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LOCAL UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT

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ABSTRACT. We prove local uniqueness of a domain D entering the conductivity equation $\operatorname{div}((1+\chi(D))\nabla u) = 0$ in a bounded planar domain Ω given the Cauchy data for u on a part of $\partial\Omega$. The main assumption is that ∇u has zero index on $\partial\Omega$ which is easy to guarantee by choosing special boundary data for u. To achieve our goals we study index of critical points of u on $\partial\Omega$.

We are interested in finding a simply connected domain D entering the conductivity equation $\operatorname{div}((1+\chi(D))\nabla u) = 0$ from one exterior boundary measurement (the Cauchy data for u on a piece of boundary of a given planar domain Ω containing D). This problem is fundamental for electrical and seismical prospection, but not much is known about it at present. The only available uniqueness results are for convex polyhedrons [FI] or cylinders [IP]. There are more general results when D is assumed to be close to a given domain D_0. In [CV], a local uniqueness theorem is proved, under the technical assumption which involves a condition on the interior values of ∇u. In [BFI], [P], under assumptions on the boundary data which guarantee that ∇u has zero index on $\partial\Omega$, and under the restrictive assumption that ∂D_0, ∂D are (piecewise) analytic, local uniqueness results have been obtained.

In this paper we remove this last assumption and we show that solutions $D \in C^{1+\lambda}$ of the inverse problem are isolated.

Notation. $B(a; r)$ is the ball of radius r centered at a; ν is the (exterior) unit normal to the boundary, $\partial/\partial z = 1/2(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$, $|\ |_{k+\lambda}(\Omega)$ is the norm in the Hölder space $C^{k+\lambda}(\Omega)$ of functions whose partial derivatives up to order k are Hölder continuous of exponent λ, $0 < \lambda < 1$. C denote positive constants which depend only on Ω, D_0 and M.

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1. Main results

Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{2+\lambda}$-boundary $\partial\Omega$, $0 < \lambda < 1$. For a domain D, $\overline{D} \subset \Omega$, we consider the refraction problem

$$\operatorname{div}((1+\chi(D))\nabla u) = 0 \text{ in } \Omega, \quad \partial u/\partial\nu = g \text{ on } \partial\Omega \tag{1}$$

where $g \in C^{1+\lambda}(\partial\Omega)$ and $\int_{\partial\Omega} g = 0$. When $\partial D \in C^{1+\lambda}$, this problem has a unique solution $u \in H^{(1)}(\Omega)$ such that $u \in C^{1+\lambda}(\overline{D}^e)$, $u \in C^{1+\lambda}(\overline{D})$ and $\int_{\partial\Omega} u = 0$ (see [DEF, Appendix]). The differential equation (1) is equivalent to the following relations:

(1_{d})
$$\Delta u^e = 0 \text{ in } D^e, \quad \Delta u^i = 0 \text{ in } D, \quad u^e = u^i, \quad \partial u^e/\partial\nu = 2\partial u^i/\partial\nu \text{ on } \partial D$$

where $D^e = \Omega\backslash\overline{D}$, $u^e = u$ on D^e, $u^i = u$ on D.

The inverse conductivity problem with one boundary measurement is to find D given

$$u = h \quad \text{on } \Gamma \subset \partial\Omega. \tag{2}$$

To formulate the main result of this paper we consider a family D_ψ of domains close to a domain $D_0 \in C^{1+\lambda}$ which contains the origin 0 in $\mathbb{R}^2$. Let $z_0(t)$ be the conformal mapping of the unit disk B onto D_0 normalized by the conditions

$$z_0(0) = 0, \quad z_0'(0) > 0. \tag{3}$$

Consider the family Ψ_M of function ψ analytic in B and satisfying the following conditions:

$$\psi(0) = 0, \quad \operatorname{Im}\psi'(0) = 0, \quad |\psi|_{1+\lambda}(B) \le M \tag{4}$$

and such that $z_0 + \psi$ is a conformal mapping. Let D_ψ be the image of B under this mapping.

Let $\partial\Omega$ be the union of two disjoint arcs Γ_1, Γ_2 and assume that

$$0 \le g \text{ on } \Gamma_1, \quad g \le 0 \text{ on } \Gamma_2, \quad g \ne 0. \tag{5}$$

Theorem 1. *If the condition* (5) *is satisfied, then for a domain D_0 there is a number $\varepsilon_0(M)$ such that if $|\psi|_0(B) < \varepsilon_0$ and D_ψ, D_0 are solutions to the same inverse conductivity problem* (1), (2), *then $\psi = 0$.*

In other words, $C^{1+\lambda}$-solutions D_ψ are isolated. By applying a standard compactness argument, we can claim that the number of solutions is finite.

Corollary 1*. *Let Ψ_M^* be a subfamily of Ψ_M which is closed in $C^{1+\lambda}(B)$. Then the number of solutions D_ψ with $\psi \in \Psi_M^*$ is finite.*

Proof. We observe that by Theorem 1 for any $\psi \in \Psi_M^*$ there is $\varepsilon(\psi, M)$ such that in the ε-neighborhood of ψ in $C(B)$ there is no other solution of the inverse problem. These ε-neighborhoods form an open covering of Ψ_M^* in $C(B)$; since Ψ_M^* is compact in $C(B)$ we can find a finite subcovering. Since in any neighborhood there is no more than one solution, the number of all solutions in Ψ_M^* is finite.

2. Index of ∇u^e

Theorem 2. *Let $u \in H^{1,2}(\Omega)$ be a nonconstant (weak) solution to the equation $\operatorname{div}(a\nabla u) = 0$, $0 < \varepsilon < a$, $a \in L_\infty(\Omega)$, in Ω.*

Then there is a quasiconformal mapping g of Ω onto B and a harmonic function h on B such that $u = h \circ g$ on Ω.

Observe, as an immediate consequence, that u cannot vanish of infinite order at any point of Ω.

We call $z^0 \in \Omega$ a *geometric critical point* of u if in a neighborhood of z^0 the level set $\{u = u(z^0)\}$ consists of $N+1$, $N > 0$, simple arcs whose pairwise intersection is $\{z^0\}$. We call N the *geometric index* of u at $z60$. If z^0 is not a geometrical critical point, we let $N = 0$.

Theorem 3. *Let $g \in C^{1+\lambda}(\partial\Omega)$. Assume that $\partial\Omega$ is the union of two disjoint arcs Γ_1, Γ_2 and let $g \geq 0$ on Γ_1 and $g \leq 0$ on Γ_2. Let u be a solution to* (1).

Then u has no geometrical critical points in Ω.

The proof of Theorems 2, 3 (in a more general situation) is given in [AM].

Now we return to the equation (1) and relate the geometrical index at a point of $\partial D \in C^{1+\lambda}$ to the traditional index. Theorem 4 below is stated in the general case when $D \subset \Omega \subset \mathbb{R}^n$, $n \geq 2$.

We assume that $0 \in \partial D$ and D near 0 is given as $\{x_n > f(x_1, \dots, x_{n-1})\}$, $f \in C^{1+\lambda}$, $f(0) = |\nabla f(0)| = 0$. Let χ^+ be the characteristic function of the half-space $\{x_n > 0\}$.

Theorem 4. *Let u be a nonconstant solution to the equation* (1) *near* $0 \in \partial D$.

Then there is a homogeneous function H_N of degree N such that

$$\operatorname{div}((1 + \chi^+)\nabla H_N) = 0 \quad \text{in } \mathbb{R}^n \tag{6}$$

and

$$u(x) - u(0) = H_N(x) + O(|x|^{N+\lambda}). \tag{7}$$

Before proving we discuss this result and obtain some corollaries.

Any homogeneous solution to the equation (6) admits the representation

$$H_N = (1 - \tfrac{1}{2}\chi^+)O_N + E_N$$

where O_N, E_N are homogeneous harmonic polynomials of degree N, O_N is odd and E_N is even with respect to x_N. In particular, when $n = 2$, we have

$$H_N(r\cos\theta, r\sin\theta) = (1 - \tfrac{1}{2}\chi^+)Ar^N \cos N\theta + Br^N \sin N\theta \tag{8}$$

with $A^2 + B^2 \neq 0$.

Corollary 5. *Let u be a solution to the Neumann problem* (1) *where g satisfies the condition* (5); *then $|\nabla u^e| > 0$, $|\nabla u^i| > 0$ on ∂D and $\operatorname{ind}(\nabla u^e, \partial D)$ and $\operatorname{ind}(\nabla u^i, \partial D)$ are* 0.

Proof of Corollary 5. Let $z^0 \in \partial D$. We may assume $z^0 = 0$. By Theorem 2 the point 0 is not a zero of infinite order; by Theorem 4 we have the representation (7). If $N > 1$, then from (7) and (8) it follows that the set $\{u - u(0) > 0\} \cap B_\varepsilon$ for small ε consists of at least 2 connected components. Therefore the geometric index $I(u, z^0) \geq 1$ which contradicts Theorem 3. So $N = 1$ and $|\nabla u^e(z^0)| > 0$ as well as $|\nabla u^i|$.

Now, $\operatorname{ind}(\nabla u^i, \partial D)$ is well defined. It is known that it is equal to the number of zeros of ∇u^i inside D which is zero by Theorem 3. We have $\nabla u^i = u^i_\nu \nu + u^i_\tau \tau$ where τ is the (anticlockwise) tangential direction on ∂D. A similar representation is valid for ∇u^e. Due to the refraction conditions (1_d) we have $2u^i_\nu = u^e_\nu$, $u^i_\tau = u^e_\tau$. To show that both interior and exterior gradients have the same index on ∂D we introduce the homotopy $v(t) = (1+t)u^i_\nu \nu + u^e_\tau \tau$ between these vector fields. We have $v(0) = \nabla u^i$, $v(1) = \nabla u^e$ and $|v(t)| > 0$ on ∂D when $0 \leq t \leq 1$. The index is a homotopic invariant so $\operatorname{ind}(v(1), \partial D) = \operatorname{ind}(v(0), \partial D) = 0$.

The proof is complete.

Proof of Theorem 4. Our proof will be modeled on arguments originally due to Bers [Be]. By the change of variables $y_j = x_j$, $j < n$, $y_n = x_n - f(x_1, \dots, x_{n-1})$ we can reduce the case to the equation

$$\operatorname{div}((1+\chi^+)a\nabla u) = 0 \tag{1'}$$

where a is a positive symmetric matrix with C^λ-coefficients and $a(0) = I$. Let $x^* = (x_1, \dots, x_{n-1}, -x_n)$. We shall use the standard fundamental solution $E(x-y)$ to the Laplace equation which is

$$-\frac{1}{2\pi}\log|x-y| \text{ when } n = 2 \quad \text{and} \quad \frac{1}{(n-2)\omega_n}|x-y|^{2-n} \text{ when } n > 2.$$

Denoting $\Delta_+ = \operatorname{div}((1+\chi^+)\nabla)$ we observe that a fundamental solution E_+ for this operator is given by $E_+(x, y) = \Theta E(x, y)$ where we define

$$\Theta f(x, y) = \begin{cases} \frac{1}{2}f(x, y) + \frac{1}{6}f(x, y^*) & \text{when } 0 < x_n,\ 0 < y_n, \\ \frac{2}{3}f(x, y) & \text{when } 0 < x_n,\ y_n < 0 \text{ or } x_n < 0,\ 0 < y_n, \\ f(x, y) - \frac{1}{3}f(x, y^*) & \text{when } x_n < 0,\ y_n < 0. \end{cases}$$

The Taylor series expansion of $E(x, y)$ gives

$$E(x, y) = E(0, y) + \sum_{j=1}^{+\infty} Q_j(x, y)$$

when $0 \leq |x| < |y|$. Here $Q_j(x, y)$ are homogeneous of degree j in x and of degree $2-n-j$ in y. We have

$$Q_j(x, y) = \begin{cases} |x|^j|y|^{-j}j^{-1}T_j(x\cdot y/(|x|\,|y|)) & \text{when } n = 2, \\ |x|^j|y|^{2-n-j}C_j^{n/2-1}(x\cdot y/(|x|\,|y|)) & \text{when } n \geq 3. \end{cases}$$

Here T_j are the Chebyshev polynomials and $C_j^{n/2-1}$ are Gegenbauer (ultraspherical) polynomial; see [E, formulas (30), 10.11, (29), 10.9]. We obtain

$$E_+(x, y) = \frac{2}{3}E(0, y) + \sum_{j\geq 1}\Theta Q_j(x, y)$$

when $|x| < |y|$. From [E, formula (7), 10.18], and using harmonicity and homogeneity, we have

$$|\nabla_y Q_j(x, y)| \leq C_n j^{n-3}|x|^j|y|^{1-n-j}$$

and, consequently,

$$|\nabla_y \Theta Q_j(x, y)| \leq C_n j^{n-3} |x|^j |y|^{1-n-j}. \tag{9}$$

Lemma 6. *Let a vector function* $F : \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the condition*

$$|F|(x) \leq |x|^{N+\varepsilon-1} \tag{10}$$

for some $\varepsilon > 0$, $N = 1, 2, \ldots$. *Define*

$$\psi(x) = -\int_{|y|<R} \nabla_y E_+(x, y) \cdot F(y)\, dy.$$

Then

$$\psi(x) = p_N(x) + O(|x|^{N+\varepsilon}) \tag{11}$$

as $|x| \to 0$ *where* $p_N(x)$ *is the sum of homogeneous solutions* H_k *to* $\Delta_+ H_k = 0$ *of degree* $k \leq N$ *(a* Δ_+*-harmonic polynomial of degree* N*).*

Proof. Let us set

$$E_+^N(x, y) = \frac{2}{3} E(0, y) + \sum_{j\geq 1} \Theta Q_j(x, y),$$

$$R^N(x, y) = E_+(x, y) - E_+^N(x, y).$$

Define

$$p_N(x) = -\int_{|y|<R} \nabla_y E_+^N(x, y) \cdot F(y)\, dy.$$

To complete the proof it is sufficient to show that

$$\psi(x) - p_N(x) = O(|x|^{N+\varepsilon}). \tag{12}$$

We will do it by splitting $\psi - p_N$ into the sum of the three integrals

$$I_1 = -\int_{|y|<2|x|} \nabla_y E_+(x, y) \cdot F(y)\, dy,$$

$$I_2 = \int_{|y|<2|x|} \nabla_y E_+^N(x, y) \cdot F(y)\, dy,$$

$$I_3 = -\int_{2|x|<|y|<R} \sum_{N+1\leq j} \nabla_y(\Theta Q_j(x, y)) \cdot F(y)\, dy$$

and bounding I_j by $C|x|^{N+\varepsilon}$.

We have $|\nabla_y E_+| \leq C|x-y|^{1-n}$, so by using the hypothesis (10) and Hölder's inequality we obtain

$$|I_1| \leq C \left(\int_{|y|<2|x|} |x-y|^{(1-n)}\, dy \right) |x|^{\frac{n}{2}+N+\varepsilon}.$$

Observe that on the integration domain $|x - y| \leq |x| + |y| \leq 3|x|$, so

$$|I_1| \leq C \int_{0<r<3|x|} dr |x|^{N+\varepsilon-1} \leq C|x|^{N+\varepsilon}.$$

By using (9) and homogeneity we get

$$
\begin{aligned}
|I_2| &\le \int_{|y|<2|x|} |\nabla_y \Theta_3^2 E(0\,,\, y)|\,|F(y)|\,dy \\
&\quad + C \sum_{1\le j\le N} \int_{|y|<2|x|} |\nabla_y \Theta Q_j(x\,,\, y)|\,|F(y)|\,dy \\
&\le C \sum_{j\le N} j^{n-3}|x|^j \int_{|y|<2|x|} |y|^{1-n-j}|F(y)|\,dy \\
&\le C \sum_{j\le N} j^{n-3}|x|^j \int_{|y|<2|x|} |y|^{N-n-j+\varepsilon}\,dy \le CN^n|x|^{N+\varepsilon}.
\end{aligned}
$$

Finally, we bound

$$
\begin{aligned}
|I_3| &\le C \sum_{N+1\le j} j^{n-3}|x|^j \int_{2|x|<|y|<R} |y|^{1-n-j}|F(y)|\,dy \\
&\le C \sum_{N+1\le j} j^{n-3}|x|^j \int_{2|x|<|y|} |y|^{N-n-j+\varepsilon}\,dy \\
&\le C|x|^{N+\varepsilon} \sum_{N+1\le j} j^{n-3}(j-N-\varepsilon)^{-1}2^{-j+N+\varepsilon} \le C|x|^{N+\varepsilon}.
\end{aligned}
$$

Lemma 7. *Suppose that* $\Delta_+ u = \operatorname{div}(F)$ *in* $B(0\,;\,R)$ *with* F *satisfying the condition* (10).

Then u *admits the representation* (11).

Proof. We extend F outside $B(0\,;\,R)$ as zero. Consider the potential $\psi(x)$ from Lemma 6; then $v = u - \psi$ satisfies the homogeneous equation $\Delta_+ v = 0$ in $B(0\,;\,R)$.

By using reflections we conclude that $v = (1-\chi^+/2)h^o + h^e$ where h^o, h^e are harmonic functions, h^o is odd and h^e is even with respect to x_n. Expanding h^o, h^e in Taylor series around 0 we represent v as the sum of the series of homogeneous Δ_+-harmonic polynomials H_k which is convergent in $B(0\,;\,R/2)$. In particular $v = H_N^* + O(|x|^{N+1})$, where H_N^* is the sum of H_k over $k \le N$. By using Lemma 6 we complete the proof.

Lemma 8. *Let* u *be a solution to the equation* (1′) *in* $B(0\,;\,R)$. *If for some* $A > 0$ *we have* $|u(x)| \le |x|^A$, *then* $|\nabla u(x)| \le C|x|^{A-1}$.

Proof. The proof is an elementary consequence of the Caccioppoli's inequality

$$
\int_{B(0\,;\,r)} |\nabla u|^2 \le Cr^{-2} \int_{B(0\,;\,2r)} u^2
$$

and of the piecewise $C^{1+\lambda}$-regularity estimate of [DEF] which, in particular, gives us

$$
\sup_{B(0\,;\,r)} |\nabla u| \le Cr^{-n/2} \left(\int_{B(0\,;\,2r)} |\nabla u|^2 + u^2 \right)^{1/2}.
$$

Proof of Theorem 4 (end). If u is a nonconstant solution, then there is an integer N such that

$$
|u(x) - u(0)| \le C|x|^N \tag{14}
$$

and

$$\limsup |x|^{-N-1}|u(x) - u(0)| = +\infty \tag{15}$$

as $x \to 0$.

Let $F = (1+\chi^+)(I - a)\nabla u$. Apparently $\Delta_+ u = \operatorname{div} F$. Since $a \in C^\lambda$ near 0 we have

$$|F(x)| \leq C|x|r^{N-1+\lambda}$$

by Lemma 8. Therefore, by Lemma 7, we have $u(x) - u(0) = p_N(x) + O(|x|^{N+\lambda})$ where p_N is a Δ_+-harmonic polynomial of degree $\leq N$. By (14) the polynomial p_N is either identically zero or homogeneous of degree N. In the latter case the proof is complete. To finish the proof we show that the first case is impossible. If not we can assume that (14) is valid for $N + \alpha$, $0 < \alpha < \lambda$, α is irrational, and by repeating the above procedure we can conclude that $u(x) - u(0)$ is $O(|x|^{N+k\alpha})$ when $k \leq [1/\alpha]$ and $p_{N+1}(x) + O(|x|^{N+k\alpha})$ when $k = [1/\alpha] + 1$, so $u(x) - u(0)$ is $O(|x|^{N+1})$ which contradicts (15).

The proof is complete.

3. Proof of the Main Theorem

We will develop some arguments already used in [C] and [P].

Let D_0, D_ψ be two solutions to the inverse problem (1), (2). Let u_0, u_ψ be the corresponding solutions to the direct problem (1). We will pick ε_0 so small that $\Omega\backslash(\overline{D}_0 \cup D_\psi)$ and $\Omega\backslash(\overline{D}_0 \cap \overline{D}_\psi)$ are connected and $\partial(D_0 \cap D_\psi)$ satisfies a Lipschitz condition with a prescribed constant. Since $u_0^e = u_\psi^e$ there, henceforth the following definition is correct: $u^e = u_0^e$ on $\partial/\overline{D}_0$ and $u^e = u_\psi^e$ on $\Omega/\overline{D}_\psi$. Moreover u^e is harmonic outside $\overline{D}_0 \cap \overline{D}_\psi$. From Schauder estimates for transmission problems [DEF, Appendix] and from the a priori constraints $\partial D_0 \in C^{1+\lambda}$, $|\psi|_{1+\lambda} \leq M$, we conclude that $|u^e|_{1+\lambda}(\Omega\backslash(D_0 \cap D_\psi)) \leq C$.

From the condition (1) on g and from Green's formula it follows that $\int_\gamma \partial u^e/\partial N \, d\gamma = 0$ for any cycle γ in $\Omega\backslash(\overline{D}_0 \cap \overline{D}_\psi)$, so there is a harmonic conjugate v^e to u^e. We will normalize it by the condition $v^e(z^o) = 0$ for some $z^o \in \partial\Omega$. Let $U^e = u^e + iv^e$. From the estimate in the previous item we have

$$|U^e|_{1+\lambda}(\Omega\backslash(\overline{D}_0 \cap \overline{D}_\psi)) \leq C. \tag{16}$$

We will show that the inverse problem with respect to the unknown domain D_ψ is equivalent to the following nonlinear boundary value problem for analytic functions ϕ, ψ in the unit disk B:

$$A(\phi, \psi)(t) = B\psi(t) \quad \text{when } |t| = 1 \tag{17}$$

where $A(\phi, \psi) = \phi - 3a\psi - \overline{a\psi}$, $a(t) = u_z^e(z_0(t))$ and $B\psi = 3B_1\psi + \overline{B_1\psi}$, $B_1\psi = U^e(z_0 + \psi) - U^e(z_0) - u_z^e(z_0)\psi$. Here A is considered as an operator from $\Phi \times \Psi$ into $C^\lambda(\partial B)$. We define Φ as the space of functions $\phi \in C^\lambda(\overline{B})$ analytic in B and Ψ as the space of functions $\psi \in \Phi$ satisfying the conditions (4). The operator A is continuous from $\Phi \times \Psi$ onto its range $\mathfrak{R} \subset C^\lambda(\partial B)$.

To derive (17) we need the following form of the Cauchy-Riemann system for the real and imaginary parts u, v of a complex analytic function

$$\partial u/\partial\tau = \partial v/\partial\nu, \quad \partial v/\partial\tau = -\partial u/\partial\nu \tag{CR}$$

where τ is the unit tangent to ∂D such that the pair (τ, ν) is oriented as the coordinate vectors of the x- and y-axes.

Let D_ψ be a solution to the inverse conductivity problem. From the conditions (CR) and from the boundary conditions (1_d) we conclude that $\partial v^e/\partial\tau = 2\partial v^i/\partial\tau$. Therefore, $v^e = 2v^i + C$. This relation and the continuity of u yield

$$U^e + \overline{U}^e = U^i + \overline{U}^i, \quad U^e - \overline{U}^e = 2(U^i - \overline{U}^i) + iC, \qquad C \in \mathbb{R}, \text{ on } \partial D.$$

Substituting U^i from the first equality into the second one and letting $z = z(t)$, $\phi(t) = 4U^i(z(t)) + iC$ we obtain the boundary condition

$$(17_a) \qquad 3U^e(z(t)) + \overline{U^e(z(t))} = \phi(t) \quad \text{when } |t| = 1$$

where ϕ is complex analytic in the unit disc and is contained in $C^{1+\lambda}$ $(|t| \le 1)$.

On other hand, let (17_a) hold. Let U_D^e be the exterior function constructed from the exterior part of the solution to the direct conductivity problem and $z(t)$ be the normalized conformal map of the unit disk onto D. Then we have the relation (17_a) with U^e replaced by U_D^e, and ϕ replaced by $4U_D^i(z(t)) + iC_1$. Subtracting the relations (17_a) for U and for U_D, defining $U^i(z) = \phi(t(z))/4$ and letting $V = U - U_D$ we obtain $3V^e + V^e = 4V^i + iC$ on ∂D. Since $\Re U^e$ and $\Re U_D^e$ have the same Dirichlet data, we have $\Re V = 0$ on $\partial\Omega$. Subtracting the relations on ∂D for V and for V and adding these relations yield

$$2(V^e - \overline{V}^e) = 4(V^i - \overline{V}^i) + iC, \quad 4(V^e + \overline{V}^e) = 4(V^i + \overline{V}^i) \quad \text{on } \partial D.$$

Letting $u^* = \Re V$, $v^* = \Im V$, we will have

$$u^{*e} = u^{*i}, \quad v^{*e} = 2v^{*i} + C \quad \text{on } \partial D.$$

Differentiating the second equality in the tangential direction and using (CR) we conclude that $\partial u^{*e}/\partial\nu = 2\partial u^{*i}/\partial\nu$ on ∂D. So u^* solves the direct conductivity problem with zero Dirichlet data on $\partial\Omega$. By the maximum principle $u^* = 0$, and hence $u^e = u_D^e$.

We proved that (17_a) is equivalent to our inverse problem. The relation (17) is a form of (17_a) obtained by letting $z = z_0 + \psi$ and using Taylor's Formula.

This equivalence is established in the papers [C] and [P]; we proved it only for the reader's convenience.

Theorem 9. *The operator A has the continuous inverse A^{-1} from $\Re$ onto $\Phi \times \Psi$.*

To prove this result we transform the operator A to some canonical form, make use of the theory of index of one-dimensional singular integral equations by Mushelishvili [Mu] and of some known estimates for the Cauchy integral operators.

To transform A we will use the formula

$$(18) \qquad 3u_z^e(z_0(t)) = 4u_z^i(z_0(t)) + \overline{u_z^e(z_0(t))z_0'(t)\bar{t}}/(z_0'(t)t).$$

To prove it we make use of the parametrization $z_0(t)$, $t = e^{i\theta}$ of ∂D_0. By differentiating the composition we obtain $(\partial/\partial\theta)u^i = u_z^i z_{0t} ie^{i\theta}$ and the similar formula for u^e. From the refraction conditions (1_d) as in [P, (4.5)], we have the equality $4U^i(z_0) = 3U^e(z_0) + \overline{U^e(z_0)} + \text{const}$ when $z_0 \in \partial D$. Differentiate both parts with respect to θ, utilize the above formulae for U_θ^i, U_θ^e and multiply both parts by $|z_0'|/(itz_0')$ to obtain (18).

Substitute (18) into (17) to replace A by

$$(17_1) \qquad A_*(\phi^*, \psi)(t) = \phi^*(t) - \overline{a(t)}(\overline{z_0}(t)\bar{t}/(z_0'(t)t))\psi(t) - \overline{a(t)}\,\overline{\psi(t)}$$

where $\phi^* = \phi - 4u_z^i(z_0)\psi$. We can prove Theorem 9 with A_* instead of A. From now on we drop $*$.

As in [P, section 7], the equation $A(\varphi, \psi) = f$ is equivalent to the system of two Riemann-Hilbert boundary value problems

$$(19) \qquad \begin{aligned} \phi(t) &= (\overline{ac}/(ac))(t)\overline{\phi(t)} + F(t), \\ F(t) &= f(t) - (\overline{ac}/ac)(t)\overline{f(t)}, \end{aligned}$$

and

$$(20) \qquad \begin{aligned} \psi(t) &= -(c/\bar{c})(t)\bar{\psi}(t) + G(t), \\ G(t) &= (\psi(t) - f(t))c(t)/(\overline{ac})(t) \end{aligned}$$

where $c(t) = z_0'(t)t$. By Corollary 5 we have $|a| \neq 0$ on ∂B and $\operatorname{ind}(u_z^e; \partial D_0) = 0$, so (19) and (20) are the Riemann-Hilbert problems of index $\lambda = -2$, $\mu = 2$ respectively (see [Mu, sec. 40]). Hence the homogeneous problem (19) with $F = 0$ has only the solution $\phi = 0$; then $G = 0$. This homogeneous problem (20) with index $\mu = 2$ has solutions $\psi(t) = X(t)(C_0t^2 + C_1t + C_2)$, $C_0 = \overline{C_2}$, $C_1 \in \mathbb{R}$, where $X(t)$ is the so-called fundamental function for the homogeneous Hilbert (20) (see [Mu, §3.5, pp. 103–104]). The condition $\psi(0) = 0$ gives $C_2 = 0 = C_0$. Now a solution to the homogeneous (20) is $\psi(t) = ic(t)$ which does not satisfy $\operatorname{Im}\psi'(0) = 0$. Finally we conclude that the kernel of A is trivial, so A has the inverse on $\mathfrak{R}$.

To show that A^{-1} is continuous it suffices to be convinced that

$$(21) \qquad |\phi|_\lambda(B) + |\psi|_\lambda(B) \le C|f|_\lambda(\partial B).$$

To estimate ϕ we recall the formula [Mu, (40.20)]:

$$\phi(z) = \frac{1}{\pi i}X(z)S(z) \quad \text{where } S(z) = \int_{\partial B} h(t)/(t-z)\,dt$$

for $|z| < 1$. Here $h(t) = i(\operatorname{Re}(-iacf))/(acX^+)(t)$ where X^+ is the limit of X at ∂B from inside B. The norms $|a|_\lambda(\partial B)_1$, $|c|_\lambda(\partial B)$ and $|X|_\lambda(B)$ (see [V, Theorem 4.1]) are bounded by C. In addition $|X| > 1/C$ on B. So the bound of ϕ follows from the well-known estimates

$$|S|_\lambda(B) \le C|h|_\lambda(\partial B)$$

for the Cauchy integral operators (see [V, Theorem 1.10]).

To estimate ψ we make use of the representation

$$\begin{aligned} \psi(z) = {} & \frac{X(z)}{2\pi i}\left(\int_{\partial B} \frac{k(t)\,dt}{\overline{c(t)}X^+(t)(t-z)} + z^2\int_{\partial B} \frac{t^{-2}k(t)\,dt}{\overline{c(t)}X^+(t)(t-z)}\right) \\ & - \frac{z^2X(z)}{2\pi i}\int_{\partial B} \frac{t^{-2}k(t)\,dt}{\overline{c(t)}X^+(t)t} + X(z)(C_0t^2 + C_1z + C_2) \end{aligned}$$

where $2k = (\varphi - f)c/\bar{a}$ and the constants C_0, C_1, C_2 are chosen so that the conditions (4) are satisfied. By repeating the argument for φ we bound the first three terms in the representation for ψ. To bound the fourth term it is sufficient to recall that

$$C_2 = -\frac{1}{2\pi i}\int_{\partial B}\frac{k(t)}{\overline{c(t)}X^+(t)}\frac{dt}{t}, \quad C_1 = \frac{1}{2\pi}\frac{\operatorname{Re}X(0)}{\operatorname{Im}X(0)}\int_{\partial B}\frac{k(t)}{\overline{c(t)}X^+(t)}\frac{dt}{t^2}.$$

The proof of Theorem 9 is complete.

Lemma 10. *For any* μ, $0 < \mu < \lambda^2/(1+\lambda)$ *we have*

$$|B\psi|_\mu \le C|\psi|_\mu^{1+\delta} \tag{22}$$

with $\delta = \lambda - \mu(1+\lambda)/\lambda$.

Proof. It suffices to prove (22) when B is replaced with B_1. Recalling that $U_z^e = 0$ in $\Omega\backslash(D_0 \cup D_\psi)$, we see that $U^e(z_0+\psi) - B_1\psi$ equals the first order Taylor polynomial for $U^e(z_0+\psi)$ centered at $\psi = 0$. Using the Lipschitz regularity of $\partial(\Omega\backslash(D_0 \cap D_\psi))$ and recalling (16) we obtain

$$|B_1\psi|_0 \le C|\psi|_0^{1+\lambda}.$$

Notice that, again by (16) and the a priori bounds on z_0, ψ, we have $|B_1(\psi)|_\lambda \le C$; hence, by the standard interpolation inequality

$$|f|_\mu \le 2|f|_0^{1-\mu/\lambda}|f|_\lambda^{\mu/\lambda}$$

we obtain (22).

End of the Proof of Theorem 1. Assume that there is a solution D_ψ then ψ satisfies the equation (17), so $B\psi \in \mathfrak{R}$. By using Theorem 9 we can write this equation as $(\phi, \psi) = A^{-1}B\psi$ in $\Phi \times \Psi$ with the norm $\|(\phi, \psi)\| = |\phi|_\mu(B) + |\psi|_\mu(B)$, $0 < \mu < \lambda$. By using Lemma 10 we obtain

$$\|A^{-1}B\psi\| \le C|\psi|_\mu^{1+\delta} \le \tfrac{1}{2}\|(\phi, \psi)\|$$

as soon as $|\psi|_\mu < \varepsilon_1$. By using an interpolation inequality ([I, Theorem 1.1.1]) and the a priori constraint $|\psi|_{1+\lambda} \le M$ we conclude that $|\psi|_\mu < \varepsilon_1$ as soon as $|\psi|_0 < \varepsilon_0$ for some $\varepsilon_0(M)$. For such ε_0 we have $\|(\phi, \psi)\| \le \frac{1}{2}\|(\phi, \psi)\|$, so $\psi = 0$.

The proof is complete.

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