from an earlier paper [3]. At a point x of $\mathbb{P}^1$ where $f(x) \neq g(x)$, there is a unique line joining $f(x)$ to $g(x)$, which corresponds to a point $f \wedge g(x)$ in $\mathbb{P}^{2*}$. Take reduced polynomial representations $(\varphi_0, \varphi_1, \varphi_2)$ for f and (χ_0, χ_1, χ_2) for g. Then the Plücker coordinates (5) of $f \wedge g(x)$ are given by

$$((\varphi_1\chi_2 - \varphi_2\chi_1)(x), (\varphi_2\chi_0 - \varphi_0\chi_2)(x), (\varphi_0\chi_1 - \varphi_1\chi_0)(x)). \tag{13}$$

The coordinate expression (13) defines an algebraic map $f \wedge g \colon \mathbb{P}^1 \to \mathbb{P}^2$ but there may be common zeros of the coordinates. We let the polynomial θ be the greatest common factor of the coordinates (13) and write

$$\varphi_1\chi_2 - \varphi_2\chi_1 = \theta\psi_0, \quad \varphi_2\chi_0 - \varphi_0\chi_2 = \theta\psi_1, \quad \varphi_0\chi_1 - \varphi_1\chi_0 = \theta\psi_2. \tag{14}$$

We let $h = f \wedge g$, so that (ψ_0, ψ_1, ψ_2) is a reduced polynomial representation for h.

A special case occurs when h is constant. Then the images of f and g both lie in a line which is the dual of the constant image of h. The conclusion of the theorem follows by the Adams-Straus theorem on rational functions, given in the introduction.

Now assume that h is not constant. Let $N_{\text{equal}}(L_i)$ be the number of points x in $\mathbb{P}^1$ such that $f(x) = g(x)$ and $f(x)$ lies on L_i. Let $N_{\text{unequal}}(L_i)$ be the number of points x in $\mathbb{P}^1$ such that $h(x) = L_i^*$. Then

$$N_{\text{shared}}(L_i) \le N_{\text{equal}}(L_i) + N_{\text{unequal}}(L_i). \tag{15}$$

We let

$$N^0_{\text{unequal}}(L_i) = \min(N_{\text{unequal}}(L_i), 1)$$

and write

$$N_{\text{unequal}}(L_i) = N^0_{\text{unequal}}(L_i) + N^1_{\text{unequal}}(L_i). \tag{16}$$

The map h will be used to estimate N_{unequal}. Write $h = h_2 \circ h_1$, where h_1 is a rational function and h_2 does not factor through a rational function of degree greater than one. Let $d = \deg h_2$, so that

$$\text{de}\, h_1 = \frac{1}{d} \deg h \tag{17}$$

because of (3). In the argument that follows, it is essential to use one estimate when d is small and another estimate when d is large. In practice we use three estimates in different ranges of d, for the sake of a smaller value of n.

We shall use h_2 to denote the variety in $\mathbb{P}^2$ that is the image of the map h_2. By the hypothesis of the theorem, at most $\frac{1}{2}(d^2 + 3d)$ of the n points L_i^* lie on the variety h_2. Therefore

$$\begin{aligned} \sum_{i=1}^{n} N^0_{\text{unequal}}(L_i) &\le \min\left(\frac{1}{2}(d^2+3d)\,,\, n\right) \times \deg h_1 \\ &= \min\left(\frac{1}{2}(d+3)\,,\, \frac{n}{d}\right) \times \deg h\,, \end{aligned} \tag{18}$$

because of equation (17). If $N^1_{\text{unequal}}(L_i)$ is positive then h_2 has a singular point at L_i^*. The total multiplicity at singular points of h_2 can be estimated, for small d, by the genus formula (8) and, for large d, by the inequality (9) of the lemma. Therefore

$$\begin{aligned} \sum_{i=1}^{n} N^1_{\text{unequal}}(L_i) &\le \min\left(\frac{1}{2}(d^2-3d+2)\,,\, n^{1/2}(d-1)\right) \times \deg h_1 \\ &= \min\left(\frac{1}{2}\left(d-3+\frac{2}{d}\right)\,,\, n^{1/2}\left(1-\frac{1}{d}\right)\right) \times \deg h\,, \end{aligned} \tag{19}$$

because of equation (17). Next we estimate N_{equal}. The polynomial θ in formulas (14) vanishes at precisely those x in $\mathbb{P}^1$ for which $f(x) = g(x)$. Now, if $f(x) = g(x)$, it is possible for $f(x)$ to lie on two of the lines L_i. It cannot happen that $f(x)$ lies on three or more of the L_i, because then three of the L_i would satisfy a linear relation, which is not allowed by the hypothesis of the theorem. Therefore

$$\sum_{i=1}^{n} N_{\text{equal}}(L_i) \le 2\deg\theta. \tag{20}$$

We now assume that $n \ge 4$, which allows us to absorb the estimate for N_{equal} into the estimate for N_{unequal}. From the definition (14) of h and θ we have

$$\deg h + \deg\theta \le \deg f + \deg g\,, \tag{21}$$

so that, combining (15), (16), (18), (19), (20) and (21) we have

$$\sum_{i=1}^{n} N_{\text{shared}}(L_i) \le A_n(\deg f + \deg g)\,, \tag{22}$$

where

(23)

$$A_n = \max_d\left(\min\left(\frac{1}{2}(d+3)\,,\, \frac{n}{d}\right) + \min\left(\frac{1}{2}\left(d-3+\frac{2}{d}\right)\,,\, n^{1/2}\left(1-\frac{1}{d}\right)\right)\right),$$

in which the maximum is taken over all positive integers d.

Now we consider f separately and give an estimate for $N_{\mathrm{multiple}}(f, \cdot)$. The same argument will yield an estimate for $N_{\mathrm{multiple}}(g, \cdot)$.

Considering L_i as a linear form, we need to account for all the multiple zeros of $L_i \circ f$. We write $f = f_2 \circ f_1$, where f_1 is a rational function and f_2 cannot be factored through a rational function of degree greater that one, and we use f_2 to denote the variety in $\mathbb{P}^2$ that is the image of the map f_2. Roughly speaking, multiple zeros of $L_i \circ f$ arise in three different ways: when L_i is tangent to f_2, when f_2 has a cusp on L_i and when f_1 has a branch point that maps to a point of L_i. These three sorts of multiple zero may of course occur in combination.

We shall first obtain estimates for f_2 instead of f. This is equivalent to considering the special case in which $\deg f_1 = 1$. After that we shall modify our estimates to allow for arbitrary values of $\deg f_1$.

We have already assumed that the image of f is not one of the lines L_i. Hence, if $\deg f_2 = 1$, then f_2 intersects each of the lines L_i with multiplicity 1, and $N_{\mathrm{multiple}}(f_2, L_i) = 0$ for $i = 1, \dots, n$.

Now assume that $\deg f_2 \geq 2$. To each x in $\mathbb{P}^1$ there corresponds a place of the singular curve f_2 which has order $r(x)$ and class $s(x)$. If $f_2(x)$ lies on L_i, then the multiplicity of the zero of $L_i \circ f_2$ at x is $r(x) + s(x)$ if L_i is the tangent to f_2 at the place corresponding to x and $r(x)$ otherwise. We have to estimate the sum of $r(x) + s(x) - 1$ over all x at which L_i is the tangent to the place corresponding to x and $r(x) - 1$ over all x such that $f_2(x)$ lies on L_i but L_i is not the tangent.

If L_i is the tangent to the place of f_2 corresponding to x, then $f_2^*(x) = L_i^*$ and the order of the corresponding place of f_2^* is $s(x)$. The multiplicity of the singularity of f_2^* at L_i^* is the sum over all such x of $s(x)$. To estimate the sum of the quantities $N_{\mathrm{multiple}}(f_2, L_i)$, observe that, by the hypothesis of the theorem, $f_2(x)$ may lie on at most two of the lines $L_1, \dots, L_n$. At most one line, say L_i, can be tangent to the place of f_2 corresponding to x, and so there is at most one contribution of $r(x) + s(x) - 1$, but another line passing through $f_2(x)$, say L_j, can contribute an additional $r(x) - 1$. Let

$$R = \sum_{x \in \mathbb{P}^1} r(x) - 1, \quad S = \sum_{x \in \Phi} s(x),$$

where Φ is the set of x in $\mathbb{P}^1$ such that $f_2^*(x)$ is one of $L_1^*, \dots, L_n^*$. Then

$$\tag{24} \begin{aligned} \sum_{i=1}^{n} N_{\mathrm{multiple}}(f_2, L_i) &\leq \sum_{x \in \Phi} (2r(x) + s(x) - 2) + \sum_{x \in \mathbb{P}^1 - \Phi} (2r(x) - 2) \\ &\leq S + 2R. \end{aligned}$$

Let S^0 be the number of points L_i^* that lie on f_2^*, each counted once, and write

$$\tag{25} S = S^0 + S^1.$$

Let $\delta = \deg f_2$, so that

$$\tag{26} \deg f_1 = \frac{1}{\delta} \deg f.$$

We shall bound S by different estimates for small and large values of δ, treating δ in the same way as d in the preceding argument.

The first Plücker formula (6) gives

$$\deg f_2^* = 2\delta - 2 - R. \tag{27}$$

By the hypothesis of the theorem we have

$$\begin{aligned} S^0 &\le \frac{1}{2}\deg f_2^*(\deg f_2^* + 3) \\ &\le \frac{1}{2}(2\delta - 2)(2\delta + 1) \quad \text{by (27)} \\ &= 2\delta^2 - \delta - 1. \end{aligned}$$

Therefore

$$S^0 \le \min(2\delta^2 - \delta - 1\,,\, n). \tag{28}$$

By the genus formula (8) we have

$$\begin{aligned} S^1 &\le \frac{1}{2}(\deg f_2^* - 1)(\deg f_2^* - 2) \\ &= \frac{1}{2}(2\delta - 3 - R)(2\delta - 4 - R) \quad \text{by (27)} \\ &\le 2\delta^2 - 7\delta + 6 - \frac{1}{2}R\max(2\delta - 3\,,\, 4\delta - 7 - R). \end{aligned} \tag{29}$$

Now, if $\delta = 1$ or 2, the curve f_2 has no cusps and so $R = 0$. If $\delta = 3$ then $R \le 1$ and the second term in the maximum of (29) is at least 4. If $\delta \ge 4$ then the first term is at least 5. Hence, for any positive δ, the estimate (29) implies

$$S^1 \le 2\delta^2 - 7\delta + 6 - 2R. \tag{30}$$

By the lemma we have

$$\begin{aligned} S^1 &\le n^{1/2}(\deg f_2^* - 1) \\ &= n^{1/2}(2\delta - 3 - R) \quad \text{by (27)} \\ &\le n^{1/2}(2\delta - 3) - 2R, \end{aligned} \tag{31}$$

under the assumption that $n \ge 4$. The estimates (30) and (31) combine to give

$$S^1 \le \min(2\delta^2 - 7\delta + 6 - 2R\,,\, n^{1/2}(2\delta - 3) - 2R). \tag{32}$$

From (24), (25), (28) and (32) we obtain

$$\begin{aligned} \sum_{i=1}^{n} N_{\text{multiple}}(f_2\,,\, L_i) \le S^0 + S^1 + 2R &\le \min(2\delta^2 - \delta - 1\,,\, n) \\ &\quad + \min(2\delta^2 - 7\delta + 6\,,\, n^{1/2}(2\delta - 3)). \end{aligned} \tag{33}$$

This estimate for f_2 must now be converted into an estimate for f. If x is a point of $\mathbb{P}^1$ that is not the image of a branch point of f_1, then $\operatorname{card} f_1^{-1}(x) = \deg f_1$. An estimate for $N_{\text{multiple}}(f\,,\,\cdot)$ can therefore be obtained by multiplying the estimate (33) for $N_{\text{multiple}}(f_2\,,\,\cdot)$ by $\deg f_1$. If $f(z)$ lies on L_i and z is a branch point of multiplicity μ of f_1, then $N_{\text{multiple}}(f\,,\,L_i)$ is increased by an additional contribution of $\mu - 1$. Since $f(z)$ may lie on two of the lines

L_i, the total additional contribution from the branch point at z may be as much as $2(\mu - 1)$. Using the Riemann-Hurwitz formula (1), we can estimate these additional contributions from branch points by the quantity $4 \deg f_1$. Therefore, using (26) and (33), we have

$$\sum_{i=1}^{n} N_{\text{multiple}}(f, L_i) \leq B_n \deg f, \tag{34}$$

where

(35)
$$B_n = \max_{\delta} \left(\frac{4}{\delta} + \min\left(2\delta - 1 - \frac{1}{\delta}, \frac{n}{\delta}\right) + \min\left(2\delta - 7 + \frac{6}{\delta}, n^{1/2}\left(2 - \frac{3}{\delta}\right)\right)\right),$$

in which the maximum is taken over all integers δ greater than or equal to 2.

Combining (11), (12) and (22) with (34) and the corresponding inequality for g, we obtain

$$n(\deg f + \deg g) \leq (2A_n + B_n)(\deg f + \deg g), \tag{36}$$

where A_n and B_n are defined by (23) and (35). If we can obtain a contradiction from the inequality (36), then the assumption that f and g are not identically equal is false. The theorem follows. The inequality (36) yields a contradiction if

$$n > 2A_n + B_n. \tag{37}$$

A computer search reveals that the least such n is 26, and it may be checked by hand that $n = 26$ does indeed imply (37).

4. A UNIQUENESS THEOREM COUNTING MULTIPLICITIES UP TO TWO

Theorem 3. *Let $\{L_i\colon i = 1, \ldots, 12\}$ be a set of distinct lines in $\mathbb{P}^2$ such that for no positive integer d do $\frac{1}{2}(d^2 + 3d + 2)$ of the L_i satisfy an irreducible algebraic relation of degree d and genus 0. Let $f, g\colon \mathbb{P}^1 \to \mathbb{P}^2$ be nonconstant algebraic maps such that, for $i = 1, \ldots, 12$, the inverse images $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are equal, counting multiplicities up to 2. Then $f \equiv g$.*

Proof. The hypothesis that $f^{-1}(L_i)$ and $g^{-1}(L_i)$ are equal, counting multiplicities up to 2, means that $L_i \circ f$ and $L_i \circ g$ have the same simple zeros and the same multiple zeros. The main difference from Theorem 2 is that simple tangents do not have to be included in the estimate for multiple intersections.

Define $N(f, L_i)$ and $N(g, L_i)$ as before, so that the estimate (11) holds. Write $\widetilde{N}_{\text{shared}}(L_i)$ for the number of points in $f^{-1}(L_i)$, with simple zeros of $L_i \circ f$ counted once and multiple zeros of $L_i \circ f$ counted twice. Let

$$\begin{aligned} \widetilde{N}_{\text{multiple}}(f, L_i) &= N(f, L_i) - \widetilde{N}_{\text{shared}}(L_i), \\ \widetilde{N}_{\text{multiple}}(g, L_i) &= N(g, L_i) - \widetilde{N}_{\text{shared}}(L_i). \end{aligned} \tag{38}$$

Let $\widetilde{N}_{\text{equal}}(L_i)$ be the number of points x in $\mathbb{P}^1$ such that $f(x) = g(x)$ and $f(x)$ lies on L_i, counting multiplicities up to 2. Let $\widetilde{N}_{\text{unequal}}(L_i)$ be the number of points x in $\mathbb{P}^1$ such that $h(x) = L_i^*$, with each stationary point h counted twice. If $f(x) \neq g(x)$ and each of $L_i \circ f$ and $L_i \circ g$ has a multiple zero at x, then the curve h with components (13) has a stationary point at x, so that x is counted twice in $\widetilde{N}_{\text{unequal}}(L_i)$. Therefore

$$\widetilde{N}_{\text{shared}}(L_i) \le \widetilde{N}_{\text{equal}}(L_i) + \widetilde{N}_{\text{unequal}}(L_i). \tag{39}$$

The estimate for $\widetilde{N}_{\text{unequal}}$ is the same as before but the estimate for $\widetilde{N}_{\text{equal}}$ needs to be modified to take account of multiple intersections. When $f(x) = g(x)$ it is possible for $f(x)$ to lie on two of the lines, say L_i and L_j. If f and g are tangent to L_i but transverse to L_j, then x may be only a simple zero of the polynomial θ, so that the right-hand member of (20) must become $3\deg\theta$. If f and g are tangent to both L_i and L_j, then x is a stationary point of f and g; therefore x is a multiple zero of θ and the estimate by $2\deg\theta$ is sufficient. We thus obtain

$$\sum_{i=1}^{n} \widetilde{N}_{\text{equal}}(L_i) \le 3\deg\theta, \tag{40}$$

corresponding to (20). (A similar estimate is discussed in more detail in an earlier paper [3].) It follows that

$$\begin{aligned}\sum_{i=1}^{n} \widetilde{N}_{\text{shared}}(L_i) &\le \sum_{i=1}^{n}(\widetilde{N}_{\text{equal}}(L_i) + \widetilde{N}_{\text{unequal}}(L_i)) \quad \text{by (39)}\\ &\le A_n(\deg f + \deg g),\end{aligned} \tag{41}$$

where A_n is defined by equation (23). Here we have assumed $n \ge 9$, so that $A_n \ge 3$ and the estimate (40) can be absorbed.

In estimating $\widetilde{N}_{\text{multiple}}$, we begin with $\widetilde{N}_{\text{multiple}}(f_2, \cdot)$. For x in $\mathbb{P}^1$, we write $r(x)$ for the order and $s(x)$ for the class of the corresponding place of f_2. We have to estimate the sum of $r(x) + s(x) - 2$ over all x at which L_i is the tangent to the place corresponding to x and $r(x) - 2$ over all x such that $f(x)$ lies on L_i but L_i is not the tangent. Let

$$T = \sum_{x\in\mathbb{P}^1}(s(x) - 1), \quad R = \sum_{x\in\mathbb{P}^1}(r(x) - 1).$$

By the second Plücker formula (7)

$$\begin{aligned}\sum_{i=1}^{n} \widetilde{N}_{\text{multiple}}(f_2, L_i) &\le \sum_{x\in\mathbb{P}^1}(r(x) + s(x) - 2) + \sum_{x\in\mathbb{P}^1}(r(x) - 2)\\ &\le T + 2R = 3\deg f_2 - 6.\end{aligned} \tag{42}$$

If x is a point of $\mathbb{P}^1$ that is not the image of a branch point of f_1, then an estimate for $\widetilde{N}_{\text{multiple}}(f, \cdot)$ can be obtained by multiplying the estimate (42) for $\widetilde{N}_{\text{multiple}}(f_2, \cdot)$ by $\deg f_1$. For branch points of f_1 we need a correction. Suppose that z is a branch point of f_1 of multiplicity μ and L_i is the tangent to f_2 at $f(z)$. Then z contributes 2 to $\widetilde{N}_{\text{shared}}$. The contribution of z to the estimate obtained by multiplying (42) by $\deg f_1$ is $\mu(r(f_1(z)) + s(f_1(z)) - 2)$. The multiplicity of the intersection of f with L_i is $\mu(r(f_1(z)) + s(f_1(z)))$. Therefore the required correction is $2\mu - 2$. If $f(z)$ lies on L_i but L_i is not the tangent to f_2 at z then the correction is $\mu - 2$. At the intersection of two of the lines L_i we may therefore require a correction of $3\mu - 4$.

Using the Riemann-Hurwitz formula (1) we can estimate these additional contributions from branch points by the quantity $6 \deg f_1$. Inequality (42) then yields

$$\sum_{i=1}^{n} \widetilde{N}_{\text{multiple}}(f, L_i) \leq 3 \deg f. \tag{43}$$

Combining (11), (38) and (41) with (43) and the corresponding inequality for g, we obtain

$$n(\deg f + \deg g) \leq (2A_n + 3)(\deg f + \deg g),$$

where A_n is defined by (23). To obtain a contradiction, and thus deduce the theorem, we need n to be at least 12.

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Department of Mathematics, University of Alabama, Tuscaloosa, Alabama 35487-0350