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FLOWS IN FIBERS

JUN-ICHI TANAKA

Dedicated to Professor Morisuke Hasumi on his 60th birthday

ABSTRACT. Let $H^\infty(\Delta)$ be the algebra of all bounded analytic functions on the open unit disc Δ, and let $\mathfrak{M}(H^\infty(\Delta))$ be the maximal ideal space of $H^\infty(\Delta)$. Using a flow, we represent a reasonable portion of a fiber in $\mathfrak{M}(H^\infty(\Delta))$. This indicates a relation between the corona theorem and the individual ergodic theorem. As an application, we answer a question of Forelli [3] by showing that there exists a minimal flow on which the induced uniform algebra is not a Dirichlet algebra. The proof rests on the fact that the closure of a nonhomeomorphic part in $\mathfrak{M}(H^\infty(\Delta))$ may contain a homeomorphic copy of $\mathfrak{M}(H^\infty(\Delta))$. Taking suitable factors, we may construct a lot of minimal flows which are not strictly ergodic.

1. INTRODUCTION

If $|\alpha| = 1$, then *the fiber* $\mathfrak{M}_\alpha$ *of* $\mathfrak{M}(H^\infty(\Delta))$ *over* α is defined to be

$$\mathfrak{M}_\alpha = \{\xi \in \mathfrak{M}(H^\infty(\Delta))\,;\, \xi(z) = \alpha\},$$

where z is the coordinate function. Then we have the decomposition

$$\mathfrak{M}(H^\infty(\Delta))\backslash\Delta = \bigcup_{|\alpha|=1} \mathfrak{M}_\alpha.$$

Since the various fibers $\mathfrak{M}_\alpha$ are homeomorphic to one another, we restrict our attention to the fiber $\mathfrak{M}_1$ over 1 to look into the fringe $\mathfrak{M}(H^\infty(\Delta))\backslash\Delta$.

Let Ω be a compact Hausdorff space on which the real line $\mathbf{R}$ acts as a topological transformation group. This means that there is a one-parameter group $\{U_t\}_{t\in\mathbf{R}}$ of homeomorphisms of Ω onto itself such that the map $(\omega, t) \to U_t\omega$ is continuous on $\Omega\times\mathbf{R}$. The pair $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is referred to as a (*continuous*) *flow*. For each ω in Ω and t in $\mathbf{R}$, the translate $U_t\omega$ of ω by t is denoted by $\omega + t$. We denote by $O(\omega)$ the *orbit* of ω, that is,

$$O(\omega) = \{\omega + t\,;\, t \in \mathbf{R}\}.$$

Then an ω in Ω is said to be *fixed* if $O(\omega)$ equals $\{\omega\}$. A flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is called *minimal* if each orbit is dense in Ω, so there are no nontrivial closed invariant sets.

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Let $H^\infty(\mathbf{R}^2_+)$ denote the space of all bounded analytic functions in the upper half-plane $\mathbf{R}^2_+$, and the space of their boundary-value functions is denoted by $H^\infty(\mathbf{R})$. They are identified each other by Fatou's theorem. Let $C(\Omega)$ be the space of all continuous complex-valued functions on Ω. A function ϕ in $C(\Omega)$ is *analytic* if the function $t \to \phi(\omega + t)$ lies in $H^\infty(\mathbf{R})$ for each ω in Ω. Let $A(\Omega)$ be the space of all analytic functions in $C(\Omega)$. Then $A(\Omega)$ is a uniformly closed subalgebra of $C(\Omega)$ containing constants. When there are no fixed points, $A(\Omega)$ separates the points on Ω, and Ω is the Choquet boundary of $A(\Omega)$ [13, Theorem I]. Throughout the paper, we shall always assume that $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is an arbitrary flow without fixed points. Then $A(\Omega)$ becomes a uniform algebra on Ω, which is called the *induced uniform algebra* on $(\Omega, \{U_t\}_{t\in\mathbf{R}})$. Let $\mathfrak{M}(A(\Omega))$ be the maximal ideal space of $A(\Omega)$. Although the structure is intractable, it has been investigated with the aid of ergodic theory (see, for instance, [12, 16 and 15]).

Let $\mathbf{A}$ be a uniform algebra on a compact Hausdorff space $\mathbf{Y}$. Then $\mathbf{A}$ is called a *Dirichlet algebra* on $\mathbf{Y}$ if the real part $\operatorname{Re}\mathbf{A}$ of $\mathbf{A}$ is uniformly dense in the space $C_{\mathbf{R}}(\mathbf{Y})$ of all real-valued functions in $C(\mathbf{Y})$. We say that $\mathbf{A}$ is a *logmodular algebra* on $\mathbf{Y}$ if

$$\log|\mathbf{A}^{-1}| = \{\log|f|\,;\, f \in \mathbf{A}^{-1}\}$$

is uniformly dense in $C_{\mathbf{R}}(\mathbf{Y})$, where $\mathbf{A}^{-1}$ denotes the set of all invertible elements of $\mathbf{A}$.

The purpose of this paper is to show how flows can be applied usefully to study the structure of $\mathfrak{M}(H^\infty(\Delta))$. We first represent a portion of the fiber $\mathfrak{M}_1$ as a subset of the maximal ideal space of the induced uniform algebra on a flow. Then we point out that there appears a relation between the corona theorem and the individual ergodic theorem. Together with the characterization of analytic structure in $\mathfrak{M}(H^\infty(\Delta))$, due to Hoffman, we second construct a minimal flow on which the induced uniform algebra is not a Dirichlet algebra. This provides a negative answer to a question posed by Forelli [3]:

When $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is minimal, is $A(\Omega)$ a Dirichlet algebra on Ω?

In the next section, we establish the notation and present some lemmas. Our representation of the fiber $\mathfrak{M}_1$ over 1 is given in §3 together with some related facts. After preparing some lemmas, we show in §4 the existence of a desired minimal flow. We close with some remarks in §5.

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2. Preliminaries

We begin by collecting some notation and facts about a flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ which we shall use throughout. Let $M(\Omega)$ be the space of all bounded complex Borel regular measures on Ω. A measure in $M(\Omega)$ is said to be *quasi-invariant* if every translate of each null set is also a null set. A quasi-invariant measure is said to be *ergodic* if every invariant subset of Ω is either negligible or has negligible complement. Let μ be a positive quasi-invariant measure. A function ϕ in $L^\infty(\mu)$ is said to be *analytic* if the function $t \to \phi(\omega + t)$ lies in $H^\infty(\mathbf{R})$ for μ-a.e. ω in Ω.

Let us consider the representing measures for the induced uniform algebra $A(\Omega)$ on $(\Omega, \{U_t\}_{t\in\mathbf{R}})$. The Poisson kernel $P_z(t)$ for $\mathbf{R}^2_+$ is defined by $P_z(t) = s/\pi(s^2+(t-u)^2)$, where $z = u+is$. For each $s>0$ and for a point $\omega+u$ in the orbit $O(\omega)$ of ω, we define a homomorphism of $A(\Omega)$ by

$$\xi_z(\phi) = \phi * P_{is}(\omega+u) = \int_{-\infty}^{\infty} \phi(\omega+t)P_z(t)\,dt. \tag{2.1}$$

Then the product $O(\omega)\times(0,\infty)$ may be regarded as a subset of $\mathfrak{M}(A(\Omega))$ by (2.1). Since the representing measure for each point in $O(\omega)\times(0,\infty)$ is absolutely continuous to another one, $O(\omega)\times(0,\infty)$ makes a nontrivial Gleason part. On the other hand, there are representing measures of a different kind. Indeed, the Markov-Kakutani theorem shows that there is at least one invariant probability measure on Ω. Together with the Krein-Milman theorem, this assures the existence of an invariant, ergodic, probability measure m on Ω. Then m is a representing measure for $A(\Omega)$ [11, Theorem I]. It is known that every representing measure for $A(\Omega)$ is quasi-invariant if it is not a point mass [11, Theorem III].

Let $\beta\mathbf{Z}$ be the Stone-Čech compactification of the group $\mathbf{Z}$ of integers, and let S_0 be the shift operator on $\mathbf{Z}$, which is defined by $S_0 n = n+1$ for each n in $\mathbf{Z}$. It then follows from the Banach-Stone theorem [1, Chapter V, 8.8] that S_0 extends to a homeomorphism $\mathbf{S}$ of $\beta\mathbf{Z}$ onto itself. We denote by $\mathbf{X}$ the quotient space obtained from $\beta\mathbf{Z}\times[0,1]$ by identifying $(y,1)$ with $(\mathbf{S}y,0)$ for each y in $\beta\mathbf{Z}$. Then $\mathbf{X}$ is a compact Hausdorff space. Let us define a flow $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$ by

$$\mathbf{S}_t(y,s) = (\mathbf{S}^{[t+s]}y,\, t+s-[t+s]), \tag{2.2}$$

where $[t]$ denotes the largest integer not exceeding t. We notice that the flow is far from a minimal one, since there exist closed invariant sets in profusion. Then there are many invariant, ergodic, probability measures on $M(\mathbf{X})$. We not only see that there are no fixed points in $\mathbf{X}$ but also that there are no periodic orbits in $\mathbf{X}$. Then $A(\mathbf{X})$ is a uniform algebra on $\mathbf{X}$.

Let X_0 be the screw with pitch 1 and radius $1/2\pi$, that is, the quotient space obtained from $\mathbf{Z}\times[0,1]$ by identifying each $(n,1)$ with $(n+1,0)$. Since $\mathbf{Z}$ is dense $\beta\mathbf{Z}$, X_0 is also dense in $\mathbf{X}$. The map $t\to([t], t-[t])$ is a homeomorphism of $\mathbf{R}$ onto the subspace X_0 of $\mathbf{X}$. In what follows, we shall sometimes identify X_0 and $X_0\times(0,\infty)$ with $\mathbf{R}$ and $\mathbf{R}^2_+$, respectively. Then $\mathbf{R}^2_+$ may be considered as a subset of $\mathfrak{M}(A(\mathbf{X}))$ by (2.1). Let $C_b(\mathbf{R})$ be the space of all bounded continuous functions on $\mathbf{R}$. Then each function in $C(\mathbf{X})$ determines a function in $C_b(\mathbf{R})$ by restricting it to X_0.

Every function ϕ on $\mathbf{X}$ has the *automorphic extention* $\phi^\#$ to $\beta\mathbf{Z}\times\mathbf{R}$ by

$$\phi^\#(y,t) = \phi(\mathbf{S}^{[t]}y,\, t-[t]). \tag{2.3}$$

A function f on $\beta\mathbf{Z}\times\mathbf{R}$ is the automorphic extension of a function on $\mathbf{X}$ if and only if $f(y,t) = f(\mathbf{S}y, t-1)$ on $\beta\mathbf{Z}\times\mathbf{R}$.

Lemma 2.1. *Let ψ_0 be a function in $C_b(\mathbf{R})$. Then ψ_0 extends to a function ψ in $C(\mathbf{X})$ if and only if ψ_0 is uniformly continuous on $\mathbf{R}$.*

Proof. Suppose that ψ_0 is uniformly continuous on $\mathbf{R}$. For each s in $[0,1)$, the sequence $\psi_0(n,s) = \psi_0(n+s)$, $n\in\mathbf{Z}$, lies in $l^\infty(\mathbf{Z})$. Then $\psi_0(n,s)$

extends to $\psi(y, s)$ in $C(\beta\mathbf{Z})$. Since $\mathbf{X}$ is identified with $\beta\mathbf{Z}\times[0, 1)$, we may regard $\psi(y, s)$ as a function on $\mathbf{X}$. Let $\psi^\sharp(y, t)$ be the automorphic extension of $\psi(y, s)$ to $\beta\mathbf{Z}\times\mathbf{R}$. It follows from the uniform continuity of ψ_0 that the family $\{\psi^\#(y, t); y \in \beta\mathbf{Z}\}$ of the functions of t is equicontinuous on $\mathbf{R}$. This implies that $\psi^\sharp$ is continuous on $\beta\mathbf{Z}\times\mathbf{R}$. Thus ψ lies in $C(\mathbf{X})$.

The converse follows from a general fact. Indeed, it is an immediate consequence of the continuity of the map $(x, t) \to x+t$ on $\mathbf{X}\times\mathbf{R}$. $\square$

Let $C_{bu}(\mathbf{R})$ be the space of bounded uniformly continuous functions on $\mathbf{R}$. Then $C_{bu}(\mathbf{R})$ is a Banach algebra with the supremum norm, which is not separable. Lemma 2.1 shows that $\mathbf{X}$ is the maximal ideal space of $C_{bu}(\mathbf{R})$ and that the flow $\{\mathbf{S}_t\}_{t\in\mathbf{R}}$ on $\mathbf{X}$ is that which is induced by the natural translations of the functions in $C_{bu}(\mathbf{R})$. Such observation may provide an approach to almost periodic functions.

Lemma 2.2. *Let $A(\mathbf{X})$ be the induced uniform algebra on $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$, and let ϕ_0 be a function in $H^\infty(\mathbf{R})$. If $r > 0$, then the convolution $\phi_0 * P_{ir}$, in the usual sense, extends to a function $\phi_r(x)$ in $A(\mathbf{X})$. Furthermore, the function $(x, r) \to \phi_r(x)$ is continuous on $\mathbf{X}\times(0, \infty)$.*

Proof. Since

$$|\phi_0 * P_{ir}(t-s) - \phi_0 * P_{ir}(t)| \le \|\phi_0\|_\infty \cdot \|P_{s+ir} - P_{ir}\|_1,$$

$t \to \phi_0 * P_{ir}(t)$ is uniformly continuous on $\mathbf{R}$. By Lemma 2.1, $\phi_0 * P_{ir}$ extends to a $\phi_r(x)$ in $C(\mathbf{X})$. Let us consider the automorphic extension $\phi_r^\#(y, t)$ of ϕ_r to $\beta\mathbf{Z}\times\mathbf{R}$. Then $t \to \phi_r^\#(y, t)$ extends to $\mathbf{R}_+^2$ by its Poisson integral:

$$\phi_r^\#(y, z) = \phi_r^\#(y, t+is) = \phi_r^\# * P_{is}(y, t). \tag{2.4}$$

Since $\phi_0 * P_{ir}$ lies in $H^\infty(\mathbf{R})$, $z \to \phi_r^\#(n, z)$ is bounded analytic on $\mathbf{R}_+^2$ for each n in $\mathbf{Z}$. Then, since $\mathbf{Z}$ is dense in $\beta\mathbf{Z}$, $z \to \phi_r^\#(y, z)$ lies also in $H^\infty(\mathbf{R}_+^2)$ for each y in $\beta\mathbf{Z}$. This shows that $t \to \phi_r(x+t)$ lies in $H^\infty(\mathbf{R})$. Thus ϕ_r is a function in $A(\mathbf{X})$.

Since $\phi_0 * P_{ir} * P_{is} = \phi_0 * P_{i(r+s)}$ for $s > 0$, we see that $\phi_r^\#(y, t+is) = \phi_{r+s}^\#(y, t)$ by (2.4). Then the family $\{\phi_r^\#(y, t+is);\ y \in \beta\mathbf{Z}\}$ of functions of $t+is$ is equicontinuous on $\mathbf{R}_+^2$. This implies that $(x, s) \to \phi_{r+s}(x)$ is continuous on $\mathbf{X}\times(0, \infty)$ for each $r > 0$. From this, the continuity of $(x, r) \to \phi_r(x)$ follows. $\square$

Proposition 2.3. *Let $A(\mathbf{X})$ be the induced uniform algebra on $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$. Then $A(\mathbf{X})$ is a logmodular algebra on $\mathbf{X}$ which is not a Dirichlet algebra.*

Proof. Suppose ϕ lies in $C_{\mathbf{R}}(\mathbf{X})$. For a given $\varepsilon > 0$, we choose an $r > 0$ such that $\|\phi - \phi * P_{ir}\|_\infty < \varepsilon$ by [2, Lemma 1]. On the other hand, the restriction ϕ_0 of ϕ to X_0 lies in $C_b(\mathbf{R})$. Let $H\phi_0$ be the Hilbert transform of ϕ_0 defined by

$$H\phi_0(t) = \lim_{\varepsilon\to+0} \frac{1}{\pi} \int_{|t-s|>\varepsilon} \phi_0(s) \left[\frac{1}{t-s} + \frac{s}{1+s^2}\right] ds. \tag{2.5}$$

Then $\psi_0 = \exp(\phi_0 + iH\phi_0)$ is an outer function in $H^\infty(\mathbf{R})$, which is invertible in $H^\infty(\mathbf{R})$. Observe that $\psi_0^{-1} * P_{ir}(t)$ is the inverse of $\psi_0 * P_{ir}(t)$ in $H^\infty(\mathbf{R})$. Since $\log|\psi_0| = \phi_0$, we see that

$$\log|\psi_0 * P_{ir}(t)| = \phi_0 * P_{ir}(t). \tag{2.6}$$

It follows from Lemma 2.2 that $\psi_0 * P_{ir}(t)$ and $\psi_0^{-1} * P_{ir}(t)$ extend to θ and θ^{-1} in $A(\mathbf{X})$. We see easily that $\log|\theta| = \phi * P_{ir}$ by (2.6). So $A(\mathbf{X})$ is a logmodular algebra on $\mathbf{X}$.

We now show that $A(\mathbf{X})$ is not a Dirichlet algebra. Let $u_0(t) = \frac{2}{\pi}\tan^{-1}(t)$. Since u_0 is uniformly continuous on $\mathbf{R}$, u_0 extends to a function u in $C_{\mathbf{R}}(\mathbf{X})$. However, we observe by (2.5) that

$$\operatorname{dist}(u_0, \operatorname{Re} H^\infty(\mathbf{R})) = \inf\{\|u_0 - \operatorname{Re}\psi_0\|_\infty ; \psi_0 \in H^\infty(\mathbf{R})\} = 1$$

(see [6, Chapter IV, Example 1.5 and Chapter V, Exercise 15]). Thus u cannot belong to the closure of $\operatorname{Re} A(\mathbf{X})$ in $C_{\mathbf{R}}(\mathbf{X})$. □

Let ν_0 be the measure on $\mathbf{Z}$ defined by $\nu_0(\{n\}) = 1/(n^2+1)$, and let m_I be the restriction of Lebesgue measure to $I = [0, 1]$. By identifying $\mathbf{R}$ with X_0, $dt/(1+t^2)$ and $\nu_0 \times m_I$ are mutually absolutely continuous. Since $l^\infty(\mathbf{Z}) = L^\infty(\nu_0)$, $L^\infty(\nu_0)$ is isometrically isomorphic to $C(\beta\mathbf{Z})$ and ν_0 extends to a normal measure ν in $M(\beta\mathbf{Z})$ (see [5, Chapter I, §9]). We then see that ν is quasi-invariant with respect to $\mathbf{S}$, to be exact, ν-null sets are preserved under the translation by $\mathbf{S}$. Therefore the measure $\nu \times m_I$ in $M(\mathbf{X})$ is a quasi-invariant measure on $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$ by the definition (2.2).

Let $H^\infty(\mathbf{X})$ be the space of all analytic functions in $L^\infty(\nu \times m_I)$. Since $\nu \times m_I$ is concentrated on X_0, that is, $\nu \times m_I(\mathbf{X}\backslash X_0) = 0$, $H^\infty(\mathbf{R})$ is isometrically isomorphic to $H^\infty(\mathbf{X})$ by

$$\Phi_0\phi_0(n, s) = \phi_0(n+s), \qquad \phi_0 \in H^\infty(\mathbf{R}), \tag{2.7}$$

for (n, s) in X_0. Since the restriction of each function in $H^\infty(\mathbf{X})$ to X_0 determines a function in $H^\infty(\mathbf{R})$, Φ_0 maps $H^\infty(\mathbf{R})$ onto $H^\infty(\mathbf{X})$. Let $\mathfrak{M}(H^\infty(\mathbf{R}))$ and $\mathfrak{M}(H^\infty(\mathbf{X}))$ be the maximal ideal spaces of $H^\infty(\mathbf{R})$ and $H^\infty(\mathbf{X})$. Then the adjoint Φ_0^* of Φ_0 maps homeomorphically $\mathfrak{M}(H^\infty(\mathbf{X}))$ onto $\mathfrak{M}(H^\infty(\mathbf{R}))$. We shall modify Φ_0 in a little while.

The important feature of $H^\infty(\mathbf{X})$ is that each element has a useful Borel version.

Lemma 2.4. *Let ϕ be a function in $H^\infty(\mathbf{X})$. Then there is a Borel function $\tilde{\phi}$ on $\mathbf{X}$ with the following properties*:

(i) *The equation $\phi(x) = \tilde{\phi}(x)$ holds for $\nu \times m_I$-a.e. x in $\mathbf{X}$.*

(ii) *The function $x \to \tilde{\phi} * P_{ir}(x)$ lies in $A(\mathbf{X})$ for each $r > 0$. Furthermore, the function $(x, r) \to \tilde{\phi} * P_{ir}(x)$ is continuous on $\mathbf{X} \times (0, \infty)$.*

(iii) *Let μ be a positive quasi-invariant measure on $\mathbf{X}$. Then we obtain*

$$\lim_{r\to+0} \int_{\mathbf{X}} \tilde{\phi} * P_{ir}(x)\, d\mu(x) = \int_{\mathbf{X}} \tilde{\phi}(x)\, d\mu(x). \tag{2.8}$$

Proof. We define the Borel function $\tilde{\phi}$ for a given ϕ in $H^\infty(\mathbf{X})$. Let ϕ_0 be the restriction of ϕ to X_0. Then ϕ_0 is regarded as a function in $H^\infty(\mathbf{R})$. By Lemma 2.2, $\phi_0 * P_{ir}$ extends to a function $\phi_r(x)$ in $A(\mathbf{X})$, and $(x, r) \to \phi_r(x)$ is continuous on $\mathbf{X} \times (0, \infty)$. We set

$$f(y, z) = f(y, t+ir) = \phi_r^\#(y, t), \tag{2.9}$$

where $\phi_r^\#(y, t)$ is the automorphic extension of ϕ_r to $\beta\mathbf{Z} \times \mathbf{R}$. So f is continuous on $\beta\mathbf{Z} \times \mathbf{R}_+^2$, and $z \to f(y, z)$ lies in $H^\infty(\mathbf{R}_+^2)$ for each y in $\beta\mathbf{Z}$.

Let $\{r_n\}$ be a decreasing sequence that converges to 0. We then define the boundary-value function $f(y, t)$ by

$$f(y, t) = \limsup_{n\to\infty} \operatorname{Re} f(y, t + ir_n) + i \limsup_{n\to\infty} \operatorname{Im} f(y, t + ir_n).$$

Since $(y, t) \to f(y, t + ir_n)$ is continuous on $\beta\mathbf{Z} \times \mathbf{R}$, $f(y, t)$ is a Borel function on $\beta\mathbf{Z} \times \mathbf{R}$. On the other hand, it follows from Fatou's theorem that, for each y in $\beta\mathbf{Z}$,

$$f(y, t) = \lim_{r\to+0} f(y, t + ir) \tag{2.10}$$

for dt-a.e. t in $\mathbf{R}$. Since $f(y, t) = f(\mathbf{S}y, t-1)$ by (2.9), there is a Borel function $\tilde{\phi}$ on $\mathbf{X}$ whose automorphic extension is $f(y, t)$.

Since (2.10) shows that if n lies in $\mathbf{Z}$, then

$$\lim_{r\to+0} \phi_0 * P_{ir}(n + t) = \phi_0(n + t) = f(n, t)$$

for dt-a.e. t, we see that $\phi^\#(y, t) = f(y, t)$ for $d\nu \times dt$-a.e. (y, t). Thus (i) holds. We see easily that $\tilde{\phi} * P_{ir}(x) = \phi_r(x)$ on $\mathbf{X}$. Then (ii) follows from Lemma 2.2. To show (iii), we define the measure μ_1 in $M(\beta\mathbf{Z})$ by $\mu_1(E) = \mu(E \times [0, 1])$. Then μ_1 is quasi-invariant with respect to $\mathbf{S}$. It is easy to see that $\mu_1 \times m_I$ is a quasi-invariant measure on $\mathbf{X}$ which is mutually absolutely continuous with respect to μ. Identifying $\mathbf{X}$ with $\beta\mathbf{Z} \times [0, 1)$,

$$\lim_{r\to+0} \tilde{\phi} * P_{ir}(y, s) = \tilde{\phi}(y, s)$$

holds for m_I-a.e. s in $[0, 1)$ and for each y in $\beta\mathbf{Z}$. Thus (2.8) follows from Fubini's theorem and the bounded convergence theorem. □

Let Φ_0 be the isometric isomorphism of $H^\infty(\mathbf{R})$ onto $H^\infty(\mathbf{X})$ by (2.7). For a given ϕ_0 in $H^\infty(\mathbf{R})$, we set $\phi = \Phi_0(\phi_0)$. Strictly speaking, ϕ is an equivalence class of functions modulo null functions. Then we assign the Borel function $\tilde{\phi}$ by Lemma 2.4 to $\Phi_0(\phi_0)$:

$$\Phi_0(\phi_0) = \tilde{\phi}, \qquad \phi_0 \in H^\infty(\mathbf{R}). \tag{2.11}$$

From now on, Φ_0 will denote this modified isomorphism.

By virtue of this specialization, certain homomorphisms of $A(\mathbf{X})$ become the ones of $H^\infty(\mathbf{R})$. Since $H^\infty(\mathbf{R})$ and $H^\infty(\mathbf{X})$ are isometrically isomorphic via Φ_0, it is a consequence of the following lemma, in which the uniqueness of extension depends on the corona theorem.

Lemma 2.5. *Suppose that ξ_0 lies in $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$. Then ξ_0 extends uniquely to an element ξ of $\mathfrak{M}(H^\infty(\mathbf{X}))$.*

Proof. Let μ be the representing measure for ξ_0. Since μ is not a point mass, μ is quasi-invariant on $\mathbf{X}$. Using the Borel version by Lemma 2.4, we define

$$\xi(\phi) = \int_{\mathbf{X}} \tilde{\phi}\, d\mu, \qquad \phi \in H^\infty(\mathbf{X}). \tag{2.12}$$

Since $(\phi\psi)^\sim = \tilde{\phi}\tilde{\psi}$ if ϕ and ψ lie in $H^\infty(\mathbf{X})$, it follows from (2.8) that ξ is a homomorphism of $H^\infty(\mathbf{X})$ which is an extension of ξ_0.

The uniqueness of ξ follows from the corona theorem. Since $H^\infty(\mathbf{R})$ is isometrically isomorphic to $H^\infty(\mathbf{X})$, the corona theorem asserts that $X_0 \times (0, \infty)$

is dense in $\mathfrak{M}(H^\infty(\mathbf{X}))$ by (2.1). Observe that $t \to e^{it}$ is uniformly continuous on $\mathbf{R}$. Then Lemma 2.1 implies that e^{it} extends to a function $s_1(x)$ in $A(\mathbf{X})$ such that $|s_1(x)| = 1$ on $\mathbf{X}$. Since $t \to s_1(x+t)$ is periodic, we see that s_1 is not constant as a function in $L^1(\mu)$. Then there is an r with $0 < r < 1$ such that

$$|\xi_0(s_1)| = \left| \int_{\mathbf{X}} s_1 \, d\mu \right| \le r. \tag{2.13}$$

Suppose ξ' in $\mathfrak{M}(H^\infty(\mathbf{X}))$ satisfies that $\xi' = \xi_0$ on $A(\mathbf{X})$. Since $A(\mathbf{X})$ is a logmodular algebra on $\mathbf{X}$, the representing measure μ for ξ_0 is unique. By the corona theorem and (2.13), there is a net $\{\xi_\alpha\}$ in $X_0 \times [-\log r, \infty)$ such that $\{\xi_\alpha\}$ converges to ξ' in $\mathfrak{M}(H^\infty(\mathbf{X}))$. By setting $\xi_\alpha = (x_\alpha, t_\alpha)$, (2.1) and (ii) of Lemma 2.4 show that

$$\int_{-\infty}^{\infty} \tilde{\phi} * P_{iu}(x_\alpha + t) P_{it_\alpha}(t)\, dt \to \int_{\mathbf{X}} \tilde{\phi} * P_{iu}\, d\mu$$

for each ϕ in $H^\infty(\mathbf{X})$ and for $u > 0$. Furthermore, we see that if $\{t_\alpha\}$ is bounded, then μ is the representing measure for $\xi_0 = (x, t)$ in $\mathbf{X} \times [-\log r, \infty)$. When $\{t_\alpha\}$ diverges, μ is an invariant measure on $\mathbf{X}$. In both cases, we obtain easily that

$$\int_{-\infty}^{\infty} \tilde{\phi}(x_\alpha + t) P_{it_\alpha}(t)\, dt \to \int_{\mathbf{X}} \tilde{\phi}\, d\mu,$$

by the condition $t_\alpha \ge -\log r$. This implies that $\{\xi_\alpha\}$ also converges to ξ in $\mathfrak{M}(H^\infty(\mathbf{X}))$. Hence $\xi' = \xi$, so the extension of ξ_0 is unique. □

3. In relation to fibers

In this section, we represent a portion of the fiber $\mathfrak{M}_1$ of $\mathfrak{M}(H^\infty(\Delta))$ over 1 as a subset of the maximal ideal space $\mathfrak{M}(A(\mathbf{X}))$ of $A(\mathbf{X})$. This enable us to make clearer the structure of fibers and to give information about Gleason parts in them.

Let $\mathbf{A}$ be a uniform algebra with the maximal ideal space $\mathfrak{M}(\mathbf{A})$. We usually identify each function in $\mathbf{A}$ with its Gelfand transform on $\mathfrak{M}(\mathbf{A})$. A continuous map F of Δ into $\mathfrak{M}(\mathbf{A})$ is *analytic* if $f \circ F$ is analytic on Δ when f lies in $\mathbf{A}$. A one-to-one analytic map is said to be an *analytic disc.* With an analytic disc, we do not distinguish between the map F and its image $F(\Delta)$. Let $\mathfrak{M}(A)^\Delta$ be the set of all maps of Δ into $\mathfrak{M}(\mathbf{A})$. Then $\mathfrak{M}(\mathbf{A})^\Delta$ is a compact Hausdorff space in the product topology. If a net $\{F_\beta\}$ of analytic maps converges to F in $\mathfrak{M}(\mathbf{A})^\Delta$, then F is also analytic.

Observe that the function

$$w(z) = i\frac{1+z}{1-z}, \qquad z \in \Delta, \tag{3.1}$$

maps conformally Δ onto $\mathbf{R}^2_+$. Then w maps $\{-1, 0, 1\}$ to $\{0, i, \infty\}$. We next define the isometric isomorphism of $H^\infty(\Delta)$ onto $H^\infty(\mathbf{R}^2_+)$ by

$$\Phi_1(f) = f \circ w^{-1}, \qquad f \in H^\infty(\Delta). \tag{3.2}$$

To investigate the fiber $\mathfrak{M}_1$, we look into the behavior of functions in $H^\infty(\mathbf{R}^2_+)$ around at infinity.

Let $s(z)$ be the singular function in $H^\infty(\Delta)$ defined by

$$s(z) = \exp iw(z), \qquad z \in \Delta. \tag{3.3}$$

Then $s(z)$ has a singularity at $z = 1$. Identifying s with its Gelfand transform on $\mathfrak{M}(H^\infty(\Delta))$, we set

$$H = \{\xi \in \mathfrak{M}(H^\infty(\Delta))\,;\, |s(\xi)| < 1\}. \tag{3.4}$$

Then H is an open set in $\mathfrak{M}(H^\infty(\Delta))$ that contains Δ. Observe that $H\backslash\Delta$ is contained in $\mathfrak{M}_1$. More precisely, ξ in $\mathfrak{M}_1$ lies in $H\backslash\Delta$ if and only if ξ lies in the closure of an open disc inside Δ which is tangent to the unit circle at 1, that is, ξ is orocycular or nontangential (see [9, §6]).

By using the flow $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$ in §2, the open set $\mathfrak{M}_1 \cap H$ in $\mathfrak{M}_1$ can be analyzed rather completely. Recall that we identify $X_0 \times (0, \infty)$ with $\mathbf{R}^2_+$.

Theorem 3.1. *Let H be the open set in $\mathfrak{M}(H^\infty(\Delta))$ defined by* (3.4). *Then there is a homeomorphism π of H onto $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$ with the following properties*:

(i) *Let π_1 denote the restriction of π to $\mathfrak{M}_1 \cap H$. Then π_1 maps homeomorphically $\mathfrak{M}_1 \cap H$ onto $\mathfrak{M}(A(\mathbf{X}))\backslash(\mathbf{X} \cup \mathbf{R}^2_+)$.*

(ii) *The homeomorphism π preserves Gleason metrics. Furthermore, a continuous map F of Δ into H is analytic if and only if so is the map $\pi \circ F$ of Δ into $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$.*

Proof. Let Φ_0 and Φ_1 be the isometric isomorphisms by (2.11) and (3.2), respectively. Setting $\Phi = \Phi_0 \circ \Phi_1$, we obtain an isometric isomorphism of $H^\infty(\Delta)$ onto $H^\infty(\mathbf{X})$ (see Figure 1). Let Φ^* denote the adjoint of Φ. Then Φ^* maps homeomorphically $\mathfrak{M}(H^\infty(\mathbf{X}))$ onto $\mathfrak{M}(H^\infty(\Delta))$. We see also that Φ^* is an isometry of $H^\infty(\mathbf{X})^*$ onto $H^\infty(\Delta)^*$, where $H^\infty(\mathbf{X})^*$ and $H^\infty(\Delta)^*$ denote the dual spaces of $H^\infty(\mathbf{X})$ and $H^\infty(\Delta)$. So Φ^* maps each Gleason part of $\mathfrak{M}(H^\infty(\mathbf{X}))$ to the one of $\mathfrak{M}(H^\infty(\Delta))$.

As in the proof of Lemma 2.5, $s_1(x)$ denotes the extension of $t \to e^{it}$ to a function in $A(\mathbf{X})$. We then set

$$H_1 = \{\xi \in \mathfrak{M}(H^\infty(\mathbf{X}))\,;\, |s_1(\xi)| < 1\}.$$

Then H_1 is an open set in $\mathfrak{M}(H^\infty(\mathbf{X}))$. Since $|s_1(\xi)| = 1$ on $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash H_1$, H_1 is a union of Gleason parts. Since $\Phi_1 s(t) = e^{it}$ by (3.3), we see that $\Phi s = \Phi_0 \circ \Phi_1 s = s_1$. This implies that Φ^* maps homeomorphically H_1 onto H.

We next show that H_1 is homeomorphic to $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$. When ξ lies in H_1, we write $\sigma(\xi)$ for the restriction of ξ to $A(\mathbf{X})$. Then $\sigma(\xi)$ cannot lie in $\mathbf{X}$, since $|s_1(\xi)| < 1$. Therefore σ maps continuously H_1 into $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$. If ξ_0 lies in $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$, then $|s_1(\xi_0)| < 1$, since the representing measure for ξ_0 is quasi-invariant. It follows from Lemma 2.5 that σ is a one-to-one map of H_1 onto $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$.

To show the continuity of σ^{-1}, we take a neighborhood V of ξ in H_1. Since H_1 is open in $\mathfrak{M}(H^\infty(\mathbf{X}))$, we notice that $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash V$ is compact in $\mathfrak{M}(H^\infty(\mathbf{X}))$. For any η in $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash V$, there is a function ϕ in $A(\mathbf{X})$ such that $\phi(\eta) = 1$ and $\phi(\xi) = 0$. Indeed, if η lies in $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash H_1$, this follows from the fact that $|s_1| = 1$ only on $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash H_1$. On the other hand, since σ is one-to-one, we see that $A(\mathbf{X})$ separates the points on H_1. This shows the existence of such a function when η lies in $H_1\backslash V$. Since $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash V$

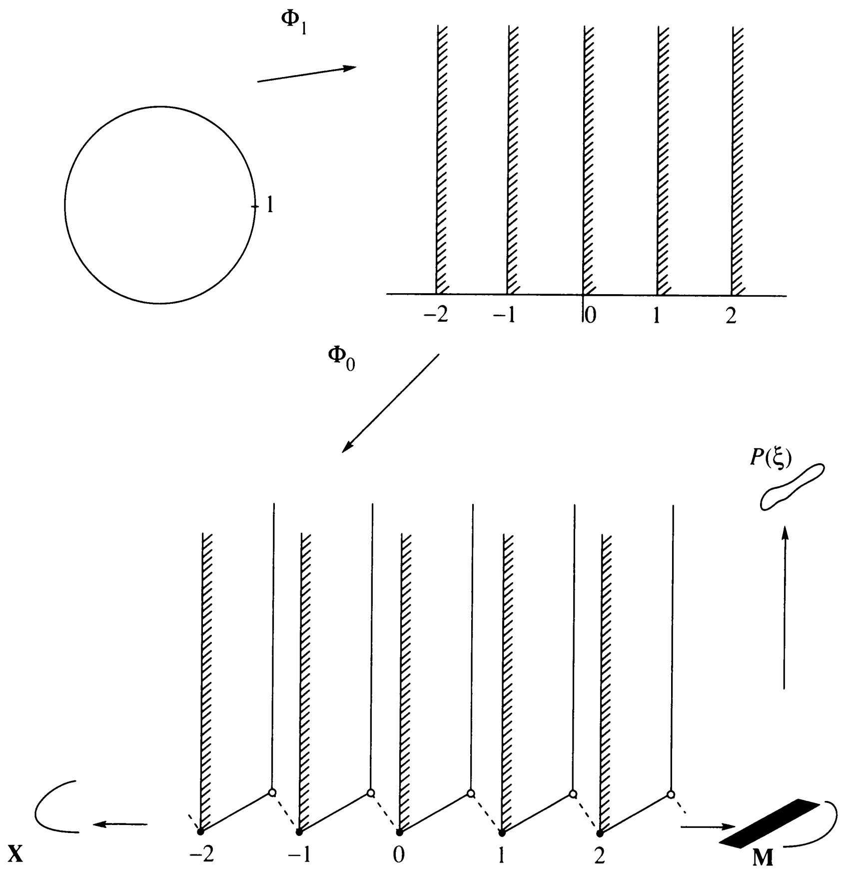

FIGURE 1

is compact, we choose such functions $\phi_1, \dots, \phi_n$ in $A(\mathbf{X})$ so that the open subsets $\{|\phi_j| > 2/3\}$, $1 \le j \le n$, make a covering of $\mathfrak{M}(H^\infty(\mathbf{X}))\backslash V$. Define the neighborhood of $\sigma(\xi)$ in $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$ by

$$W = \{\eta_0 \in \mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X};\ |\phi_j(\eta_0)| < 1/3,\ 1 \le j \le n\}.$$

Then we see $\sigma^{-1}W$ is contained in V. Thus σ is a homeomorphism.

Let $\pi = \sigma \circ (\Phi^*)^{-1}$. Then π maps homeomorphically H onto $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$. It is easy to see that $\pi(\Delta) = \mathbf{R}^2_+$. Since $\mathfrak{M}_1 \cap H = H\backslash\Delta$, the restriction π_1 maps $\mathfrak{M}_1 \cap H$ onto $\mathfrak{M}(A(\mathbf{X}))\backslash(\mathbf{X} \cup \mathbf{R}^2_+)$. Thus (i) follows.

It follows from (iii) of Lemma 2.4 and Lemma 2.5 that

$$\|\xi - \eta\| = \|\sigma(\xi) - \sigma(\eta)\|$$

whenever ξ and η lie in H_1. Since Φ^* is isometric, the map π preserves Gleason metrics. We notice that H and $\mathfrak{M}_1 \cap H$ are unions of Gleason parts.

Suppose that F is an analytic map of Δ into H. If ϕ lies in $A(\mathbf{X})$, then $\Phi^{-1}\phi(F(z))$ is analytic on Δ, since $\Phi^{-1}\phi$ lies in $H^\infty(\Delta)$. This implies that $\pi \circ F$ is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}$. Conversely, suppose that $\pi \circ F$ is analytic. Let ϕ be a function in $H^\infty(\mathbf{X})$. Then $\tilde{\phi}$ denotes the Borel version by Lemma 2.4. Since $\tilde{\phi} * P_{ir}$ is in $A(\mathbf{X})$ for $r > 0$, $\tilde{\phi} * P_{ir}(\pi \circ F(z))$ is analytic on Δ. Then we see by (iii) of Lemma 2.4 that

$$\lim_{r\to+0} \tilde{\phi} * P_{ir}(\pi \circ F(z)) = \phi((\Phi^*)^{-1} \circ F(z)),$$

since the representing measure for $\pi \circ F(z)$ is quasi-invariant on $\mathbf{X}$. This shows that $\Phi^{-1}\phi(F(z))$ is analytic on Δ. Since Φ^{-1} maps $H^\infty(\mathbf{X})$ onto $H^\infty(\Delta)$, $F(z)$ is an analytic map of Δ into H. This completes the proof. $\square$

Let μ be an invariant, ergodic, probability measure on $\mathbf{X}$. Then μ is a representing measure for $A(\mathbf{X})$. As we mentioned earlier, there are many such measures. We identify here μ with the homomorphism of $A(\mathbf{X})$ by μ. By Theorem 3.1, μ determines a homomorphism of $H^\infty(\Delta)$ lying in $\mathfrak{M}_1$. We claim that $\{\mu\}$ is a one-point part in $\mathfrak{M}(A(\mathbf{X}))$. Indeed, since $A(\mathbf{X})$ is a log-modular algebra by Proposition 2.3, each Gleason part is either a one-point part or an analytic disc. Let $H^2(\mu)$ be the closure of $A(\mathbf{X})$ in $L^2(\mu)$, and let $H_0^2(\mu)$ be the space of all functions in $H^2(\mu)$ which vanish at μ. Then the orthogonal complement of $H_0^2(\mu) \oplus \overline{H_0^2}(\mu)$ consists of constants by [11, Theorem I]. Recall that μ is represented as $\mu_0 \times m_I$, where μ_0 is an invariant measure on $\beta\mathbf{Z}$ with respect to $\mathbf{S}$. Suppose $\{\mu\}$ is not a one-point part. Then there is an inner function q in $H^2(\mu)$ such that $H_0^2(\mu) = qH^2(\mu)$ (see [5, Chapter V, §7]). Let $q^\#(y, t)$ be the automorphic extension of q to $\beta\mathbf{Z} \times \mathbf{R}$. Since the inner function $t \to q^\#(y, t)$ is not constant for μ_0-a.e. y, we choose easily a non-negative function u in $L^\infty(d\mu_0 \times dt)$ satisfying $t \to q^\#(y, t)u(y, t)(1+t^2)^{-1}$ lies in $H^\infty(\mathbf{R})$. By setting $g(y, t) = u(y, t)(1+t^2)^{-1}$, a suitable choice of u shows that

$$\phi^\#(y, t) = \sum_{j=-\infty}^{\infty} g(\mathbf{S}^j y, t-j)$$

for some nonconstant function ϕ in $L^\infty(\mu)$. Then ϕ is orthogonal to $qH^2(\mu) \oplus \overline{qH^2}(\mu)$, thus we have a contradiction.

The following proposition is, of course, an immediate consequence of the corona theorem, so the point is the direct proof of it.

Proposition 3.2. *Let μ be an invariant, ergodic, probability measure on $\mathbf{X}$. Then the homomorphism in $\mathfrak{M}(A(\mathbf{X}))$ by μ is an accumulating point of $\mathbf{R}_+^2$.*

Proof. For each ϕ in $A(\mathbf{X})$, the individual ergodic theorem shows that

$$\int_{\mathbf{X}} \phi\, d\mu = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \phi(x+t)\, dt \tag{3.5}$$

for μ-a.e. x in $\mathbf{X}$. It follows from Wiener's Tauberian theorem that the right side of (3.5) is equal to $\lim_{r\to\infty} \phi * P_{ir}(x)$ for each x such that (3.5) holds (see [14, Proof of Lemma 2.6]). By regarding $\mathbf{X} \times (0, \infty)$ as a subset of $\mathfrak{M}(A(\mathbf{X}))$, $\phi * P_{ir}(x)$ is the value of ϕ at the point (x, r) by (2.1). Let $\varepsilon > 0$, and let $\phi_1, \dots, \phi_n$ lie in $A(\mathbf{X})$. Since the union of finite null sets is also null, we find a point (x, r) in $\mathbf{X} \times (0, \infty)$ which belongs to the neighborhood

$$W = \{\eta \in \mathfrak{M}(A(\mathbf{X}));\ |\phi_j(\eta) - \phi_j(\mu)| < \varepsilon,\ 1 \le j \le n\}$$

of μ in $\mathfrak{M}(A(\mathbf{X}))$. Since $\mathbf{R}_+^2$ is dense in $\mathbf{X} \times (0, \infty)$, there is a point in $\mathbf{R}_+^2$ lying in W. Thus μ is an accumulating point of $\mathbf{R}_+^2$. $\square$

Let $\{\alpha_n\}$ be an interpolating sequence in Δ. We define

$$F_n(z) = \frac{z + \alpha_n}{1 + \overline{\alpha}_n z}, \qquad z \in \Delta,$$

which is an analytic map of Δ into $\mathfrak{M}(H^\infty(\Delta))$, since Δ is regarded as a subset of $\mathfrak{M}(H^\infty(\Delta))$. If F is an accumulating point of $\{F_n\}$ in $\mathfrak{M}(H^\infty(\Delta))^\Delta$, then F is an analytic disc. Conversely, all analytic structure in $\mathfrak{M}(H^\infty(\Delta))\backslash\Delta$ comes about in this manner. This is the characterization of nontrivial parts in $\mathfrak{M}(H^\infty(\Delta))$ due to Hoffman (see [9] or [6, Chapter X]). We define similarly

$$L_n(z) = \frac{w_n - \overline{w}_n z}{1 - z}, \qquad z \in \Delta,$$

for an interpolating sequence $\{w_n\}$ in $\mathbf{R}^2_+$. Then L_n is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{X}))$ and lies in $[\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}]^\Delta$. Suppose that an accumulating point L of $\{L_n\}$ in $\mathfrak{M}(A(\mathbf{X}))^\Delta$ lies in $[\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}]^\Delta$. It then follows from Hoffman's theorem and Theorem 3.1 that L is an analytic disc in $\mathfrak{M}(A(\mathbf{X}))$. This fact will be used in §4.

Recall that an interpolating Blaschke product $B(z)$ on $\mathbf{R}^2_+$ with zeros $\{w_n = t_n + is_n\}$ is *thin* if $B(z)$ satisfies that

$$\lim_{n\to\infty} 2s_n|B'(w_n)| = 1.$$

Suppose that $\{s_n\}$ is bounded away from zero. Let $\{L_n\}$ and L be as in above. Then L is an analytic disc which is homeomorphic to Δ and L^{-1} is the product cB of B by a unimodular constant c (see [6, Chapter X, Exercise 8]). From this fact, we obtain the following:

Proposition 3.3. *There is a nontrivial part P in $\mathfrak{M}(A(\mathbf{X}))$ satisfying the following properties*:

(i) *Let L be the analytic map of Δ onto P. Then L is a homeomorphism.*

(ii) *Let μ be the representing measure for $L(0)$. Then μ is invariant on $\mathbf{X}$. Furthermore, there is a measurable function $K(z, x)$ on $\Delta \times \mathbf{X}$ such that $x \to K(z, x)$ is invariant on $\mathbf{X}$ and*

$$\phi(L(z)) = \int_{\mathbf{X}} \phi(x)K(z, x)\,d\mu(x), \qquad \phi \in A(\mathbf{X}).$$

Proof. Let us consider the case that $\{s_n\}$ diverges, and let $P = L(\Delta)$. Then we see that the representing measure for $L(z)$ is invariant for each z in Δ. Since $A(\mathbf{X})$ is logmodular, each representing measure for $L(z)$ is mutually absolutely continuous to μ. Let $\theta(x)$ be the invariant function by the right side of (3.5) with cB in place of ϕ. It is easy to see that the function

$$K(z, x) = \frac{1}{2\pi}\frac{1 - |z|^2}{|\theta(x) - z|^2}$$

satisfies the desired property. □

4. On a question of Forelli

Let $\mathscr{C}$ be the class of all closed, nonempty, invariant subsets in a flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$. A set in $\mathscr{C}$ is *minimal* in $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ if it is minimal with respect to the inclusion relation of $\mathscr{C}$. Then Zorn's lemma guarantees the existence of a minimal set. We notice that $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is a minimal flow if and only if Ω is minimal by itself. A minimal flow is said to be *strictly ergodic* if there is exactly one invariant probability measure.

As a generalization of Wermer's maximality theorem [8, Chapter 6], Forelli [4] showed that if $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is minimal, then the induced uniform algebra $A(\Omega)$ is a maximal algebra. In connection with this result, he has asked the question stated in Introduction (see [12, §6] for a nice account of related topics). In response to this question, Muhly [11, Theorem II] gave a sufficient condition: If $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is strictly ergodic, then $A(\Omega)$ is a Dirichlet algebra on Ω. Thus, under the condition of strictly ergodicity, many interesting results have been obtained.

Our objective in this section is to give a negative answer to the question:

Theorem 4.1. *There exists a minimal flow on which the induced uniform algebra is not a Dirichlet algebra.*

Let $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$ be the flow in §2, and let $\mathbf{M}$ be a minimal set in it. We denote the restriction of $\{\mathbf{S}_t\}_{t\in\mathbf{R}}$ to $\mathbf{M}$ by $\{T_t\}_{t\in\mathbf{R}}$. Then $(\mathbf{M}, \{T_t\}_{t\in\mathbf{R}})$ is a minimal flow (see Figure 1 for the situation). We shall show the induced uniform algebra $A(\mathbf{M})$ is not a Dirichlet algebra on $\mathbf{M}$.

The outline of our proof runs as follows. As usual, $\mathbf{M}\times(0, \infty)$ is regarded as a subset of the maximal ideal space $\mathfrak{M}(A(\mathbf{M}))$ of $A(\mathbf{M})$. For a given x_0 in $\mathbf{M}$, we choose a suitable divergent sequence $\{u_n\}$ such that an accumulating point ξ of $\{(x_0, u_n)\}$ lies in a nontrivial part $P(\xi)$ that is homeomorphic to Δ. This follows from Hoffman's theorem and Theorem 3.1. Furthermore, by virtue of Marshall's theorem [10], we see that the closure $\overline{P(\xi)}$ of $P(\xi)$ is a homeomorphic replica of $\mathfrak{M}(H^\infty(\Delta))$ and that the restriction $A|_{\overline{P(\xi)}}$ of $A(\mathbf{M})$ to $\overline{P(\xi)}$ is isometrically isomorphic to $H^\infty(\Delta)$. Let Γ_0 be the Šilov boundary of $A|_{\overline{P(\xi)}}$. Since $H^\infty(\Delta)$ is not a Dirichlet algebra on its Šilov boundary, a real measure on Γ_0 is orthogonal to $A|_{\overline{P(\xi)}}$. Although Γ_0 lies off $\mathbf{M}$, the Riesz representation theorem assures the existence of a real invariant measure on $\mathbf{M}$ which is orthogonal to $A(\mathbf{M})$. Consequently, $A(\mathbf{M})$ cannot be a Dirichlet algebra.

We begin by constructing artificial Blaschke products. Let N be a positive integer, and fix $\gamma = t + iu$ in $\mathbf{R}^2_+$. We then set

$$\zeta_j = Nj + \gamma, \qquad j \in \mathbf{Z}. \tag{4.1}$$

Then the points $\{\zeta_j\}$ are periodically distributed on the horizontal line $\{\operatorname{Im} z = u\}$ and form an interpolating sequence in $\mathbf{R}^2_+$. We denote by $B(N, \gamma)(z)$ the Blaschke product with zeros $\{\zeta_j\}$ (see [6, Chapter II, (2.3)]). It is useful to represent $B(n, \gamma)(z)$ by another form. Observe that $z \to e^{2\pi i z/N}$ is an inner function in $H^\infty(\mathbf{R}^2_+)$ with period N. Then we have that

$$B(N, \gamma)(z) = c\frac{e^{2\pi i z/N} - e^{2\pi i\gamma/N}}{1 - e^{2\pi i z/N}\overline{e^{2\pi i\gamma/N}}}, \tag{4.2}$$

where c is a unimodular constant. In particular, if $u > 1$, then we write

$$z_j = Nj + iu, \qquad j \in \mathbf{Z}. \tag{4.3}$$

Since z_j and z_{-j} are axial symmetric with respect to the imaginary axis, $B(N, iu)$ has a simple form

$$B(N, iu)(z) = -\prod_{j=-\infty}^{\infty}\frac{z - z_j}{z - \overline{z}_j} = -\frac{z - iu}{z + iu}\prod_{j=1}^{\infty}\frac{(z - iu)^2 - N^2j^2}{(z + iu)^2 - N^2j^2}. \tag{4.4}$$

Setting $\gamma = t+iu$, we see that $B(N, \gamma)(z) = cB(N, iu)(z-t)$ for a unimodular constant c.

The pseudo-hyperbolic distances ρ_1 and ρ on Δ and $\mathbf{R}^2_+$ are defined by

$$(4.5)\qquad \begin{cases} \rho_1(z, w) = \left|\dfrac{z-w}{1-\overline{w}z}\right| & \text{on } \Delta, \text{ and} \\[2ex] \rho(z, w) = \left|\dfrac{z-w}{z-\overline{w}}\right| & \text{on } \mathbf{R}^2_+, \end{cases}$$

respectively. Let L be a linear fractional map of Δ onto $\mathbf{R}^2_+$. Then we see that $\rho_1(z, w) = \rho(L(z), L(w))$ on $\Delta \times \Delta$. Let $0 < \lambda < 1$, and set

$$U(w, \lambda) = \{z \in \mathbf{R}^2_+ ; \rho(z, w) < \lambda\}$$

for $w = t + is$ in $\mathbf{R}^2_+$. Then an easy calculation shows that $U(w, \lambda)$ is an open disc with center $(t, s(1+\lambda^2)/(1-\lambda^2))$ and radius $2s\lambda/(1-\lambda^2)$. It follows from [6, Chapter I, Lemma 1.4] that if ζ lies in $U(w, \lambda)$, then $U(\zeta, \lambda)$ is contained in $U(w, 2\lambda/(1+\lambda^2))$.

Lemma 4.2. *Let $K > 0$, and let $0 < \delta < \varepsilon < 1$. Then there are a positive integer $N = N(K, \varepsilon, \delta)$ and a real number $u = u(K, \varepsilon, \delta)$, $u > 1$, satisfying the following properties*:

(i) *Let $\{z_j\}$ be the sequence by* (4.3) *with these N and u, and let*

$$U_j = \{z ; \rho(z, z_j) < 2(1-\varepsilon)/(1+(1-\varepsilon)^2)\}$$

for each j in $\mathbf{Z}$. Then the discs $\{U_j\}$ are pairwise disjoint, and each U_j is contained in the half-plane $\{\operatorname{Im} z > K\}$ (see Figure 2 on the next page).

(ii) *Let γ in $\mathbf{R}^2_+$ satisfy that $\rho(\gamma, iu) < 1-\varepsilon$, and let $\{\zeta_j\}$ be the sequence by* (4.1). *If $B(N, \gamma)(z)$ denotes the Blaschke product with zeros $\{\zeta_j\}$, then*

$$(4.6)\qquad |B(N, \gamma)(z)| \geq 1-\delta \quad \textit{on } \{0 \leq \operatorname{Im} z \leq K\},$$

and

$$(4.7)\qquad \big||B(N, \gamma)(z)| - \rho(z, \zeta_j)\big| < \varepsilon \quad \textit{on } U_j.$$

(iii) *To each $\delta' > 0$, there correspond $a = a(\gamma)$, $1-\delta < a < 1$, and K', $K' > K$, such that*

$$(4.8)\qquad \sup\{\big||B(N, \gamma)(z)| - a\big| ; \operatorname{Im} z > K'\} < \delta',$$

that is, the left side of (4.8) *converges to* 0 *as $K' \to \infty$. Furthermore, for each square $Q = [t_0, t_0 + l] \times [0, l]$, $l > K'$, the inequality*

$$(4.9)\qquad \sum_{\zeta_j \in Q} \operatorname{Im} \zeta_j \leq \varepsilon l$$

holds.

Proof. By setting $\lambda = 2(1-\varepsilon)/(1+(1-\varepsilon)^2)$, U_j is the disc with center $(Nj, u(1+\lambda^2)/(1-\lambda^2))$ and radius $2u\lambda/(1-\lambda^2)$. Thus the discs $\{U_j\}$ have the property (i) whenever N and u satisfy that

$$(4.10)\qquad u > K(1+\lambda)/(1-\lambda) \quad \text{and} \quad N > 4u\lambda/(1-\lambda^2).$$

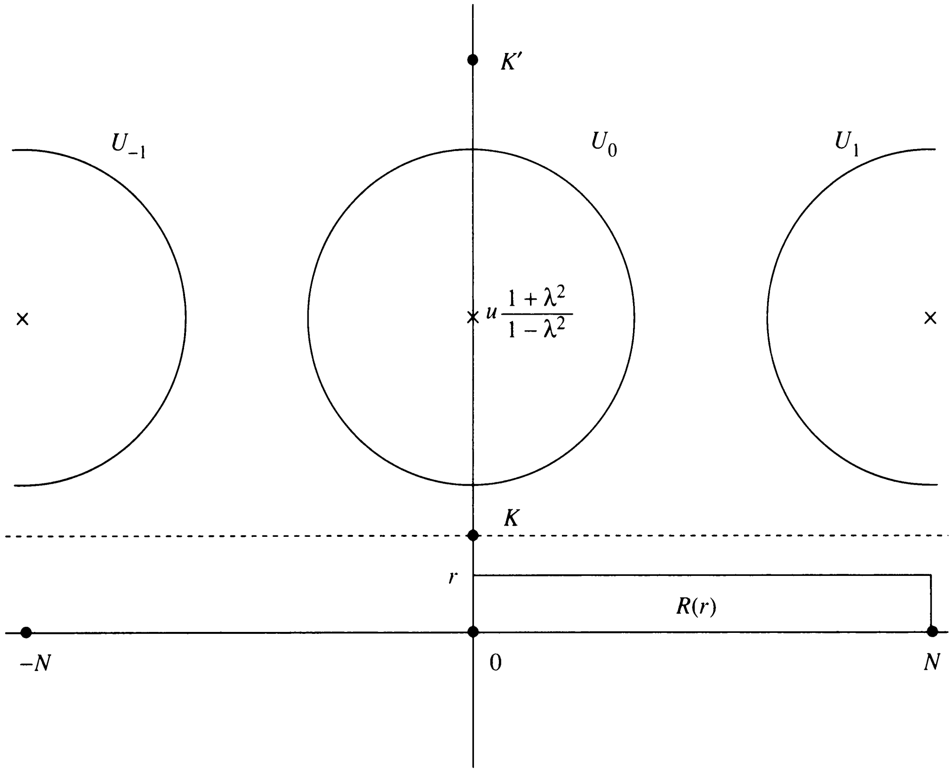

FIGURE 2

Let (N, u) be a pair with the property (i). We notice that u and N/u may be chosen as large as desired. Define that

$$W_0 = \{\gamma ;\ \rho(\gamma , iu) \leq 1 - \varepsilon\}.$$

Then W_0 is a compact disc contained in U_0. Let $R(r)$, $0 \leq r \leq K$, be the rectangle $[0, N] \times [0, r]$ (see Figure 2). We claim that $(z, \gamma) \to |B(N, \gamma)(z)|$ is uniformly continuous on $R(K) \times W_0$. Indeed, we find easily a constant $c > 0$ such that

$$(4.11) \qquad 1 - \rho(z, Nj + \gamma)^2 < c/j^2, \qquad j \neq 0,$$

on $R(K) \times W_0$. Then the product $|B(N, \gamma)(z)| = \prod_{j=-\infty}^{\infty} \rho(z, Nj + \gamma)$ converges uniformly on $R(K) \times W_0$. Since each $(z, \gamma) \to \rho(z, Nj + \gamma)$ is continuous on $R(K) \times W_0$, the uniform continuity of $|B(N, \gamma)(z)|$ follows from the compactness of $R(K) \times W_0$. Consequently, since $|B(N, \gamma)(z)| = 1$ on $R(0) \times W_0$, there is an r, $0 < r < K$, such that $|B(N, \gamma)(z)| > 1 - \delta$ on $R(r) \times W_0$. Since $\rho(z, Nj + \gamma) = \rho(z, \zeta_j) = \rho(kz, k\zeta_j)$ we see that

$$|B(N, \gamma)(z)| = \prod_{j=-\infty}^{\infty} \rho(kz, k\zeta_j) = |B(kN, k\gamma)(kz)|$$

for each positive integer k. Then $|B(kN, k\gamma)(z)| > 1-\delta$ on $[0, kN]\times[0, kr]$. Therefore, if we replace N and u by larger ones, we make (N, u) satisfy $|B(N, \gamma)(z)| > 1 - \delta$ on $R(K) \times W_0$. Since $|B(N, \gamma)(z)|$ is periodic with period N, (4.6) follows.

We see similarly that (4.11) holds on $U_0 \times W_0$ with a suitable constant $c > 0$. Since the partial product $\prod_{j=-m}^{m} \rho(z, Nj + \gamma)$ converges uniformly to $|B(N, \gamma)(z)|$ on $U_0 \times W_0$, there is an m such that

$$\left| \prod_{|j| \geq m+1} \rho(z, Nj + \gamma) - 1 \right| < \varepsilon$$

on $U_0 \times W_0$. Replacing N with one larger than mN, we may assume

$$\| B(N, \gamma)(z)| - \rho(z, \gamma)| < \varepsilon$$

on $U_0 \times W_0$. Then (4.7) follows from the periodicity of $|B(N, \gamma)(z)|$. Thus we obtain the property (ii).

We write $\gamma = t + is$. Since $\rho(\gamma, iu) < 1 - \varepsilon$, we have easily the inequality,

$$u \frac{1 - (1 - \varepsilon)}{1 + (1 - \varepsilon)} < s < u \frac{1 + (1 - \varepsilon)}{1 - (1 - \varepsilon)}.$$

Since N/u is chosen as large as desired, we may assume that $e^{-2\pi s/N} > 1 - \delta$. Define $a(\gamma) = e^{-2\pi s/N}$. By (4.2), we see that (4.8) holds for some $K' > K$.

If we make N/u and K' sufficiently large, (4.9) holds for each square $Q = [t_0, t_0 + l] \times [0, l]$, $l > K'$, so the proof is complete. □

Let $\{\varepsilon_n\}$ be a decreasing sequence such that $0 < \varepsilon_n < 1$ and $\sum \varepsilon_n < \infty$. We then choose a sequence $\{\delta_n\}$, $0 < \delta_n < 1$, such that

$$\prod_{k=n}^{\infty} (1 - \delta_k) > 1 - \varepsilon_n, \qquad n \geq 1. \tag{4.12}$$

For a given $K_n > 0$, let N_n, u_n, γ_n, and a_n be as in Lemma 4.2 with respect to the above ε_n and δ_n. We also set $z_{nj} = N_n j + iu_n$ and $\zeta_{nj} = N_n j + \gamma_n$, $j \in \mathbf{Z}$.

By induction, we obtain immediately the following:

Lemma 4.3. *There is an increasing sequence $\{K_n\}$ with the following properties:*

(i) *Let $n \geq 2$. Then*

$$\left| \prod_{k=m}^{n-1} |B(N_k, \gamma_k)(z - t_k)| - \prod_{k=m}^{n-1} a_k \right| < \varepsilon_n \tag{4.13}$$

holds on $\{\operatorname{Im} z > K_n\}$, where $0 \leq t_k \leq N_k - 1$ and $m = 1, 2, \ldots, n-1$.

(ii) *Let $U_{nj} = \{z ; \rho(z, z_{nj}) < 2(1 - \varepsilon_n)/(1 + (1 - \varepsilon_n)^2)\}$. Then each U_{nj}, $j \in \mathbf{Z}$, is contained in $\{K_n < \operatorname{Im} z < K_{n+1}\}$.*

(iii) *If $Q = [t_0, t_0 + l] \times [0, l]$, $l > K_{n+1}$, then the inequality*

$$\sum_{\zeta_{nj} \in Q} \operatorname{Im} \zeta_{nj} \leq \varepsilon_n l, \qquad n \geq 1,$$

holds.

Recall our notation $\mathbf{M}$ for a minimal set in $(\mathbf{X}, \{\mathbf{S}_t\}_{t \in \mathbf{R}})$. Identifying $\mathbf{X}$ with $\beta\mathbf{Z} \times [0, 1)$, we fix an x_0 in $\mathbf{M}$ of the form $(y_0, 0)$ in $\beta\mathbf{Z} \times [0, 1)$. For each $n \geq 1$, we set

$$Z_n(k) = \{N_n j + k ; j \in \mathbf{Z}\},$$

where $k = 0, 1, \dots, N_n - 1$. Since $\mathbf{Z}$ is the disjoint union of $Z_n(k)$, there is a unique k_n such that the closure of $Z_n(k_n)$ in $\beta\mathbf{Z}$ contains the above point y_0. We then define

$$w_{nj} = z_{nj} + k_n = (N_n j + k_n) + iu_n. \tag{4.14}$$

Then $\{w_{nj}\,;\, j \in \mathbf{Z}\}$ is an interpolating sequence for each $n \geq 1$. We notice that $B(N_n, k_n + iu_n)(z)$ is the Blaschke product with zeros $\{w_{nj}\,;\, j \in \mathbf{Z}\}$. Recall that $\mathbf{X} \times [0, \infty)$ is embedded in $\mathfrak{M}(A(\mathbf{X}))$ by (2.1). By identifying $X_0 \times [0, \infty)$ with $\mathbf{R}^2_+$, the point $(x_0, u_n) = ((y_0, 0), u_n)$ lies in the closure of $\{w_{nj}\,;\, j \in \mathbf{Z}\}$ in $\mathfrak{M}(A(\mathbf{X}))$. We notice that (x_0, u_n) lies also in $\mathfrak{M}(A(\mathbf{M}))$, since $\mathbf{M} \times [0, \infty)$ is a subset of $\mathfrak{M}(A(\mathbf{M}))$. It will be shown that the accumulating point ξ of $\{(x_0, u_n)\,;\, n \geq 1\}$ in $\mathfrak{M}(A(\mathbf{M}))$ lies in a nontrivial part $P(\xi)$ that is homeomorphic to Δ.

Let us show that $\{w_{nj}\,;\, n \geq 1,\, j \in \mathbf{Z}\}$ is an interpolating sequence in $\mathbf{R}^2_+$. These points are clearly separated by (ii) of Lemma 4.3. Since $\sum \varepsilon_n < \infty$, it follows from (iii) of Lemma 4.3 that

$$\sum_{w_{nj} \in Q} \operatorname{Im} w_{nj} \leq \left(1 + \sum \varepsilon_n\right) l \tag{4.15}$$

for each square $Q = [t_0, t_0 + l] \times [0, l]$. Therefore, by [6, Chapter VII, Theorem 1.1], $\{w_{nj}\,;\, n \geq 1,\, j \in \mathbf{Z}\}$ is an interpolating sequence.

Let $B_m(z)$ be the Blaschke product with zeros $\{w_{nj}\,;\, n \geq m,\, j \in \mathbf{Z}\}$. Since

$$B_m(z) = \prod_{n=m}^{\infty} B(N_n, k_n + iu_n)(z), \tag{4.16}$$

$B_m(z)$ has the periodic zeros $\{w_{nj}\,;\, j \in \mathbf{Z}\}$ on each horizontal line $\{\operatorname{Im} z = u_n\}$, $n \geq m$, (see Figure 3 for the distribution of zeros of $B_1(z)$). We see

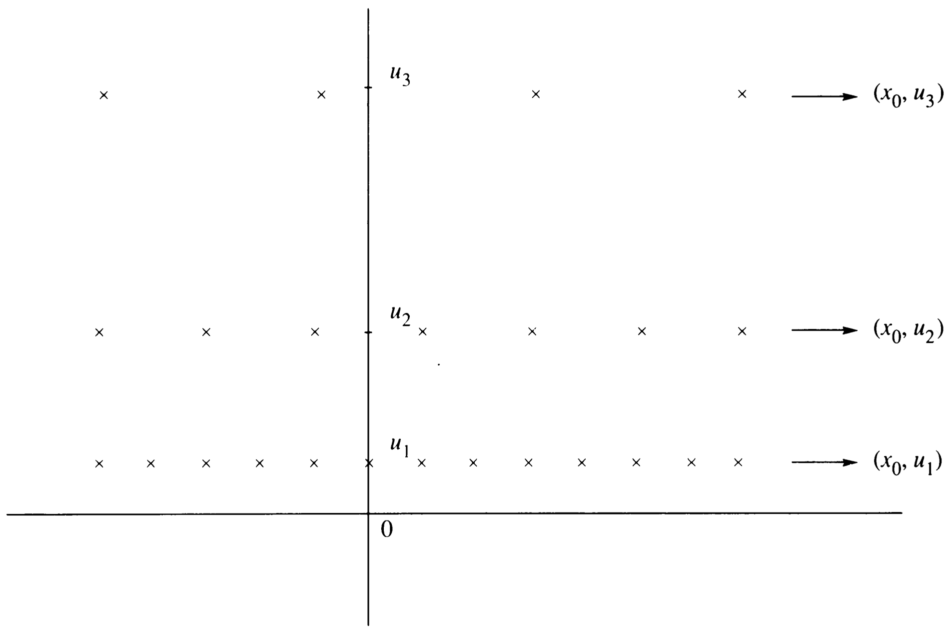

FIGURE 3

by (4.16) and [6, Chapter II, Theorem 6.1] that $B_m(z)$ extends analytically to $\{\operatorname{Im} z \geq -K_m\}$. It follows from (4.12) and (ii) of Lemma 4.2 that $|B_m(z)| \leq (1-\varepsilon_m)^{-1}$ on $\{\operatorname{Im} z \geq -K_m\}$. Let $0 < s < K_m$, and define the bounded function F on $\mathbf{R}$ by $F(t) = B_m(t - is)$. Since $B_m(t) = F * P_{is}(t)$, we see that $t \to B_m(t)$ is uniformly continuous on $\mathbf{R}$. Hence Lemma 3.1 implies that $B_m(t)$ extends to a function $B_m(x)$ in $A(\mathbf{X})$. Recall that $O(x_0) \times (0, \infty)$ is a nontrivial part in $\mathfrak{M}(A(\mathbf{X}))$. However, since x_0 is in the minimal set $\mathbf{M}$, $O(x_0) \times (0, \infty)$ is not homeomorphic to $\mathbf{R}^2_+$. On the other hand, $O(x_0) \times (0, \infty)$ contains every (x_0, u_n), which is an accumulating point of $\{w_{nj}; j \in \mathbf{Z}\}$.

Let $\{\varepsilon_n\}$, $\{\gamma_n\}$, and $\{U_{nj}\}$ be as in Lemma 4.3, and let $\{w_{nj}\}$ be the sequence defined by (4.14). We then set

$$\begin{aligned} V_{nj} &= \{z; \rho(z, w_{nj}) < 2(1-\varepsilon_n)/(1+(1-\varepsilon_n)^2)\} \\ &= U_{nj} + k_n. \end{aligned}$$

Observe that $B(N_n, \gamma_n)(z - k_n)$ is (a unimodular constant multiple of) the interpolating Blaschke product with zeros $\{\zeta_{nj} + k_n; j \in \mathbf{Z}\}$, where $\zeta_{nj} = N_n j + \gamma_n$. By (iii) of Lemma 4.3, (4.15) holds with $\zeta_{nj} + k_n$ in place of w_{nj}. Then $\{\zeta_{nj} + k_n; n \geq 1, j \in \mathbf{Z}\}$ is also an interpolating sequence. Let $C_m(z)$ be the Blaschke product with zeros $\{\zeta_{nj} + k_n; n \geq m, j \in \mathbf{Z}\}$. So we may write

$$C_m(z) = \prod_{n=m}^{\infty} B(N_n, k_n + \gamma_n)(z) \tag{4.17}$$

(compare with (4.16)).

Let $\{\delta_n\}$, $\{K_n\}$, and $\{a_n\}$ be also as in Lemma 4.3. Then we see that $|B(N_n, k_n + \gamma_n)(z)| > 1 - \delta_n$ on $\{0 < \operatorname{Im} z < K_n\}$ and $1 - \delta_n < a_n < 1$ by Lemma 4.2. It follows from (4.12) that the product

$$a^{(m)} = \prod_{n=m}^{\infty} a_n \tag{4.18}$$

converges and satisfies that $1 - \varepsilon_m < a^{(m)} < 1$.

Lemma 4.4. *Let V_{nj} and $C_m(z)$ be as in above. Then*

$$\sup\{\,||C_m(z)| - a^{(m)}\rho(z, \zeta_{nj} + k_n)|; z \in V_{nj}\}$$

converges to 0 *as* $n \to \infty$.

Proof. We set, at our convenience, that

$$E_n(z) = |B(N_n, k_n + \gamma_n)(z)|,$$

and

$$F_n(z) = a^{(n+1)} E_n(z) \prod_{k=m}^{n-1} E_k(z).$$

Then $|C_m(z)| = \prod_{k=m}^{\infty} E_k(z)$ by (4.17). Since V_{nj} is contained in $\{K_n < \operatorname{Im} z < K_{n+1}\}$ by (ii) of Lemma 4.3, it follows from (4.6) that

$$\prod_{k=n+1}^{\infty} (1 - \delta_n) < \prod_{k=n+1}^{\infty} E_k(z) < 1, \qquad z \in V_{nj}.$$

Observe that

$$|F_n(z) - a^{(n+1)}\rho(z\,,\,\zeta_{nj} + k_n)\prod_{k=m}^{n-1} E_k(z)| \le |E_n(z) - \rho(z\,,\,\zeta_{nj} + k_n)|.$$

Then we see by (4.7), (4.13), and (4.18) that

$$\begin{aligned}
&||C_m(z)| - a_n^{-1}a^{(m)}\rho(z\,,\,\zeta_{nj} + k_n)| \\
&\quad \le ||C_m(z)| - F_n(z)| + \left|F_n(z) - a^{(n+1)}\rho(z\,,\,\zeta_{nj} + k_n)\prod_{k=m}^{n-1} a_k\right| \\
&\quad \le \left||C_{n+1}(z)| - a^{(n+1)}\right| + |E_n(z) - \rho(z\,,\,\zeta_{nj} + k_n)| + \left|\prod_{k=m}^{n-1} E_k(z) - \prod_{k=m}^{n-1} a_k\right| \\
&\quad < \left(1 - \prod_{k=n+1}^{\infty}(1-\delta_k)\right) + \varepsilon_n + \varepsilon_n \\
&\quad < 3\varepsilon_n
\end{aligned}$$

for each z in V_{nj}. Since a_n^{-1} tends to 1 as $n \to \infty$, the conclusion follows. □

We denote by ξ_n the homomorphism $(x_0\,,\,u_n)$ in $\mathfrak{M}(A(\mathbf{X}))$. To each $n \ge 1$, there corresponds a subnet $\{w_{n\beta}\}$ of $\{w_{nj}\,;\,j \in \mathbf{Z}\}$ converging to ξ_n in $\mathfrak{M}(A(\mathbf{X}))$. Let ξ be an accumulating point of $\{\xi_n\}$. Then ξ is also an accumulating point of the interpolating sequence $\{w_{nj}\,;\,n \ge 1\,,\,j \in \mathbf{Z}\}$. Therefore, by Hoffman's theorem and Theorem 3.1, ξ belongs to a nontrivial part $P(\xi)$. Precisely, let L_{nj} be the linear fractional map of Δ onto $\mathbf{R}_+^2$ by

$$L_{nj}(z) = \frac{w_{nj} - \overline{w}_{nj}z}{1-z}, \qquad z \in \Delta. \tag{4.19}$$

Then L_{nj} is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{X}))$. Taking a finer subnet $\{L_{n\beta}\}$ of $\{L_{nj}\}$, if necessary, we may assume that

$$\lim_{\beta} L_{n\beta} = L_n\,, \qquad n \ge 1\,, \tag{4.20}$$

in $\mathfrak{M}(A(\mathbf{X}))^{\Delta}$, where L_n is in $[\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}]^{\Delta}$ and satisfies that $L_n(0) = \xi_n$. Let us determine the explicit form of L_n. For simplicity, we write $x + t + is$ for $(x+t\,,\,s)$ in $\mathbf{X} \times [0\,,\,\infty)$. It then follows from (4.19) that

$$L_n(z) = x_0 + \frac{iu_n(1+z)}{1-z}, \qquad z \in \Delta\,, \tag{4.21}$$

which maps Δ onto the nontrivial part $O(x_0) \times (0\,,\,\infty)$. We notice that $L_{n\beta}$ and L_n lie in $[\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}]^{\Delta}$, so the convergence in (4.20) is the same one in $\mathfrak{M}(H^\infty(\mathbf{X}))^{\Delta}$ by Theorem 3.1. As we mentioned earlier, L_n is not a homeomorphism, since x_0 lies in the minimal set $\mathbf{M}$. We take again a subnet $\{L_\alpha\}$ of $\{L_n\}$ such that

$$\lim_{\alpha} L_\alpha = L_0 \tag{4.22}$$

for a map L_0 in $\mathfrak{M}(A(\mathbf{X}))^{\Delta}$ with $L_0(0) = \xi$. Since $\lim_\alpha u_\alpha = \infty$, we see by (4.21) that L_0 is also in $[\mathfrak{M}(A(\mathbf{X}))\backslash\mathbf{X}]^{\Delta}$. It then follows from Theorem 3.1 and Hoffman's theorem that L_0 is an analytic map and the range $L_0(\Delta)$ is the nontrivial part $P(\xi)$ containing ξ. We recall that the meaning of (4.22) is that if z lies in Δ, then

$$\lim_{\alpha} \phi \circ L_\alpha(z) = \phi \circ L_0(z) \tag{4.23}$$

for each ϕ in $A(\mathbf{X})$.

We next show that L_0 may be considered as an analytic map of Δ into $\mathfrak{M}(A(\mathbf{M}))$. Let $E_\mathbf{X}$ be the closure of $\mathbf{M}\times[0,\infty)$ in $\mathfrak{M}(A(\mathbf{X}))$. Since $L_\alpha(z)$, $z\in\Delta$, is in $\mathbf{M}\times[0,\infty)$ by (4.21) and $\lim_\alpha u_\alpha=\infty$, we see that $L_0(z)$ lies in $E_\mathbf{X}\backslash\mathbf{M}\times[0,\infty)$. Then the whole $P(\xi)$ is contained in $E_\mathbf{X}\backslash\mathbf{M}\times[0,\infty)$. On the other hand, $\mathbf{M}\times[0,\infty)$ is also considered as a subset of $\mathfrak{M}(A(\mathbf{M}))$. We denote similarly by $E_\mathbf{M}$ the closure of $\mathbf{M}\times[0,\infty)$ in $\mathfrak{M}(A(\mathbf{M}))$. We have to examine the relation between $E_\mathbf{X}$ and $E_\mathbf{M}$.

Let $A|_\mathbf{M}$ be the space of the restrictions to $\mathbf{M}$ of functions in $A(\mathbf{X})$. Then $A|_\mathbf{M}$ is a subalgebra of $A(\mathbf{M})$ containing constants. So there is a natural restriction map σ_1 of $E_\mathbf{M}$ to $E_\mathbf{X}$ which is the identity map on $\mathbf{M}\times[0,\infty)$. Let us show that σ_1 is a homeomorphism of $E_\mathbf{M}$ onto $E_\mathbf{X}$. Since $A(\mathbf{X})$ is a logmodular algebra, σ_1 is a one-to-one continuous map. If η lies in $E_\mathbf{X}$, then there is a net $\{\eta_\beta\}$ in $\mathbf{M}\times[0,\infty)$ converging to η in $E_\mathbf{X}$. Since $E_\mathbf{M}$ is compact, we may assume that $\{\eta_\beta\}$ converges to η_1 in $E_\mathbf{M}$. We then have that $\sigma_1(\eta_1)=\eta$. Since $E_\mathbf{M}$ is compact, $E_\mathbf{M}$ and $E_\mathbf{X}$ are homeomorphic via the map σ_1.

We see that the closure of $A|_\mathbf{M}$ is a logmodular algebra on $\mathbf{M}$, then so is $A(\mathbf{M})$. If $\sigma_1(\eta_1)=\eta$, then η_1 and η have the same representing measures. Identifying L_0 with $\sigma_1^{-1}\circ L_0$, we see that (4.23) holds for each ϕ in $A(\mathbf{M})$. This implies that L_0 is an analytic map of Δ into $\mathfrak{M}(A(\mathbf{M}))$.

Since the representing measures for the points in $P(\xi)$ are mutually absolutely continuous, $P(\xi)$ is also a nontrivial part with respect to $A(\mathbf{M})$ by [5, Chapter VI, §1.2].

Let μ be the representing measure for a point η in $E_\mathbf{M}\backslash\mathbf{M}\times[0,\infty)$. We show that μ is invariant on $(\mathbf{M}, \{T_t\}_{t\in\mathbf{R}})$, although the similar fact has been used tacitly in the proofs of Lemma 2.5 and (ii) of Proposition 3.3. Indeed, there is a net (x_β, u_β) in $\mathbf{M}\times[0,\infty)$ converging to η, where $\lim_\beta u_\beta=\infty$. Let μ_β be the representing measure for (x_β, u_β). Then $\{\mu_\beta\}$ converges to μ in the weak-$*$ topology on $M(\mathbf{M})$. Since

$$|P_{iu}(t-s)-P_{iu}(t)|\le \tfrac{s}{u}(1+\tfrac{s}{u})P_{iu}(t-s),$$

we see that

$$\int_\mathbf{M}\phi(x+s)\,d\mu(x)=\int_\mathbf{M}\phi(x)\,d\mu(x),\qquad \phi\in C(\mathbf{M}),$$

for each s in $\mathbf{R}$. Then μ is invariant on $\mathbf{M}$. In particular, the representing measure for each point in $P(\xi)$ is invariant on $\mathbf{M}$.

Lemma 4.5. *Let ξ_n denote the homomorphism (x_0, u_n) defined above, and let ξ be an accumulating point of $\{\xi_n\}$ in $\mathfrak{M}(A(\mathbf{M}))$. Then ξ lies in a nontrivial part $P(\xi)$ in $\mathfrak{M}(A(\mathbf{M}))$ satisfying the following properties:*

(i) *Let $\overline{P(\xi)}$ be the closure of $P(\xi)$ in $\mathfrak{M}(A(\mathbf{M}))$. Then each point in $\overline{P(\xi)}$ has a unique representing measure which is invariant on $\mathbf{M}$.*

(ii) *The part* $P(\xi)$ *is homeomorphic to* Δ. *Precisely, the analytic map* L_0 *by* (4.22) *has its continuous inverse.*

Proof. We have shown that ξ lies in a nontrivial part $P(\xi)$, in which every representing measure is invariant on $\mathbf{M}$. Since the set of all invariant probability measures on $\mathbf{M}$ is weak-$*$ compact in $M(\mathbf{M})$, each point in $\overline{P(\xi)}$ has also an invariant representing measure. Thus (i) follows.

Let L_{nj}, L_n, and L_0 be as in (4.19), (4.21), and (4.22), respectively. Let $B_m(z)$, $m \geq 1$, be the Blaschke product defined by (4.16). Since $B_m(z)$ extends to a function $B_m(x)$ in $A(\mathbf{X})$, we may regard $B_m(x)$ as a function in $A(\mathbf{M})$ by identifying with its restriction to $\mathbf{M}$. Since

$$\rho(L_{nj}(z), L_{nj}(0)) = \rho_1(z, 0) = |z|, \qquad z \in \Delta,$$

by (4.5), Lemma 4.4 implies that

$$\sup\{||B_m \circ L_{nj}(z)| - a^{(m)}|z||\,;\, |z| \leq 1 - \varepsilon_n\}$$

converges to 0 as $n \to \infty$, where $a^{(m)}$ is the constant by (4.18). By (4.20) and (4.22), we obtain that $|B_m \circ L_0(z)| = a^{(m)}|z|$ on Δ. It then follows from Schwarz's lemma that $(c_m B_m) \circ L_0(z) = z$ for a constant c_m with $|c_m| = (a^{(m)})^{-1}$. Since B_m is continuous on $\mathfrak{M}(A(\mathbf{M}))$, L_0 has the continuous inverse. □

To extend (ii) of Lemma 4.5 (see Lemma 4.8), we need the following:

Lemma 4.6. *Let* L_0 *be the analytic map of* Δ *into* $\mathfrak{M}(A(\mathbf{X}))$ *by* (4.22), *and let* $b(z)$ *be an arbitrary Blaschke product on* Δ *with zeros* $\{\alpha_k\}$, *that is,*

$$b(z) = \prod_{k=1}^{\infty} \frac{\overline{\alpha}_k}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha}_k z}$$

(*we consider* $\overline{\alpha_k}/|\alpha_k| = -1$, *if* $\alpha_k = 0$). *If* $0 < \varepsilon < 1$, *then there is a Blaschke product* $B(z)$ *on* $\mathbf{R}^2_+$ *such that* $B(z)$ *extends to a function* $B(x)$ *in* $A(\mathbf{X})$ *satisfying that*

$$cB \circ L_0(z) = b(z), \qquad z \in \Delta, \tag{4.25}$$

for some constant c *with* $|c| < 1 + \varepsilon$.

Proof. We have already done the hard work back in Lemma 4.4. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be as in (4.12), and let $\{r_k\}$ be a sequence such that $0 < r_k < 1$ and $\prod r_k > (1 + \varepsilon)^{-1}$. We assume that $|\alpha_k| \leq |\alpha_{k+1}|$ for all $k \geq 1$.

Let w_{nj} and L_{nj} be as in (4.14) and (4.19), respectively. Then we see by (4.5) that $\rho_1(z, w) = \rho(L_{nj}(z), L_{nj}(w))$ on $\Delta \times \Delta$. Since $L_{nj}(0) = w_{nj}$, to each $k \geq 1$, there corresponds $m_1(k)$ such that

$$\rho(w_{nj}, L_{nj}(\alpha_k)) = \rho_1(0, \alpha_k) = |\alpha_k| < 1 - \varepsilon_n$$

for all $n \geq m_1(k)$. Observe that $L_{nj}(\alpha_k) = N_n j + L_{n0}(\alpha_k)$ by (4.19). By (iii) of Lemma 4.3, $\{L_{nj}(\alpha_k)\,;\, n \geq m_1(k),\ j \in \mathbf{Z}\}$ is an interpolating sequence in $\mathbf{R}^2_+$. Making $m_1(k)$ large, we may assume that

$$\sum_{n=m_1(k)}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\operatorname{Im} L_{nj}(\alpha_k)}{1 + |L_{nj}(\alpha_k)|^2} < \left(\frac{1}{2}\right)^k. \tag{4.26}$$

Let $m(k) \geq m_1(k)$. We denote by $C_{m(k)}(z)$ the Blaschke product on $\mathbf{R}^2_+$ with zeros $\{L_{nj}(\alpha_k); n \geq m(k), j \in \mathbf{Z}\}$. Then $C_{m(k)}(z)$ is the Blaschke product in (4.17) with $L_{n0}(\alpha_k)$ in place of γ_n. Let $a^{(m(k))}$ be the constant by (4.18) with respect to $C_{m(k)}(z)$. If we make $m(k)$ sufficiently large, then $a^{(m(k))}$ satisfies that $r_k < a^{(m(k))} < 1$. By Lemma 4.4, we see that

$$\sup\{||C_{m(k)}(z)| - a^{(m(k))}\rho(z, L_{nj}(\alpha_k))|; z \in V_{nj}\}$$

converges to 0 as $n \to \infty$. By induction, we choose an increasing sequence $\{m(k)\}$, $m(k) \geq m_1(k)$, such that

$$\sup\left\{\left|\left|\prod_{i=1}^{k} C_{m(i)}(z)\right| - \prod_{i=1}^{k} a^{(m(i))}\rho(z, L_{nj}(\alpha_i))\right| ; z \in V_{nj}\right\} < \varepsilon_k$$

for each n with $m(k+1) \geq n > m(k)$, and that

$$|C_{m(k)}(z)| \geq \prod_{i=m(k)}^{\infty} (1 - \delta_i) > 1 - \delta_k$$

on $\{0 \leq \operatorname{Im} z < K_{m(k)}\}$ by (4.6) and (4.17). We then set

$$B(z) = \prod_{i=1}^{\infty} C_{m(i)}(z) \quad \text{and} \quad a = \prod_{i=1}^{\infty} a^{(m(i))}.$$

If $m(k+1) \geq n > m(k)$, then we have by (4.12) that

$$\left||B(z)| - \left|\prod_{i=1}^{k} C_{m(i)}(z)\right|\right| \leq \left|\left|\prod_{i=k+1}^{\infty} C_{m(i)}(z)\right| - 1\right| < 1 - \prod_{i=k+1}^{\infty} (1 - \delta_i) < \varepsilon_k$$

on V_{nj}. This implies that

$$\sup\left\{\left||B(z)| - \prod_{i=1}^{k} a^{(m(i))}\rho(z, L_{nj}(\alpha_i))\right| ; z \in V_{nj}\right\} < 2\varepsilon_k.$$

Let $b_k(z)$ be the partial product of $b(z)$:

$$b_k(z) = \prod_{i=1}^{k} \frac{\overline{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \overline{\alpha}_i z}.$$

Then, whenever $m(k+1) \geq n > m(k)$, we obtain that

$$\sup\left\{\left||B \circ L_{nj}(z)| - \prod_{i=1}^{k} a^{(m(i))}|b_k(z)|\right| ; |z| \leq 1 - \varepsilon_n\right\} < 2\varepsilon_k.$$

Consequently, it follows from (4.20) and (4.22) that

$$|B \circ L_0(z)| = a|b(z)|, \qquad z \in \Delta.$$

Since $a > (1 + \varepsilon)^{-1}$, the maximum principle shows that (4.25) holds for a constant c with $|c| = a^{-1}$. □

Recall that we identify ϕ in $A(\mathbf{M})$ with its Gelfand transform on $\mathfrak{M}(A(\mathbf{M}))$. Then $\phi|_{P(\xi)}$ denotes the restriction of ϕ to $P(\xi)$. Let $A|_{P(\xi)}$ denote the algebra of all $\phi|_{P(\xi)}$ for ϕ in $A(\mathbf{M})$. Then the analytic map L_0 in (4.22) induces naturally the isometric isomorphism Ψ of $A|_{P(\xi)}$ to $H^\infty(\Delta)$ by

$$\Psi(\phi|_{P(\xi)})(z) = \phi \circ L_0(z), \qquad z \in \Delta. \tag{4.28}$$

If $b(z)$ is a Blaschke product, let us agree to also call $cb(z)$ a Blaschke product when c is a unimodular constant. Together with Bernard's trick, Marshall has shown that the unit ball of $H^\infty(\Delta)$ is the norm closed convex hull of the set of Blaschke products (see [10] or [6, Chapter V, Corollary 2.6]).

Lemma 4.7. *Let f be a function in $H^\infty(\Delta)$ with $\|f\|_\infty = 1$, and let $\varepsilon > 0$. Then there is a function ψ in $A(\mathbf{M})$ with $\|\psi\|_\infty < 1+\varepsilon$ such that $\psi \circ L_0(z) = f(z)$ on Δ. Consequently, the isomorphism Ψ by (4.28) maps $A|_{P(\xi)}$ onto $H^\infty(\Delta)$.*

Proof. It follows from Marshall's theorem that there is a sequence $\{q_n\}$ of convex combinations of Blaschke products on Δ such that

$$f = q_0 + \frac{\varepsilon}{3}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n q_n.$$

On the other hand, Lemma 4.6 shows that we may choose a convex combination Q_n of Blaschke products on $\mathbf{R}^2_+$ which extends to a function in $A(\mathbf{X})$ such that $(1+\varepsilon/3)Q_n \circ L_0 = q_n$, $n \geq 0$. Since $\|Q_n\|_\infty \leq 1$, the series

$$\psi(x) = \left(1+\frac{\varepsilon}{3}\right)\left(Q_0(x) + \frac{\varepsilon}{3}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n Q_n(x)\right)$$

converges uniformly on $\mathbf{X}$, and belongs to $A(\mathbf{X})$. Of course, the restriction of ψ to $\mathbf{M}$ lies in $A(\mathbf{M})$. It is easy to see that $\|\psi\|_\infty < 1+\varepsilon$ and $\psi \circ L_0(z) = f(z)$ on Δ. □

Let $A|_{\overline{P(\xi)}}$ similarly be the algebra of all restrictions to the closure $\overline{P(\xi)}$ of functions in $A(\mathbf{M})$, and let $\widehat{H}^\infty(\Delta)$ be the uniform algebra on $\mathfrak{M}(H^\infty(\Delta))$ of all Gelfand transforms of functions in $H^\infty(\Delta)$.

The following lemma is an easy consequence of the corona theorem and Lemma 4.7, so the proof is omitted.

Lemma 4.8. *Let Γ and Γ_0 be the Šilov boundaries of $\widehat{H}^\infty(\Delta)$ and $A|_{\overline{P(\xi)}}$, respectively. Let L_0 be the analytic map of Δ onto $P(\xi)$ by (4.22). Then L_0 extends to a homeomorphism of $\mathfrak{M}(H^\infty(\Delta))$ onto $\overline{P(\xi)}$, which maps Γ onto Γ_0.*

Observe that Γ_0 is contained in $\overline{P(\xi)}\backslash P(\xi)$. We denote by λ_ω the representing measure for ω in Γ_0. Then λ_ω is invariant on the flow $(\mathbf{M}, \{T_t\}_{t\in\mathbf{R}})$ by (i) of Lemma 4.5. For each ϕ in $C(\mathbf{M})$, we set

$$\tau\phi(\omega) = \int_{\mathbf{M}} \phi(x)\, d\lambda_\omega(x), \qquad \omega \in \Gamma_0.$$

Then the map $\omega \to \tau\phi(\omega)$ is continuous on Γ_0. Indeed, by identifying each ω in Γ_0 with its unique representing measure λ_ω in $M(\mathbf{M})$, Γ_0 becomes a subset of $M(\mathbf{M}) = C(\mathbf{M})^*$, the dual space of $C(\mathbf{M})$. The weak-$*$ closure of $\{\lambda_\omega; \omega \in \Gamma_0\}$ consists of invariant probability measures which are multiplicative on $A(\mathbf{M})$. Since Γ_0 is closed in $\mathfrak{M}(A(\mathbf{M}))$ and since representing measures are unique, the weak-$*$ closure of $\{\lambda_\omega\,;\,\omega\in\Gamma_0\}$ has to be itself. Then the map $\lambda_\omega\to\omega$ is continuous from one compact space to another, so the continuity of $\omega\to\tau\phi(\omega)$ follows. We observe easily that τ is a positive linear operator of $C(\mathbf{M})$ into $C(\Gamma_0)$ with $\|\tau\|=1$.

Let ν be a positive measure in $M(\Gamma_0)$. We then define the linear functional F on $C(\mathbf{M})$ by

$$F(\phi)=\int_{\Gamma_0}\tau\phi(\omega)\,d\nu(\omega)\,,\qquad \phi\in C(\mathbf{M}).$$

We see that $|F(\phi)|\le\|\nu\|\cdot\|\phi\|_\infty$ and $F(T_t\phi)=F(\phi)$, where $T_t\phi(x)=\phi(x+t)$. We also see that F is positive and $F(1)=\nu(\Gamma_0)=\|\nu\|$. Then the Riesz representation theorem assures the existence of a positive invariant measure μ in $M(\mathbf{M})$ such that

$$F(\phi)=\int_{\mathbf{M}}\phi(x)\,d\mu(x)\,,\qquad \phi\in C(\mathbf{M}). \tag{4.29}$$

It is clear that μ is uniquely determined and $\|\mu\|=\|\nu\|$. However, ν may not be determined uniquely by μ, which causes a difficulty.

We next extend τ to a positive linear operator of $L^1(\mu)$ to $L^1(\nu)$ with $\|\tau\|=1$. Let ϕ be a function in $C_{\mathbf{R}}(\mathbf{M})$. We write $\phi=\phi_1-\phi_2$, where ϕ_1 and ϕ_2 are respectively the positive and negative parts of ϕ. Since $|\tau\phi_1-\tau\phi_2|\le\tau\phi_1+\tau\phi_2$ and $|\phi|=\phi_1+\phi_2$, we have that $\|\tau\phi\|_1\le\|\phi\|_1$. From this fact, the assertion follows easily. We denote also by τ this extended operator of $L^1(\mu)$ to $L^1(\nu)$.

Let τ^* be the adjoint of the extended operator τ. Then τ^* is a positive linear operator of $L^\infty(\nu)$ to $L^\infty(\mu)$, and its range consists of invariant functions. When ϕ is in $L^1(\mu)$, we put

$$\begin{cases} m_T\phi(x)=\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\phi(x+t)\,dt\,,\ \text{and}\\ m\phi(x)=\displaystyle\lim_{T\to\infty}m_T\phi(x).\end{cases} \tag{4.30}$$

Then the mean ergodic theorem shows that $m\phi$ lies in $L^1(\mu)$ and $\tau(m\phi)=\tau\phi$. We see also that $m_T\phi$ lies in $C(\mathbf{M})$ if ϕ is in $C(\mathbf{M})$.

Let $b(z)$ be a Blaschke product on Δ. By Lemma 4.6, we may choose a decreasing sequence $\{r_n\}$ converging to 1 and a sequence $\{B_n\}$ of Blaschke products on $\mathbf{R}^2_+$ such that

$$r_nB_n\circ L_0(\eta)=b(\eta)\,,\qquad \eta\in\mathfrak{M}(H^\infty(\Delta))\,,$$

where L_0 is identified with its extension to $\mathfrak{M}(H^\infty(\Delta))$ by Lemma 4.8. This yields that $\tau(r_nB_n)(\omega)=b\circ L_0^{-1}(\omega)$ on Γ_0. Since $b\circ L_0^{-1}$ lies in $L^\infty(\nu)$, $\tau^*(b\circ L_0^{-1})(x)$ is an invariant function in $L^\infty(\mu)$. We claim that

$$\tau\circ\tau^*(b\circ L_0^{-1})(\omega)=b\circ L_0^{-1}(\omega)\,,\qquad \omega\in\Gamma_0.$$

Indeed, if we set $\phi_n=m(r_nB_n)$ by (4.30), then ϕ_n is an invariant function in $L^\infty(\mu)$ with $\|\phi_n\|_\infty\le r_n$ and satisfies that $\tau\phi_n=b\circ L_0^{-1}$ on Γ_0. So we have that

$$\|\nu\|=\int_{\Gamma_0}|b\circ L_0^{-1}|^2\,d\nu=\int_{\Gamma_0}(\tau\overline{\phi}_n)(b\circ L_0^{-1})\,d\nu=\int_{\mathbf{M}}\overline{\phi}_n\tau^*(b\circ L_0^{-1})\,d\mu.$$

Since $\|\tau^*(b\circ L_0^{-1})\|_\infty \le 1$ and $\|\mu\| = \|\nu\|$, this shows that

$$\|\phi_n - \tau^*(b\circ L_0^{-1})\|_2^2 = \|\phi_n\|_2^2 - 2\,\mathrm{Re}\int_{\mathbf{M}} \overline{\phi}_n \tau^*(b\circ L_0^{-1})\,d\mu + \|\tau^*(b\circ L_0^{-1})\|_2^2$$
$$\le r_n^2\|\mu\| - 2\|\mu\| + \|\mu\| = (r_n^2 - 1)\|\mu\|.$$

Since $\|\phi_n - \tau^*(b\circ L_0^{-1})\|_1 \le (r_n^2-1)^{1/2}\|\mu\|$ and $\tau\phi_n = b\circ L_0^{-1}$, we see that $\tau\circ\tau^*(b\circ L_0^{-1}) = b\circ L_0^{-1}$. Thus we obtain the following:

Lemma 4.9. *Let L_0, τ, and τ^* be as in above. If b_1 and b_2 are Blaschke products on Δ, then we have the equation*

$$\int_{\Gamma_0} (b_1\circ L_0^{-1})(\overline{b}_2\circ L_0^{-1})\,d\nu = \int_{\mathbf{M}} \tau^*(b_1\circ L_0^{-1})\tau^*(\overline{b}_2\circ L_0^{-1})\,d\mu.$$

Setting $b_1 = b_2 = b$ in Lemma 4.9, we see that $\tau^*(b\circ L_0^{-1})$ is a unimodular function.

Under these observations, we may show that $A(\mathbf{M})$ is not a Dirichlet algebra on $\mathbf{M}$. The following proof is due to the referee, which improves our former one so much.

Proof of Theorem 4.1. It suffices to show that there is a nonzero real measure in $M(\mathbf{M})$ which is orthogonal to $A(\mathbf{M})$. It is well known that $\widehat{H}^\infty(\Delta)$ is not a Dirichlet algebra on its Šilov boundary Γ (see, for example, [8, Chapter 10]). By Lemma 4.7, we see that $A|_{\overline{P(\xi)}}$ is not a Dirichlet algebra on Γ_0. Consequently, we may find a nonzero real measure ν in $M(\Gamma_0)$ which is orthogonal to $A|_{\overline{P(\xi)}}$. By the Jordan decomposition theorem, ν is represented as $\nu = \nu_1 - \nu_2$, where ν_1 and ν_2 are positive and negative variations of ν. We may assume that both ν_1 and ν_2 are probability measures on Γ_0. It follows from (4.29) that there are invariant probability measures μ_i, $i = 1, 2$, such that

$$\int_{\Gamma_0} \tau\phi\,d\nu_i = \int_{\mathbf{M}} \phi\,d\mu_i, \qquad \phi\in C(\mathbf{M}).$$

Let τ_i denote the extension of τ to $L^1(\mu_i)$, as described in above. The adjoint of τ_i is similarly denoted by τ_i^*. Let $0 < \varepsilon < \frac{1}{3}$. Since ν_1 and ν_2 are mutually singular, we may choose a unimodular continuous function θ on Γ_0 such that

$$\left|\int_{\Gamma_0} \theta d(\nu_1 - \nu_2)\right| \ge \|\nu_1\| + \|\nu_2\| - \varepsilon = 2 - \varepsilon.$$

Let $L^\infty(\mathbf{T})$ be the space of bounded (Lebesgue) measurable functions on the unit circle $\mathbf{T}$. Since Γ is the maximal ideal space of $L^\infty(\mathbf{T})$, $\theta\circ L_0$ is regarded as a unimodular function in $L^\infty(\mathbf{T})$. Then the Douglas-Rudin theorem and Frostman's theorem show that there are Blaschke products b_1 and b_2 on Δ such that $\|\theta\circ L_0 - b_1\overline{b}_2\|_\infty < \frac{1}{2}\varepsilon$ (see [6, Chapter 5, Theorem 2.1]). This implies that

$$\left|\int_{\Gamma_0} (b_1\circ L_0^{-1})(\overline{b}_2\circ L_0^{-1}) d(\nu_1 - \nu_2)\right| \ge 2 - 2\varepsilon.$$

By Lemma 4.9, we see that

$$\left|\int_{\mathbf{M}} \tau_1^*(b_1\circ L_0^{-1})\tau_1^*(\overline{b}_2\circ L_0^{-1})\,d\mu_1 - \int_{\mathbf{M}} \tau_2^*(b_1\circ L_0^{-1})\tau_2^*(\overline{b}_2\circ L_0^{-1})\,d\mu_2\right| \ge 2 - 2\varepsilon.$$

There are two sequences of Blaschke products $\{B_n^{(1)}\}$ and $\{B_n^{(2)}\}$ on $\mathbf{R}_+^2$ and two decreasing sequences $\{r_n^{(1)}\}$ and $\{r_n^{(2)}\}$ converging to 1 such that each $B_n^{(j)}$, $j = 1, 2$, extends to be continuous on $\mathbf{X}$ and $r_n^{(j)} B_n^{(j)} \circ L_0 = b_j$ on $\mathfrak{M}(H^\infty(\Delta))$. Let m_T and m_i be the operators by (4.30) with respect to $L^1(\mu_i)$. Recall that, for ϕ in $C(\mathbf{M})$, $m_T(\phi)$ lies in $C(\mathbf{M})$ and $\|m_T(\phi) - m_i(\phi)\|_2$ converges to 0 as $T \to \infty$. On the other hand, from the paragraph preceding Lemma 4.9, we see that

$$\lim_{n\to\infty} \int_{\mathbf{M}} \left| m_i(r_n^{(j)} B_n^{(j)}) - \tau_i^*(b_j \circ L_0^{-1}) \right|^2 d\mu_i = 0$$

for $i, j = 1, 2$. Hence there exists a k such that

$$\left| \int_{\mathbf{M}} m_1(r_k^{(1)} B_k^{(1)}) m_1(r_k^{(2)} \overline{B}_k^{(2)})\, d\mu_1 - \int_{\mathbf{M}} m_2(r_k^{(1)} B_k^{(1)}) m_2(r_k^{(2)} \overline{B}_k^{(2)})\, d\mu_2 \right| \geq 2 - 3\varepsilon.$$

Therefore there is a $T > 0$ such that

$$\left| \int_M m_T(r_k^{(1)} B_k^{(1)}) m_T(r_k^{(2)} \overline{B}_k^{(2)}) d(\mu_1 - \mu_2) \right| \geq 2 - 4\varepsilon > \frac{2}{3}.$$

Since $m_T(r_k^{(j)} B_k^{(j)})$ is in $C(\mathbf{M})$, this shows that $\mu_1 \neq \mu_2$. Since $\nu = \nu_1 - \nu_2$ is orthogonal to $A|_{\overline{P(\xi)}}$, we obtain a nonzero real measure $\mu = \mu_1 - \mu_2$ in $M(\mathbf{M})$ which is orthogonal to $A(\mathbf{M})$. This completes the proof. □

5. Remarks

Let $(\mathbf{M}, \{T_t\}_{t\in\mathbf{R}})$, $A(\mathbf{M})$, and $P(\xi)$ be as in §4, and let μ be the representing measure for ξ. Then μ is an invariant probability measure in $M(\mathbf{M})$ which is not ergodic. We denote by $H^\infty(\mu)$ the weak-$*$ closure of $A(\mathbf{M})$ in $L^\infty(\mu)$. Then $H^\infty(\mu)$ is a weak-$*$ Dirichlet algebra in $L^\infty(\mu)$, since $A(\mathbf{M})$ is a logmodular algebra.

(i) Recall that, since $(\mathbf{M}, \{T_t\}_{t\in\mathbf{R}})$ is minimal, $A(\mathbf{M})$ is a maximal subalgebra of $C(\mathbf{M})$. By contrast to this fact, it follows from [11, Corollary 3.1] that $H^\infty(\mu)$ is not a weak-$*$ maximal algebra in $L^\infty(\mu)$.

(ii) For each ϕ in $L^1(\mu)$, $m\phi$ denotes the invariant function by (4.30). We now set

$$\mathscr{H} = \{m\phi;\, \phi \in A(\mathbf{M})\}.$$

We then see that $\mathscr{H}$ is isometrically isomorphic to $H^\infty(\Delta)$. Of course, $m\phi$ cannot be continuous on $\mathbf{M}$ except for constants.

(iii) Let $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ be a minimal flow. Recall that if $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ is strictly ergodic, then $A(\Omega)$ is a Dirichlet algebra on Ω. Then it is natural to ask whether strict ergodicity is necessary when $A(\Omega)$ is a Dirichlet algebra. We may, however, construct a minimal flow $(\mathbf{Y}, \{V_t\}_{t\in\mathbf{R}})$ not being strictly ergodic, on which $A(\mathbf{Y})$ is a Dirichlet algebra. Indeed, let $B_1(z)$ be the Blaschke product in (4.16) with $m = 1$. Since $B_1(z)$ extends to be continuous on $\mathbf{X}$, it determines a unimodular function $B_1(x)$ in $A(\mathbf{M})$. Let $\mathfrak{B}$ be the closed subalgebra of $C(\mathbf{M})$ generated by $\{T_t B_1, \overline{T_s B_1};\, t, s \in \mathbf{R}\}$. Taking a suitable factor of $(\mathbf{M}, \{T_t\}_{t\in\mathbf{R}})$, we have a minimal flow $(\mathbf{Y}, \{V_t\}_{t\in\mathbf{R}})$ such that $\mathfrak{B}$ is isometricaly isomorphic to $C(\mathbf{Y})$. We may consider that $A(\mathbf{Y})$ contains the uniform algebra generated by $\{T_t B_1;\, t \in \mathbf{R}\}$ and all the constant functions. Then it is not difficult to show that $(\mathbf{Y}, \{V_t\}_{t\in\mathbf{R}})$ is the desired one. We mention that since $C(\mathbf{Y})$ is separable, $\mathbf{Y}$ is metrizable.

Incidentally, the referee has pointed out that there can be a minimal flow $(\Omega, \{U_t\}_{t\in\mathbf{R}})$ where Ω is metrizable, and yet $A(\Omega)$ is not a Dirichlet algebra. In this situation, Choquet's theorem works well. So we would expect some information about the invariant measures representing homomorphisms of $A(\Omega)$.

Note added in proof. One application of the flow $(\mathbf{X}, \{\mathbf{S}_t\}_{t\in\mathbf{R}})$ should be noted: Let μ be an invariant, ergodic, probability measure in $M(\mathbf{X})$, and let $H^2(\mu)$ and $H_0^2(\mu)$ be as ian the paragraph preceding Proposition 3.2. Then the invariant subspace $H_0^2(\mu)$ is generated by one of its elements. This answers an old question in the setting of weak-* Dirichlet algebras (see [7, Chapter 5, §4]). For the proof, see our subsequent note *Single generator problem.*

References

1. N. Dunford and J. Schwartz, *Linear operators. Part* I, Wiley Interscience, New York, 1958.
2. F. Forelli, *Analytic and quasi-invariant measures*, Acta Math. **118** (1967), 33–59.
3. ——, *Fourier theory and flows*, Actes, Congrès Internat. Math. 2 (1970), 475–476.
4. ——, *A maximal algebra*, Math. Scand. **30** (1970), 152–158.
5. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
6. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
7. H. Helson, *Analyticity on compact abelian groups*, Algebras in Analysis, Academic Press, London, 1975, pp. 1–62.
8. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
9. ——, *Bounded analytic functions and Gleason parts*, Ann. of Math. (2) **86** (1967), 74–111.
10. D. Marshall, *Blaschke products generate* H^∞, Bull. Amer. Math. Soc. **82** (1967), 494–496.
11. P. Muhly, *Function algebras and flows*, Acta Sci. Math. (Szeged) **35** (1973), 111–121.
12. ——, *Function algebras and flows.* II, Ark. Mat. **11** (1973), 203–213.
13. ——, *Function algebras and flows.* IV, Trans. Amer. Math. Soc. **203** (1975), 55–66.
14. ——, *Isometries of ergodic Hardy spaces*, Israel J. Math. **36** (1980), 50–74.
15. J. Tanaka, *Corona problem and flows*, J. Funct. Anal. **102** (1991), 360–378.
16. J. Xia, *Conditional expectations and the corona problem of ergodic Hardy spaces*, J. Funct. Anal. **64** (1985), 251–274.

Department of Mathematics, Tsuru University, Tsuru City, Yamanashi 402, Japan