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ABEL'S THEOREM FOR TWISTED JACOBIANS

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ABSTRACT. A twisted version of the Abel-Jacobi map, associated to a local system with finite monodromy on a smooth projectve complex curve, is introduced. An analogue of Abel's theorem characterizing the kernel of this map is proved. The proof, which is new even in the classical case, involves reinterpreting the Abel-Jacobi map in the language of mixed Hodge structures and their extensions.

1. INTRODUCTION

The Jacobian of a smooth complex projective curve X is the complex torus $J(X) = H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$. Integration defines the Abel-Jacobi homomorphism $\alpha : \mathrm{Div}^0(X) \to J(X)$ from the group of degree zero divisors. Abel's theorem states that the kernel coincides with the subgroup of principal divisors. The classical proof of the hard half of the theorem proceeds in two steps [GH, pp. 232–235]: Given a divisor D with $\alpha(D) = 0$, construct a differential form η with integral periods and prescribed singularities depending on D. Then the function

$$f(x) = \exp\left(2\pi\sqrt{-1}\int_{x_0}^{x}\eta\right)$$

is a (single valued) meromorphic function such that $D = (f)$. We can reinterpret the first step using the language of extensions of mixed Hodge structures [C]. To every divisor D in $\mathrm{Div}^0(X)$, we can associate an extension of $\mathbb{Z}(-1)$ by $H^1(X, \mathbb{Z})$, which splits exactly when $\alpha(D) = 0$. Furthermore the existence of the above form η is implied by and in fact equivalent to the splitting of this extension.

Aside from the improvement in conceptual clarity, the language of extensions of mixed Hodge structures lends itself to the treatment of various generalized Abel-Jacobi maps. In this paper, we study the Abel-Jacobi map associated to certain twisted Jacobians over curves. Given a local system of finitely generated abelian groups L, with finite monodromy over a smooth projective curve X, $H^1(X, L)$ carries a polarizable pure Hodge structure of weight one. We define

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L-valued Jacobian (or 'twisted Jacobian') as the associated torus

$$J(X, L) = \frac{H^1(X, L \otimes \mathbb{C})}{H^1(X, L) + H^0(X, \Omega^1_X \otimes L)}$$

and we prove an analogue of Abel's theorem for it. The twisted Jacobian is isogenous to a subtorus of the Jacobian of an étale covering of X, and so it can be viewed as a generalized Prym variety. In fact, if $L = f_*(\mathbb{Z})$ where $f : Y \to X$ is an étale Galois covering, then $J(X, L) = J(Y, \mathbb{Z})$ and also our Abel's theorem is equivalent to the usual one for Y.

Here is a brief description of the layout of the paper. In the next two sections, we review some notions from Hodge theory for the convenience of the reader. In §4, we define the Jacobian $J(X, L)$ of a local system L on a compact Kähler manifold X. When X is a smooth complex projective curve, we show that $J(X, L)$ can be identified with an analogue of the Albanese variety $\mathrm{Alb}(X, L)$. In §5, We study the mixed Hodge structure on the cohomology $H^1(U, L)$ of a smooth open curve U with coefficients in a local system L. In §6, we introduce the notion of an L-valued divisor for a local system L on a curve X. Then we construct an Abel-Jacobi map of a certain group of L-valued divisors $\mathrm{Div}^0(X, L)$ to $\mathrm{Alb}(X, L)$. Finally in the last section, we obtain an analogue of Abel's theorem that the kernel of this map is the group of (suitably defined) principal L-valued divisors.

2. Hodge structures

Definition 2.1. A (pure) Hodge structure H of weight m consists of a finitely generated abelian group $H_{\mathbb{Z}}$ and a decreasing filtration $F^\bullet$ of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$ such that $H_{\mathbb{C}} = F^p \oplus \overline{F^{m-p+1}}$.

Example 1. The Hodge structure of Tate $\mathbb{Z}(-1)$ is defined to be the Hodge structure of weight 2 with $H_{\mathbb{Z}} = \frac{1}{2\pi\sqrt{-1}}\mathbb{Z} \subset \mathbb{C} = F^1 H_{\mathbb{C}}$.

The most natural example of Hodge structure of weight k is the kth integral cohomology of a compact Kähler manifold. A differential form lies in F^p if in local coordinate it has at least p "dz's". To extend Hodge theory to any (singular or nonprojective) complex algebraic varieties X, Deligne [D] introduced the notion of a mixed Hodge structure. He showed that the cohomology of any variety carries such a structure.

Definition 2.2. A *mixed Hodge structure* (MHS) H consists of a triple $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$, where

(1) $H_{\mathbb{Z}}$ is a finitely generated abelian group. (In practice $H_{\mathbb{Z}}$ will be free and we will identify it with a lattice in $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes \mathbb{Q}$.)
(2) $W_\bullet$ is an increasing filtration of $H_{\mathbb{Q}}$, called the *weight filtration.*
(3) $F^\bullet$ is a decreasing filtration of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$, called the *Hodge filtration*.

The Hodge filtration $F^\bullet$ is required to induce a (pure) Hodge structure of weight m on each of the graded pieces $Gr_m^{W_\bullet} = W_m / W_{m-1}$.

A morphism of mixed Hodge structures $\phi : A \to B$ is given by a homomorphism of the underlying abelian groups which preserves both filtrations. Note

that an abelian group $\mathrm{Hom}(A, B)_{\mathbb{Z}} := \mathrm{Hom}(A_{\mathbb{Z}}, B_{\mathbb{Z}})$ with filtrations

$$W_m\mathrm{Hom}_{\mathbb{Q}} = \{\phi \mid \phi(W_r A_{\mathbb{Q}}) \subset W_{r+m} B_{\mathbb{Q}} \text{ for all } r\},$$
$$F^p\mathrm{Hom}_{\mathbb{C}} = \{\phi \mid \phi(F^r A_{\mathbb{C}}) \subset F^{r+p} B_{\mathbb{C}} \text{ for all } r\}$$

forms a mixed Hodge structure.

Given two mixed Hodge structures A and B, we write $B > A$ if there exists m_0 such that $W_m A_{\mathbb{Q}} = A_{\mathbb{Q}}$ for all $m \geq m_0$ and $W_m B_{\mathbb{Q}} = 0$ for all $m < m_0$.

Finally, we define the p-th *Jacobian* of a mixed Hodge structure of H to be the generalized torus $J^p H = H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^p H_{\mathbb{C}}$.

3. Extensions

The category of mixed Hodge structures is abelian. Thus one can form the abelian group of extension classes of two objects. Carlson [C] described the structure of this extension group in terms of the Jacobian.

Theorem 3.1 (Carlson). *Let A and B be mixed Hodge structures with $B > A$ and B torsion free. Then there is a natural isomorphism.*

$$\mathrm{Ext}^1_{MHS}(B, A) \cong J^0\mathrm{Hom}(B, A).$$

Proof. For the proof, we refer to [C]. Here we merely describe the correspondence.

Given $f \in \mathrm{Hom}_{\mathbb{C}}(B, A)$, define

$$E_f = A \oplus B \quad \text{as a group}, \quad W_m E_f = W_m A \oplus W_m B$$

and

$$F^p E_f = \{(a, b) \in A \oplus B \mid b \in F^p B, a - f(b) \in F^b A\}.$$

In this way, we get a mixed Hodge structure which fits into a sequence of mixed Hodge structures.

$$0 \longrightarrow A \longrightarrow E_f \longrightarrow B \longrightarrow 0$$

Given $g \in \mathrm{Hom}_{\mathbb{Z}}(B, A)$, there is an isomorphism between E_f and E_{f+g} via $(a, b) \mapsto (a + g(b), b)$. If $g \in \mathrm{Hom}_{\mathbb{C}}(B, A)$ preserves $F^\bullet$, then E_f and E_{f+g} are identical. Therefore the extension class associated to E_f depends only on the class of $f \in J^0\mathrm{Hom}(B, A)$.

Conversely, given an extension

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} B \longrightarrow 0,$$

choose a $\mathbb{C}$-linear section $s : B \to E$ preserving $F^\bullet$, and a $\mathbb{Z}$-linear retraction $r : E_{\mathbb{Z}} \to A_{\mathbb{Z}}$ preserving $W_\bullet$. Then $f := r \circ s$ is an element in $J^0\mathrm{Hom}(B, A)$, and $E \xrightarrow{r \oplus \pi} E_f$ is an equivalence. □

4. Local systems and Jacobians

Let X be a compact connected Kähler manifold with a base point x. A local system L on X is defined to be a locally constant sheaf of free abelian groups on X. It is well known that there is one-to-one correspondence between the local systems L on X and the monodromy representations $\rho_L \in$

$\mathrm{Hom}(\pi_1(X, x), \mathrm{Aut}(L_x))/\mathrm{Aut}(L_x)$ up to conjugacy. The local system $\check{L}$ associated to the dual representation $\rho_{\check{L}} := \rho_L^{-1}$ is called the *dual* system to L.

We say that a local system L is *polarizable* if it has finitely generated stalks and $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$ carries a positive definite symmetric pairing

$$\langle\ ,\ \rangle : L_{\mathbb{Q}} \otimes L_{\mathbb{Q}} \longrightarrow \mathbb{Q}_X.$$

The last condition is equivalent to requiring that the associated monodromy representation ρ_L is orthogonal. This is possible when the monodromy group of L is finite. The polarizable variations of Hodge structure of weight 0 with trivial Hodge filtration are natural examples of polarizable local systems.

The sheaf of the sections of the vector bundle $\mathscr{O}_X \otimes L$ carries a natural integrable connection $\nabla : \mathscr{O}_X \otimes L \to \Omega^1_X \otimes L$. By integrability, we can form a complex $(\Omega^\bullet_X \otimes L, \nabla)$, which resolves $L_{\mathbb{C}} := L \otimes \mathbb{C}$. In an unpublished manuscript (see [Z] for a published account), Deligne proved

Theorem 4.1 (Deligne). *Let X be a Kähler manifold and L be a polarizable local system on X. Then $(L, \Omega^\bullet_X \otimes L, F^\bullet)$ is a polarizable cohomological Hodge complex of weight 0, where $F^\bullet$ is the stupid filtration on $\Omega^\bullet_X \otimes L$.*

In particular, $H^1(X, L)$ carries a pure Hodge structure of weight 1 with $F^1 H^1(X, L) = H^0(X, \Omega^1_X \otimes L)$. Since the Tate structure $\mathbb{Z}(-1)$ is pure of weight 2, we have $\mathbb{Z}(-1) > H^1(X, L)$. Hence by Theorem 3.1, we have

$$\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1),\ H^1(X, L)) \cong J^0\mathrm{Hom}(\mathbb{Z}(-1),\ H^1(X, L)).$$

Remark 1. The second term $J^0\mathrm{Hom}(\mathbb{Z}(-1),\ H^1(X, L))$, which we denote by $J(X, L)$, can be identified with

$$\frac{H^1(X, L_{\mathbb{C}})}{H^1(X, L) + H^0(X, \Omega^1_X \otimes L)}$$

via the map $f \in J(X, L) \mapsto f(1/2\pi\sqrt{-1})$. In particular, $J(X, \mathbb{Z})$ coincides with the classical Jacobian of X.

From now on, X will be a smooth complex projective curve and L will be a polarizable local system on X. Furthermore, we will assume that $H^1(X, L)$ is torsion free. In order to get another description of the Jacobian $J(X, L)$, consider the following exact sequence associated to the deRham complex $\Omega^\bullet_X \otimes L$.

$$\begin{aligned} 0 &\longrightarrow H^0(X, L_{\mathbb{C}}) \longrightarrow H^0(X, \mathscr{O}_X \otimes L) \longrightarrow H^0(X, \Omega^1_X \otimes L) \\ &\xrightarrow{\delta} H^1(X, L_{\mathbb{C}}) \xrightarrow{i} H^1(X, \mathscr{O}_X \otimes L) \longrightarrow H^1(X, \Omega^1_X \otimes L) \\ &\longrightarrow H^2(X, L_{\mathbb{C}}) \longrightarrow 0 \end{aligned} \tag{1}$$

Lemma 4.2. *In the above exact sequence* (1), *the map δ is injective and the map i is surjective. Thus the following sequence is exact.*

$$0 \longrightarrow H^0(X, \Omega^1_X \otimes L) \xrightarrow{\delta} H^1(X, L_{\mathbb{C}}) \xrightarrow{i} H^1(X, \mathscr{O}_X \otimes L) \longrightarrow 0 \tag{2}$$

Remark 2. This is a special case of the degeneration of the (generalized) Hodge to De Rham spectral sequence. We give a more elementary proof.

Proof. For injectivity of δ, it suffices to establish an isomorphism:

$$H^0(X, L_{\mathbb{C}}) \cong H^0(X, \mathscr{O}_X \otimes L).$$

The polarization on L induces a pointwise norm $\| \ \|$ on $H^0(X, \mathscr{O}_X \otimes L)$. Given a global section v of $\mathscr{O}_X \otimes L$, $\|v\|^2$ must attain a maximum at some point $0 \in X$. Let z be a local parameter about 0 and let $\{\lambda_i\}$ be a unitary local frame of $L_{\mathbb{C}}$. We can expand v about 0,

$$v = \sum_i f_i(z)\lambda_i$$

with the f_i's holomorphic. By assumption, we have

$$\|v(0)\|^2 = \sum |f_i(0)|^2 \geq \sum |f_i(z)|^2.$$

On the other hand, by the maximum modulus principle:

$$\sum |f_i(0)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \sum |f_i(re^{i\theta})|^2 \, d\theta$$

for all sufficiently small $r > 0$. Therefore v is constant near 0 and hence by analytic continuation, locally constant everywhere. This establishes the above isomorphism.

Dually, to establish the surjectivity of i, it suffices to show that the surjection

$$H^1(X, \Omega_X^1 \otimes L) \longrightarrow H^2(X, L_{\mathbb{C}})$$

is an isomorphism. Serre and Poincaré duality give isomorphisms:

$$H^1(X, \Omega_X^1 \otimes L) \cong H^0(X, \mathscr{O}_X \otimes \check{L})^*,$$
$$H^2(X, L_{\mathbb{C}}) \cong H^0(X, \check{L}_{\mathbb{C}})^*.$$

The previous argument, applied to $\check{L}$, shows that these spaces are isomorphic. □

It will be convenient to make the Poincaré and Serre duality isomorphisms a little more explicit. Given an L-valued p-form ξ and an $\check{L}$-valued q-form ω, the product $\xi \wedge \omega$ is an $L \otimes \check{L}$-valued $(p+q)$-form. Taking its trace results in ordinary $(p+q)$-form, which we denote by $\langle \xi, \omega \rangle$. The Poincaré duality map

$$D : H^1(X, L_{\mathbb{C}}) \to H^1(X, \check{L}_{\mathbb{C}})^*$$

is given on the C^∞-form level by

$$D(\xi)(\omega) = \int_X \langle \xi, \omega \rangle.$$

We denote the composite

$$H^1(X, L_{\mathbb{C}}) \xrightarrow{D} H^1(X, \check{L}_{\mathbb{C}})^* \longrightarrow H^0(X, \Omega_X^1 \otimes \check{L})^* \tag{3}$$

by p. Observe that $p(H^0(X, \Omega_X^1 \otimes L)) = 0$ because the integrand for $D(\xi)(\omega)$, in this case, is a holomorphic 2-form on X. Therefore p factors as

$$H^1(X, L_{\mathbb{C}}) \xrightarrow{i} H^1(X, \mathscr{O}_X \otimes L) \xrightarrow{\cong} H^0(X, \Omega_X^1 \otimes \check{L})^*.$$

The last map is an isomorphism because, by Serre duality, both spaces have the same dimension. This provides an explicit Serre duality isomorphism. The image Λ of $H^1(X, L)$ under p will be called the *period lattice.* The ratio

$$\mathrm{Alb}(X, L) = \frac{H^0(X, \Omega^1_X \otimes \check{L})^*}{\Lambda}$$

will be called the *L-valued Albanese torus.* Under the above identification, we have

$$\mathrm{Alb}(X, L) \cong \frac{H^1(X, \mathscr{O}_X \otimes L)}{H^1(X, L)} \cong J(X, L)$$

by Lemma 4.2. Although we will not use this, it is worth noting that these tori are in fact Abelian varieties. A Riemann form can be produced from a polarization on $H^1(X, L)$.

5. Hodge theory of an open curve

Let L be a polarizable local system on a smooth projective curve X. Let U be the complement of a finite set $S \subset X$, then using standard techniques, we prove $H^1(U, L_{|U})$ carries a mixed Hodge structure which fits into a "Thom-Gysin" exact sequence.

Proposition 5.1. *The cohomology group $H^1(U, L_{|U})$ carries a mixed Hodge structure. Moreover, there is an exact sequence of Hodge structures:*

$$(4)\quad 0 \longrightarrow H^1(X, L) \longrightarrow H^1(U, L_{|U}) \longrightarrow \bigoplus_{x \in S} L_x(-1) \xrightarrow{Gysin} H^2(X, L) \longrightarrow 0$$

where $L_x(-1)$ denotes L_x with trivial (weight 0) Hodge structures tensored by $\mathbb{Z}(-1)$.

Proof. First we will show that this sequence is exact as a sequence of abelian groups. Let j be the inclusion map from U to X. From the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q j_* j^{-1} L) \Rightarrow H^{p+q}(U, j^{-1}L),$$

we get an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(X, j_* j^{-1} L) \longrightarrow H^1(U, j^{-1}L) \\ &\longrightarrow H^0(X, R^1 j_* j^{-1} L) \longrightarrow H^2(X, L) \longrightarrow H^2(U, j^{-1}L) = 0. \end{aligned}$$

Since there is no local monodromy around any point $x \in S$, we have $j_* j^{-1} L \cong L$ and $R^1 j_* j^{-1} L \cong \bigoplus_{x \in S} L_x$.

Now we will briefly indicate that the Thom Gysin sequence (4) is in fact an exact sequence of mixed Hodge structures. A proof of this can be found in [Z]. First, let us note that the Thom-Gysin sequence is just the long exact sequence associated to the triangle:

$$\begin{tikzcd}
L \cong \tau_{\leq 0}\mathbb{R}j_*j^{-1}L \arrow[rr, "\phi"] & & \mathbb{R}j_*j^{-1}L \arrow[dl] \\
& R^1j_*j^{-1}L[-1] \cong \bigoplus_{x\in S} L_x[-1] \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

This triangle can be realized on the level of differential forms with the help of the exact sequence

$$0 \longrightarrow \Omega_X^\bullet \otimes L \longrightarrow \Omega_X^\bullet(\log S) \otimes L \xrightarrow{\text{Res}} \bigoplus_{x\in S}(L_x \otimes \mathbb{C})[-1] \longrightarrow 0 \tag{5}$$

where

$$\text{Res} = \sum_{x\in S} \text{Res}_x \quad \textit{and} \quad \text{Res}_x\left(f(z)\frac{dz}{z} \otimes l\right) = f(0)l_x$$

for some local system coordinate z at x with $z(x) = 0$. There is a morphism in the derived category, given by the composite:

$$\mathbb{R}j_*j^{-1}L \longrightarrow \mathbb{R}j_*(\Omega_U^\bullet \otimes L) \xrightarrow{\cong} \Omega_X^\bullet(\log S) \otimes L$$

It becomes an isomorphism after tensoring the left side by $\mathbb{C}$. This induces a morphism of triangles:

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathbb{R}j_*j^{-1}L \arrow[r] \arrow[d] & \bigoplus L_x[-1] \arrow[r] \arrow[d, "\frac{1}{2\pi\sqrt{-1}}"] & L[1] \arrow[d] \\
\Omega_X^\bullet \otimes L \arrow[r] & \Omega_X^\bullet(\log S) \otimes L \arrow[r, "\text{Res}"] & \bigoplus(L_x \otimes \mathbb{C})[-1] \arrow[r] & (\Omega_X^\bullet \otimes L)[1]
\end{tikzcd}$$

Therefore we get compatible exact sequences on cohomology. If we define:

$W_0H^1(U, L) = 0$,
$W_1H^1(U, L) =$ the image of $(H^1(X, L) \to H^1(U, L))$,
$W_2H^1(U, L) = H^1(U, L)$,
$F^0H^1(U, L) \otimes \mathbb{C} = H^1(U, L) \otimes \mathbb{C}$,
$F^1H^1(U, L) \otimes \mathbb{C} =$ the image of $(H^0(X, \Omega^1(\log S) \otimes L) \to H^1(U, L) \otimes \mathbb{C})$,
$F^2H^1(U, L) \otimes \mathbb{C} = 0$,

then we get a mixed Hodge structure on $H^1(U, L)$, which fits into an exact sequence of Hodge structures

$$0 \longrightarrow H^1(X, L) \longrightarrow H^1(U, L) \longrightarrow \bigoplus_{x\in S} L_x(-1) \xrightarrow{\text{Gysin}} H^2(X, L) \longrightarrow 0. \quad \square$$

6. Abel-Jacobi map

Definition 6.1. The group of L-valued divisors $\mathrm{Div}(X, L)$ is a free abelian group $\bigoplus_{x\in X} L_x$. We write a divisor as $D = \sum D_x$ where $D_x \in L_x$. The support of D is the finite set $|D| := \{x \in X \mid D_x \neq 0\}$.

Note that our divisor group is the same as the classical divisor group when L is a constant sheaf $\mathbb{Z}$. The Gysin map associated to $U = X - \{x\}$ gives a homomorphism $L_x \to H^2(X, L)$. This extends to a homomorphism

$$\deg : \mathrm{Div}(X, L) \to H^2(X, L). \tag{6}$$

When $L = \mathbb{Z}$, we get the usual degree after identifying $H^2(X, L)$ with $\mathbb{Z}$. We denote the kernel of the map deg in (6) by $\mathrm{Div}^0(X, L)$.

Let $D = \sum D_x \in \mathrm{Div}^0(X, L)$ be a L-valued divisor with degree 0 and $|D|$ be the support of D. Let U be the complement of $|D|$. From the sequence (6), we get an extension of mixed Hodge structures

$$0 \longrightarrow H^1(X, L) \longrightarrow H^1(U, L) \longrightarrow K \longrightarrow 0 \tag{7}$$

where $K = \mathrm{Ker}[\bigoplus_{x\in|D|} L_x(-1) \xrightarrow{\text{Gysin}} H^2(X, L)]$. Let

$$\phi_D : \mathbb{Z}(-1) \longrightarrow \bigoplus_{x\in|D|} L_x(-1)$$

be a morphism of Hodge structures defined by $\phi_D(\frac{1}{2\pi\sqrt{-1}}) = \sum_x D_x$. As $\deg D = 0$, ϕ_D factors through K. By pulling back the above extension (7) along ϕ_D, we get a new extension of mixed Hodge structures:

$$0 \longrightarrow H^1(X, L) \longrightarrow E_D \longrightarrow \mathbb{Z}(-1) \longrightarrow 0, \tag{8}$$

where $E_D = H^1(U, L) \times_K \mathbb{Z}(-1)$. An element of $F^1 E_D$ can be identified with a form $\omega \in H^0(X, \Omega^1(\log|D|) \otimes L)$ such that for each $p \in |D|$, $\mathrm{Res}_p\omega = \lambda \otimes D_p$ for some constant $\lambda \in \mathbb{C}$ depending on ω but not on p.

The extension class of E_D is an element of $\mathrm{Ext}^1(\mathbb{Z}(-1), H^1(X, L))$. Thus by Theorem 3.1 and the discussion at the end of §4, it is an element of the Albanese torus $\mathrm{Alb}(X, L) = H^0(X, \Omega^1_X \otimes \check{L})^*/\Lambda$. The map

$$\alpha : \mathrm{Div}^0(X, L) \longrightarrow \mathrm{Alb}(X, L) \tag{9}$$

obtained in this way will be called the *Abel-Jacobi map.*

To give a more concrete description of the Abel-Jacobi map, we need a generalized version of the classical reciprocity law.

First, we construct a retraction $r\colon E_D \to H^1(X, L)$ to the natural inclusion $H^1(X, L) \hookrightarrow E_D$ (cf. the sequence (8)) in the following way. Choose a set $\{\xi_1, \dots, \xi_m\}$ of $\check{L}$-valued differential 1-forms on X representing a basis of $H^1(X, \check{L})$ such that ξ_i vanishes in a neighborhood $N(D)$ of $|D|$. Let $\{\xi^1, \dots, \xi^m\}$ be the dual basis of $H^1(X, L)$. We now set

$$r(\eta) = \sum_i \int_X \langle \eta, \xi_i \rangle \xi^i,$$

where η is a L-valued differential 1-form on U representing an element of E_D.

Second, let Δ be a fundamental domain in the universal covering of X whose boundary does not contain any point of $|D|$. Since Δ is simply connected, every closed form is exact. Thus for every $\omega \in H^0(X, \Omega^1_X \otimes \check{L})$, there exists a $\check{L}$-valued function $\int \omega \in \Gamma(\Delta, \mathscr{O}_X \otimes \check{L})$ with $d \int \omega = p^* \omega$ where $p : \Delta \to X$ is the natural map. Such a function $\int \omega$ is called a primitive of ω. With this notation, we now give a generalized version of the classical reciprocity law [GH, p. 230].

Lemma 6.1. *Let $\eta \in F^1 E_D$ and ω be a $\check{L}$-valued holomorphic 1-form representing a class in $H^0(X, \Omega^1_X \otimes \check{L})$.*

$$\int_X \langle r(\eta), \omega \rangle = 2\pi\sqrt{-1} \sum_{x \in |D|} \operatorname{Res}_x \left\langle \eta, \int \omega \right\rangle \tag{10}$$

where $\int \omega$ is a primitive of ω.

Proof. Let B_x be a small disk in X of $x \in |D|$ such that $B_x \cap |D| = \{x\}$ and the closure of $B(D) := \bigcup_{x \in |D|} B_x$ is contained in $N(D)$. We can write $\omega = \sum_{i=1}^m c_i \xi_i + df$ where $f \in \Gamma(X, \check{L})$. Now we have

$$\begin{aligned}
\int_X \langle r(\eta), \omega \rangle &= \int_X \left\langle r(\eta), \sum_i c_i \xi_i \right\rangle = \int_X \left\langle \sum_i \langle \eta, \xi_i \rangle \xi^i, \sum_i c_i \xi_i \right\rangle \\
&= \int_X \left\langle \eta, \sum_i c_i \xi_i \right\rangle = \int_{X - B(D)} \langle \eta, \omega - df \rangle \\
&= \int_{X - B(D)} -\langle \eta, df \rangle \\
&\quad \text{(since both } \eta \text{ and } \omega \text{ are holomorphic on } X - B(D), \\
&\qquad \langle \eta, \omega \rangle \text{ is a holomorphic 2-form on } X - B(D)) \\
&= \int_{\partial B(D)} \langle \eta, f \rangle \quad \text{(by Stokes' Theorem)} \\
&= \int_{\partial B(D)} \left\langle \eta, \int \omega \right\rangle \quad (\text{since } \omega = df \text{ on } B(D)) \\
&= 2\pi\sqrt{-1} \sum_{x \in |D|} \operatorname{Res}_x \left\langle \eta, \int \omega \right\rangle. \quad \square
\end{aligned}$$

To each divisor $D = \sum D_x \in \operatorname{Div}^0(X, L)$ of degree 0, one can associate a form $\eta_D \in H^0(X, \Omega^1_X(\log |D|) \otimes L) = F^1 H^1(U, L_{\mathbb{C}})$ with $\operatorname{Res}_x \eta_D = D_x$ for all $x \in |D|$ since the map in the sequence (7) strictly preserves the Hodge filtration $F^\bullet$.

Theorem 6.2. *The Abel-Jacobi map*

$$\alpha : \operatorname{Div}^0(X, L) \longrightarrow \operatorname{Alb}(X, L)$$

is given by

$$\alpha(D)(\omega) = \sum_{x \in |D|} \operatorname{Res}_x \left\langle \eta_D , \int \omega \right\rangle$$

where $\omega \in H^0(X, \Omega^1_X \otimes \check{L})$ *and* $\int \omega$ *is the primitive of* ω.

Proof. Recall that D corresponds to an extension class (cf. the sequence (8))

$$0 \longrightarrow H^1(X, L) \longrightarrow E_D \longrightarrow \mathbb{Z}(-1) \longrightarrow 0. \tag{11}$$

By Theorem 3.1 and Remark 1, this extension class gives an element $\frac{1}{2\pi\sqrt{-1}} r(\eta_D) \in H^1(X, L)$. Finally, through the period map p (cf. the sequence (3)), α is given by

$$\alpha(D)(\omega) = \frac{1}{2\pi\sqrt{-1}} \int_X \langle r(\eta), \omega \rangle$$

Now the previous lemma finishes the proof. □

Corollary 6.3. *Let* X *be a smooth projective curve over* $\mathbb{C}$ *of genus* g *and* $D = \sum(q_i - p_i) \in \operatorname{Div}^0(X, \mathbb{Z})$ *a divisor of degree* 0 *on* X. *Then the Abel-Jacobi map is given by*

$$\alpha(D)(\omega) = \sum \int_{p_i}^{q_i} \omega, \qquad \omega \in H^0(X, \Omega^1_X). \tag{12}$$

In other words, the Abel-Jacobi map α *is identical to the classical Abel-Jacobi map when* $L = \mathbb{Z}$.

Proof. The holomorphic function $w(z) = \int_{x_0}^{z} \omega$ is the primitive of ω. Thus Corollary 6.3 follows from the equality

$$\operatorname{Res}_x \left\langle \eta_D , \int_{x_0}^{z} \omega \right\rangle = (\operatorname{Res}_x \eta_D) \cdot \int_{x_0}^{x} \omega. \quad \square$$

7. Abel's theorem

We use the same notation as in §6.

Definition 7.1. The quotient sheaf $\overline{\mathscr{L}} := L \otimes \mathscr{O}_X / L$ is called the *sheaf of* L*-valued multiplicative functions.* The associated exact sequence

$$0 \longrightarrow L \longrightarrow L \otimes \mathscr{O}_X \xrightarrow{e} \overline{\mathscr{L}} \longrightarrow 0 \quad (e \text{ is the quotient map.})$$

is called the *exponential sequence* of L.

The multiplication by $2\pi\sqrt{-1}$ induces a morphism between the exponential sequence and the deRham sequence.

Diagram (1)

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "2\pi\sqrt{-1}"] & L \otimes \mathscr{O}_X \arrow[r, "e"] \arrow[d, "2\pi\sqrt{-1}"] & \overline{\mathscr{L}} \arrow[r] \arrow[d, "dl"] & 0 \\
0 \arrow[r] & L_{\mathbb{C}} \arrow[r] & L \otimes \mathscr{O}_X \arrow[r] & \Omega^1_X \otimes L \arrow[r] & 0
\end{tikzcd}$$

where dl is the induced map.

Since the lifting of the sheaf $L \otimes \mathscr{O}_X$ is isomorphic to $\mathscr{O}^n_{\tilde{X}}$ $(n = \operatorname{rank} L)$ under the universal covering $p : \tilde{X} \to X$, the sheaf $p^{-1}(\overline{\mathscr{L}})$ can be identified with $(\mathscr{O}^*_{\tilde{X}})^n$. Thus Diagram (1) lifts to the following diagram.

Diagram (2)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n_{\tilde{X}} \arrow[r] \arrow[d, "2\pi\sqrt{-1}"] & \mathscr{O}^n_{\tilde{X}} \arrow[r, "\exp 2\pi\sqrt{-1}"] \arrow[d, "2\pi\sqrt{-1}"] & (\mathscr{O}^*_{\tilde{X}})^n \arrow[r] \arrow[d, "d\log"] & 0 \\
0 \arrow[r] & \mathbb{C}^n_{\tilde{X}} \arrow[r] & \mathscr{O}^n_{\tilde{X}} \arrow[r] & (\Omega^1_{\tilde{X}})^n \arrow[r] & 0
\end{tikzcd}$$

Let $U \subset X$ be a Zariski open set and $f \in \Gamma(U, \overline{\mathscr{L}})$. We denote by $\tilde{f} \in \Gamma(p^{-1}(U), (\mathscr{O}^*_{\tilde{X}})^n)$ the lifting of f to the universal covering $\tilde{X}$. We say that f has *meromorphic singularities* if $\tilde{f}$ can be extended to an n-tuple of meromorphic functions on $\tilde{X}$.

Definition 7.2. Let $D = \sum D_x \in \operatorname{Div}^0(X, L)$ be a divisor of degree 0. D is called a *principal divisor* if there exists $f \in \Gamma(X - |D|, \overline{\mathscr{L}})$ with meromorphic singularities such that $\operatorname{Res}_x dl(f) = D_x$. f is said to be a *defining function* of D.

Remark 3. If $L = \mathbb{Z}$, a section $f \in \Gamma(U, \overline{\mathscr{L}})$ with meromorphic singularities is the same thing as a meromorphic function on X without zeros or poles in U. A divisor $D = \sum n_x x$ is principal if there exists a meromorphic function f with $\operatorname{ord}_x f = \operatorname{Res}_x d\log f = n_x$.

It is convenient to introduce the following notation. Let $A = (a_{ij})$ be an $n \times n$ matrix and $v = (v_j)$ a vector. We denote by v^A the vector $(\prod_j v_j^{a_{ij}})_{i=1,\dots,n}$.

Theorem 7.1. *Suppose that $D = \sum D_x \in \operatorname{Div}^0(X, L)$ be a divisor of degree 0. Then $\alpha(D) = 0$ if and only if D is a principal divisor. Moreover when*

$\alpha(D) = 0$, we can find a defining function f of D such that its lifting $\tilde{f}$ satisfies the functional equation $\tilde{f}(\delta \cdot \tilde{x}) = \tilde{f}(\tilde{x})^{\rho_L(\delta)}$ where $\delta \in \pi_1(X, x_0)$.

Proof. If D is principal, then there exists $f \in \Gamma(X - |D|, \overline{\mathscr{L}})$ with meromorphic singularities such that $\operatorname{Res} dl(f) = D$ by definition. Furthermore, it can be checked that the form $dl(f)$ lies in $H^1(U, L)$. Thus the extension (8)

$$0 \longrightarrow H^1(X, L) \longrightarrow E_D \longrightarrow \mathbb{Z}(-1) \longrightarrow 0,$$

splits in the category of mixed Hodge structures. Thus $\alpha(D) = 0$.

Conversely, if $\alpha(D) = 0$, this extension splits. Hence there is a form $\eta_D \in H^0(X, \Omega^1_X(\log|D|) \otimes L)$ such that $\operatorname{Res}_x \eta_D = D_x$ and $\eta_D \in H^1(U, L)$. Consider the following commutative diagram of cohomologies associated to Diagram (1).

$$\begin{tikzcd}
H^0(U, \overline{\mathscr{L}}) \arrow[r, "\delta"] \arrow[d, "dl"] & H^1(U, L) \arrow[d, "2\pi\sqrt{-1}"] \\
H^0(U, \Omega^1_U \otimes L) \arrow[r] & H^1(U, L_{\mathbb{C}})
\end{tikzcd}$$

The connecting map δ is surjective since U is Stein. Thus there is a function $f \in H^0(U, \overline{\mathscr{L}})$ such that $2\pi\sqrt{-1}\delta(f) = dl(f) = 2\pi\sqrt{-1}\eta_D$.

To see that f has meromorphic singularities, note that $\tilde{\eta}_D := \eta_D \circ p \in H^0(\tilde{X}, (\Omega^1_{\tilde{X}})^n(\log(p^{-1}(|D|))))$ has only simple poles. Thus the system of differential equations

$$d \log w = 2\pi\sqrt{-1}\tilde{\eta}_D \tag{13}$$

has regular singularities along $p^{-1}(|D|)$. By a theorem of Fuchs [B, Theorem 1.1.1 on p. 130], its solution $\tilde{f}$ is meromorphic at $p^{-1}(|D|)$.

Finally to get a functional equation, observe that $\tilde{f}(\tilde{x}) = \exp(2\pi\sqrt{-1}\int_{\tilde{x}_0}^{\tilde{x}} \tilde{\eta}_D)$ satisfies the system of differential equations (13). Hence we have

$$\begin{aligned}
\tilde{f}(\delta \cdot \tilde{x}) &= \exp\left(2\pi\sqrt{-1}\int_{\tilde{x}_0}^{\delta\cdot\tilde{x}} \tilde{\eta}_D\right) \\
&= \exp\left(2\pi\sqrt{-1}\int_{\tilde{x}_0}^{\delta\cdot\tilde{x}_0} \tilde{\eta}_D + 2\pi\sqrt{-1}\int_{\delta\cdot\tilde{x}_0}^{\delta\cdot\tilde{x}} \tilde{\eta}_D\right) \\
&= \exp\left(2\pi\sqrt{-1}\int_{\tilde{x}_0}^{\delta\cdot\tilde{x}_0} \tilde{\eta}_D + \rho_L(\delta)(2\pi\sqrt{-1}\int_{\tilde{x}_0}^{\tilde{x}} \tilde{\eta}_D)\right) \\
&= \exp\left(\rho_L(\delta)\left(2\pi\sqrt{-1}\int_{\tilde{x}_0}^{\tilde{x}} \tilde{\eta}_D\right)\right) \quad (\text{since } \eta_D \in H^1(U, L)) \\
&= \tilde{f}(\tilde{x})^{\rho_L(\delta)}. \quad \square
\end{aligned}$$

When $L = \mathbb{Z}$, we obtain the classical Abel theorem.

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