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# NOETHERIAN PROPERTIES OF SKEW POLYNOMIAL RINGS WITH BINOMIAL RELATIONS

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ABSTRACT. In this work we study standard finitely presented associative algebras over a fixed field $K$. A restricted class of skew polynomial rings with quadratic relations considered in an earlier work of M. Artin and W. Schelter will be studied. We call them binomial skew polynomial algebras. We establish necessary and sufficient conditions for such an algebra to be a Noetherian domain.

## 1. INTRODUCTION

In this paper we work with graded associative algebras over a fixed field $K$. Given a nonempty set $X = \{x_1, \dots, x_n\}$, $\langle X \rangle$ will denote the free monoid with unit generated by $X$, $K\langle X \rangle$ will denote the free associative algebra (with 1) generated by $X$. We fix the degree-lexicographic order $<*$ on $\langle X \rangle$ (we set $x_1 <* x_2 <* \cdots <* x_n$).

(1.1) An *ordered monomial* in $\{x_1, \dots, x_n\}$ is a monomial of the type $x_1^{t_1} \cdots x_n^{t_n}$, $t_i \geq 0$. By $N$ we shall denote the set of all ordered monomials. Given a polynomial $f$ in $K\langle X \rangle$, $HM(f)$ will denote its highest monomial. For any subset $F$ of $K\langle X \rangle$, $(F)$ will denote the two-sided ideal generated by $F$.

(1.2) Let $F_0$ be a set of polynomials in $K\langle X \rangle$ of the type

$$F_0 = \{x_j x_i - f_{ji} | 1 \leq i < j \leq n\},$$

where for $1 \leq i < j \leq n$, $f_{ji}$ is a linear combination of ordered monomials of degree 2, and $HM(f_{ji}) <* x_j x_i$. We shall recall the following

(1.3) **Definition.** A monomial $u$ is *normal* (modulo $F_0$) if it does not contain as a segment any of the monomials $x_j x_i$, $1 \leq i < j \leq n$.

It is clear that a monomial is normal $(\operatorname{mod} F_0)$ if and only if it is an ordered monomial.

We shall recall now some facts extracted from Bergman's Diamond Lemma [Berg] in the particular case, when $\langle X \rangle$ is ordered by the degree-lexicographic ordering, and the set $F_0$ is as in (1.2).

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Consider the $K$-linear operators (we call them *reductions*)

$$r_{u,j,i,v}, \quad \text{for } u, v \in \langle X\rangle, \ 1 \le i < j \le n,$$

defined on the underlying vector space of $K\langle X\rangle$ by the formulas

$$r_{u,j,i,v}(ux_jx_iv) = uf_{ji}v,$$
$$r_{u,j,i,v}(w) = w, \quad \text{for all } w \ne ux_jx_iv.$$

It is well known, cf. [Berg], that for any polynomial $f$ in $K\langle X\rangle$ there exists a finite sequence of reductions, $r_1, r_2, \dots, r_s$, such that

$$f \xrightarrow{r_1} f_1 \xrightarrow{r_2} f_2 \to \cdots \xrightarrow{r_s} f_s,$$

where $f - f_s$ is in the ideal $(F_0)$, and $f_s$ is a linear combination of ordered (or equivalently normal) monomials. In general $f_s$ is not uniquely determined. It follows from Bergman's Diamond Lemma that

(1.4) [Berg]. *The following conditions are equivalent*:

(a) *There is an isomorphism of vector spaces* $K\langle X\rangle \cong \operatorname{Span} N \oplus (F_0)$;
(b) *For any triple* $(k, j, i)$, *where* $n \ge k > j > i \ge 1$, *the difference* $f_{kj}x_i - x_kf_{ji}$ *can be reduced to zero* (*by means of a finite sequence of reductions*).

In particular, if this is the case for $F_0$, the set of ordered monomials $N$ projects to a basis (as a vector space) of the algebra $A = K\langle X\rangle/(F_0)$.

(1.5) It is clear that in this case for any polynomial $f$ in $K\langle X\rangle$, one has $f = \operatorname{Nor}(f) + h$, where $\operatorname{Nor}(f) \in \operatorname{Span} N$, and $h \in (F_0)$ are uniquely determined. The element $\operatorname{Nor}(f)$ is called *the normal form of* $f$.

(1.6) **Definition.** The set $F_0$ is called a *Groebner basis for the ideal* $(F_0)$ if it satisfies the equivalent conditions (1.4(a), (b)).

For general references on Groebner bases see [Berg, Buch, Gol, G-I$_1$, Mor$_1$, Mor$_2$ and K-R-W].

(1.7) **Definition** [Art-S]. An algebra $A$ is a *skew polynomial ring* if it can be presented as $A = K\langle X\rangle/(F_0)$, where the set of relations $F_0$ is as in (1.2) and is a Groebner basis of the ideal $(F_0)$.

In this paper we shall work with a particular case of skew polynomial rings, namely with "binomial" skew polynomial rings.

(1.8) A skew polynomial ring $A = K\langle X\rangle/(F)$ is called *binomial* if the Groebner basis $F$ is of the form:

$$(1.9) \quad F = \{x_jx_i - a_{ij}x_{i'}x_{j'} \mid 1 \le i < j \le n\}, \quad \text{where } 0 \ne a_{ij} \in K, \ 1 \le i' \le j' \le n, \ x_{i'}x_{j'} <* \, x_jx_i.$$

(1.10) We shall always assume that $i' < j'$, excluding the case $i' = j'$.

Note that in general the set $\{x_{i'}x_{j'} \mid 1 \le i < j \le n\}$ contains at most $\binom{n}{2}$ elements; i.e., we do not assume that all the monomials $x_{i'}x_{j'}$, for $1 \le i < j \le n$, are pairwise different.

**Examples.**

(1.11) $$A = K\langle x_1, x_2, x_3\rangle/(x_3x_2 - x_2x_3, x_3x_1 - x_1x_2, x_2x_1 - x_1x_2),$$

(1.12) $$A = K\langle x_1, x_2, x_3\rangle/(x_3x_2 - x_1x_3, x_3x_1 - x_1x_3, x_2x_1 - x_1x_2),$$

(1.13) $$\begin{aligned}A = K\langle x_1, x_2, x_3, x_4\rangle/(x_4x_3 - x_3x_4, x_4x_2 - x_2x_4, x_4x_1 - x_1x_3,\\ x_3x_2 - x_2x_3, x_3x_1 - x_1x_2, x_2x_1 - x_1x_4).\end{aligned}$$

We are interested in the question when a binomial skew polynomial algebra $A$ is Noetherian.

There are various results on the Noetherianness of algebras with quadratic relations [Ap, G-I$_1$, G-I$_2$, Mor$_1$, K-R-W, Sm-St].

Before formulating the main results of this paper we need some more notation. From now on we shall always assume that the set $F$ is fixed. It is clear that the normal form of any monomial $w$ in $\langle X\rangle$ is of the type $cw_0$, where $c \in K$ and $w_0 \in N$.

(1.14) Given two normal monomials $u$ and $v$, by $u \star v$ we shall denote the monomial which appears in the normal form of $u \cdot v$. (In other words we ignore the coefficient appearing in the normal form of $u \cdot v$.) Clearly, for any pair $(j, i)$, $1 \le i < j \le n$, one has

(1.15) $$x_j \star x_i = x_{i'} \cdot x_{j'}, \text{ where } i' \text{ and } j' \text{ are as in (1.9).}$$

The main results of this paper are contained in Theorems A, B, and C.

(1.16) **Theorem A.** *Let $A = K\langle x_1, \dots, x_n\rangle/(F)$ be a binomial skew polynomial ring without zero divisors, with reduced Groebner basis*

$$F = \{x_jx_i - a_{ij}x_{i'}x_{j'} | 1 \le i < j \le n\},$$

*where for $1 \le i < j \le n$ one has $1 \le i' < j' \le n$, and $x_{i'}x_{j'} <\ast\ x_jx_i$. Suppose furthermore that*

(1.17) *The set $\{x_{i'}x_{j'} | 1 \le i < j \le n\}$ contains precisely $\binom{n}{2}$ elements.*

*Then the algebra $A$ is cyclic, i.e., the following condition is satisfied:*

(1.18) *For any $j$ and $k$, $1 \le k < j \le n$, there exists a $p > k$ and a cycle $\sigma = (k, k_1, k_2, \dots, k_s)$ in the symmetric group $S_j$, where all $k_i < p, j$, such that*

$$\begin{aligned} x_j \star x_k &= x_{k_1} \cdot x_p, \\ x_j \star x_{k_1} &= x_{k_2} \cdot x_p, \\ &\vdots \\ x_j \star x_{k_s} &= x_k \cdot x_p. \end{aligned}$$

(1.19) **Theorem B.** *Suppose $A = K\langle x_1, \dots, x_n\rangle/(F)$ is a binomial skew polynomial ring, with reduced Groebner basis*

$$F = \{x_jx_i - a_{ij}x_{i'}x_{j'} | 1 \le i < j \le n\}.$$

*Suppose, furthermore, that for some positive integer $P$, the following condition is satisfied:*

(1.20) *For any pair of integers* $q, k$, $1 \leq k < q \leq n$, *there exists a* $j$, $k < j \leq n$, *such that*

$$(x_j)^P \star x_k = x_k \cdot (x_q)^P.$$

*Then* $A$ *is left Noetherian.*

(1.21) **Theorem C.** *Let* $A = K\langle x_1, \dots, x_n\rangle/(F)$ *be a binomial skew polynomial ring without zero divisors, with reduced Groebner basis*

$$F = \{x_j x_i - a_{ij} x_{i'} x_{j'} | 1 \leq i < j \leq n\},$$

*where for* $1 \leq i < j \leq n$ *one has* $1 \leq i' < j' \leq n$, *and* $x_{i'} x_{j'} < * \, x_j x_i$. *Then the following three conditions are equivalent:*

(i) *The set* $\{x_{i'} x_{j'} | 1 \leq i < j \leq n\}$ *contains precisely* $\binom{n}{2}$ *elements.*
(ii) $A$ *is left Noetherian.*
(iii) $A$ *is right Noetherian.*

(1.22) *Remark.* Further results on Noetherian binomial rings are obtained in [G-I$_3$].

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## 2. The cyclic condition

In this section we shall prove Theorem A. All conventions and definitions made in the introduction shall be in force throughout the paper. We begin with some preliminary results, and we assume the hypotheses of the theorem.

(2.1) *Remark.* Since the Groebner basis $F$ consists of binomial relations only, it is clear that the normal form of any monomial $u$ is of the type $c \cdot v$ where $c$ is a nonzero coefficient in $K$ and $v$ is a normal monomial. It follows from the Diamond Lemma that both $c$ and $v$ are uniquely determined. We can ignore $c$ and write $u \to v$ to denote that $v$ is the result of "almost normalisation" of $u$. It is clear that in order to obtain $v$ it will be enough to use "almost reductions" instead of usual reductions, more precisely: For any $i, j$, $1 \leq i < j \leq n$, we shall actually need a simpler "almost reduction" replacing $x_j \cdot x_i$ with $x_{i'} \cdot x_{j'}$ instead of replacing it by $a_{ij} \cdot x_{i'} \cdot x_{j'}$, which the usual reduction does.

For arbitrary monomials $v$ and $w$, we shall write $v \cdot [x_j \cdot x_i] \cdot w \to v \cdot (x_{i'} \cdot x_{j'}) \cdot w$ to denote that we have replaced the monomial $x_j \cdot x_i$ by $x_{i'} \cdot x_{j'}$ (or equivalently by $x_j \star x_i$).

(2.2) *Remark.* Under the hypothesis of Theorem A it is clear that for any pair of integers $i, j$, $1 \leq i < j \leq n$, there exist uniquely determined $q$ and $p$, $1 \leq p < q \leq n$, such that $x_q \star x_p = x_i \cdot x_j$.

(2.3) **Lemma.** *Let* $i$ *and* $j$ *be integers,* $1 \leq i < j \leq n$. *Then for* $i'$ *and* $j'$ *as in* (1.9), (1.10) *one has* $i' < j$, *and* $i < j'$.

*Proof.* The inequality $x_{i'} x_{j'} < * \, x_j x_i$ implies $i' \leq j$. Thus $i' < j$, since $A$ has no zero divisors. We shall prove now that $i < j'$. Note first that $j' \neq i$, since $A$ has no zero divisors. Assume that $j' < i$. Consider the two possible ways of "almost normalisation" of the monomial $x_j \cdot x_i \cdot x_{j'}$:

(2.4) $$[x_j \cdot x_i] \cdot x_{j'} \to (x_{i'} \cdot x_{j'}) \cdot x_{j'} = u \in N,$$

and

(2.5) $$x_j \cdot [x_i \cdot x_{j'}] \to x_j \cdot (x_r \cdot x_s) \to \cdots \to u \in N,$$

where

(2.6) $$r < i, s.$$

It follows from (1.10) that the final replacement in (2.5) should be of the type

$$[x_m \cdot x_k] \cdot x_{j'} \to (x_{i'} \cdot x_{j'}) \cdot x_{j'},$$

for some $m$ and $k$, $m > k$. It follows from here that

(2.7) $$x_m \star x_k = x_{i'} \cdot x_{j'} = x_j \star x_i.$$

The inequality (2.6) and the fact that after any replacement the result is a monomial which is strictly less than the previous one, imply that the pairs $(m, k)$ and $(j, i)$ are different, thus (2.7) contradicts Remark (2.2).

Lemma (2.3) has been proved.

(2.8) **Lemma** (prohibiting an approach from below). *Suppose* $x_j \star x_k = x_m \cdot x_p$, *where* $j > k$, $p > m$. *Suppose also that* $x_a \star x_b = x_k \cdot x_p$, *for some* $a$ *and* $b$, $a > b$. *Then* $a \geq j$.

*Proof*. Assume the contrary, i.e., $a < j$. Consider the following replacements:

(2.9) $$x_j \cdot [x_a \cdot x_b] \to x_j \cdot (x_k \cdot x_p) = [x_j \cdot x_k] \cdot x_p \to (x_m \cdot x_p) \cdot x_p = u \in N$$

and

(2.10) $$[x_j \cdot x_a] \cdot x_b \to f = (x_v \cdot x_w) \cdot x_b \to \cdots \to u,$$

where

(2.11) $$v < j.$$

It follows from (1.10) that as a final replacement in (2.10) one has

$$g = [x_s \cdot x_t] \cdot x_p \to (x_m \cdot x_p) \cdot x_p,$$

where $s > t$, and

(2.12) $$x_s \star x_t = x_m \cdot x_p.$$

Since the monomial $g$ is obtained from $f$ as a result of a finite sequence of replacements (or $g = f$), one has

$$f = x_v \cdot x_w \cdot x_q \ast\!\geq x_s \cdot x_t \cdot x_p = g,$$

which implies $s \leq v$, and by (2.11) one has

(2.13) $$s < j.$$

By the hypothesis of the lemma we have

$$x_j \star x_k = x_m \cdot x_p,$$

which together with (2.12) implies

$$x_j \star x_k = x_s \star x_t\,, \tag{2.14}$$

The pairs $(j, k)$ and $(s, t)$ are different by (2.13). Thus, by Remark (2.2) the equality (2.14) is impossible, a contradiction, due to the assumption that $a < j$. We have proved the lemma.

(2.15) **Inductive Lemma.** *Under the hypothesis of Theorem* A *let* $j_0$ *be an integer,* $1 < j_0 < n$, *such that*

(2.16) *For any* $j > j_0$ *and* $k$, $1 \le k < j$, *there exists a* $p > k$ *and a cycle* $(k, k_1, \dots, k_s)$ *such that*

$$\begin{aligned} x_j \star x_k &= x_{k_1} \cdot x_p\,, \\ x_j \star x_{k_1} &= x_{k_2} \cdot x_p\,, \\ &\vdots \\ x_j \star x_{k_s} &= x_k \cdot x_p. \end{aligned}$$

*Then the condition* (2.16) *holds for* $j = j_0$.

Under the hypothesis of (2.15) we shall first prove some facts.

(2.17) **Lemma.** *Let* $k, j_1, j_2$, *be integers such that* $k \le j_0 < j_1, j_2$, $j_1 \ne j_2$ *and let*

$$x_{j_1} \star x_k = x_{k_1} \cdot x_{p_1} \quad \textit{and} \quad x_{j_2} \star x_k = x_{k_2} \cdot x_{p_2}\,,$$

*for some* $k_1, k_2, p_1$, *and* $p_2$. *Then* $p_1 \ne p_2$.

*Proof*. It follows from (2.16) that there exist $m_1$ and $m_2$ such that

$$x_{j_1} \star x_{m_1} = x_k \cdot x_{p_1} \quad \text{and} \quad x_{j_2} \star x_{m_2} = x_k \cdot x_{p_2}.$$

The pairs $(j_1, m_1)$, $(j_2, m_2)$ are different since by hypothesis $j_1 \ne j_2$. Thus, by Remark (2.2.), $p_1 \ne p_2$.

(2.18) **Corollary.** *For any* $j > j_0$ *there exists an* $a > j_0$, *such that*

$$x_a \star x_{j_0} = x_k \cdot x_j\,,$$

*for some* $k$, $k < j$.

*Proof*. For $j_0 < j \le n$, let $p_j > j_0$ be the integer, determined by the equality

$$x_j \star x_{j_0} = x_{k_j} \cdot x_{p_j} \in N\,, \quad \text{for some } k_j.$$

It follows from the previous lemma that all the elements $p_n, \dots, p_{(j_0+1)}$ are pairwise different. Hence there is an equality of sets:

$$\{p_{(j_0+1)}, \dots, p_n\} = \{j_0 + 1, \dots, n\}\,,$$

which proves the corollary.

(2.19) **Lemma** (taking the preimage). *Let* $k < j_0$, *and let* $x_{j_0} \star x_k = x_{k_1} \cdot x_p$, *for some* $p$ *and* $k_1$, $k_1 < j_0$. *Then there exists a uniquely determined* $s < j_0$ *such that*

$$x_{j_0} \star x_s = x_k \cdot x_p.$$

*Proof*. By Remark (2.2), there exist uniquely determined $j$ and $s$, $j > s$, such that $x_j \star x_s = x_k \cdot x_p$. It follows from Lemma (2.8) that $j \geq j_0$. Assume $j > j_0$. It follows then from (2.16) and (2.3) that

(2.20) $$x_j \star x_k = x_{k'} \cdot x_p ,$$

for some $k'$,

(2.21) $$k' < j, \quad k < p.$$

By (2.18) there exists an $a > j_0$, such that

(2.22) $$x_a \star x_{j_0} = x_q \cdot x_j , \quad \text{for some } q.$$

It follows from (2.16) that

(2.23) $$x_a \star x_q = x_{q_1} \cdot x_j , \quad \text{for some } q_1.$$

Consider now the sequences of replacements:

(2.24) $[x_a \cdot x_{j_0}] \cdot x_k \to (x_q \cdot x_j) \cdot x_k = x_q \cdot [x_j \cdot x_k] \to x_q \cdot x_{k'} \cdot x_p$ by (2.20), (2.22),

and

(2.25) $$x_a \cdot [x_{j_0} \cdot x_k] \to x_a \cdot x_{k_1} \cdot x_p ,$$

by the hypothesis of the lemma. It follows from (2.24) and (2.25) that

$$(x_q \star x_{k'}) \star x_p = (x_a \star x_{k_1}) \star x_p ,$$

which implies

(2.26) $$x_q \star x_{k'} = x_a \star x_{k_1} ,$$

since $A$ has no zero divisors.

Clearly, the pairs $(q, k')$ and $(a, k_1)$ are different. By the hypothesis of the lemma, $k_1 < j_0$, hence, since $a > j_0$, the equality (2.26) is possible only in case that $q < k'$. Thus (2.26) can be written as

$$x_a \star x_{k_1} = x_q \cdot x_{k'}.$$

This, together with (2.16) gives

$$x_a \star x_q = x_{q'} \cdot x_{k'} , \quad \text{for some } q'.$$

Then it follows from (2.23) that

$$x_{q_1} \cdot x_j = x_{q'} \cdot x_{k'}.$$

Note that this is an equality of two normal monomials. Thus, $j = k'$, which contradicts (2.21). We have proved that the inequality $j > j_0$ is impossible, hence $j = j_0$ and we are done.

(2.27) *Proof of the inductive lemma.* Assume $j_0 > 1$. Let $k < j_0$, and let

$$x_{j_0} \star x_k = x_m \cdot x_p$$

for some $m$ and $p$, $p > k, m$. We shall prove that there exists a finite sequence of pairwise different integers

(2.28) $$c_0 = k, c_1, \ldots, c_{s-1}, c_s = m,$$

such that

$$1 \le c_i < j_0 , \qquad s < j_0 , \tag{2.29}$$

$$x_{j_0} \star x_{c_i} = x_{c_{(i-1)}} \cdot x_p \quad \text{for } 1 \le i \le s , \tag{2.30}$$

and

$$x_{j_0} \star x_k = x_{c_s} \cdot x_p. \tag{2.31}$$

For $i = 1$ it follows from Lemma (2.19) (taking the preimage) that there exists a $c_1$, uniquely determined by the equalities (2.30) and (2.29). If $c_1 = k$ we are done. Otherwise, we shall use the following procedure. Assume a sequence of pairwise different integers $c_1 , \dots , c_{(r-1)}$ has already been found such that

$$c_i \neq k , \quad \text{for } 1 \le i < r - 1 , \tag{2.32}$$

and (2.29), (2.30) hold for all $i$, $1 \le i \le r - 1$. Furthermore, assume that

$$c_{(r-1)} \neq k. \tag{2.33}$$

Then, by Lemma (2.19) there exists a $c_r$, uniquely determined by the equality (2.30), where $i = r$. Note that

$$c_r \neq c_i \quad \text{for all } i, \ 1 \le i \le r - 1. \tag{2.34}$$

Indeed, from the equality $c_r = c_i$ and from (2.30) it follows that

$$x_{c_{(r-1)}} \cdot x_p = x_{j_0} \star x_{c_r} = x_{j_0} \star x_{c_i} = x_{c_{(i-1)}} \cdot x_p ,$$

thus, since the algebra $A$ has no zero divisors, one has $c_{(r-1)} = c_{(i-1)}$, a contradiction with the choice of $c_1 , \dots , c_{(r-1)}$. Hence (2.34) holds. If $c_r = k$, we are done. Otherwise we continue applying the above procedure. Recall that all the $c_i$'s are pairwise different, and $c_i < j_0$. It is clear then that in finitely many steps we should find a $c_s$, such that $c_s = k$. This proves the Inductive Lemma.

(2.35) *Proof of Theorem* A (1.16). We shall use decreasing induction on $j$. The base of the induction: $n = j$. Similarly to the proof of Lemma 2.23 one can show that if $k < n$ and $x_n \star x_k = x_m \cdot x_p$, for some $m$ and $p$ then there is a uniquely determined $c$, such that $x_n \star x_c = x_k \cdot x_p$. It is now clear that using the same argument as in the proof of the Inductive Lemma (cf. (2.27)) one can show that the cyclic condition (1.18) holds, but with $j = n$. It is now enough to apply the Inductive Lemma (2.15). □

Similarly to the proof of Lemma (2.17) and Corollary (2.18) one can easily see that

(2.36) *Remark.* Let $k$ be an integer, $k < n$. Then for any $p$, $k < p \le n$, there exists a $j$, $j > k$, such that

$$x_j \star x_k = x_m \cdot x_p , \qquad \text{for some } m.$$

(2.37) **Corollary.** *Let* $k , j , p , s = s(k , j)$ *be as in* (1.18). *Then*

$$(x_j)^{s+1} \star x_k = x_k \cdot (x_p)^{s+1} ,$$

*and therefore, since* $s+1$ *divides* $n!$,

$$(x_j)^P \star x_k = x_k \cdot (x_p)^P, \tag{2.38}$$

*where* $P = n!$.

(2.39) **Corollary.** *Suppose that the hypothesis of Theorem* A (1.16) *is satisfied. Then for* $P = n!$ *the following two conditions hold:*

(i) *For any pair of integers* $q, k$, $1 \le k < q \le n$, *there exists a* $j$, $k < j \le n$, *such that*

$$(x_j)^P \star x_k = x_k \cdot (x_q)^P.$$

(ii) *For any pair of integers* $j, k$, $1 \le k < j \le n$, *there exists a* $p$, $k < p \le n$, *such that*

$$(x_j)^P \star x_k = x_k \cdot (x_p)^P.$$

*Proof.* Assertion (i) follows from Remark (2.36) and Corollary (2.37). The cyclic condition (1.18) and Corollary (2.37) give (ii).

## 3. Sufficient conditions for left Noetherianness

In this section we shall prove Theorem B of the Introduction. Here the restriction that $A$ has no zero divisors is not necessary. It will be enough to assume as in (1.8), (1.9) that all the coefficients $a_{ij}$ are nonzero. We begin with some preliminary technical results, assuming, as before that the hypotheses of the theorem to be proved are in force, i.e.,

There exists a positive integer $P$, so that the following condition is satisfied:

(3.1) *For any pair* $q, k$, $1 \le k < q \le n$ *there exists a* $j$, $k < j \le n$, *such that*

$$(x_j)^P \star x_k = x_k \cdot (x_q)^P.$$

It follows immediately from (3.1) that

(3.2) *For any pair* $j, k$, $1 \le k < j \le n$ *there exists a* $q$, $k < q \le n$, *such that*

$$(x_j)^P \star x_k = x_k \cdot (x_q)^P.$$

(3.3) **Definition.** (a) Let $u$ and $v$ be two ordered monomials,

$$u = x_1^{s_1} \cdots x_n^{s_n}, \qquad s_i \ge 0,$$

and

$$v = x_1^{t_1} \cdots x_n^{t_n}, \qquad t_i \ge 0.$$

We say that $v$ is a *P-multiple of* $u$ if for all $i$, $1 \le i \le n$, one has

$$t_i = s_i + r_i \cdot P, \quad \text{for some } r_i \ge 0.$$

(b) We call a normal monomial $w$ a *P-monomial* if $w = x_1^{r_1 P} \cdots x_n^{r_n P}$, for some $r_1, \dots, r_n$, $r_i \ge 0$.

Applying (3.1) and (3.2) one can easily see that

(3.4) **Lemma.** *If $u$ and $v$ are two normal monomials, and $v$ is a $P$-multiple of $u$, then there exists a $P$-monomial $W$ such that $W \star u = v$.*

(3.5) **Lemma.** *Let $u$ and $v$ be two normal monomials. Assume that $u <* v$. Let $j$ be an integer, $1 \le j \le n$. Then*

$$(x_j)^P \star u <* (x_j)^P \star v. \tag{3.6}$$

*Proof*. Since the algebra $A$ is graded, it is obvious that (3.6) holds in case that $\deg u < \deg v$. Assume now that $\deg u = \deg v$. It follows from the inequality $u <* v$ that $u = w \cdot x_i \cdot f$, $v = w \cdot x_k \cdot g$ for some integers $i, k$, $1 \le i < k \le n$, and normal monomials $w, f, g$. ($w = 1$, or $f = g = 1$ is also possible.) Let $x_j^P \star w = w_1$. Two cases arise.

*Case* 1. The monomial $w_1 \cdot x_i$ is normal. Then, obviously, the monomial $w_1 \cdot x_k$ is normal as well and one has

$$(x_j)^P \star u = w_1 \cdot x_i \cdot f <* w_1 \cdot x_k \cdot g = (x_j)^P \star v.$$

*Case* 2. The monomial $w_1 \cdot x_i$ is not normal. Applying (3.2), one can easily see in this case that there exists an integer $a$, $1 < a \le n$, such that

$$w_1 = (x_j)^P \star w = w \cdot (x_a)^P. \tag{3.7}$$

This gives $a > i$. It follows again from (3.2) that

$$(x_a)^P \star x_i = x_i \star (x_b)^P, \quad \text{for some } b > i. \tag{3.8}$$

Consider the equalities

$$\begin{aligned} (x_j)^P \star u &= ((x_j)^P \star w) \star (x_i \cdot f) \\ &= w \cdot (((x_a)^P \star x_i) \star f) \quad \text{by (3.7)} \\ &= w \cdot x_i \cdot ((x_b)^P \star f) \quad \text{by (3.8)} \\ &= w \cdot x_i \cdot f_1, \quad \text{for } f_1 = (x_b)^P \star f. \end{aligned}$$

Hence

$$(x_j)^P \star u = w \cdot x_i \cdot f_1 \in N. \tag{3.9}$$

*Case* 2.a. $k \ge a > i$. Then

$$\begin{aligned} (x_j)^P \star v &= ((x_j)^P \star w) \star (x_k \cdot g) \\ &= w \cdot (((x_a)^P \star x_k) \star g) \quad \text{by (3.7)} \\ &= w \cdot (x_a)^P \cdot x_k \cdot g \\ &= w \cdot x_a \cdot g_1, \end{aligned}$$

for $g_1 = (x_a)^{P-1} \cdot x_k \cdot g$. Thus

$$(x_j)^P \star v = w \cdot x_a \cdot g_1. \tag{3.10}$$

Obviously,

$$w \cdot x_i \cdot f_1 <* w \cdot x_a \cdot g_1.$$

This together with (3.9) and (3.10) gives us the desired inequality (3.6).

*Case* 2.6. $a > k$. It follows from (3.2) that there is a $c$, $c > k$, such that

$$(x_a)^P \star x_k = x_k \cdot (x_c)^P$$

and

(3.11) $$(x_j)^P \star v = w \cdot x_k \cdot ((x_c)^P \star g) = w \cdot x_k \cdot g_1 ,$$

where $g_1 = ((x_c)^P \star g)$. If we compare the right-hand sides of the equalities (3.11) and (3.9) we again obtain (3.6). □

(3.12) **Corollary.** *Let $u$ and $v$ be normal monomials, such that $u <* v$. Let $W$ be a P-monomial. Then*

$$W \star u <* W \star v.$$

(3.13) *Warning.* Note that it is not clear whether for $W$, $u$, and $v$ as in (3.12) one has $v \star W* > u \star W$.

The following result can easily be obtained from the original Dickson Lemma, cf. [K-R-W].

(3.14) **A "$P$-generalisation" of the Dickson Lemma.** *Let $u_1, u_2, \dots, u_s, \dots$ be a sequence of ordered monomials. Then there exists an integer $i$ such that for any $k > i$, there is a $j = j(k)$, $j \leq i$, such that $u_k$ is a $P$-multiple of $u_j$.*

(3.15) *Proof of Theorem* B (1.19). Let $J$ be a left ideal in $A$. We shall prove that $J$ is finitely generated as a left ideal. Let $U$ be the set of all highest monomials of the elements of $J$. It is clear that $U$ is a countable set. We can always assume that $U = \{u_1, u_2, \dots, u_k, \dots\}$, where $u_1 <* u_2 <* \cdots <* u_k <* \cdots$.

By Lemma (3.14), there exists an integer $i_0$, such that for any $k > i_0$, there is an $i = i(k)$, $i \leq i_0$, such that $u_k$ is a $P$-multiple of $u_i$. Let $f_1, f_2, \dots, f_{i_0}$ be elements of $J$, with highest monomials respectively $u_1, u_2, \dots, u_{i_0}$. We can always assume that $f_i = u_i + g_i$, for some polynomial $g_i$, such that $HM(g_i) <* u_i$. We shall prove that the polynomials $f_1, \dots, f_{i_0}$ generate $J$ as a left ideal. Let $J_0$ be the left ideal, generated by $f_1, \dots, f_{i_0}$. Obviously $J_0 \subseteq J$. Assume that $J_0 \neq J$. Let $f \in J \backslash J_0$ be a polynomial with minimal highest monomial. By the definition of $U$ there is a $k$, such that $u_k$ is the highest monomial of $f$, i.e., $f = c \cdot u_k + g$, where $c \in K$, $g = a_1 \cdot v_1 + \cdots + a_s \cdot v_s$, and $v_j <* u_k$, for $1 \leq j \leq s$. We can always assume that $c = 1$. By the choice of $i_0$, there exists an $i$, $i \leq i_0$, such that $u_k$ is a $P$-multiple of $u_i$.

It follows from Lemma (3.4) that there exists a normal $P$-monomial $W$ such that $W \star u_i = u_k$. Note that by Corollary (3.12), $W \star u_i$ is the highest monomial of the polynomial $\mathrm{Nor}(W \cdot f_i)$, where $\mathrm{Nor}(W \cdot f_i)$ is the normal form of the element $W \cdot f_i$. Consider now the polynomial

$$h = f - \frac{1}{\alpha} \mathrm{Nor}(W \cdot f_i),$$

where $\alpha$ is the coefficient of the highest monomial of $\mathrm{Nor}(W \cdot f_i)$. It is clear that $h \in J \backslash J_0$, and $h$ is a nonzero polynomial with highest monomial strictly less than $u_k$. This contradicts the choice of $f$. We have proved that $J_0 = J$. Hence, $A$ is left Noetherian. □

(3.16) *Remark.* If $A$ has no zero divisors then conditions (1.17) and (1.20) from the introduction are equivalent.

*Proof.* Clearly (1.20) implies (1.17). The implication (1.17) $\Rightarrow$ (1.20) follows from Corollary (2.39).

Note that, if $A$ has no zero divisors and satisfies (1.20), then $A$ is cyclic. (This follows from Remark (3.16) and Theorem A (1.16).)

(3.17) **Corollary.** *Let* $A = K\langle x_1, \dots, x_n\rangle/(F)$ *be a binomial skew polynomial ring without zero divisors, with reduced Groebner basis*

$$F = \{x_j x_i - a_{ij} x_{i'} x_{j'} \mid 1 \le i < j \le n\},$$

*where for* $1 \le i < j \le n$ *one has* $1 \le i' < j' \le n$, *and* $x_{i'} x_{j'} <* \, x_j x_i$. *Suppose furthermore that*

(3.18) *The set* $\{x_{i'} x_{j'} \mid 1 \le i < j \le n\}$ *contains precisely* $\binom{n}{2}$ *elements.*

*Then* $A$ *is left Noetherian.*

*Proof.* It follows from Corollary (2.39) and Remark (3.16) that for $P = n!$, $A$ satisfies the hypotheses of Theorem B. Hence $A$ is left Noetherian.

## 4. Left and right Noetherianness

In this section we shall prove Theorem C (1.21) of the Introduction. We assume that the hypotheses of the theorem to be proved are in force, i.e.,

(4.1) $A = K\langle x_1, \dots, x_n\rangle/(F)$ *is a binomial skew polynomial ring without zero divisors, with reduced Groebner basis*

$$F = \{x_j x_i - a_{ij} x_{i'} x_{j'} \mid 1 \le i < j \le n\},$$

*where for* $1 \le i < j \le n$ *one has* $1 \le i' < j' \le n$, *and* $x_{i'} x_{j'} <* \, x_j x_i$.

We begin with some technical results.

The following lemma is true even if the condition (1.10) is not satisfied.

(4.2) **Lemma.** *Let* $A$ *be a binomial skew polynomial ring without zero divisors. Then the following holds:*

(a) *If* $A$ *is left Noetherian then for any* $i, j$, $1 \le i < j \le n$, *there exists a normal monomial* $w$ *such that*

$$w \star x_i = x_i \cdot x_j^{\deg w}.$$

(b) *If* $A$ *is right Noetherian, then for any* $i, j$, $1 \le i < j \le n$, *there exists a normal monomial* $w$ *such that*

$$x_j \star w = x_i^{\deg w} \cdot x_j.$$

*Proof.* (a) Assume $A$ is left Noetherian and $j > i$. Consider the increasing chain of left ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$, where for $k \ge 1$, $I_k$ is the left ideal generated by the elements $x_i x_j$, $x_i x_j^2, \dots, x_i x_j^k$. It follows from the left Noetherianness of $A$ that there exists a $k$ such that $I_k = I_{k+1} = \cdots$. This implies that $x_i x_j^{k+1} \in I_k$. Hence the following equality holds:

$$x_i x_j^{k+1} - \sum_q \sum_{1 \le r \le k} b_{rq} w_{rq} \star x_i x_j^r = 0, \tag{4.3}$$

where the $w_{rq}$ are normal monomials and the $b_{rq}$ are nonzero elements of $K$. This equality gives that a linear combination of nonzero normal monomials is zero. This is possible only in case all the coefficients in the equality (4.3) are zero, which implies that $x_i x_j^{k+1} = w_{rq} \star x_i x_j^r$ for some $q$ and $r$, $r \le k$. Since

$A$ has no zero divisors this implies that $x_i x_j^{k+1-r} = w_{rq} \star x_i$. We have proved (4.2(a). A similar argument shows that (4.2(b)) also holds. □

(4.4) **Lemma.** *Let $f$ be an arbitrary monomial which is not normal. Assume the normal form of $f$ is either* (a) $w = x_i \cdot (x_j)^t$, *or* (b) $w = (x_i)^t \cdot x_j$, *where in both cases $1 \le i < j \le n$ and $t > 1$. Then there exists a pair of integers $p, q$, $1 \le p < q \le n$, such that $x_q \star x_p = x_i \cdot x_j$.*

*Proof*. This follows from condition (1.10).

(4.5) **Corollary.** *Let $A = K\langle x_1, \dots, x_n\rangle/(F)$ be a binomial skew polynomial ring, with reduced Groebner basis*

$$F = \{x_j x_i - a_{ij} x_{i'} x_{j'} | 1 \le i < j \le n\},$$

*where for $1 \le i < j \le n$ one has $1 \le i' < j' \le n$, and $x_{i'} x_{j'} < * \; x_j x_i$. Suppose furthermore that $A$ is left* (*or right*) *Noetherian. Then the following two equivalent conditions hold*:

(a) *The set $\{x_{i'} x_{j'} | 1 \le i < j \le n\}$ contains precisely $\binom{n}{2}$ elements.*
(b) *For any pair of integers $i, j$, $1 \le i < j \le n$, there exist uniquely determined $q$ and $p$, $1 \le p < q \le n$, such that $x_q \star x_p = x_i \cdot x_j$.*

*Proof*. *Case* 1. $A$ is left Noetherian. Consider the sets

$$P_1 = \{x_j \star x_i | 1 \le i < j \le n\} = \{x_i' \cdot x_j' | 1 \le i < j \le n\},$$
$$P_2 = \{x_i \cdot x_j | 1 \le i < j \le n\}.$$

Obviously

$$P_1 \subseteq P_2. \tag{4.6}$$

We shall show that there is an equality of sets in (4.6). Indeed, take an arbitrary pair $i, j$, $1 \le i < j \le n$. It follows from Lemma (4.2a) that there exists a normal monomial $w$ such that

$$w \star x_i = x_i \cdot (x_j)^{\deg w}.$$

It then follows from Lemma (4.4) that $x_i \cdot x_j = x_q \star x_p$, for some $p$ and $q$, $1 \le p < q \le n$. Thus $P_2 \subseteq P_1$, which together with (4.6) gives $P_1 = P_2$ and

$$\mathrm{Card}(P_1) = \mathrm{Card}(P_2) = \binom{n}{2}.$$

This proves (4.5(a)) and (4.5(b)).

*Case* 2. $A$ is right Noetherian. Apply Lemmas (4.2(b)), (4.4(b)). □

Recall that given an algebra $A$ with product written as $u \cdot v$, $A^{\mathrm{op}}$ is the algebra based on the same underlying vector space, with product written as $u \circ v = v \cdot u$. Clearly, $A$ is left Noetherian if and only if $A^{\mathrm{op}}$ is right Noetherian. Also, if $A = K\langle X\rangle/(F)$, where $(F)$ is as in (4.1), is a binomial skew polynomial ring, then $A^{\mathrm{op}} = K\langle x\rangle/(F^{\mathrm{op}})$, where

$$F^{\mathrm{op}} = \{x_{j'} x_{i'} - a_{ij}^{-1} x_i x_j | 1 \le i < j \le n\}. \tag{4.7}$$

In general, the fact that $F$ is a Groebner basis does not imply that $F^{\mathrm{op}}$ is a Groebner basis, as one can check in example (1.12).

(4.8) **Proposition.** *Let $A = K\langle x_1, \dots, x_n\rangle/(F)$ be a binomial skew polynomial ring, with reduced Groebner basis*

$$F = \{x_j x_i - a_{ij} x_{i'} x_{j'} | 1 \le i < j \le n\},$$

*where for $1 \le i < j \le n$ one has $1 \le i' < j' \le n$, and $x_{i'}x_{j'} <* x_j x_i$. Suppose furthermore that*

(4.9) *The set $\{x_{i'}x_{j'} | 1 \le i < j \le n\}$ contains precisely $\binom{n}{2}$ elements.*

*Then the following three conditions hold:*

(a) *$F^{\mathrm{op}}$ is the reduced Groebner basis of the ideal $(F^{\mathrm{op}})$.*
(b) *$A^{\mathrm{op}}$ is a binomial skew polynomial ring.*
(c) *$A^{\mathrm{op}}$ is left Noetherian.*

*Proof*. It follows from (4.7) that all the elements of $F^{\mathrm{op}}$ are binomials of the type

$$g_{i'j'} = x_{j'}x_{i'} - a_{ij}^{-1} x_i x_j,$$

where $1 \le i < j \le n$, $1 \le i' < j' \le n$. Lemma (2.3) shows that for any pair $i'$, $j'$, $1 \le k' < j' \le n$, one has $i < j'$, hence

(4.10) $$x_{j'}x_{i'} = HM(g_{i'j'}),$$

where $HM(g_{i'j'})$ is the highest monomial of $g_{i'j'}$, cf. (1.1). Recall now (cf. [G-I$_1$]) that given an ideal $I$ in $K\langle X\rangle$ a monomial $u$ is *normal modulo I* if it does not contain as a subword any of the monomials $HM(f)$, where $f \in I$. Let $N(I)$ be the set of all normal modulo $I$ monomials. It is known, cf. [Berg, G-I$_1$], that $N(I)$ projects to a $K$-basis of the algebra $B = K\langle X\rangle/I$. Now for $I = (F^{\mathrm{op}})$ it is clear that

(4.11) $$N(I) \subseteq N,$$

where as in (1.1), $N$ is the set of ordered monomials. We shall prove that equality holds in (4.11). Recall that

(4.12) The algebras $A$ and $A^{\mathrm{op}}$ have the same Hilbert series.

(4.13) $N$ projects to a $K$-basis of $A = K\langle X\rangle/(F)$, $N(I)$ projects to a $K$-basis of $A^{\mathrm{op}} = K\langle X\rangle/(I)$.

It follows from (4.10), (4.12), and (4.13) that

$$N(F^{\mathrm{op}}) = N(I) = N.$$

It then follows from the Diamond Lemma [Berg] (cf. also (1.6)) that $F^{\mathrm{op}}$ is a Groebner basis of the ideal $(F^{\mathrm{op}})$. This proves (4.8a). Obviously (4.8a) implies (4.8b).

Consider now the set

$$F^{\mathrm{op}} = \{x_{j'}x_{i'} - a_{ij}^{-1} x_i x_j | 1 \le i < j \le n,\ 1 \le i' < j' \le n\}.$$

It follows from (4.10) that there is an equality

$$x_{(i')'}x_{(j')'} = x_i x_j,$$

for all $i'$, $j'$, $1 \le i' < j' \le n$. Obviously, the set

$$\{x_{(i')'}x_{(j')'} | 1 \le i' < j' \le n\} = \{x_i x_j | 1 \le i < j \le n\}$$

contains precisely $\binom{n}{2}$ elements. It follows from here that $A^{\mathrm{op}}$ satisfies the hypotheses of Corollary (3.17), hence, it is left Noetherian. □

(4.14) *Proof of Theorem* C (1.21).

Corollary (3.17) gives the implication (i) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (iii). It follows from Proposition (4.8(c)) that $A^{\mathrm{op}}$ is left Noetherian, hence $A$ is right Noetherian.

Corollary (4.5(a)) shows that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). □

(4.15) **Definition.** For the set of semigroup relations

$$G = \{x_j \cdot x_i = x_{i'} x_{j'} \mid 1 \le i < j \le n\}$$

where for $1 \le i < j \le n$ one has $1 \le i' < j' \le n$, and $i' < j$, we say that $G$ is a *semigroup Groebner basis* if and only if the corresponding set of relations

$$F(G) = F = \{x_j \cdot x_i - x_{i'} x_{j'} \mid 1 \le i < j \le n\}$$

is a Groebner basis of the ideal $(F)$ in $K\langle X\rangle$.

(4.16) **Corollary.** *Let $S = \langle X | G\rangle$ be a semigroup with set of generators $X$ and set of relations*

$$G = \{x_j x_i = x_{i'} x_{j'} \mid 1 \le i < j \le n\},$$

*where for $1 \le i < j \le n$ one has $1 \le i' < j' \le n$, $i' < j$. Suppose $G$ is a semigroup Groebner basis. The the following three conditions are equivalent:*

(i) *The set $\{x_{i'} x_{j'} \mid 1 \le i < j \le n\}$ contains precisely $\binom{n}{2}$ elements.*
(ii) *$S$ is left Noetherian.*
(iii) *$S$ is right Noetherian.*

We shall finish with a full list of left and right Noetherian binomial skew polynomial rings with three and four generators. It turns out that

(4.17) In this case the condition (1.10), $i' < j'$ for any $i < j$ appears as a consequence of left Noetherianness.

By $F_1$ we shall denote a subset of $K\langle X\rangle$ of the type

$$F_1 = \{x_j x_i - a_{ji} x_{i'} x_{j'} \mid 1 \le i < j \le n,\ 1 \le i' \le j' \le n\}. \tag{4.18}$$

Note that in difference with $F$, cf. (1.9), (1.10), we do not assume $i' \ne j'$. One can easily check using the Groebner basis property (1.4(b)) that

(4.19) **Lemma.** *Let $A = K\langle x_1, x_2, x_3\rangle/(F_1)$ be a binomial skew polynomial ring. Then $A$ is left (respectively right) Noetherian if and only if one of the following conditions hold:*

(1) $F_1 = \{x_3x_2 - ax_2x_3,\ x_3x_1 - bx_1x_3,\ x_2x_1 - cx_1x_2 \mid abc \ne 0\}$.
(2) $F_1 = \{x_3x_2 - ax_1x_3,\ x_3x_1 - bx_2x_3,\ x_2x_1 - cx_1x_2 \mid c^2 = 1,\ ab \ne 0\}$.
(3) $F_1 = \{x_3x_2 - ax_2x_3,\ x_3x_1 - bx_1x_2,\ x_2x_1 - cx_1x_3 \mid a^2 = 1,\ bc \ne 0\}$.

Some more delicate combinatorial arguments show that the following proposition holds.

(4.20) **Proposition.** *Let* $A = K\langle x_1, x_2, x_3, x_4\rangle/(F_1)$ *be a binomial skew polynomial ring. Then* $A$ *is left* (*respectively right*) *Noetherian if and only if one of the following conditions holds*:

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_1x_4,\ x_4x_2 - bx_3x_4,\ x_4x_1 - cx_2x_4,\ x_3x_2 - dx_2x_3, \\ &x_3x_1 - d^{-1}x_1x_3,\ x_2x_1 - dx_1x_2 | abcd \neq 0\},\end{aligned} \tag{1}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_2x_4,\ x_4x_2 - bx_1x_4,\ 4x_4x_1 - cx_3x_4, \\ &x_3x_2 - dx_2x_3,\ x_3x_1 - d^{-1}x_1x_3, \\ &x_2x_1 - dx_1x_2 | abcd \neq 0\},\end{aligned} \tag{2}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_2x_4,\ x_4x_2 - bx_3x_4,\ x_4x_1 - cx_1x_4,\ x_3x_2 - dx_2x_3, \\ &x_3x_1 - ex_1x_2,\ x_2x_1 - fx_1x_3 | d^2 = 1, \\ &af = be,\ abcdef \neq 0\},\end{aligned} \tag{3}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_2x_4,\ x_4x_2 - bx_3x_4,\ x_4x_1 - cx_1x_4,\ x_3x_2 - dx_2x_3, \\ &x_3x_1 - ex_1x_3,\ x_2x_1 - ex_1x_2 | d^2 = 1,\ abcde \neq 0\},\end{aligned} \tag{4}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_1x_4,\ x_4x_2 - bx_2x_4, \\ &x_4x_1 - cx_3x_4,\ x_3x_2 - dx_2x_3, \\ &x_3x_1 - ex_1x_3,\ x_2x_1 - d^{-1}x_1x_2 | e^2 = 1,\ abcd \neq 0\},\end{aligned} \tag{5}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_3x_4,\ x_4x_2 - bx_1x_3,\ x_4x_1 - cx_2x_3,\ x_3x_2 - dx_1x_4, \\ &x_3x_1 - ex_2x_4,\ x_2x_1 - fx_1x_2 | abcdef \neq 0, \\ &a^2 = f^2 = be/cd = cd/be\},\end{aligned} \tag{6}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_3x_4,\ x_4x_2 - bx_2x_3,\ x_4x_1 - cx_1x_3,\ x_3x_2 - dx_2x_4, \\ &x_3x_1 - ex_1x_4,\ x_2x_1 - fx_1x_2 | a^2 = 1, \\ &be = cd,\ cdf \neq 0\},\end{aligned} \tag{7}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_3x_4,\ x_4x_2 - bx_1x_4,\ x_4x_1 - cx_2x_4,\ x_3x_2 - dx_1x_3, \\ &x_3x_1 - ex_2x_3,\ x_2x_1 - fx_1x_2 | f^2 = 1, \\ &be = cd,\ acd \neq 0\},\end{aligned} \tag{8}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_3x_4,\ x_4x_2 - bx_1x_4,\ x_4x_1 - cx_2x_4,\ x_3x_2 - dx_2x_3, \\ &x_3x_1 - dx_1x_3,\ x_2x_1 - fx_1x_2 | f^2 = 1,\ abcd \neq 0\},\end{aligned} \tag{9}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_3x_4,\ x_4x_2 - bx_2x_4,\ x_4x_1 - cx_1x_2,\ x_3x_2 - ax_2x_3, \\ &x_3x_1 - ex_1x_4,\ x_2x_1 - fx_1x_3 | ab = 1,\ cef \neq 0\},\end{aligned} \tag{10}$$

$$\begin{aligned}F_1 = \{&x_4x_3 - ax_3x_4,\ x_4x_2 - bx_2x_4,\ x_4x_1 - cx_1x_3,\ x_3x_2 - ax_2x_3, \\ &x_3x_1 - ex_1x_2,\ x_2x_1 - fx_1x_4 | ab = 1,\ cef \neq 0\},\end{aligned} \tag{11}$$

$$F_1 = \{x_4x_3 - ax_3x_4\,,\ x_4x_2 - bx_2x_4\,,\ x_4x_1 - cx_1x_2\,,\ x_3x_2 - dx_2x_3\,, x_3x_1 - ex_1x_3\,,\ x_2x_1 - fx_1x_4 | ad = b^2 = 1\,,\ cef \neq 0\}\,, \tag{12}$$

$$F_1 = \{x_4x_3 - ax_3x_4\,,\ x_4x_2 - bx_2x_4\,, x_4x_1 - cx_1x_3\,,\ x_3x_2 - bx_2x_3\,,\ x_3x_1 - ex_1x_4\,,\ x_2x_1 - fx_1x_2 | a^2 = 1\,,\ bcef \neq 0\}\,, \tag{13}$$

$$F_1 = \{x_4x_3 - ax_3x_4\,,\ x_4x_2 - bx_2x_4\,,\ x_4x_1 - cx_1x_4\,,\ x_3x_2 - dx_2x_3\,,\ x_3x_1 - ex_1x_3\,, x_2x_1 - fx_1x_2 | abcdef \neq 0\}\,, \tag{14}$$

$$F_1 = \{x_4x_3 - ax_3x_4\,,\ x_4x_2 - bx_2x_4\,,\ x_4x_1 - bx_1x_4\,,\ x_3x_2 - dx_1x_3\,,\ x_3x_1 - ex_2x_3,\ x_2x_1 - fx_1x_2 | f^2 = 1,\ abde \neq 0\}\,, \tag{15}$$

$$F_1 = \{x_4x_3 - ax_3x_4\,,\ x_4x_2 - ax_2x_4\,, x_4x_1 - cx_1x_4\,,\ x_3x_2 - dx_2x_3\,,\ x_3x_1 - ex_1x_2\,,\ x_2x_1 - fx_1x_3 | d^2 = 1\,,\ acf \neq 0\}\,, \tag{16}$$

$$F_1 = \{x_4x_3 - ax_3x_4\,,\ x_4x_2 - bx_2x_3\,, x_4x_1 - cx_1x_4\,,\ x_3x_2 - dx_2x_4\,,\ x_3x_1 - cx_1x_3\,,\ x_2x_1 - fx_1x_2 | a^2 = 1\,,\ bcdf \neq 0\}. \tag{17}$$

## Bibliography


[Ap] J. Apel, *Gröbnerbasen in nichtkommutativen Algebren und ihre anwendung*, Dissertation A, Karl-Marx-Universität, Leipzig.

[Art-S] M. Artin and W. Schelter, *Graded algebras of global dimension* 3, Adv. Math. **66** (1987), 171–216.

[Berg] G. M. Bergman, *The diamond lemma of ring theory*, Adv. Math. **29** (1978), 178–218.

[Buch] B. Buchberger, *An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal*, Ph.D. thesis, Univ. Innsbruck (Austria), Math. Inst., 1965, Aequationes Math. **4** (1970), 374–383. (German)

[G-I$_1$] T. Gateva-Ivanova, *On the Noetherianity of some associative finitely presented algebras*, J. Algebra **138** (1991), 13–35.

[G-I$_2$] ——, *Noetherian properties and growth of some associative algebras*, Progr. Math., vol. 94, Birkhäuser, Basel and Boston, 1991, pp. 143–158.

[G-I$_3$] ——, *Groebner bases in skew polynomial rings*, preprint, Dept. of Math. Reports No. 8, University of Stockholm, 1992, pp. 1–33.

[Gol] E. S. Golod, *Standard bases and homology*, Lecture Notes in Math., vol. 1352, Springer-Verlag, Berlin and New York, 1988, pp. 88–95.

[K-R-W] A. Kandri-Rody and V. Weispfenning, *Non-commutative Gröbner bases in algebras of solvable type*, J. Symbolic Comput. **9** (1990), 1–26.

[Mor$_1$] T. Mora, *Groebner bases for non-commutative algebras*, ISSAC 88, Lecture Notes in Computer Science, vol. 358, Springer-Verlag, Berlin and New York, 1989, pp. 150–161.

[Mor$_2$] T. Mora, *Seven variations on standard bases*, Univ. di Genova, Dipartimento di Matematica, N. 45 Preprint, 1988.

[Sm-St] S. P. Smith and J. T. Stafford, *Regularity of the four dimensional Sklyanin algebra*, preprint, 1990.



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