

ON THE SET OF PERIODS FOR σ MAPS

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ABSTRACT. Let σ be the topological graph shaped like the letter σ . We denote by 0 the unique branching point of σ , and by \mathbf{O} and \mathbf{I} the closures of the components of $\sigma \setminus \{0\}$ homeomorphic to the circle and the interval, respectively. A continuous map from σ into itself satisfying that f has a fixed point in \mathbf{O} , or f has a fixed point and $f(0) \in \mathbf{I}$ is called a σ map. These are the continuous self-maps of σ whose sets of periods can be studied without the notion of rotation interval. We characterize the sets of periods of all σ maps.

1. INTRODUCTION AND MAIN RESULTS

Let E be a topological space. Our goal in this work will be to describe the structure of the set of periods for some class of self-maps on E .

The set of natural numbers, real numbers and complex numbers will be denoted by \mathbb{N} , \mathbb{R} and \mathbb{C} respectively.

For a map $f: E \rightarrow E$ we use the symbol f^n to denote $f \circ f \circ \dots \circ f$ ($n \in \mathbb{N}$ times), f^0 denotes the identity map on E . Then, for a point $x \in E$ we define the orbit of x , denoted by $\text{Orb}_f(x)$, as the set $\{f^n(x) : n = 0, 1, 2, \dots\}$. We say that x is a fixed point of f if $f(x) = x$. We say that x is a periodic point of f of period $k \in \mathbb{N}$ (or k -periodic) if $f^k(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < k$. In this case we say that the orbit of x is a periodic orbit of period k . Note that if x is a periodic point of period k , then $\text{Orb}_f(x)$ has exactly k elements, each of which is a periodic point of period k . We denote by $\text{Per}(f)$ the set of periods of all periodic points of f .

From now on, the topological space E will denote one of the following spaces:

$$\begin{aligned}\mathbf{I} &= \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } y = 0\}, \\ \mathbf{Y} &= \{z \in \mathbb{C} : z^3 \in [0, 1]\}, \\ \mathbf{O} &= \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 1)^2 = 1\}, \\ \sigma &= \mathbf{I} \cup \mathbf{O}.\end{aligned}$$

The point $0 \in \mathbf{Y}$ or the $0 = (0, 0) \in \sigma$ are called *branching points*.

An *interval map* is a continuous self-map on the interval \mathbf{I} . Similarly we define *Y maps*, *circle maps* and *sigma maps*.

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We remark that any interval map or any Y map always has a fixed point, but there are circle maps and sigma maps without fixed points.

Our objective in this section is to describe the structure of the set $\text{Per}(f)$ for a class of sigma maps. To this end, we need to introduce some orderings in the set of natural numbers adding or removing some few elements.

The *Sarkovskii ordering* $>_s$ on the set $N_s = \mathbb{N} \cup \{2^\infty\}$ is given by

$$3 >_s 5 >_s 7 >_s \dots >_s 2 \cdot 3 >_s 2 \cdot 5 >_s 2 \cdot 7 >_s \dots >_s 2^2 \cdot 3 >_s 2^2 \cdot 5 >_s 2^2 \cdot 7 >_s \dots >_s 2^n \cdot 3 >_s 2^n \cdot 5 >_s 2^n \cdot 7 >_s \dots >_s 2^\infty >_s \dots >_s 2^n >_s \dots >_s 2^4 >_s 2^3 >_s 2^2 >_s 2 >_s 1.$$

More precisely, if $k = k' \cdot 2^p$ where $p \in \mathbb{N}$ and k' is odd, we have:

- (i) $k >_s 2^\infty$ if $k' > 1$,
- (ii) $2^\infty >_s k$ if $k' = 1$,

and if $n = n' \cdot 2^q$ where $q \in \mathbb{N}$ and n' is odd, then we have $n >_s k$ if and only if one of the following cases occurs:

- (iii) $k' > 1$, $n' > 1$ and $p > q$,
- (iv) $k' > n' > 1$ and $p = q$,
- (v) $k' = 1$ and $n' > 1$,
- (vi) $k' = 1$, $n' = 1$ and $p < q$.

We shall use the symbols \geq_s , $<_s$ and \leq_s in the natural way. We have to include the symbol 2^∞ to ensure the existence of supremum of every subset with respect to the ordering $>_s$. For $n \in N_s$ we denote $S(n) = \{k \in \mathbb{N} : n \geq_s k\}$. So $S(2^\infty) = \{2^i : i = 0, 1, 2, \dots\}$.

Now we state the Sarkovskii Theorem, which is proved in [Sa] (see also [St], [BGM] and [ALM2]).

Theorem 1.1 (Interval Theorem).

- (a) If f is an interval map, then $\text{Per}(f) = S(n)$ for some $n \in N_s$.
- (b) If $n \in N_s$, then there exists an interval map f such that $\text{Per}(f) = S(n)$.

If we want to get a similar result for the Y space, we need two new orderings. The *green ordering* $>_g$ on $\mathbb{N} \setminus \{2\}$ is given by

$$5 >_g 8 >_g 4 >_g 11 >_g 14 >_g 7 >_g 17 >_g 20 >_g 10 >_g \dots >_g 3 \cdot 3 >_g 3 \cdot 5 >_g 3 \cdot 7 >_g \dots >_g 3 \cdot 2 \cdot 3 >_g 3 \cdot 2 \cdot 5 >_g 3 \cdot 2 \cdot 7 >_g \dots >_g 3 \cdot 2^2 \cdot 3 >_g 3 \cdot 2^2 \cdot 5 >_g 3 \cdot 2^2 \cdot 7 >_g \dots >_g 3 \cdot 2^3 >_g 3 \cdot 2^2 >_g 3 \cdot 2 >_g 3 \cdot 1 >_g 1.$$

The first part of this ordering can be rewritten as

$$6 - 1 >_g 6 + 2 >_g 3 + 1 >_g 2 \cdot 6 - 1 >_g 2 \cdot 6 + 2 >_g 2 \cdot 3 + 1 >_g 3 \cdot 6 - 1 >_g 3 \cdot 6 + 2 >_g 3 \cdot 3 + 1 >_g \dots$$

To be more precise, denote by the symbol \equiv congruences modulus 3. We have $k >_g n$ for $k, n \in \mathbb{N} \setminus \{2\}$ if and only if $n > 1$ and one of the following cases occurs:

- (i) $k \not\equiv 0$, $n \not\equiv 0$, $k \equiv n$, $k > n$,
- (ii) $k \equiv 1$, $n \equiv 2$, $2k \geq n$,
- (iii) $k \equiv 2$, $n \equiv 1$, $k > 2n$,
- (iv) $k \equiv 0$, $n \not\equiv 0$,
- (v) $k \equiv 0$, $n \equiv 0$, $k/3 >_s n/3$,
- (vi) $k = 1$.

The *red ordering* $>_r$ on $\mathbb{N} \setminus \{2, 4\}$ is given by

$7 >_r 10 >_r 5 >_r 13 >_r 16 >_r 8 >_r 19 >_r 22 >_r 11 >_r \dots >_r 3 \cdot 3 >_r 3 \cdot 5 >_r 3 \cdot 7 >_r \dots >_r 3 \cdot 2 \cdot 3 >_r 3 \cdot 2 \cdot 5 >_r 3 \cdot 2 \cdot 7 >_r \dots >_r 3 \cdot 2^2 \cdot 3 >_r 3 \cdot 2^2 \cdot 5 >_r 3 \cdot 2^2 \cdot 7 >_r \dots >_r 3 \cdot 2^3 >_r 3 \cdot 2^2 >_r 3 \cdot 2 >_r 3 \cdot 1 >_r 1$.

The first part of this ordering can be rewritten as

$6 + 1 >_r 6 + 4 >_r 3 + 2 >_r 2 \cdot 6 + 1 >_r 2 \cdot 6 + 4 >_r 2 \cdot 3 + 2 >_r 3 \cdot 6 + 1 >_r 3 \cdot 6 + 4 >_r 3 \cdot 3 + 2 >_r \dots$

We have $k >_r n$ for $k, n \in \mathbb{N} \setminus \{2, 4\}$ if and only if $n > 1$ and one of the following cases occurs:

- (i) $k \neq 0, n \neq 0, k \equiv n, k > n$,
- (ii) $k \equiv 2, n \equiv 1, 2k \geq n$,
- (iii) $k \equiv 1, n \equiv 2, k > 2n$,
- (iv) $k \equiv 0, n \neq 0$,
- (v) $k \equiv 0, n \equiv 0, k/3 >_s n/3$,
- (vi) $k = 1$.

For $n \in \mathbb{N} \setminus \{2\}$ denote $G(n) = \{k \in \mathbb{N} : n \geq_g k\}$, for $n \in \mathbb{N} \setminus \{2, 4\}$ denote $R(n) = \{k \in \mathbb{N} : n \geq_r k\}$ and additionally $G(3 \cdot 2^\infty) = R(3 \cdot 2^\infty) = \{1\} \cup \{3n : n \in S(2^\infty)\}$. We also denote $\mathbb{N}_g = (\mathbb{N} \setminus \{2\}) \cup \{3 \cdot 2^\infty\}$ and $\mathbb{N}_r = (\mathbb{N} \setminus \{2, 4\}) \cup \{3 \cdot 2^\infty\}$.

The next theorem is due to Alsedà, Llibre and Misiurewicz [ALM1] for Y maps with the branching point fixed and to Baldwin for arbitrary Y maps [Ba].

Theorem 1.2 (Y Theorem).

- (a) If f is an Y map, then $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ for some $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$.
- (b) If $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$, then there exists an Y map f having the branching point fixed such that $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$.

We define the *Block ordering* $>_b$ on $\mathbb{N}_b = \mathbb{N} \setminus \{1\}$ as the converse of the natural ordering on \mathbb{N} ; i.e. $2 >_b 3 >_b 4 >_b \dots$. For $n \in \mathbb{N}_b$, we denote $B(n) = \{k \in \mathbb{N} : n \geq_b k\} \cup \{1\}$. Sharkovskii's Theorem has been generalized by Block to the circle maps having a fixed point in [Bl].

Theorem 1.3 (Circle Theorem).

- (a) If f is a circle map having a fixed point, then $\text{Per}(f) = S(n_s) \cup B(n_b)$ for some $n_s \in \mathbb{N}_s$ and $n_b \in \mathbb{N}_b$.
- (b) If $n_s \in \mathbb{N}_s$ and $n_b \in \mathbb{N}_b$, then there exists a circle map f having a fixed point such that $\text{Per}(f) = S(n_s) \cup B(n_b)$.

The study of the set of periods for circle maps without fixed points is more difficult, and needs the notion of rotation interval. Here we do not consider maps without fixed points, for more details see [ALM2].

In this paper we extend the previous three theorems to a class of sigma maps. As for circle maps if we want to avoid in a first study of the sigma maps the problems related with the computation of the set of periods from the rotation interval, we must add some additional assumptions to the continuity of the map. While for circle maps it is sufficient to add the assumption that they have a fixed point, this is not the case for sigma maps. More concretely, in order to avoid the rotation interval associated to a sigma map in the computation of its set of periods, we must restrict our attention to the subclass of sigma maps f such that f has a fixed point and $f(0) \in \mathbb{I}$, or f has a fixed point in \mathbb{O} . In what follows such sigma maps will be denoted σ maps. We remark that our

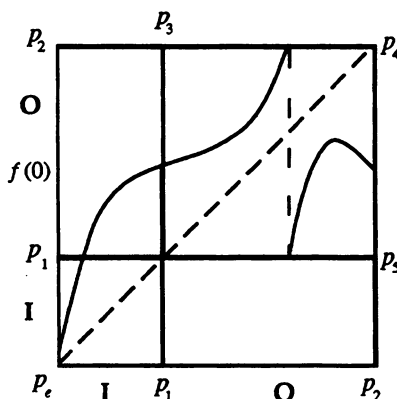


FIGURE 1.1. This sigma map f satisfies that $\text{Per}(f) = \text{Per}(f|_O) \cup \{1\}$, and $f|_O$ is a circle map of degree 1 without fixed points (see [ALM2]).

σ maps are the sigma maps such that $f(0) \in I$ or f has a fixed point in O , because in the first case f always has a fixed point in I . In short the unique sigma maps having a fixed point that are not σ maps satisfy that the fixed point does not belong to O and $f(0) \notin I$; such maps need the rotation interval for describing the set of periods, see Figure 1.1 and [ALM2].

The graph of a sigma map f is the subset $\{(x, f(x)) : x \in \sigma\}$ of the cartesian product $\sigma \times \sigma$, and it can be represented as in Figure 1.1. More precisely, if in the closed square $[p_e, p_2] \times [p_e, p_2]$ of Figure 1.1 we identify the vertical straight lines p_1p_3 and p_2p_4 , and the horizontal straight lines p_1p_5 and p_2p_4 , we get the space $\sigma \times \sigma$. The segment p_ep_1 represents I , and the segment p_1p_2 with the points p_1 and p_2 identified to the branching point O represents O . Roughly speaking we think in the graph of a sigma map like the graph of an interval map from $[p_e, p_2]$ into itself with the above identifications.

Our main result is the following one.

Theorem 1.4 (σ Theorem).

- (a) If f is a σ map, then $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r) \cup B(n_b)$ for some $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$, $n_r \in \mathbb{N}_r$ and $n_b \in \mathbb{N}_b$.
- (b) If $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$, $n_r \in \mathbb{N}_r$ and $n_b \in \mathbb{N}_b$, then there exists a σ map f having the branching point fixed such that $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r) \cup B(n_b)$.

Theorem 1.4 when the branching point is fixed was proved simultaneously and in a different way by Llibre, Paraños and Rodríguez see [LPR]. As far as we know Theorem 1.4 is the first result on the characterization of the set of periods of continuous self-maps on topological graphs which are not trees without having fixed all the branching points.

2. INTERVALS AND BASIC INTERVALS

A closed (respectively open, half-open or half-closed) interval J of E is a subset of E homeomorphic to the closed interval $[0, 1]$ (resp. $(0, 1)$, $[0, 1)$). Notice that an interval cannot be a single point.

Let J be a closed interval of E , and let $h: [0, 1] \rightarrow J$ be a homeomorphism. Then $h(0) = a$ and $h(1) = b$ are called the *endpoints* of J . If either $E \in \{I, Y\}$ or $E = \sigma$ and $a, b \in I$, then J will be denoted by $[a, b]$ or $[b, a]$. If $E = O$ and $a, b \in O$, then we write $[a, b]$ to denote the closed interval from a counterclockwise to b . If $E = \sigma$, $a \in O$ and $b \in I$, then $[a, b]$ will denote $[a, 0] \cup [0, b]$ or $[a, 0] \cup [b, 0]$, and $[b, a]$ will denote $[b, 0] \cup [0, a]$ or $[0, b] \cup [0, a]$.

Notice that it is possible that two different intervals of O or σ have the same endpoints. But two different points of I or Y always determine a unique closed interval.

Now we define an special class of subintervals of E . Let $Q = \{q_1, q_2, \dots, q_n\}$ be a finite subset of σ . For each pair q_i, q_j such that $q_i \neq q_j$ we say that the interval $[q_i, q_j]$ (respectively $[q_j, q_i]$) is *basic* if and only if $(q_i, q_j) \cap Q = \emptyset$ (respectively $(q_j, q_i) \cap Q = \emptyset$). The set of all these basic intervals is called the *set of basic intervals associated to Q* .

Assume that $E = \sigma$, $0 \notin Q$, $Q \cap I \neq \emptyset$ and $Q \cap O \neq \emptyset$. Then we consider $Q' = Q \cup \{0\}$. Clearly there are exactly three basic intervals A, B and C associated to Q' with 0 as endpoint. Without loss of generality we can assume that $A \subset I$. The endpoint of A, B and C different from 0 is denoted by x_a, x_b and x_c respectively. We define L_1, L_2, L_3 as the three basic intervals associated to Q containing 0 and such that $L_1 = A \cup B$, $L_2 = A \cup C$, $L_3 = B \cup C$. Notice that L_3 is an interval if and only if $x_b \neq x_c$. In what follows, the intervals A, B, C and the endpoints x_a, x_b, x_c are called the *intervals A, B, C* and the *endpoints x_a, x_b, x_c* associated to Q' ; the intervals L_1, L_2 and L_3 are called the *basic intervals L_1, L_2 and L_3* associated to Q .

3. LOOPS AND f -GRAPHS

Let $f: E \rightarrow E$ be an E map. If K and J are intervals of E , then we say that K *f -covers J* or $K \rightarrow J$ (or $J \leftarrow K$), if there is a closed subinterval M of K such that $f(M) = J$. If K does not f -cover J we write $K \nrightarrow J$.

A *path of length m* is any sequence $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_m$, where J_0, J_1, \dots, J_m are closed subintervals of E (in general, basic intervals). Furthermore, if $J_0 = J_m$, then this path is called a *loop of length m* . Such a loop will be called *non-repetitive* if there is no integer i , $0 < i < m$, such that i divides m and $J_{j+i} = J_j$ for all j , $0 \leq j \leq m-i$. This means that the loop is not a repetition of a shorter loop.

Let Q be a finite subset of E . An *f -graph of Q* is a graph with the basic intervals as vertices, and such that if K and J are basic intervals and K f -covers J , then there is an arrow from K to J . Note that the f -graph of Q is unique up to labeling of the basic intervals. Hence from now on we shall talk about the *f -graph of Q* (or just the *f -graph* for short).

Lemma 3.1. *Let f be an E -map and let K, J, L be closed subintervals of E . If $L \subset J$ and K f -covers J , then K f -covers L .*

Proof. By hypotheses, there exists an interval $M \subset K$ such that $f(M) = J$. First we suppose that $E = I$. Let $L = [c, d]$. Then there are $a, b \in M$ such that $f(a) = c$ and $f(b) = d$. We can assume that $a < b$. Then set $e = \sup\{x \in [a, b] : f(x) = f(a)\}$ and $l = \inf\{x \in [e, b] : f(x) = f(b)\}$. Clearly $f([e, l]) = L$. So $K \rightarrow L$.

Finally assume that $E \neq I$. There exist homeomorphisms $h_M : M \rightarrow I \equiv [0, 1]$ and $h_J : J \rightarrow I$. Note that $g = h_J \circ f|_M \circ h_M^{-1}$ is an interval map. Since $g(I) = I$, I f -covers I . By the first part of this proof, I g -covers the interval $h_J(L) \subset I$. Then there exists a closed subinterval N of I such that $g(N) = h_J(L)$. Therefore $h_M^{-1}(N)$ is a closed subinterval of M and $f(h_M^{-1}(N)) = h_J^{-1} \circ g \circ h_M(h_M^{-1}(N)) = L$. So $K \rightarrow L$. \square

Lemma 3.2. *Let f be an E map and let J be a subinterval of E such that J f -covers J . Then f has a fixed point in J .*

Proof. First we consider the case $E = I$. Set $J = [a, b]$. Then there are points $c, d \in J$ such that $f(c) = a$ and $f(d) = b$. We obtain $f(c) - c \leq 0$ and $f(d) - d \geq 0$. Then by continuity of f , there is a point x between c and d such that $f(x) - x = 0$; i.e. $f(x) = x$.

Finally suppose that $E \neq I$. Since $J \rightarrow J$, there is a closed interval L of J such that $f(L) = J$. Then there exists a homeomorphism $h : J \rightarrow I \equiv [0, 1]$ and we can consider an interval map g defined as $g = h \circ f|_L \circ h^{-1}$. Since $g(I) = I$, by the first part of this proof there exists $y \in I$ such that $g(y) = y$. Consequently for $x = h^{-1}(y) \in L$, we have $f(x) = h^{-1} \circ g \circ h(x) = h^{-1} \circ g(y) = h^{-1}(y) = x$, and we are done. \square

Lemma 3.3. *Let f be an E map and let J_0, J_1, \dots, J_{n-1} be closed subintervals of E such that $J_i \rightarrow J_{i+1}$ for $i = 0, 1, \dots, n-2$ and $J_{n-1} \rightarrow J_0$. Then there exists a fixed point x of f^n in J_0 such that $f^i(x) \in J_i$ for $i = 1, 2, \dots, n-1$.*

Proof. We shall use backward induction. Let $K_{n-1} \subset J_{n-1}$ be a closed interval such that $f(K_{n-1}) = J_0$, and suppose we have constructed $K_i \subset J_i$ for some $i > 0$, $i \leq n-1$ such that $f(K_i) = K_{i+1}$ if $i < n-1$ and $f(K_i) = J_0$ if $i = n-1$. Then, by Lemma 3.1, J_{i-1} f -covers K_i and therefore there exists an interval $K_{i-1} \subset J_{i-1}$ such that $f(K_{i-1}) = K_i$. Let g be as follows: $g = f|_{K_{n-1}} \circ \dots \circ f|_{K_1} \circ f|_{K_0}$. Then $K_0 \subset J_0$ and $g(K_0) = J_0$. Consequently $f^n(K_0) = J_0$. By continuity of f^n and Lemma 3.2 f^n has a fixed point $x \in K_0 \subset J_0$, such that $f^i(x) \in K_i \subset J_i$ for $i = 1, 2, \dots, n-1$. \square

4. BASIC ASSUMPTIONS: f , P AND k

Let f be a σ map. Notice that for proving Theorem 1.4(a) for f it is sufficient to show that if $k \in \text{Per}(f)$, then at least one of the sets $S(k)$, $G(k)$, $R(k)$ or $B(k)$ is contained in $\text{Per}(f)$. So in what follows we fix the σ map f and a $k \in \text{Per}(f)$ and we will prove the claim: at least one of the sets $S(k)$, $G(k)$, $R(k)$ or $B(k)$ is contained in $\text{Per}(f)$. This fixed σ map will be called the *standard σ map*.

Since a σ map have fixed points, we can assume that $k > 1$. Furthermore, since $S(2) = \{1, 2\}$ and $G(3) = R(3) = \{1, 3\}$ we can assume that $k > 3$. Due to the fact that $k \in \text{Per}(f)$, there exists a periodic orbit $P = \{x_1, \dots, x_k\}$ of period k for f . Also in what follows P always will denote a periodic orbit of period k for f .

5. REDUCTION TO THE CASE $P \cap O \setminus \{0\} \neq \emptyset$ AND $P \cap I \setminus \{0\} \neq \emptyset$

The next proposition shows that if P is contained either in O or in I , then the claim of Section 4 follows.

Proposition 5.1. *Let f be the standard σ map. If either $P \subset \mathbf{O}$ or $P \subset \mathbf{I}$, then $S(k)$ or $B(k)$ is contained in $\text{Per}(f)$.*

Proof. First assume that $P \subset \mathbf{O}$. We define $g : \mathbf{O} \rightarrow \mathbf{O}$ by $g(x) = f(x)$ if $f(x) \in \mathbf{O}$, and $g(x) = 0$ otherwise. Then g is continuous and P is a periodic orbit of period k for g . Furthermore g has a fixed point, because if $f(0) \in \mathbf{I}$, then $g(0) = 0$; otherwise f has a fixed point z in \mathbf{O} and so $g(z) = z$. From the Circle Theorem it follows that $\text{Per}(g) = S(n_s) \cup B(n_b)$ for some $n_s \in \mathbb{N}_s$, $n_b \in \mathbb{N}_b$. Clearly $\text{Per}(g) \subset \text{Per}(f)$. Since $k \in \text{Per}(g)$, we get that $S(k)$ or $B(k)$ is contained in $\text{Per}(g)$. Consequently $S(k)$ or $B(k)$ is contained in $\text{Per}(f)$.

Now suppose that $P \subset \mathbf{I}$. We define $g : \mathbf{I} \rightarrow \mathbf{I}$ by $g(x) = f(x)$ if $f(x) \in \mathbf{I}$, and $g(x) = 0$ otherwise. Then g is an interval map. Furthermore P is a periodic orbit of period k for g . By the Interval Theorem $\text{Per}(g) = S(n_s)$ for some $n_s \in \mathbb{N}_s$. Clearly $\text{Per}(g) \subset \text{Per}(f)$. Since $k \in \text{Per}(g)$, we get $S(k) \subset \text{Per}(g)$. Therefore $S(k) \subset \text{Per}(f)$. \square

From now on we can assume that the periodic orbit P has points in $\mathbf{O} \setminus \{0\}$ and points in $\mathbf{I} \setminus \{0\}$.

It is important to remark that Proposition 5.1 is not true if $P \subset \mathbf{O}$, $f(0) \in \mathbf{O}$ and f has no fixed points in \mathbf{O} . This is due to the fact that in such a case the circle map $g : \mathbf{O} \rightarrow \mathbf{O}$ defined in the first part of the proof of Proposition 5.1 has no fixed points, and consequently its set of periods depends on the rotation interval of g (see [ALM2]). Since in this paper we do not deal with the rotation interval, this fact is the main reason which forces us to restrict our attention to the class of σ maps instead of the class of all sigma maps.

6. REDUCTION TO THE CASE THE ENDPOINT OF σ BELONGS TO P

Denote by q_e the endpoint of σ , and let $p_e \in P \cap \mathbf{I}$ be such that $(q_e, p_e) \cap P = \emptyset$. Notice that such a p_e exists in virtue of Section 5.

Lemma 6.1. *Let f be the standard σ map. Suppose that $f(0) \in \mathbf{I} \setminus \{0\}$. Then there exists a fixed point z of f such that $z \in [p_e, 0] \subset \mathbf{I}$.*

Proof. Since $(q_e, p_e) \cap P = \emptyset$, $p_e \in P \cap \mathbf{I}$ and P is a periodic orbit, we have that $f(p_e) \notin [q_e, p_e]$. Moreover $f(0) \in \mathbf{I}$. Hence by the continuity of f , the subinterval $[p_e, 0] \subset \mathbf{I}$ contains a fixed point of f . \square

In a similar way as in Section 4 where we have fixed f , P and k , now we shall fix a fixed point of f .

If $f(0) = 0$ then let $x_0 = 0$.

Suppose that $f(0) \in \mathbf{O} \setminus \{0\}$. Then by the definition of σ map, f has fixed points in \mathbf{O} . We consider the following statements:

- (a) $[0, f(0)] \cap P = \emptyset$ and the subinterval $[0, f(0)]$ contains fixed points of f .
- (b) There is a fixed point $z \in \mathbf{O}$ of f such that $[0, z] \cap P = \emptyset$ and $(0, z)$ does not contain fixed points of f .

If statement (a) is satisfied, then let $x_0 \in [0, f(0)]$ be such that $[0, x_0] \subset [0, f(0)]$ and $[0, x_0]$ does not contain fixed points. Suppose that (a) is not satisfied. If statement (b) holds, let $x_0 = z$ satisfying (b); otherwise let x_0 be an arbitrary fixed point of f in \mathbf{O} .

Suppose that $f(0) \in \mathbf{I} \setminus \{0\}$. From Lemma 6.1 there is a fixed point z of f such that $z \in [p_e, 0]$. Then let $x_0 \in \mathbf{I}$ be the fixed point of f such that $[p_e, x_0)$ does not contain fixed points.

Remark 6.2. In what follows x_0 always will be a fixed point of f satisfying the above assumptions. Moreover we note that if $f(0) \neq 0$, then we have either $\{f(0), x_0\} \subset \mathbf{I}$ or $\{f(0), x_0\} \subset \mathbf{O}$.

Proposition 6.3. *Let f be the standard σ map. We can suppose that $q_e = p_e$.*

Proof. If $q_e \neq p_e$, consider the new topological space σ' obtained by shrinking the interval $[q_e, p_e]$ to the point p_e . Note that σ' is homeomorphic to σ . We define $g: \sigma' \rightarrow \sigma'$ by $g(x) = f(x)$ if $f(x) \in \sigma'$, and $g(x) = p_e$ if $f(x) \notin \sigma'$. By Lemma 6.1, g has a fixed point x_0 which verifies the assumptions of Remark 6.2. Then g is a σ map such that P is a periodic orbit of period k for g and the endpoint of σ belongs to P . Since $\text{Per}(g) \subset \text{Per}(f)$, it is sufficient to prove Theorem 1.4(a) for the σ maps satisfying that $q_e = p_e$. \square

Remark 6.4. From now on we can assume that $q_e = p_e$.

7. LOOPS AND PERIODS

In this section we shall study the relation between the periodic points of f and the loops of some f -graphs.

As usual if $U \subset \sigma$ we denote by $\text{Int}(U)$, $\text{Cl}(U)$ and by ∂U , the interior, the closure and the boundary of U respectively. If U is a finite subset of σ we denote by $\text{Card}(U)$ the cardinality of U .

We will use the following proposition for the study of σ maps with $0 \in P \cup \{x_0\}$.

Proposition 7.1. *Let f be the standard σ map. Suppose that $0 \in P \cup \{x_0\}$. Let $J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_n = J_0$ be a non-repetitive loop of length n in the f -graph where the J_i 's are basic intervals associated to $P \cup \{x_0\}$. If at least one J_i does not contain x_0 , then f has a periodic point of period n .*

Proof. By Lemma 3.3, there exists $x \in J_0$ such that $f^n(x) = x$, and $f^i(x) \in J_i$ for $i \in \{1, 2, \dots, n-1\}$. If x has period n , we are done. So suppose that x has period s with s a proper divisor of n . Notice that $x \neq x_0$, because x_0 is a fixed point and at least one J_i does not contain x_0 . If $x \in \text{Int}(J_0)$, then $\text{Orb}_f(x) \cap (P \cup \{x_0\}) = \emptyset$. Since 0 is not a periodic point having orbit different from P , each $f^i(x)$ is on a unique basic interval and consequently the loop is repetitive, in contradiction with the hypotheses. Hence, the only possibility is $\text{Orb}_f(x) = P$. So $\text{Orb}_f(x) = P$ and $k = s > 3$. Furthermore, by Remark 6.4 we may assume that x is the endpoint of σ .

Let $K_0 \subset J_0$ be the subinterval constructed in the proof of Lemma 3.3. Then $f^i(x) \in f^i(K_0) \subset J_i$ for $i \in \{0, 1, \dots, n-1\}$ and $f(K_0) = K_1 \subset J_1$. Since $x = f^s(x) \in f^s(K_0) \subset J_s$ it follows that J_s and J_0 have a common endpoint x , which is the endpoint of σ , so $J_0 = J_s$. Both sets K_0 and $f^s(K_0)$ are contained in J_0 and contain x , an endpoint of J_0 . Therefore $L = K_0 \cap f^s(K_0)$ must be a closed interval. In fact, L is either K_0 or $f^s(K_0)$. Clearly $f^i(L) \subset f^i(K_0) \subset J_i$, $f^i(L) \subset f^{s+i}(K_0) \subset J_{s+i}$ and $f^i(L)$ is a closed interval for $i \in \{0, 1, \dots, s-1\}$. Thus $J_i = J_{s+i}$ for $i \in \{0, 1, \dots, s-1\}$. Repeating this process we get that the loop $J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_n = J_0$ is repetitive, because s divides n , in contradiction with the assumptions. \square

A sequence of subsets $W_0, W_1, \dots, W_n = W_0$ of σ is *non-repetitive* if there is no integer i , $0 < i < n$, such that i divides n and $W_{j+i} = W_j$ for all j , $0 \leq j \leq n - i$.

We will use the following result for the σ maps with $0 \notin P \cup \{x_0\}$.

Proposition 7.2. *Let f be the standard σ map. Suppose that $0 \notin P \cup \{x_0\}$. Let $J_0, J_1, \dots, J_n = J_0$ be closed subintervals of σ , each one contained in a basic interval associated to $P \cup \{x_0\}$ and such that $J_i \rightarrow J_{i+1}$ for $0 \leq i \leq n - 1$. For each i , $0 \leq i \leq n$, let V_i the component of $\sigma \setminus (P \cup \{x_0\})$ containing $\text{Int}(J_i)$ and let $W_i = \text{Cl}(V_i)$. Suppose that at least one W_i does not contain 0 (i.e. W_i is an interval) and at least one W_i does not contain x_0 . If the sequence $W_0, W_1, \dots, W_n = W_0$ is non-repetitive, then f has a periodic point of period n .*

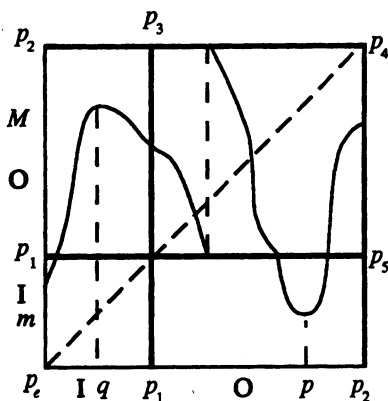
Proof. Without loss of generality we can suppose that $J_0 \subset W_0$ and that W_0 is an interval (i.e., W_0 does not contain 0). By Lemma 3.3, f^n has a fixed point x such that $f^i(x) \in K_i \subset J_i$ for $i \in \{0, 1, \dots, n - 1\}$, where the subintervals K_i 's are defined in the proof of Lemma 3.3. If x has period n , we are done. So suppose that x has period s , s a proper divisor of n . Notice that $x \neq x_0$ because x_0 is a fixed point and at least one W_i does not contain x_0 . If $x \in \text{Int}(J_0)$, then each $f^i(x)$ belongs to a unique W_i and therefore the sequence $W_0, W_1, \dots, W_n = W_0$ is repetitive, because s divides n . So we must have $\text{Orb}_f(x) = P$ and $k = s$.

By Remark 6.4 we can suppose that x is the endpoint of σ . Then, we note that now W_0 is not necessarily an interval. Since $x = f^s(x) \in J_s \subset W_s$ and $x \in J_0 \subset W_0$, x is a common endpoint of W_0 and W_s . So $W_0 = W_s$. We have $x \in K_0 \subset J_0 \subset W_0$, $f^s(x) = x \in K_s \subset J_s \subset W_s = W_0$. Note that K_0 and $f^s(K_0) = K_s$ are intervals and have x as common endpoint. Therefore $L = K_0 \cap f^s(K_0)$ is an interval, either K_0 or $f^s(K_0)$. Furthermore $f^i(L)$ is an interval for $0 \leq i \leq s$. Clearly $f^i(L) \subset f^i(K_0) \subset J_i$, $f^i(L) \subset f^{s+i}(K_0) \subset J_{s+i}$ and so $J_i \cap J_{s+i}$ contains an interval. Consequently $W_i = W_{s+i}$ for $0 \leq i \leq s$. Repeating this process we get that the sequence $W_0, W_1, \dots, W_n = W_0$ is repetitive, in contradiction with the hypotheses. \square

8. REDUCTION TO MONOTONE MAPS

The graph of a σ map g is the subset $\{(x, g(x)) : x \in \sigma\}$ of $\sigma \times \sigma$, see Figure 8.1. The segment $p_e p_1$ represents **I**, and the segment $p_1 p_2$ with the points p_1 and p_2 identified to the branching point 0 represents **O**. Roughly speaking we think in the graph of a σ map like the graph of an interval map from $[p_e, p_2]$ into itself with the above identifications. This allows us to talk about local or absolute maximum or minimum for a σ map in the same way as for interval maps. Thus, for instance in the points p and q the σ map g represented in Figure 8.1 has a local minimum and maximum with values m and M respectively.

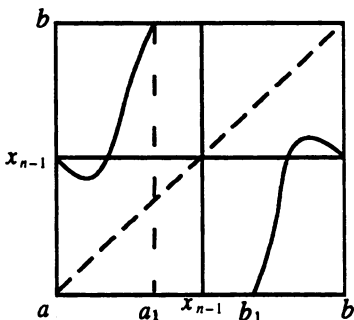
Let g be a σ map and let $[a, b]$ a closed subinterval of a basic interval associated to $P \cup \{x_0\} \cup \{0\}$. We will say that g is *non-monotone* in $[a, b]$ if there exist two different points $c, d \in [a, b]$ such that $g(c) = g(d)$, and g is not constant in the subinterval $[c, d] \subset [a, b]$. Otherwise, we say that g is *monotone* in $[a, b]$.

FIGURE 8.1. The graph of a σ map.

Proposition 8.1. *Let f be the standard σ map. Let U be an open subinterval of a basic interval associated to $P \cup \{x_0\} \cup \{0\}$. Suppose that $U = (a, b)$ with $f(a) = f(b) = c$, f non-monotone in U , and $f(d) \neq c$ for $d \in (a, b)$. Let g be the σ map defined by $g|_{\sigma \setminus U} = f|_{\sigma \setminus U}$, and $g|_U$ is constant and equal to c . Then $\text{Per}(g) \subset \text{Per}(f)$.*

Proof. Let x be a periodic point of g of period $n > 1$. If $c \notin \text{Orb}_g(x)$, then x is a periodic point of period n for f and we are done. So, suppose that $c \in \text{Orb}_g(x)$; i.e., c is a periodic point of period n for g . We shall prove that $n \in \text{Per}(f)$. Set $x_i = g^i(c)$ for $0 \leq i \leq n$. Therefore $g^i(c) \notin U$ for $1 \leq i \leq n-2$ (otherwise c has not period n for g). Thus $x_i = f(x_{i-1}) = f^i(c)$ for $0 \leq i \leq n-1$. If $f(x_{n-1}) = c$, then c is a periodic point of period n for f and we are done. So, suppose that $f(x_{n-1}) \neq c$. Then $x_{n-1} \in U$. We claim that $g^{n-1} \circ f$ has a fixed point in $\text{Cl}(U)$. If $(g^{n-1} \circ f)(\text{Cl}(U)) \subset \text{Cl}(U)$, the claim is clear because $\text{Cl}(U)$ is a closed interval and $g^{n-1} \circ f$ is continuous (see Lemma 3.2). Now we assume that $(g^{n-1} \circ f)(\text{Cl}(U)) \not\subset \text{Cl}(U)$. Also we suppose that there are no fixed points of $(g^{n-1} \circ f)|_{\text{Cl}(U)}$. So in order to prove the claim we must obtain a contradiction.

Since $(g^{n-1} \circ f)(a) = (g^{n-1} \circ f)(b) = x_{n-1}$, $(g^{n-1} \circ f)(\text{Cl}(U)) \not\subset \text{Cl}(U)$ and $(g^{n-1} \circ f)|_{\text{Cl}(U)}$ has no fixed points, the graph of $(g^{n-1} \circ f)|_{\text{Cl}(U)}$ must be as in Figure 8.2.

FIGURE 8.2. The graph of $(g^{n-1} \circ f)|_{\text{Cl}(U)}$.

Now we consider $a_1 \in \text{Cl}(U)$ such that $(g^{n-1} \circ f)(a_1) = b$ and $[a, a_1]$ does not contain points d satisfying $(g^{n-1} \circ f)(d) = b$. Set $J_1 = [a, a_1]$. We denote by J_2 the subinterval of $\text{Cl}(U)$ of the form $[b_1, b]$ such that $f(J_1) = f(J_2)$, such interval exists because $f|_U$ does not contain any neighbourhood of c and $f(\text{Cl}(U)) \not\subseteq \text{Cl}(U)$. Consequently $(g^{n-1} \circ f)(J_1) = (g^{n-1} \circ f)(J_2)$. A contradiction with Figure 8.2. Hence the claim is proved.

Let x be the fixed point of $(g^{n-1} \circ f)|_{\text{Cl}(U)}$. We shall prove that x is a periodic point of period n for f . We define the sets $Z_i = g^{-(n-1-i)}(\text{Cl}(U))$ for $0 \leq i \leq n-1$. Since $g(\text{Cl}(U)) = c$, we have $g^{n-i}(Z_i) = g^{n-i-n+1+i}(\text{Cl}(U)) = c$ and $g^n(Z_i) = x_i$. Therefore, the sets Z_i must be pairwise disjoint. Since $x \in \text{Cl}(U)$, from $x = (g^{n-1} \circ f)(x) = g^{(n-1-i)}((g^i \circ f)(x)) \in \text{Cl}(U)$ we get $(g^i \circ f)(x) \in Z_i$ for $0 \leq i \leq n-1$. Then, the points $(g^i \circ f)(x)$ are all different for $0 \leq i \leq n-1$. Note that f and g coincide in $\sigma \setminus U$, thus f and g coincide in every Z_i for $0 \leq i \leq n-2$. Therefore $(g^i \circ f)(x) = f^{i+1}(x) \in Z_i$. In particular, for $i = n-2$ we have $(g^{n-2} \circ f)(x) = f^{n-1}(x) \in Z_{n-2}$ and so $f^n(x) = (g^{n-1} \circ f)(x) = x$; i.e., $n \in \text{Per}(f)$.

Now to finish the proof of the proposition we must show that g is a σ map. Since $\{0, x_0\} \not\subseteq U$, $f(0) = g(0)$ and $f(x_0) = g(x_0) = x_0$, so g has fixed points. If $f(0) \in I$ then $g(0) \in I$ and g is a σ map. If $f(0) \notin I$ then $x_0 \in O$. Hence, again g is a σ map. \square

We consider σ as a metric space. We assume for convenience that our metric, called δ on σ satisfies the property that if z is in the interval $[x, y]$, then $\delta(x, y) = \delta(x, z) + \delta(z, y)$. Let $(f_i)_{i \in \mathbb{N}}$ be a sequence of σ maps, and let f be the standard σ map. The Cauchy's Criterion for uniform convergence says that the sequence $(f_i)_{i \in \mathbb{N}}$ converges uniformly in σ if and only if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\delta(f_i(x), f_j(x)) < \varepsilon$ for all $x \in \sigma$, whenever $i, j \geq n_0$ (see Theorem 7.8 of [R]). Another well-known result is the following: If $(f_i)_{i \in \mathbb{N}}$ converges uniformly to g in σ , then g is a sigma map (see Theorem 7.12 of [R]).

Proposition 8.2. *Let f be the standard σ map. There exists a σ map g such that $g|_{P \cup \{x_0\} \cup \{0\}} = f|_{P \cup \{x_0\} \cup \{0\}}$, $g|_J$ is monotone for each basic interval J associated to $P \cup \{x_0\} \cup \{0\}$, and $\text{Per}(g) \subset \text{Per}(f)$.*

Proof. We define inductively σ maps f_i as follows. Set $f_1 = f$ and suppose that f_i has been defined such that $f_i|_{P \cup \{x_0\} \cup \{0\}} = f|_{P \cup \{x_0\} \cup \{0\}}$ and $\text{Per}(f_i) \subset \text{Per}(f)$. If f_i is monotone on each basic interval associated to $P \cup \{x_0\} \cup \{0\}$ for some i , then we are done. Otherwise, there exist two different points $a, b \in \text{Int}(J)$, for some basic interval J such that $f_i(a) = f_i(b) = c$, f_i is not constant on $[a, b] \subset J$ and $f([a, b])$ does not contain any neighbourhood of c . We can suppose that $(a, b) = (a_i, b_i)$ is the largest subinterval of J with the above properties. Let f_{i+1} be the result of applying Proposition 8.1 to $U = (a_i, b_i)$, so f_{i+1} is constant on $[a_i, b_i]$. Note that since the interval $[a_i, b_i]$ was selected as large as possible, the intervals $[a_i, b_i]$ are pairwise disjoint. Furthermore, f_i differs from f_{i+1} only on (a_i, b_i) . Notice that the set of subintervals where f is not monotone is numerable (we can associate a rational number to each one such subintervals). In other words, the set of local maxima and minima of f is numerable. Notice that each f_i is a σ map from Proposition 8.1. Now we will prove that $(f_i)_{i \in \mathbb{N}}$ converges uniformly to a sigma map g . If not, there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ if $i, j \geq n$ then $\delta(f_i, f_j) \geq \varepsilon$. This means

that for each $n \in \mathbb{N}$, there exists a subinterval K where f is not monotone and $\max \{\delta(f(x), f(y)) : x, y \in K\} > \varepsilon$. So there is an infinite set of pairwise disjoint subintervals K such that $f(K)$ has length larger than ε . There is an accumulation point p of such subintervals, then f is not continuous at p , in contradiction with the assumptions. Hence $(f_i)_{i \in \mathbb{N}}$ converges uniformly to a sigma map g . Since $f_i(0) = f(0)$ and $f_i(x_0) = f(x_0) = x_0$ for all i , it follows that $g(0) = f(0)$ and $g(x_0) = x_0$. Hence since f is a σ map, also g is a σ map.

We shall prove that $\text{Per}(g) \subset \text{Per}(f)$. Let x be a periodic point of g of period n . Denote by D the set of points $c_i \in \sigma$ with $i \in \mathbb{N}$ for which $f_i(a_i) = f_i(b_i) = c_i$, f_i non-monotone on $[a_i, b_i]$ and $f_{i+1}|_{[a_i, b_i]} = c_i$. This means that D is the set where we "cut" f_i and obtain the new map f_{i+1} . If $c_i \notin \text{Orb}_g(x)$ for any $c_i \in D$, then clearly x is a periodic point of period n for f_i for all $i \in \mathbb{N}$ and we are done. So suppose that $c_i \in \text{Orb}_g(x)$ for some $c_i \in D$; i.e., c_i is a periodic point of period n for g . Then as in the proof of Proposition 8.1 we obtain that $n \in \text{Per}(f_{i+1}) \subset \text{Per}(f_i) \subset \text{Per}(f)$. This completes the proof. \square

From now on we will assume that f is monotone on each basic interval associated to $P \cup \{x_0\} \cup \{0\}$. Notice that after this process x_0 verifies the properties of Section 6. We also remark that if f is monotone on each basic interval J associated to $P \cup \{x_0\} \cup \{0\}$, then $f(J)$ cannot contain \mathbf{O} .

9. REDUCTION TO σ MAPS SUCH THAT THERE EXISTS A CLOSED SUBINTERVAL OF σ f -COVERING \mathbf{O}

Eventually we shall consider the circle as a closed subinterval with the end-points identified to a point.

Let f be the standard σ map. Let $V = [v_1, v_2]$ be a subinterval of σ , V the largest one such that $f(V) \subset \mathbf{O}$, $f(V) \neq \mathbf{O}$ and $f(d) \neq 0$ for $d \in (v_1, v_2)$. We say that V is an *upper subinterval* if $f(V) \cap (p_1, p_1 + \varepsilon) = \emptyset$ for some $\varepsilon > 0$ where $p_1 + \varepsilon \in \mathbf{O}$. We say that V is a *down subinterval* if $f(V) \cap (p_2 - \varepsilon, p_2) = \emptyset$ for some $\varepsilon > 0$, with $p_2 - \varepsilon \in \mathbf{O}$.

Let K be a closed subinterval of σ such that there exists a closed subinterval $[c, d] \subset K$ with $f([c, d]) = \mathbf{O}$, $f(c) = f(d) = 0$, and $f(x) \neq 0$ for all $x \in (c, d)$, then we say that K *f-covers* \mathbf{O} , or $K \rightarrow \mathbf{O}$ (or $\mathbf{O} \leftarrow K$).

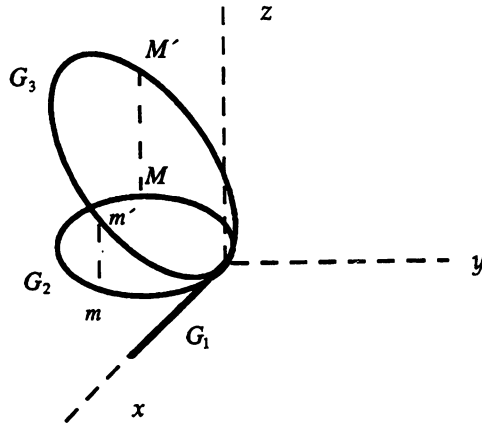
From now on in this section we assume that there are no closed intervals of σ *f-covering* \mathbf{O} , and under this assumption we prove Theorem 1.4(a).

Since f is monotone on each basic interval associated to $P \cup \{x_0\} \cup \{0\}$, f has finitely many local extrema (see Section 8); and consequently finitely many upper and down subintervals. Let U_1, \dots, U_r and D_1, \dots, D_s be the upper and down subintervals of f respectively. Let m_i be the absolute minimum of $f|_{U_i}$ and M_i the absolute maximum of $f|_{D_i}$. We denote by $m = \min\{m_1, \dots, m_r\}$ and $M = \max\{M_1, \dots, M_s\}$.

Lemma 9.1. *With the above notation we have that $m \neq 0$ and $M \neq 0$.*

Proof. Suppose $m = 0$, then by continuity of f , there exists an $i \in \{1, \dots, r\}$ such that $f(U_i) = \mathbf{O}$, in contradiction with the hypotheses. In the same way, we get that $M \neq 0$. \square

Now we shall consider an unfolding of σ as follows. Define the graph $\tilde{\sigma} = G_1 \cup G_2 \cup G_3$ where

FIGURE 9.1. The unfolding of σ .

$$G_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, \quad y = 0, \quad 0 \leq x \leq 1\},$$

$$G_2 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, \quad x^2 + (y + 1)^2 = 1\},$$

$$G_3 = \{(x, y, z) \in \mathbb{R}^3 : y = z, \quad x^2 + (y + 1)^2 = 1\};$$

see Figure 9.1.

Clearly $G_1 \cup G_2$ is homeomorphic to σ , so we identify σ with $G_1 \cup G_2$. Consider the projection $\pi : \tilde{\sigma} \rightarrow \sigma$ defined by $\pi(x, y, z) = (x, y, 0)$. We denote by m' and M' the points of G_3 such that $\pi(m') = m$ and $\pi(M') = M$. For the standard σ map f we define $\tilde{f} : \tilde{\sigma} \rightarrow \tilde{\sigma}$ as follows. Since there are no closed subintervals of σ f -covering \mathbf{O} , and f is monotone in each basic interval associated to $P \cup \{x_0\} \cup \{0\}$, there exists a finite partition of σ into closed subintervals V_i for $i = 1, \dots, n$ (i.e., $\sigma = \bigcup_{i=1}^n V_i$ and $\text{Int}(V_i) \cap \text{Int}(V_j) = \emptyset$ if $i \neq j$) such that

- (i) either $f(V_i) \subset \mathbf{I}$,
- (ii) or $f(V_i) \subset \mathbf{O}$, and $f(d) \neq 0$ for $d \in \text{Int}(V_i)$,
- (iii) the subintervals V_i are the largest ones satisfying (i) or (ii).

If V_i is one of the subintervals of σ satisfying (i), then define $\tilde{f}|_{V_i} = f|_{V_i}$. Let V_i be a subinterval of σ satisfying (ii). Then V_i is either an upper subinterval or a down subinterval. Suppose that V_i is an upper subinterval. Then for every $r \in V_i$ we define $\tilde{f}(r) = q' \in G_3$ such that $\pi(q') = f(r) \in G_2$. Now assume that V_i is a down subinterval. Then for every $r \in V_i$, we define $\tilde{f}(r) = f(r) \in G_2$. Clearly \tilde{f} is well-defined. Notice that $f = \pi \circ \tilde{f} : \sigma \rightarrow \sigma$. Define $F = \tilde{f} \circ \pi : \tilde{\sigma} \rightarrow \tilde{\sigma}$.

Lemma 9.2. *The map $F : \tilde{\sigma} \rightarrow \tilde{\sigma}$ satisfies that $\text{Per}(F) = S(n_s) \cup G(n_g) \cup R(n_r)$ for some $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$.*

Proof. By Lemma 9.1, $\mathbf{O} \notin \{m, M\}$. Furthermore, since there are no closed intervals of σ f -covering \mathbf{O} , we have that $F(\tilde{\sigma})$ is homeomorphic to \mathbf{I} or \mathbf{Y} . So, from the Interval Theorem or the \mathbf{Y} Theorem, we can write $\text{Per}(F) = S(n_s) \cup G(n_g) \cup R(n_r)$ for some $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$. \square

Now we shall relate the periodic orbits of f with the periodic orbits of F .

Lemma 9.3. *The map $F : \tilde{\sigma} \rightarrow \tilde{\sigma}$ satisfies that $k \in \text{Per}(F)$.*

Proof. Let $x \in P$. Then x is a periodic point of period k for f . Also we can assume, by Section 5, that $x \in I$ (so $\pi(x) = x$). We have $f^k(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < k$. This means that $(\pi \circ \tilde{f})^k(\pi(x)) = x$. So $\pi \circ (\tilde{f} \circ \pi)^k(x) = x$, and by definition $\pi(F^k(x)) = \pi(x)$. Thus $F^k(x) = x$. Moreover $F^i(x) \neq x$ for $1 \leq i < k$, otherwise $(\tilde{f} \circ \pi)^i(x) = x$ for some $i < k$, and so $(\pi \circ \tilde{f})^i \circ \pi(x) = \pi(x)$. Hence $f^i(\pi(x)) = \pi(x)$ for some $1 \leq i < k$ in contradiction with the hypotheses. Therefore x is a periodic point of period k for F . \square

Lemma 9.4. *If $x \in \tilde{\sigma}$ is a periodic point of period n for F , then $\pi(x) \in \sigma$ is periodic of period n for f .*

Proof. Set $\text{Orb}_F(x) = \{p_1, p_2, \dots, p_n\}$ such that $F(p_i) = p_{i+1}$ for $1 \leq i \leq n-1$ and $F(p_n) = p_1$. Let $q_i = \pi(p_i)$ for $1 \leq i \leq n$. Notice that $\pi(p_i) \neq \pi(p_j)$ if $i \neq j$, otherwise $\pi(p_i) = \pi(p_j)$ and consequently $F(p_i) = (\tilde{f} \circ \pi)(p_i) = (\tilde{f} \circ \pi)(p_j) = F(p_j)$ in contradiction with the fact that $\{p_1, \dots, p_n\}$ is a periodic orbit of period n . Then the q_i 's are pairwise different and $f(q_i) = (\pi \circ \tilde{f})(\pi(p_i)) = \pi(F(p_i)) = \pi(p_{i+1}) = q_{i+1}$ for $1 \leq i \leq n-1$ and $f(q_n) = q_1$. So $\{q_1, \dots, q_n\} = \text{Orb}_f(\pi(x))$ is a periodic orbit of period n for f . \square

Proposition 9.5. *Let f be the standard σ map. Then $S(k)$, $G(k)$ or $R(k)$ is contained in $\text{Per}(f)$.*

Proof. By Lemma 9.3, $k \in \text{Per}(F)$; and by Lemma 9.4, $\text{Per}(F) \subset \text{Per}(f)$. Then from Lemma 9.2 $S(k)$, $G(k)$ or $R(k)$ is contained in $\text{Per}(f)$. \square

Remark 9.6. *From now on we can suppose that there exists a closed subinterval K of σ such that K f -covers \mathbf{O} . The interval K is not necessarily basic. Eventually K is the circle.*

We remark that the idea of the unfolding of σ when there is no closed interval f -covering \mathbf{O} comes from the Ph.D. thesis of Paraños [P]. In fact the unfolding of σ is essentially equivalent to work with the lifting of f . Thus the ideas of Section 3.5 of [ALM2] are closed to the ones developed here.

10. REDUCTION TO σ MAPS SUCH THAT EVERY BASIC INTERVAL OF T IS f -COVERED BY SOME BASIC INTERVAL OF T

From now on we denote by S or S' the set of all basic intervals associated to $P \cup \{x_0\}$ or $P \cup \{x_0\} \cup \{0\}$ respectively.

We shall work with the basic intervals A, B, C, L_1, L_2 and L_3 associated to $P \cup \{x_0\} \cup \{0\}$ or $P \cup \{x_0\}$ defined in Section 2.

In this section we can restrict our attention to the σ maps satisfying that each basic interval of S is f -covered by some basic interval of S ; or each basic interval of $S \setminus \{L_j\}$ is f -covered by some basic interval of $S \setminus \{L_j\}$ for some $j \in \{1, 2, 3\}$.

Proposition 10.1. *Let f be the standard σ map. Suppose that $f(0) \in \text{Int}(D)$ for some $D \in S'$. If D is not f -covered by any basic interval of $\{L_1, L_2, L_3\}$, then there exists a σ map g such that $g|_{P \cup \{x_0\}} = f|_{P \cup \{x_0\}}$, $g(0) \in \partial D$, D*

is not f -covered by any basic interval of $\{L_1, L_2, L_3\}$, g is monotone in each basic interval of S' and $\text{Per}(g) \subset \text{Per}(f)$.

Proof. Since $f(0) \in \text{Int}(D)$ and there is no basic interval of $\{L_1, L_2, L_3\}$ f -covering D , it follows that there are closed subintervals $A' \subset A$, $B' \subset B$ and $C' \subset C$ such that $f(A' \cup B' \cup C') \subset D$ and the image of the endpoints of A' , B' and C' is $\{f(0), z\}$, where z is an endpoint of D . Notice that $A' \cup B' \cup C'$ is homeomorphic to Y . We define $g|_{\sigma \setminus (A' \cup B' \cup C')} = f|_{\sigma \setminus (A' \cup B' \cup C')}$ and $g|_{A' \cup B' \cup C'} = z$ (in particular $g(0) = z \in \partial D$). Clearly g is a σ map because $g(0) \in D$ and $g(x_0) = x_0$. Since $g|_P = f|_P$, $k \in \text{Per}(g)$. Furthermore g is monotone on each basic interval of S' and $\text{Per}(g) \subset \text{Per}(f)$. \square

Remark 10.2. From now on, by Proposition 10.1, we can assume that there are no basic intervals $D \in S'$ satisfying the following properties: $f(0) \in \text{Int}(D)$ and D is not f -covered by any basic interval of $\{L_1, L_2, L_3\}$.

Lemma 10.3. *Let f be the standard σ map. Then each basic interval J of S (respectively S') contained in I is f -covered by some basic interval of S (respectively S').*

Proof. Since $f(\sigma)$ is a connected set, the endpoint of I belongs to P (see Remark 6.4) and the endpoints of all basic intervals belong to $P \cup \{x_0\}$ (respectively $P \cup \{x_0\} \cup \{0\}$), we obtain that every basic interval contained in I is f -covered by some basic interval of S (respectively S'). \square

Lemma 10.4. *Let f be the standard σ map. Then each basic interval of S' is f -covered by some basic interval of S .*

Proof. Notice that if $0 \in P \cup \{x_0\}$, then $S = S'$. We have that each basic interval of S' is a subinterval of some basic interval of S . So by Lemma 10.3 each basic interval of S' contained in I is f -covered by some basic interval of S . On the other hand, since there exists a closed subinterval K of σ (non-necessarily basic) such that K f -covers O we have that each basic interval of $S' \cap S$ contained in O is f -covered by some basic interval (subinterval of K) of S , this would be shown as in the proof of Lemma 3.1. \square

Remark 10.5. From now on, by Lemma 10.3 and Lemma 10.4 we can assume that each basic interval of S' is f -covered by some basic interval of S and $f(\sigma) = \sigma$.

Lemma 10.6. *Let f be the standard σ map. Suppose that $0 \notin P \cup \{x_0\}$. Then at least two intervals of $\{L_1, L_2, L_3\}$ are f -covered by some basic intervals of S .*

Proof. From Lemma 10.4 each interval A , B and C is f -covered by some basic interval of S . For each pair $i, j \in \{1, 2, 3\}$, $i \neq j$, $L_i \cap L_j$ is one interval of $\{A, B, C\}$. We claim that L_i or L_j is f -covered by some basic interval of S . Suppose that $L_i \cap L_j = A$ (the proof is analogous in the other cases). By Remark 10.5 there exists $J \in S$ such that $J \rightarrow A$. First suppose that $J \in S'$. Since f is monotone on J and the endpoints of J are not mapped in $\{0\}$, we have that $J \rightarrow B$ or $J \rightarrow C$. Hence $J \rightarrow L_i$ or $J \rightarrow L_j$.

Finally we can assume that $J \notin S'$. If $f(0) \in \text{Int}(A \cup B \cup C)$, then $f(0) \in \text{Int}(D)$ for some $D \in \{A, B, C\}$. By Remark 10.2, D is f -covered by some interval of $\{L_1, L_2, L_3\}$. Hence there is $M \in \{L_1, L_2, L_3\}$ such that $M \rightarrow A$ and either $M \rightarrow B$ or $M \rightarrow C$. So $M \rightarrow L_i$ or $M \rightarrow L_j$. Otherwise, $f(0) \notin \text{Int}(A \cup B \cup C)$. Therefore there exists $N \in \{A, B, C\}$, $N \subset J$ such

that $N \rightarrow A$. By the above arguments $N \rightarrow L_i$ or $N \rightarrow L_j$. So the claim is proved. Consequently at least two intervals of $\{L_1, L_2, L_3\}$ are f -covered by some basic intervals of S . \square

Lemma 10.7. *Let f be the standard σ map. Suppose that $0 \notin P \cup \{x_0\}$. Let J be a basic interval of S . If there is $L_j \in S$ for some $j = 1, 2, 3$ such that $L_j \rightarrow J$, then there exists L_i for some $i = 1, 2, 3$, $i \neq j$ such that $L_i \rightarrow J$.*

Proof. Suppose that $L_j \rightarrow J$ for some $j = 1, 2, 3$. If $f(0) \notin \text{Int}(J)$, then A, B or C f -covers J . Notice that each interval A, B and C belongs to two intervals of $\{L_1, L_2, L_3\}$. Hence there is L_i , $i = 1, 2, 3$, $i \neq j$, such that $L_i \rightarrow J$. So we can assume that $f(0) \in \text{Int}(J)$. Denote by J_1 and J_2 the closures of the two connected components of $J \setminus \{f(0)\}$. Set $L_j = M_1 \cup M_2$ where $M_1, M_2 \in \{A, B, C\}$, and $M_1 \rightarrow J_1$, $M_2 \rightarrow J_2$. Then by continuity, the interval $M_3 \in \{A, B, C\} \setminus \{M_1, M_2\}$ f -covers either J_1 or J_2 . So, either $M_1 \cup M_3$ or $M_2 \cup M_3$ f -covers J , and the result follows. \square

Proposition 10.8. *Let f be the standard σ map. Suppose $0 \notin P \cup \{x_0\}$. Then at least one of the following two statements hold.*

- (a) *Each basic interval of S is f -covered by some basic interval of S .*
- (b) *Each basic interval of $S \setminus \{L_j\}$ for some $j \in \{1, 2, 3\}$ is f -covered by some basic interval of $S \setminus \{L_j\}$.*

Proof. If statement (a) does not hold, by Lemma 10.4 there is $L_j \in S$ for some $j = 1, 2, 3$ such that there are no basic intervals of S f -covering L_j , and by Lemma 10.6 the above interval L_j is unique. If $L_j \rightarrow J$ for some $J \in S \setminus \{L_j\}$, then from Lemma 10.7 there exists $L_i \in S$ such that $L_i \rightarrow J$ and statement (b) holds. \square

Notice that if $0 \in P \cup \{x_0\}$, by Lemma 10.4 each basic interval of $S = S'$ is f -covered by some basic interval of $S = S'$. So from now on, if $0 \in P \cup \{x_0\}$ we denote by T the collection of all basic intervals of $S = S'$. If $0 \notin P \cup \{x_0\}$ we denote by T the collection of all basic intervals of S if (a) of Proposition 10.8 is satisfied, otherwise $T = S \setminus \{L_j\}$ for some $j \in \{1, 2, 3\}$ such that (b) of Proposition 10.8 is satisfied.

11. SOME BASIC RESULTS

In this section we obtain some results which we are using in the next sections.

Lemma 11.1. *Let f be the standard σ map. Let J_0 be a closed subinterval of I with endpoints elements of $P \cup \{x_0\}$. Notice that J_0 is not necessarily basic. Then there exists a repetitive or non-repetitive loop of length k containing J_0 formed by closed subintervals of σ , in general no basic intervals.*

Proof. Set $J_0 = [x, y]$. For each $0 < i \leq k$, we define J_i recursively as the closed subinterval with endpoints $f^i(x)$ and $f^i(y)$ such that $J_{i-1} \rightarrow J_i$. (Note that if $f^i(x)$ or $f^i(y)$ belongs to \mathbf{O} , then J_i perhaps is not unique.) Since $J_0 \subset I$, we get $J_0 = J_k$. Thus, we obtain the loop $J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_k = J_0$. Of course, in general J_i is not a basic interval and the above loop can be repetitive or non-repetitive. \square

Remark 11.2. Suppose that g is an interval map. Note that if g^2 has a periodic point z of period s , then z is a periodic point for g with period $2s$ if s is even, and period s or $2s$ if s is odd.

Lemma 11.3. *Let f be the standard σ map. Let $z \in \mathbf{I}$ be a fixed point of f . Denote by D_1 and D_2 the closures of the two connected components of $\sigma \setminus \{z\}$, such that D_1 is homeomorphic to \mathbf{I} . If $f(D_1) = D_2$ and $f(D_2) = D_1$, then $S(k) \subset \text{Per}(f)$.*

Proof. Notice that the map $f^2|_{D_1} : D_1 \rightarrow D_1$ is an interval map. Therefore f has no periodic points of odd period except the period 1. So k is even and by Remark 11.2 the points of $P \cap D_1$ have period $k/2$ for $f^2|_{D_1}$. From the Interval Theorem it follows that $S(k/2) \subset \text{Per}(f^2)$. Since all the periods of f are even except 1, $S(k) \subset \text{Per}(f)$. \square

Lemma 11.4. *Let f be the standard σ map. Suppose that each basic interval of T is f -covered by some basic interval of T different from itself. Suppose that there is $J \in T$ such that $J \rightarrow J$. Then there is a non-repetitive loop of length n for each $n \geq k + 1$ containing J . Moreover, if $N \in T$ and there is a path starting at N and ending at J , then we can assume that the loop of length n contains such a path.*

Proof. From Remark 10.5 we have $f(\sigma) = \sigma$. Set $K_i = f^i(J_0)$ for $i \geq 0$, where $J_0 = J$. Notice that each K_i is a connected set and $\text{Card}(K_1 \cap P) \geq 2$. So from the fact that P is a periodic orbit and Remark 10.5, it follows that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_r = \sigma$ and $\text{Card}(K_i \cap P) \geq i + 1$ for $i < r$. Since P has period k we have that $r \leq \text{Card}(K_{r-1} \cap P) \leq k$. Since each basic interval of T is f -covered by some basic interval different from itself, for each basic interval $J_i \in T$, $J_i \subset K_i \setminus K_{i-1}$ there exists $J_{i-1} \in T$, $J_{i-1} \subset K_{i-1} \setminus K_{i-2}$ such that $J_{i-1} \rightarrow J_i$. By hypotheses there exists $M \in T$, $M \neq J_0$ such that $M \rightarrow J_0$. Hence there is a loop of length $l \leq r + 1 \leq k + 1$ containing J_0 . By construction, this loop is formed by pairwise different basic intervals and so is non-repetitive. The above loop of length l together with the loop $J_0 \rightarrow J_0$ give us a non-repetitive loop of length n for each $n \geq k + 1$ containing J_0 . The last statement of the lemma follows easily by the construction of the non-repetitive loop of length n . \square

Lemma 11.5. *Let f be the standard σ map. Suppose that each basic interval of T is f -covered by some basic interval of T different from itself. Suppose that there are $I_1, I_2 \in T$ such that $I_1 \cap I_2 = \{x_0\}$ and $I_1 \rightleftharpoons I_2$. If $J \in T$, then there is a path of length $l \leq k - 1$ starting at one of the intervals I_1, I_2 and ending at J .*

Proof. By Remark 10.5, $f(\sigma) = \sigma$. Set $K_i = f^i(I_1 \cup I_2)$ for $i \geq 0$. Notice that K_i is connected for all $i \in \mathbb{N}$ and $\text{Card}(K_1 \cap P) \geq 3$. Since P is a periodic orbit, from Remark 10.5 it follows that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_r = \sigma$ and $\text{Card}(K_i \cap P) \geq i + 2$ for $i < r$. From the fact that P has period k we get that $r + 1 \leq \text{Card}(K_{r-1} \cap P) \leq k$, and so $r \leq k - 1$. From the assumptions, for each basic interval $J_i \in T$, $J_i \subset K_i \setminus K_{i-1}$ there exists $J_{i-1} \in T$, $J_{i-1} \subset K_{i-1} \setminus K_{i-2}$ such that $J_{i-1} \rightarrow J_i$. Hence given $J \in T$ there is a path of length $l \leq r \leq k - 1$ starting at one of the intervals I_1 or I_2 and ending at J . \square

We denote by $2\mathbb{N}$ the set of all even natural numbers.

Lemma 11.6. *Let f be the standard σ map. Let $K \subset \mathbf{I}$ and $M, N \subset \mathbf{O}$ be closed subintervals such that $M \rightleftharpoons K \rightleftharpoons N$. Suppose that K has endpoints elements of $P \cup \{x_0\}$ and K does not contain any proper closed subinterval with*

endpoints in $P \cup \{x_0\}$ f -covering M and N . Assume that M and N have disjoint interiors and do not contain any proper closed subinterval f -covering K . Then Theorem 1.4(a) holds.

Proof. First suppose that at least one interval of $\{M, N, K\}$ does not contain x_0 . Then the subgraph $M \rightleftharpoons K \rightleftharpoons N$ give us a non-repetitive loop $J_0 = K \rightarrow J_1 \rightarrow \cdots \rightarrow J_n = K$ of length n for each n even such that $J_i = K$ if i is even; $J_1 = M$ and $J_i = N$ if $i > 1$ is odd and $x_0 \in N$; and $J_1 = N$ and $J_i = M$ if $i > 1$ is odd and $x_0 \in M$. By Lemma 3.3 there exists $x \in K$ such that $f^n(x) = x$ and $f^i(x) \in J_i$ for $1 \leq i \leq n-1$. Since x_0 does not belong to some interval of $\{M, N, K\}$, and the loop is non-repetitive, x has period n . So $2\mathbb{N} \subset \text{Per}(f)$. If k is even we are done, because $S(k) \subset \text{Per}(f)$. So suppose k odd. Since $K \subset I$ and K has endpoints elements of $P \cup \{x_0\}$, by Lemma 11.1 there exists a repetitive or non-repetitive loop of length k containing K . This loop together with the loops $M \rightleftharpoons K \rightleftharpoons N$ give us a non-repetitive loop of length n for each $n > k$ odd. Such a loop can be chosen in such a way that at least one of its intervals (non necessarily basic intervals) does not contain x_0 . Hence we get that $S(k) \subset \text{Per}(f)$.

Finally suppose that $x_0 \in M \cap N \cap K$. In particular, this means that $x_0 = 0$. So we get that $A \subset K$, $B \rightarrow A \leftarrow C$ and either $A \rightarrow B$ or $A \rightarrow C$. Without loss of generality we can assume that $B \rightleftharpoons A \leftarrow C$. Also we can suppose that there exists $z \in I$ such that $f(z) \in I$ or $z \in O$ such that $f(z) \in O$; otherwise $f(I) = O$ and $f(O) = I$ and, by Lemma 11.3, $S(k) \subset \text{Per}(f)$. Therefore since f is monotone on each basic interval of S' , there exists $J \in S$ such that $0 \notin J$ and either $J \rightarrow A \cup B$ or $J \rightarrow A \cup C$. From Remark 10.5 and Lemma 11.5, there is a path of length $l \leq k-1$ starting in one of the intervals A or B and ending in J . Then we consider the non-repetitive loops $A \rightleftharpoons B$, $A \rightarrow \cdots \rightarrow J \rightarrow A$ and either $B \rightarrow \cdots \rightarrow J \rightarrow B$ or $C \rightarrow \cdots \rightarrow J \rightarrow C$ of lengths either $2, l+1 \leq k, l+2 \leq k+1$ or $2, l+2 \leq k+1, l+1 \leq k$, be the rest of this section we will assume that $0 \notin P \cup \{x_0\}$ respectively. By Proposition 7.1 we obtain $B(k) \subset \text{Per}(f)$. \square

In the rest of the section we will assume that $0 \notin P \cup \{x_0\}$.

Lemma 11.7. *Let f be the standard σ map. Let $M \in S'$ be such that $f(0) \notin \text{Int}(M)$. If $\text{Int}(M)$ contains a periodic point of period $n > 1$, then there exists $L \in S'$, $L \neq M$ such that $L \rightarrow M$.*

Proof. From the hypotheses and since f is monotone in M we can choose a periodic point z of period n such that $z \in \text{Int}(M)$ and $f^{n-1}(z) \in L$ for some $L \in S'$, $L \neq M$. Since z has period n and f is monotone, we have $L \rightarrow M$. \square

Proposition 11.8. *Let f be the standard σ map. Suppose that there exists $J \in S$ such that $J \rightarrow J$ and J is not f -covered by any basic interval of T different from itself.*

- (a) *If $J \notin \{L_1, L_2, L_3\}$ then Theorem 1.4(a) holds.*
- (b) *If $J \in \{L_1, L_2, L_3\}$ then we assume that $f(0) \notin \text{Int}(J)$. Therefore Theorem 1.4(a) holds.*

Proof. Suppose that $J \notin \{L_1, L_2, L_3\}$. Then $J \in S'$. Since J is not f -covered by any $M \in T \setminus \{J\}$, from Lemma 10.7, J is not f -covered by any $M \in S \setminus \{J\}$, and consequently J is not f -covered by any $M \in S' \setminus \{J\}$. Therefore, from Remark 10.2 we get that $f(0) \notin \text{Int}(J)$.

We separate the proof into the following three cases.

Case 1: $J \subset \mathbf{O}$ and $J \neq L_3$. Then $J \in S'$. Since $f(0) \notin \text{Int}(J)$, by Lemma 11.7, $\text{Int}(J)$ does not contain periodic points of period $n > 1$. Since f is monotone on each basic interval of S' , $J \rightarrow J$, J is not f -covered by any basic interval of S' different from itself and $f(0) \notin \text{Int}(J)$, we have that $f(\sigma \setminus \text{Int}(J)) \cap \text{Int}(J) = \emptyset$. Then we consider the Y map $g = f|_{\sigma \setminus \text{Int}(J)} : \sigma \setminus \text{Int}(J) \rightarrow \sigma \setminus \text{Int}(J)$. Clearly g is well-defined, $k \in \text{Per}(g)$ and $\text{Per}(g) = \text{Per}(f)$. By the Y Theorem the result follows.

Case 2: $J \subset \mathbf{I}$. Then $J \in S'$. Since $J \rightarrow J$ there exists a fixed point $z \in J$ of f . Denote by D_1 and D_2 the closures of the two connected components of $\sigma \setminus \{z\}$ such that D_1 is homeomorphic to \mathbf{I} . Moreover since P is a periodic orbit it is not possible that $f(D_1) \subset D_1$. So, since $J \rightarrow J$, J is not f -covered by any $M \in S' \setminus \{J\}$, $f(0) \notin \text{Int}(J)$ and from Remark 10.2 we have that $f(D_1) = D_2$ and $f(D_2) = D_1$. Hence, by Lemma 11.3, statement (a) follows.

Case 3: $J \in \{L_1, L_2, L_3\}$. Then $f(0) \notin \text{Int}(J)$. Therefore, since $J \rightarrow J$ we have that A, B or C f -covers J . In particular, J is f -covered by a basic interval $L \in \{L_1, L_2, L_3\}$ different from itself. So $L \notin T$. Set $J = M_1 \cup M_2$, $L = M_1 \cup M_3$ where $M_1 \rightarrow M_1 \rightarrow M_2$ and $\{M_1, M_2, M_3\} = \{A, B, C\}$. Therefore $M_2 \cup M_3 \in T$. We claim that M_2 does not f -cover M_1 . Otherwise $M_2 \rightarrow M_1$, and since $f(0) \notin \text{Int}(J)$ M_2 also f -covers M_2 or M_3 and consequently $M_2 \cup M_3$ f -covers J or L . If $M_2 \cup M_3 \rightarrow J$ we have a contradiction with the fact that $M_2 \cup M_3 \in T$ and J is not f -covered by any $M \in T \setminus \{J\}$. Suppose that $M_2 \cup M_3 \rightarrow L$. Since $L \notin T$, from the definition of T we get that $T = S \setminus \{L\}$ and that L is not f -covered by any basic interval of S , in contradiction with $M_2 \cup M_3 \in T$ and $M_2 \cup M_3 \in L$. So the claim is proved.

Since the only basic intervals f -covering J are J and $L \notin T$, there are no basic intervals of S' f -covering $M_1 \in S'$ different from itself. So the proof of (b) follows in the same way as the proof of Cases 1 and 2. \square

Remark 11.9. In what follows, from Proposition 11.8, we can assume that if J is a basic interval of T such that either $J \subset \mathbf{O}$ and $J \neq L_3$, or $J \subset \mathbf{I}$, or $J \in \{L_1, L_2, L_3\}$ and $f(0) \notin \text{Int}(J)$, then J is f -covered by some basic interval of T different from itself.

Proposition 11.10. *Let f be the standard σ map. Suppose that each basic interval of T is f -covered by some basic interval of T different from itself. If there exists a basic interval $J \in S \cap S'$ f -covering itself, then Theorem 1.4(a) holds.*

Proof. By Lemma 11.4 there exists a non-repetitive loop of length n for each $n \geq k+1$ containing J . Since $J \in S \cap S'$, $0 \notin J$. Moreover, if $x_0 \in J$, by the monotonicity of f on each interval of S' , J does not f -cover any basic interval of T containing x_0 different from J . Then the loop verifies the hypotheses of Proposition 7.2 and the result follows. \square

Corollary 11.11. *Let f be the standard σ map. Suppose that each basic interval of T is f -covered by some basic interval of T different from itself and that $x_0 \in \{x_a, x_b, x_c\}$.*

(a) *If $f^{k-1}(p_e) \in \mathbf{I}$ then Theorem 1.4(a) holds.*

(b) *If $x_0 \neq x_a$ and $f(x_a) \in \mathbf{I}$ then Theorem 1.4(a) holds.*

Proof. First suppose that $f^{k-1}(p_e) \in \mathbf{I}$, then we claim that the subinterval $M = [p_e, f^{k-1}(p_e)]$ contains a fixed point z . Now we prove the claim. The interval $M \subset \mathbf{I}$ contains points of P (at least p_e and $f^{k-1}(p_e)$). Since $P \cap \mathbf{O} \neq \emptyset$, and $f(f^{k-1}(p_e)) = p_e$, it follows that $M \rightarrow M$. So there is a fixed point in M , and the claim is proved. Notice that $z \neq x_0$ because $x_0 \in \{x_a, x_b, x_c\}$. Therefore, from the monotonicity of f , the basic interval of $S \cap S'$ containing z f -covers itself. By Proposition 11.10 the result follows.

Finally suppose that $x_0 \neq x_a$ and $f(x_a) \in \mathbf{I}$. Now we get that the interval $M = [p_e, x_a]$ contains a fixed point z . Since $x_0 \in \{x_b, x_c\} \subset \mathbf{O}$, we have that $z \neq x_0$. Thus the proof follows as before. \square

Lemma 11.12. *Let f be the standard σ map. If there exists a basic interval $J \in S'$, $J \subset \mathbf{O}$ such that there are no basic intervals of S' different from J f -covering J , and $f(0) \notin \text{Int}(J)$, then Theorem 1.4(a) holds.*

Proof. Since $f(0) \notin \text{Int}(J)$, f is monotone on each basic interval of S' , and there are no basic intervals of S' different from J f -covering J , we can consider the map $g : \sigma \setminus \text{Int}(J) \rightarrow \sigma \setminus \text{Int}(J)$ defined as f restricted to $\sigma \setminus \text{Int}(J)$. Clearly g is well-defined and g is either an \mathbf{I} map or a \mathbf{Y} map. By Lemma 11.7 $\text{Int}(J)$ does not contain periodic points of period $n > 1$. So $\text{Per}(g) = \text{Per}(f)$. Moreover P is a periodic orbit for g . Hence from the Interval Theorem or the \mathbf{Y} Theorem the result follows. \square

Lemma 11.13. *Let f be the standard σ map. Suppose that each basic interval of T is f -covered by some different basic interval of T and $x_0 \in \{x_a, x_b, x_c\}$. Denote by I_1 the basic interval of $S \cap S'$ with x_0 as endpoint. Let $\{i, j\} \subset \{1, 2, 3\}$ be with $i \neq j$ such that $L_i \cap L_j = \{x_0\}$. Assume that there is $J \in S \cap S' \setminus \{I_1\}$ such that one of the following subgraphs is satisfied:*

- (a) $J \rightarrow I_1 \rightleftharpoons L_i \leftarrow J$;
- (b) $J \rightarrow L_j \rightarrow I_1 \leftarrow J$ and $I_1 \rightleftharpoons L_i$.

Then Theorem 1.4(a) holds.

Proof. Clearly $I_1 \cap L_i = \{x_0\}$. Then by Lemma 11.5 there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or L_i and ending at J . Suppose that the path starts at I_1 . Then if (a) is satisfied we consider the loops $I_1 \rightarrow \cdots \rightarrow J \rightarrow I_1$, $I_1 \rightarrow \cdots \rightarrow J \rightarrow L_i \rightarrow I_1$ and $I_1 \rightleftharpoons L_i$ of lengths $l+1 \leq k$, $l+2 \leq k+1$ and 2 respectively. If (b) is satisfied, then we consider the loops $I_1 \rightarrow \cdots \rightarrow J \rightarrow I_1$, $I_1 \rightarrow \cdots \rightarrow J \rightarrow L_j \rightarrow I_1$ and $I_1 \rightleftharpoons L_i$ of lengths $l+1 \leq k$, $l+2 \leq k+1$ and 2 respectively. Suppose that the path starts at L_i . Then if (a) is satisfied we consider the loops $L_i \rightarrow \cdots \rightarrow J \rightarrow L_i \rightarrow I_1 \rightarrow L_i$, $L_i \rightarrow \cdots \rightarrow J \rightarrow I_1 \rightarrow L_i$ and $I_1 \rightleftharpoons L_i$ of lengths $l+3 \leq k+2$, $l+2 \leq k+1$ and 2 respectively. If (b) is satisfied, then we consider the loops $L_i \rightarrow \cdots \rightarrow J \rightarrow I_1 \rightarrow L_i$, $L_i \rightarrow \cdots \rightarrow J \rightarrow L_j \rightarrow I_1 \rightarrow L_i$ and $I_1 \rightleftharpoons L_i$ of lengths $l+2 \leq k+1$, $l+3 \leq k+2$ and 2 respectively. In all cases we obtain a non-repetitive loop of length n for each $n \geq k+1$ containing I_1 and J . Notice that $0 \notin I_1$ and $x_0 \notin J$. Then by the construction of this loop of length n and Proposition 7.2 the result follows. \square

Lemma 11.14. *Let f be the standard σ map. Suppose that $x_0 \in \{x_b, x_c\}$. Denote by I_1 the basic interval of $S \cap S'$ with x_0 as endpoint. Set $D_1 = [p_e, x_a] \subset \mathbf{I}$ and $D_2 = \text{Cl}(\mathbf{O} \setminus (B \cup C \cup I_1))$. Suppose that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$. Then Theorem 1.4(a) holds.*

Proof. Notice that $P \subset D_1 \cup D_2$. Since $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$ k must be even. Consider the interval map $g = f^2|_{D_1} : D_1 \rightarrow D_1$. Clearly the points of $D_1 \cap P$ have period $k/2$ under g . From the Interval Theorem it follows that $S(k/2) \subset \text{Per}(g)$. Since $1 \in \text{Per}(f)$, again from the fact that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$ we get that $S(k) \subset \text{Per}(f)$. \square

Lemma 11.15. *Let f be the standard σ map. Suppose that there exist two closed subintervals $K \subset \mathbf{I}$ and $N \subset \mathbf{O}$ such that K has endpoints elements of $P \cup \{x_0\}$, and $K \rightleftharpoons N$. If k is odd then $n \in \text{Per}(f)$ for each $n > k$ odd.*

Proof. Since $K \subset \mathbf{I}$ has endpoints elements of $P \cup \{x_0\}$, by Lemma 11.1 there is a loop of length k odd containing K . This loop of length k together with the loop $K \rightleftharpoons N$ give us a non-repetitive loop of length n for each $n > k$ odd. By the hypotheses $x_0 \notin K \cap N$, then we obtain that $n \in \text{Per}(f)$ for all $n > k$ odd. \square

12. MAPS WITH $0 \in P \cup \{x_0\}$

In this section we will prove Theorem 1.4(a) when $x_0 = 0$ or $0 \in P$. Then, under these assumptions $S = S' = T$.

Proposition 12.1. *Let f be the standard σ map. Suppose that $0 \in P \cup \{x_0\}$ and that there exists $J \in S'$ such that the unique basic interval of S' f -covering J is itself. Then Theorem 1.4(a) holds.*

Proof. We consider the two possibilities either $J \subset \mathbf{O}$ or $J \subset \mathbf{I}$. If $J \subset \mathbf{O}$, then we define the map $g = f|_{\sigma \setminus \text{Int}(J)} : \sigma \setminus \text{Int}(J) \rightarrow \sigma \setminus \text{Int}(J)$. The map g is well-defined because the unique basic interval of S f -covering J is itself and f is monotone on each basic interval of S' . Clearly $k \in \text{Per}(g)$ and since f is monotone in J , $\text{Per}(g) = \text{Per}(f)$. Moreover g is either an \mathbf{I} map or an \mathbf{Y} map. By the Interval Theorem or the \mathbf{Y} Theorem, the result follows.

Suppose that $J \subset \mathbf{I}$. By Lemma 3.2 f has a fixed point $y \in J$. Denote by D_1 and D_2 the closures of the two connected components of $\sigma \setminus \{y\}$ such that D_1 is homeomorphic to \mathbf{I} . Since $J \subset \mathbf{I}$ and the unique interval of S f -covering J is itself, we obtain easily $f(D_1) = D_2$ and $f(D_2) = D_1$. By Lemma 11.3 Theorem 1.4(a) holds. \square

Remark 12.2. In what follows we can assume that if $0 \in P \cup \{x_0\}$, then each basic interval of S is f -covered by some basic interval of S different from itself.

Lemma 12.3. *Let f be the standard σ map. Suppose that $0 \in P \cup \{x_0\}$. If there is a basic interval J f -covering itself, then Theorem 1.4(a) follows.*

Proof. By Remark 12.2 there exists a basic interval $M \in S$ different from J such that $M \rightarrow J$. By Lemma 11.4 there is a non-repetitive loop of length n for each $n \geq k + 1$ containing J . We claim that at least one basic interval of this loop does not contain x_0 . In fact, if $x_0 \in J$, by the monotonicity of f on each basic interval of S' , we have that $f(J)$ does not contain intervals of S different from J with x_0 as endpoint. By Proposition 7.1 we get $B(k) \subset \text{Per}(f)$ and the result follows. \square

Proposition 12.4. *Let f be the standard σ map. If $x_0 = 0$ then Theorem 1.4(a) holds.*

Proof. If there is a basic interval J f -covering itself, by Lemma 12.3, the result follows. So we can suppose that each basic interval does not f -cover itself. By Remark 9.6, there exists a closed subinterval K of σ such that K f -covers \mathbf{O} . We can assume that K has endpoints elements of $P \cup \{x_0\}$. Since $f(0) = 0$ and there are no basic intervals f -covering themselves, we get that $K \subset \mathbf{I}$. If $f^{k-1}(p_e) \in \mathbf{I}$, then the interval $[p_e, f^{k-1}(p_e)]$ contains a fixed point $z \neq 0$. Therefore, by the monotonicity of f , the basic interval of S containing z f -covers itself, in contradiction with the fact that there are no basic intervals f -covering themselves. Hence $f^{k-1}(p_e) \in \mathbf{O}$. So there are two closed subintervals $M, N \subset \mathbf{O}$ f -covering K in the hypotheses of Lemma 11.6. So the result follows. \square

From now on we can assume that $x_0 \neq 0$.

Proposition 12.5. *Let f be the standard σ map. Suppose that $0 \in P$, then Theorem 1.4(a) holds.*

Proof. If there is some basic interval f -covering itself, from Lemma 12.3 the result follows. So we can assume that each basic interval does not f -cover itself. Denote by I_1, I_2 the two basic intervals having x_0 as endpoint. We have $I_1 \rightleftharpoons I_2$. Notice that a basic interval $J \notin \{I_1, I_2\}$ f -covers I_1 if and only if f -covers I_2 .

First suppose that there are no basic intervals f -covering I_1 and I_2 . If $x_0 \in \mathbf{I}$, then denote by D_1 and D_2 the closures of the two connected components of $\sigma \setminus \{x_0\}$ such that D_1 is homeomorphic to \mathbf{I} . Hence $f(D_1) = D_2$ and $f(D_2) = D_1$. From Lemma 11.3 the result follows. If $x_0 \in \mathbf{O}$, then consider either the \mathbf{I} map or the \mathbf{Y} map g defined by f restricted to $\sigma \setminus \text{Int}(I_1 \cup I_2)$. Clearly $k \in \text{Per}(g)$ and $\text{Per}(g) = \text{Per}(f)$. So by the Interval Theorem or the \mathbf{Y} Theorem the result follows.

Finally suppose that there is a basic interval J such that $I_1 \leftarrow J \rightarrow I_2$. By Lemma 11.5 there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or I_2 and ending at J . By construction, this path together with $I_1 \leftarrow J \rightarrow I_2$ and $I_1 \rightleftharpoons I_2$ give us a non-repetitive loop of length n for each $n \geq k+1$. Since $x_0 \notin J$ from Proposition 7.1 it follows that $B(k) \subset \text{Per}(f)$. \square

From now on we can assume that $0 \notin P \cup \{x_0\}$.

Remark 12.6. Suppose that there is a basic interval $J = [0, x_i] \in S'$ for some $x_i \in P \cup \{x_0\}$ such that $f(0) = f(x_i)$. Notice that from the monotonicity of f on each basic interval of S' , f is constant in J . Then we consider the new topological space σ' homeomorphic to σ obtained by shrinking the interval J to the point 0, and define the σ' map g in the natural way as the σ map $f|_{\sigma \setminus \text{Int}(J)}$. So P is a periodic orbit of period k for g and clearly $\text{Per}(g) = \text{Per}(f)$. Now $0 \in P \cup \{x_0\}$. From Propositions 12.4 and 12.5 the result follows. Hence from now on we can assume that $0 \notin P \cup \{x_0\}$ and $f(0) \notin \{f(x_a), f(x_b), f(x_c)\}$.

13. MAPS WITH $f(0) \notin \text{Int}(A \cup B \cup C)$

In this section we prove Theorem 1.4(a) when $f(0) \notin \text{Int}(A \cup B \cup C)$. Notice that from now on 0 can be a periodic or non-periodic point, and if it is periodic it does not belong to P .

Proposition 13.1. *Let f be the standard σ map. Suppose that $f(0) \notin \text{Int}(A \cup B \cup C)$ and $x_0 \notin \{x_a, x_b, x_c\}$. Then Theorem 1.4(a) holds.*

Proof. Let I_1, I_2 be the two basic intervals of S with x_0 as endpoint. Such intervals exist in virtue of Remark 6.4. Moreover $I_1, I_2 \in S \cap S'$ because $x_0 \notin \{x_a, x_b, x_c\}$. By hypotheses $f(0) \notin \text{Int}(A \cup B \cup C)$, hence $f(0) \notin \text{Int}(L_1 \cup L_2 \cup L_3)$. Therefore from Remark 11.9 it follows that each basic interval of T is f -covered by some basic interval different from itself. If $I_i \rightarrow I_i$ for some $i = 1, 2$, by Proposition 11.10 the result follows. So assume that I_i does not f -cover itself for $i = 1, 2$. Then we get the loop $I_1 \rightleftharpoons I_2$.

First suppose that $x_0 \in \mathbf{I}$. Denote by D_1 and D_2 the closure of the two connected components of $\sigma \setminus \{x_0\}$ such that D_1 is homeomorphic to \mathbf{I} . If $f(D_1) = D_2$ and $f(D_2) = D_1$ by Lemma 11.3 the result follows. Otherwise, by Remark 10.2, there exists $y \in D_i \cap P$ such that $f(y) \in D_i \cap P$ for some $i = 1, 2$. Then since $I_1 \rightleftharpoons I_2$, by the continuity of f , there exists a basic interval $J \in S$ f -covering I_1 and I_2 . By Lemma 10.7, we can assume that $J \in T$. From the monotonicity of f on I_1 and I_2 we get that $J \notin \{I_1, I_2\}$.

Now suppose $x_0 \in \mathbf{O}$. If there are no basic intervals of S different from I_1 and I_2 f -covering I_1 or I_2 , then we consider the Y map g defined by f restricted to $\sigma \setminus \text{Int}(I_1 \cup I_2)$. Clearly g is well-defined, $k \in \text{Per}(g)$ and $\text{Per}(g) \subset \text{Per}(f)$. From the Y Theorem the result follows. So suppose that there exists $J \in S$, $J \notin \{I_1, I_2\}$ f -covering I_1 or I_2 . Since $I_1 \cap I_2 = \{x_0\}$, if $f(0) \notin I_1 \cup I_2$ then $I_1 \leftarrow J \rightarrow I_2$; if $f(0) \in I_1 \cup I_2$ then, by Remark 10.2 we obtain that $I_1 \leftarrow J \rightarrow I_2$. Furthermore, from Lemma 10.7, we can assume that $J \in T$.

Finally we consider the two possibilities either $x_0 \in \mathbf{I}$ or $x_0 \in \mathbf{O}$ simultaneously. Since we are in the hypotheses of Lemma 11.5, there is a path of length $l \leq k-1$ starting at one of the intervals I_1, I_2 and ending at J . Without loss of generality we can assume that the loop starts at I_1 . So we consider the loops $I_1 \rightarrow \cdots \rightarrow J \rightarrow I_1$ and $I_1 \rightarrow \cdots \rightarrow J \rightarrow I_2 \rightarrow I_1$ of lengths $l+1 \leq k$ and $l+2 \leq k+1$ respectively. By construction (see the proof of Lemma 11.5), these loops are formed by different intervals. The two above loops together with the loop $I_1 \rightleftharpoons I_2$ give us a loop of length n for each $n \geq k+1$ containing J and also containing I_1 or I_2 . Since $J \notin \{I_1, I_2\}$, we have $x_0 \notin J$. Furthermore $0 \notin I_1 \cup I_2$. By construction of the loop of length n , the associated sequence of W_i 's in Proposition 7.2 is non-repetitive. Hence from Proposition 7.2 we get that $B(k) \subset \text{Per}(f)$. \square

Proposition 13.2. *Let f be the standard σ map. Suppose that $f(0) \notin \text{Int}(A \cup B \cup C)$ and $x_0 \in \{x_a, x_b, x_c\}$. Then Theorem 1.4(a) holds.*

Proof. Denote by $I_1 = [x_0, x_1]$ the basic interval of $S \cap S'$ with x_0 as endpoint. By hypotheses $f(0) \notin \text{Int}(A \cup B \cup C)$, hence $f(0) \notin \text{Int}(L_1 \cup L_2 \cup L_3)$. Therefore from Remark 11.9 it follows that each basic interval of T is f -covered by some basic interval different from itself. By Proposition 11.10 we can assume that each $J \in S \cap S'$ does not f -cover itself. In particular, I_1 does not f -cover itself. From Corollary 11.11(a) we can suppose that $f^{k-1}(p_e) \in \mathbf{O}$ and $f(x_a) \in \mathbf{O}$.

If $x_0 = x_a$, by Remark 6.2, we have $f(0) \in \mathbf{I}$. Since I_1 does not f -cover itself we get that $I_1 \rightarrow A$. Therefore either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_2$. Without loss of generality we can assume that $I_1 \rightarrow L_1$. Denote by D_1 and D_2 the closures of the two connected components of $\sigma \setminus \{x_0\}$ such that D_1 is homeomorphic

to I . If $f(D_1) = D_2$ and $f(D_2) = D_1$, by Lemma 11.3 the result follows. Otherwise, since $f(0) \notin \text{Int}(A \cup B \cup C)$, $x_0 = x_a$ and from Remark 12.6 $f(0) \neq x_a$, it follows that there exists $z \in P \cap D_i$ such that $f(z) \in P \cap D_i$ for some $i \in \{1, 2\}$. By continuity of f , there exists $J \in S \cap S'$ such that $I_1 \leftarrow J \rightarrow A$ and so either $I_1 \leftarrow J \rightarrow L_1$ or $I_1 \leftarrow J \rightarrow L_2$.

By symmetry we only need to consider the following three cases.

Case 1: $x_0 = x_a$ and $f(0) \notin \text{Int}(I_1)$. Since $f(0) \in I$, we get $A \rightarrow I_1$, and so $L_1 \rightarrow I_1$ and $L_2 \rightarrow I_1$. Then we obtain the loop $I_1 \rightleftharpoons L_1$. Since $I_1 \cap L_1 = \{x_0\}$, by Lemma 11.5 there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or L_1 and ending at J .

First suppose that $I_1 \leftarrow J \rightarrow L_1$. Then we obtain the loops either $I_1 \rightleftharpoons L_1$, $I_1 \rightarrow \dots \rightarrow J \rightarrow I_1$ and $I_1 \rightarrow \dots \rightarrow J \rightarrow L_1 \rightarrow I_1$, or $I_1 \rightleftharpoons L_1$, $L_1 \rightarrow \dots \rightarrow J \rightarrow L_1$ and $L_1 \rightarrow \dots \rightarrow J \rightarrow I_1 \rightarrow L_1$ of lengths 2, $l+1 \leq k$ and $l+2 \leq k+1$ respectively. Then we get a loop of length n for each $n \geq k+1$ containing J . Notice that $\{x_0, 0\} \cap J = \emptyset$. By construction of the loop of length n , the associated sequence of W_i 's of Proposition 7.2 is non-repetitive. Hence $B(k) \subset \text{Per}(f)$.

Now suppose that $I_1 \leftarrow J \rightarrow L_2$. By Lemma 11.5 there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or L_1 and ending at J . If the path starts at I_1 , then we get the loops $I_1 \rightleftharpoons L_1$, $I_1 \rightarrow \dots \rightarrow J \rightarrow I_1$ and $I_1 \rightarrow \dots \rightarrow J \rightarrow L_2 \rightarrow I_1$ of lengths 2, $l+1 \leq k$ and $l+2 \leq k+1$ respectively. If the path starts at L_1 , then we obtain the loops $I_1 \rightleftharpoons L_1$, $L_1 \rightarrow \dots \rightarrow J \rightarrow I_1 \rightarrow L_1$ and $L_1 \rightarrow \dots \rightarrow J \rightarrow L_2 \rightarrow I_1 \rightarrow L_1$ of lengths 2, $l+2 \leq k+1$ and $l+3 \leq k+2$ respectively. As above, Proposition 7.2 allows us to prove that $B(k) \subset \text{Per}(f)$.

Case 2: $x_0 = x_a$ and $f(0) \in \text{Int}(I_1)$.

Suppose that $L_1 \rightarrow L_1$. By Lemma 11.4 there is a loop of length n for each $n \geq k+1$ containing L_1 . Since there exists $J \in S \cap S'$ such that $J \rightarrow I_1 \rightarrow L_1$, from Lemma 11.4 we can assume that it contains the path $J \rightarrow I_1 \rightarrow L_1$. Since $\{x_0, 0\} \cap J = \emptyset$, by the above construction the loop verifies the hypotheses of Proposition 7.2. So $B(k) \subset \text{Per}(f)$. Hence in the rest of the proof of Case 2 we can assume that L_1 does not f -cover L_1 .

Subcase (i). $L_1 \rightarrow I_1$. Then, since $f(0) \in \text{Int}(I_1)$, $f(x_b) \in I$. We have the loop $I_1 \rightleftharpoons L_1$. Suppose that $I_1 \leftarrow J \rightarrow L_1$. So by Lemma 11.5 there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or L_1 and ending at J . This path together with the loop $I_1 \rightleftharpoons L_1$ give us a loop of length n for each $n \geq k+1$. Notice that $\{x_0, 0\} \cap J = \emptyset$. By construction the above loop satisfies the hypotheses of Proposition 7.2 and consequently $B(k) \subset \text{Per}(f)$.

So we can assume that there are no basic intervals of $S \cap S'$ f -covering $I_1 \cup L_1$. Therefore we have $I_1 \leftarrow J \rightarrow L_2$. In particular $L_2 \in T$. Suppose that $L_2 \rightarrow I_1$. Then by Lemma 11.5 we obtain the loops either $I_1 \rightleftharpoons L_1$, $I_1 \rightarrow \dots \rightarrow J \rightarrow I_1$ and $I_1 \rightarrow \dots \rightarrow J \rightarrow L_2 \rightarrow I_1$, or $I_1 \rightleftharpoons L_1$, $L_1 \rightarrow \dots \rightarrow J \rightarrow I_1 \rightarrow L_1$ and $L_1 \rightarrow \dots \rightarrow J \rightarrow L_2 \rightarrow I_1 \rightarrow L_1$ of lengths either 2, $l+1 \leq k$ and $l+2 \leq k+1$, or 2, $l+2 \leq k+1$ and $l+3 \leq k+2$ respectively. Since $\{x_0, 0\} \cap J = \emptyset$ and by construction from these loops we can obtain a loop of length n for each $n \geq k+1$ in the hypotheses of Proposition 7.2. So the result follows.

Hence we can suppose that L_2 does not f -cover I_1 . Therefore $f(x_c) \in O$ and either $C \rightarrow L_1$ or $C \rightarrow L_2$ because $f(0) \in \text{Int}(I_1)$. Suppose that $C \rightarrow L_2$,

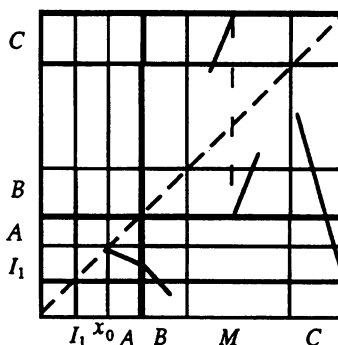


FIGURE 13.1. In Subcase (i) there is $M \in S \cap S'$, $M \subset \mathbf{O}$ such that $M \rightarrow L_3$.

then L_2 f -covers itself. Consider the path $J \rightarrow L_2$. By Lemma 11.4 there exists a non-repetitive loop of length n for each $n \geq k+1$ containing L_2 . By construction of the above loop we can assume that it contains the path $J \rightarrow L_2$. Since $\{x_0, 0\} \cap J = \emptyset$ and the associated sequence of the W'_i 's of Proposition 7.2 is non-repetitive, we get that $B(k) \subset \text{Per}(f)$. Hence we can assume that C does not f -cover L_2 . Therefore $C \rightarrow L_1$ and so $L_3 \rightarrow L_1 \cup I_1$ because $f(x_b) \in \mathbf{I}$. Since $f(x_b) \in \mathbf{I}$, $f(x_c) \in \mathbf{O}$, each basic interval of $S \cap S'$ does not f -cover itself, there are no basic intervals of $S \cap S'$ f -covering $L_1 \cup I_1$ and from the continuity of f , it follows that there exists $M \in S \cap S'$, $M \subset \mathbf{O}$ such that $M \rightarrow L_3$ (see Figure 13.1). In particular $L_3 \in T$. Therefore consider the paths $I_1 \leftarrow L_3 \rightarrow L_1 \rightleftharpoons I_1$. By Lemma 11.5 there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or L_1 and ending at L_3 . So we obtain either the loops $I_1 \rightleftharpoons L_1$, $I_1 \rightarrow \dots \rightarrow L_3 \rightarrow I_1$ and $I_1 \rightarrow \dots \rightarrow L_3 \rightarrow L_1 \rightarrow I_1$, or $I_1 \rightleftharpoons L_1$, $L_1 \rightarrow \dots \rightarrow L_3 \rightarrow L_1$ and $L_1 \rightarrow \dots \rightarrow L_3 \rightarrow I_1 \rightarrow L_1$ of lengths 2, $l+1 \leq k$ and $l+2 \leq k+1$ respectively. So we obtain a loop of length n for each $n \geq k+1$ containing I_1 and L_3 . By construction of this loop and from the facts that $0 \notin I_1$ and $x_0 \notin L_3$, the associated sequence of W'_i 's of Proposition 7.2 is non-repetitive. Hence $B(k) \subset \text{Per}(f)$.

Subcase (ii). $L_1 \rightarrow I_1$. So, since $x_b \in P$ we have that $f(x_b) \in \mathbf{O}$. By Remark 10.2 we get that $f(x_c) \in \mathbf{I}$. From the facts that there exists a closed subinterval $K \subset \sigma$ f -covering \mathbf{O} (see Remark 9.6), each basic interval of $(S \cap S') \cup \{L_1\}$ does not f -cover itself, $f(x_c) \in \mathbf{I}$ and $f(0) \in \mathbf{I}$, it follows that $K \subset \mathbf{I}$ (see Figure 13.2). Moreover we can assume that K has endpoints elements of $P \cup \{x_0\}$.

Since $f(x_b) \in \mathbf{O}$ and $f^{k-1}(p_e) \in \mathbf{O}$, there is a closed subinterval $M \subset \mathbf{O}$ such that $M \rightleftharpoons K$.

Let $p_0 \in I_1$ be such that $f(p_0) = 0$. Consider $K' \subset K$ a minimal closed subinterval f -covering \mathbf{O} . Then $K' \subset [p_e, p_0]$. Since $f^{k-1}(p_e) \in \mathbf{O}$, there are two minimal closed subintervals $M', N' \subset \mathbf{O} \setminus B$ f -covering $[p_e, f(0)]$ such that M' and N' have pairwise disjoint interiors. We have $M' \leftarrow K' \rightarrow N'$.

First suppose that $p_0 \in [p_e, f(0)]$. Then, since $K' \subset [p_e, p_0]$, we get that $M' \rightleftharpoons K \rightleftharpoons N'$. Thus we obtain a non-repetitive loop of length n for each n even. Since $x_0 \notin M' \cup N' \cup K'$ and $0 \notin K'$ we get that $2\mathbb{N} \subset \text{Per}(f)$. If k is even, then we are done because $S(k) \subset \text{Per}(f)$. So we suppose that k is odd.

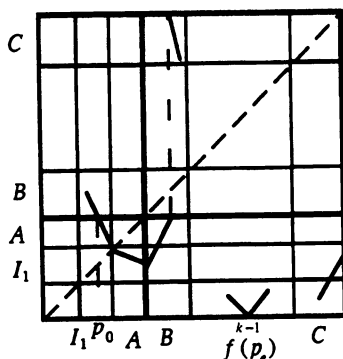


FIGURE 13.2. In Subcase (ii) the graph of f has only one fixed point, x_0 .

Since $x_0 \notin M_1$, from Lemma 11.15 we get that $n \in \text{Per}(f)$ for each $n > k$ odd. Therefore $S(k) \subset \text{Per}(f)$ and the result follows.

Finally suppose that $p_0 \notin [p_e, f(0)]$. Then we have $[p_0, 0] \rightarrow [p_0, 0] \rightarrow [f(0), p_0]$ (see Figure 13.3) and $B \rightarrow [p_0, 0]$ (see Figure 13.2). Assume that $f^2(0) \notin \text{Int}(B)$. Then, since $I_1 \rightarrow L_1$, again from Figure 13.2 we have that $[f(0), p_0] \rightarrow B$. So the loops $[p_0, 0] \rightarrow [p_0, 0] \rightarrow [f(0), p_0] \rightarrow B \rightarrow [p_0, 0]$ give us a non-repetitive loop of length n for each $n \geq k+1$. Hence $B(k) \subset \text{Per}(f)$. Now assume that $f^2(0) \in \text{Int}(B)$. Then, since $M' \cup N' \subset \mathbf{O} \setminus B$ we obtain $M' \rightleftharpoons K' \rightleftharpoons N'$. Notice that $x_0 \notin M' \cup N' \cup K'$. By the above arguments $S(k) \subset \text{Per}(f)$.

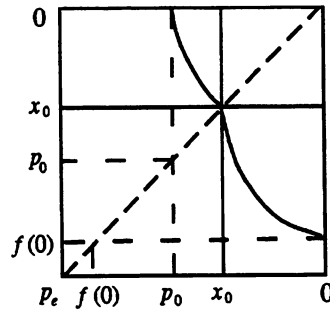
Case 3: $x_0 = x_b$. From Corollary 11.11(b) we can suppose that $f(x_a) \in \mathbf{O}$. By Remark 6.2 we have that $f(0) \in \mathbf{O}$. Since $I_1 \nrightarrow I_1$, we get $I_1 \rightarrow B$. So either $I_1 \rightarrow B \cup A = L_1$ or $I_1 \rightarrow B \cup C = L_3$.

Subcase (i). Suppose that $C \rightarrow C$. Therefore, since $f(0) \notin \text{Int}(A \cup B \cup C)$, we obtain either $C \rightarrow A$ or $C \rightarrow B$. Consequently either $C \rightarrow L_2$ or $C \rightarrow L_3$. Therefore at least one of the following statements is satisfied:

- (a) $C \rightarrow L_2$ and $L_3 \rightarrow L_2 \rightarrow L_2$,
- (b) $C \rightarrow L_3$ and $L_2 \rightarrow L_3 \rightarrow L_3$.

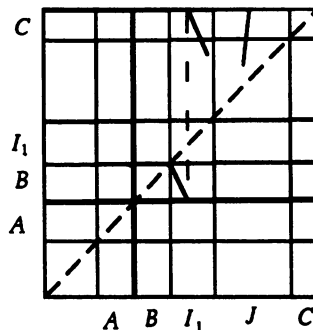
First assume that (a) holds. If there exists $J \in S \cap S'$ such that $J \rightarrow L_2$ or $J \rightarrow L_3$, then we consider the path $J \rightarrow L_2 \rightarrow L_2$ or $J \rightarrow L_3 \rightarrow L_2 \rightarrow L_2$. Given that each basic interval of T is f -covered by some different basic interval, by Lemma 11.4 there is a non-repetitive loop of length n for each $n \geq k+1$ containing L_2 . Furthermore we can assume that the above loop of length n also contains J . Clearly $0 \notin J$ and $x_0 \notin L_2$. By Proposition 7.2 the result follows.

Therefore we suppose that for each $J \in S \cap S'$, $J \nrightarrow L_2$ and $J \nrightarrow L_3$. Consequently, since either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_3$, we get that $I_1 \rightarrow L_1$. Since $f(0) \notin \text{Int}(C)$, from Lemma 11.12 we can assume that there is a basic interval $J_0 \in S'$ f -covering C . Taking into account that $C \subset L_2 \cap L_3$, and that for each $J \in S \cap S'$, $J \nrightarrow L_2$ and $J \nrightarrow L_3$, we get that $J_0 \in \{A, B\}$. In particular $L_1 \rightarrow C$. Hence $L_1 \rightarrow L_2$ or $L_1 \rightarrow L_3$. Then consider the path $I_1 \rightarrow L_1 \rightarrow L_2 \rightarrow L_2$ or $I_1 \rightarrow L_1 \rightarrow L_3 \rightarrow L_2 \rightarrow L_2$. By Lemma 11.4 there is a non-repetitive loop of length n for each $n \geq k+1$ containing L_2 and I_1 . Notice that $0 \notin I_1$ and $x_0 \notin L_2$. Then by Proposition 7.2 the result follows.

FIGURE 13.3. The graph of f when $p_0 \notin [p_e, f(0)]$.

Finally suppose that (b) holds. Since $C \rightarrow C \cup B$, by the monotonicity of f on C we get that $f(x_c) \in \mathbf{O}$. Notice that either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_3$, and so $I_1 \nrightarrow L_2$. If there is a basic interval $J \in S \cap S' \setminus \{I_1\}$ such that either $J \rightarrow L_2$ or $J \rightarrow L_3$, then we consider either the path $J \rightarrow L_2 \rightarrow L_3 \rightarrow L_3$ or the path $J \rightarrow L_3 \rightarrow L_3$. Clearly $\{0, x_0\} \cap J = \emptyset$. Therefore by Lemma 11.4 and Proposition 7.2 the result follows.

Thus we can assume that L_2 is not f -covered by any basic interval of $S \cap S'$ and L_3 is not f -covered by any basic interval of $S \cap S' \setminus \{I_1\}$. If $I_1 \rightarrow L_3$, from the fact that each basic interval of $S \cap S'$ does not f -cover itself and by the continuity of f , we have that there exists $J \in S \cap S' \setminus \{I_1\}$, $J \subset \mathbf{O}$ f -covering C (see Figure 13.4). Consequently either $J \rightarrow L_2$ or $J \rightarrow L_3$ in contradiction with the assumptions. Hence we can suppose that $I_1 \nrightarrow L_3$. Therefore $I_1 \rightarrow L_1$. In particular $I_1 \nrightarrow C$. By Lemma 11.12 we can assume that there is $J_0 \in S'$ f -covering C . Since for each $J \in S \cap S'$, $J \nrightarrow L_2$ and $J \nrightarrow L_3$, we obtain that $J_0 \in \{A, B\}$. Thus $L_1 \rightarrow L_2$ or $L_1 \rightarrow L_3$. From the facts that $f^{k-1}(p_e) \in \mathbf{O}$, $f(x_c) \in \mathbf{O}$ and by the continuity of f , it follows that there is $J \in S \cap S'$, $J \neq I_1$ such that $J \rightarrow A$. Moreover either $J \rightarrow L_1$ or $J \rightarrow L_2$. By the above assumptions we have that $J \rightarrow L_1$. Notice that $\{0, x_0\} \cap J = \emptyset$. Consider the loops $J \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow L_3$ or $J \rightarrow L_1 \rightarrow L_3 \rightarrow L_3$. From Lemma 11.4 and Proposition 7.2 the result follows.

FIGURE 13.4. The graph of f cannot cross the diagonal in $\mathbf{O} \setminus (B \cup C)$.

Subcase (ii). Suppose that $C \nrightarrow C$. First assume that $B \rightarrow B$. Since $f(0) \in \mathbf{O}$, $f(0) \notin \text{Int}(A \cup B \cup C)$ and f is monotone in B , it follows that $B \rightarrow B \cup C = L_3$, which implies that $L_1 \rightarrow L_3 \rightarrow L_3$. If there is $J \in S \cap S' \setminus \{I_1\}$ such that either $J \rightarrow L_1$ or $J \rightarrow L_3$, then we consider either the path $J \rightarrow L_1 \rightarrow L_3 \rightarrow L_3$ or the path $J \rightarrow L_3 \rightarrow L_3$. Clearly $\{0, x_0\} \cap J = \emptyset$. So by Lemma 11.4 and Proposition 7.2 the result follows. Hence we can assume that for each $J \in S \cap S' \setminus \{I_1\}$, $J \nrightarrow L_1$ and $J \nrightarrow L_3$. So we claim that $f(x_c) \in \mathbf{I}$. Otherwise, $f(x_c) \in \mathbf{O}$, and since any basic interval of $S \cap S'$ cannot f -cover itself, $f^{k-1}(p_e) \in \mathbf{O}$ and by the continuity of f , there exists $J \in S \cap S' \setminus \{I_1\}$ such that $J \rightarrow B$. Therefore either $J \rightarrow A \cup B = L_1$ or $J \rightarrow B \cup C = L_3$, in contradiction with the assumptions. Then the claim is proved.

Now, from the facts that $f(0) \in \mathbf{O}$, $f(x_c) \in \mathbf{I}$, $f(0) \notin \text{Int}(A \cup B \cup C)$ and $C \nrightarrow C$, we get that $C \rightarrow L_1$. Therefore $L_2 \rightarrow L_1$ and $L_3 \rightarrow L_1$. Then we have $L_2 \rightarrow L_1 \rightleftharpoons L_3 \rightarrow L_3$. If there is $J \in S \cap S' \setminus \{I_1\}$ such that $J \rightarrow L_2$, then we consider the path $J \rightarrow L_2 \rightarrow L_1 \rightarrow L_3 \rightarrow L_3$. Since $\{x_0, 0\} \cap J = \emptyset$, from Lemma 11.4 and Proposition 7.2 the result holds. Hence we can assume that $J \nrightarrow L_2$, for each $J \in S \cap S' \setminus \{I_1\}$. Therefore, since $f^{k-1}(p_e) \in \mathbf{O}$ and by the continuity it follows that $f(x_1) \in \mathbf{O}$ (otherwise there is $J \in S \cap S' \setminus \{I_1\}$ f -covering L_1 or L_2 or L_3). Consequently, if we take $D_1 = [p_e, x_a]$ and $D_2 = \text{Cl}(\mathbf{O} \setminus (B \cup C \cup I_1))$, we obtain that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$ (see Figure 13.5). Thus by Lemma 11.14 the result follows.

So in the rest of this proof we can suppose that $B \nrightarrow B$. Since $f(x_a) \in \mathbf{O}$ and $f(0) \in \mathbf{O}$ it follows that $A \nrightarrow A$.

First assume that $f(0) \notin \text{Int}(I_1)$. Therefore, since $B \nrightarrow B$ by Remark 12.6 $f(0) \neq x_0$ and so $B \rightarrow I_1$. Hence we have $L_1 \rightarrow I_1 \leftarrow L_3$. Moreover, since either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_3$ we obtain either $L_1 \rightleftharpoons I_1 \leftarrow L_3$ or $L_3 \rightleftharpoons I_1 \leftarrow L_1$. From the facts that $f(0) \in \text{Cl}(\mathbf{O} \setminus (B \cup C \cup I_1))$, $f^{k-1}(p_e) \in \mathbf{O}$, there are no basic intervals of S' f -covering themselves, and by the continuity of f it follows that there is $J_1 \subset \mathbf{O}$, $J_1 \in S' \setminus \{I_1\}$ such that $B \leftarrow J_1 \rightarrow I_1$ (notice that J_1 can be C and $J_1 \neq B$). Set $J = J_1$ if $J_1 \neq C$ and $J = L_2$ if $J_1 = C$. Clearly $x_0 \notin J$. We have $B \leftarrow J \rightarrow I_1$. Moreover either $J \rightarrow B \cup A = L_1$ or $J \rightarrow B \cup C = L_3$. So we obtain either $L_1 \leftarrow J \rightarrow I_1$ or $L_3 \leftarrow J \rightarrow I_1$. Hence one of the following subgraphs is satisfied:

- (a) $J \rightarrow I_1 \rightleftharpoons L_i \leftarrow J$ for $i \in \{1, 3\}$;
- (b) $J \rightarrow L_j \rightarrow I_1 \leftarrow J$ and $I_1 \rightleftharpoons L_i$ for $i, j \in \{1, 3\}$, $i \neq j$.

Clearly $I_1 \cap L_i = \{x_0\}$ for $i \in \{1, 3\}$. Then by Lemma 11.13 the result follows.

Finally assume that $f(0) \in \text{Int}(I_1)$. Denote by D_b and D_c the closures of the two components of $\mathbf{O} \setminus \{0, f^{k-1}(p_e)\}$ such that $B \subset D_b$ and $C \subset D_c$. We have two possibilities: $f(x_c) \in \mathbf{I}$ or $f(x_c) \in \mathbf{O}$.

Suppose that $f(x_c) \in \mathbf{I}$. Since $f(x_c) \in \mathbf{I}$ and $f(0) \in \text{Int}(I_1)$, we get that $C \rightarrow L_1$ and so $L_2 \rightarrow L_1 \leftarrow L_3$. From Remark 10.2 $A \rightarrow [f(0), x_1] \subset I_1$ and so $L_1 \rightarrow I_1$. If $I_1 \rightarrow L_1$, then we have $L_1 \rightleftharpoons I_1$. Notice that $I_1 \cap L_1 = \{x_0\}$. Define $D_1 = [p_e, x_a] \subset \mathbf{I}$ and $D_2 = \text{Cl}(\mathbf{O} \setminus (B \cup C \cup I_1)) \subset \mathbf{O}$. If $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$, then by Lemma 11.14 we are done. Otherwise, there is $J \in S \cap S'$ such that $J \rightarrow A$ and consequently either $J \rightarrow L_1 \cup I_1$ or $J \rightarrow L_2$. Consider $L_1 \rightleftharpoons I_1$. If $J \rightarrow L_1 \cup I_1$, from Lemma 11.5 and Proposition 7.2 the result holds. If $J \rightarrow L_2$, we have the paths $J \rightarrow L_2 \rightarrow L_1$ and $J \rightarrow L_2 \rightarrow I_1$. Then again from Lemma 11.5 and Proposition 7.2 the result follows.

Then we can suppose that $I_1 \nrightarrow L_1$. Therefore $I_1 \rightarrow L_3$ and $f(x_1) \in \mathbf{O}$.

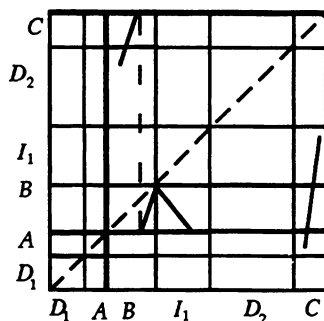


FIGURE 13.5. The map f satisfies that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$.

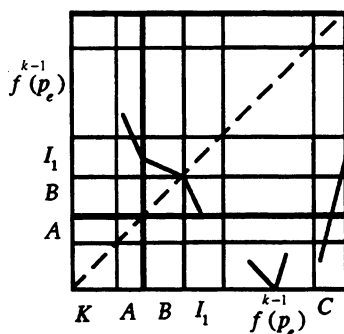


FIGURE 13.6. There are two closed subintervals $M, N \subset \mathbf{O}$ f -covering K .

Since each basic interval of S' does not f -cover itself, $f(0) \in \text{Int}(I_1)$ and there exists a closed subinterval f -covering \mathbf{O} (see Remark 9.6), we have that there is a closed subinterval $K \subset \mathbf{I}$ such that $K \rightarrow \text{Cl}(\mathbf{O} \setminus (B \cup I_1))$. From the fact that $\{f(x_1), f(0), f^{k-1}(p_e)\} \subset \mathbf{O}$ it follows that there are two closed subintervals $M, N \subset \mathbf{O} \setminus \text{Cl}(B \cup I_1)$ f -covering K (see Figure 13.6). Hence we have $M \rightleftharpoons K \rightleftharpoons N$. Without loss of generality we can assume that $M \subset D_b$ and $N \subset D_c$. Thus we obtain a non-repetitive loop of length n for each n even. Since $x_0 \notin K$ we get that $2\mathbb{N} \subset \text{Per}(f)$. If k is even $S(k) \subset \text{Per}(f)$ and we are done. So we can suppose that k is odd. If $f(x_a) \in D_b$, then there is a minimal closed subinterval $K' \subset [p_e, x_a]$ with endpoints elements of P such that $K' \rightleftharpoons N$. Since $x_0 \notin K'$, from Lemma 11.15 we obtain that $n \in \text{Per}(f)$ for each $n > k$ odd. Then $S(k) \subset \text{Per}(f)$ and the result follows.

If $f(x_a) \in D_c$, then $A \rightarrow [x_1, f^{k-1}(p_e)] \subset D_b$. By the continuity of f and from the fact that $\{f(x_1), f^{k-1}(p_e)\} \subset \mathbf{O}$ it follows that there is a minimal closed subinterval $M_1 \subset [x_1, f^{k-1}(p_e)]$ (and so $x_0 \notin M_1$) such that either $M_1 \rightarrow L_2$ or $M_1 \rightarrow L_1 \cup I_1$. Therefore $A \rightarrow M_1$ and we obtain either the loops $M_1 \rightleftharpoons L_2$ and $M_1 \rightarrow L_2 \rightarrow L_1 \rightarrow M_1$; or the loops $M_1 \rightleftharpoons L_1$ and $L_1 \rightarrow I_1 \rightarrow L_3 \rightarrow L_1$. So we have a non-repetitive loop of length n for each $n > k$ odd containing M_1 . Since $x_0 \notin M_1$ we get that $S(k) \subset \text{Per}(f)$.

Finally assume that $f(x_c) \in \mathbf{O}$. Since $C \rightarrow C$ we have that $L_3 \rightarrow I_1$. By the continuity of f and since each basic interval does not f -cover itself, it

Lemma 14.1. *Let f be the standard σ map. Suppose that $f(0) \in \text{Int}(A \cup B \cup C)$ and that each basic interval of T is f -covered by some basic interval of T different from itself. If there is $J \in T$ such that $J \rightarrow J$, then Theorem 1.4(a) holds.*

Proof. If $J \in S \cap S'$, then by Proposition 11.10 we are done. So assume that $J \in \{L_1, L_2, L_3\}$. Since $f(0) \in \text{Int}(A \cup B \cup C)$ and from the monotonicity of f on each basic interval of S' , it follows that J does not f -cover basic intervals of $\{L_1, L_2, L_3\} \setminus \{J\}$. If $x_0 \in J$, from the facts that $f(0) \in \text{Int}(A \cup B \cup C)$ and that f is monotone on each basic interval of S' , it follows that J does not f -cover basic intervals of T containing x_0 different from itself. By Lemma 11.4 there is a non-repetitive loop of length n for each $n \geq k+1$. By construction, there are at least two basic intervals of the above loop (perhaps they are the same) such that one does not contain 0, and the other does not contain x_0 . Then the associated sequence of W_i 's in Proposition 7.2 is non-repetitive and so $B(k) \subset \text{Per}(f)$. \square

Proposition 14.2. *Let f be the standard σ map. Suppose that $f(0) \in \text{Int}(A \cup B \cup C)$ and $x_0 \in \{x_a, x_b, x_c\}$. Then Theorem 1.4(a) holds.*

Proof. Denote by I_1 the basic interval with endpoint x_0 different from A, B and C . Set $I_1 = [x_0, x_1]$. By symmetry and from Remark 6.2 we need only consider three cases.

Case 1: $f(0) \in \text{Int}(A)$ and $x_0 = x_a$. Notice that from the monotonicity of f on A , A does not f -cover any basic interval. By definition of x_0 in Section 6, the interval $[p_e, x_0)$ does not contain fixed points, so $f^{k-1}(p_e) \in \mathbf{O}$ and $I_1 \rightarrow A$. Then either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_2$. Without loss of generality we can assume that $I_1 \rightarrow L_1$. In particular $L_1 \in T$, and L_1 is f -covered by some basic interval different from itself. Notice that from Remark 10.2 A is f -covered by some basic interval of $\{L_1, L_2, L_3\}$. So there is $z \in \{x_b, x_c\}$ such that $f(z) \in P \cap \mathbf{O}$. Hence there exists $J \in S \cap S'$, $J \subset \mathbf{O}$ such that $I_1 \leftarrow J \rightarrow A$. Therefore either $J \rightarrow L_1$ or $J \rightarrow L_2$. Notice that $\{x_0, 0\} \cap J = \emptyset$. Now we consider the three different possibilities for L_2 , the other basic interval containing $f(0)$.

Subcase (i). Suppose that $L_2 \notin T$. Consequently $J \rightarrow L_1$. Moreover $L_3 \in T$ and $f(0) \notin L_3$. By Remark 11.9 each basic interval of T is f -covered by some basic interval of T different from itself. If there is a basic interval of T f -covering itself, by Lemma 14.1 the result follows. So suppose that there are no basic intervals of T f -covering themselves. Hence since $L_2 \notin T$, $f(x_b) \in \mathbf{I}$, so we get $L_1 \rightarrow I_1$. By applying Lemma 11.5 to $I_1 \rightleftharpoons L_1$, there is a path of length $l \leq k-1$ starting at one of the intervals I_1 or L_1 and ending at J . This path together with the paths $L_1 \leftarrow J \rightarrow I_1 \rightleftharpoons L_1$, give us a non-repetitive loop of length n for each $n \geq k+1$ in the hypotheses of Proposition 7.2, so $B(k) \subset \text{Per}(f)$.

Subcase (ii). Suppose that $L_2 \in T$ and L_2 is not f -covered by any basic interval of T different from itself. Then $J \rightarrow L_1$. Moreover $C \rightarrow C$ (because $L_2 \rightarrow L_2$) and by Lemma 11.12 taking the interval C , we can assume that there exists $K \in S'$, $K \neq C$ such that $K \rightarrow C$, so either $K \rightarrow B$ or $K \rightarrow A$. Since L_2 only is f -covered by itself, $K \rightarrow B$. Hence $L_3 \in T$. Consequently $T = S$. Hence L_3 and L_1 do not f -cover L_2 . So $f(x_b) \in \mathbf{O}$ and consequently

$L_1 \rightarrow L_1$ (because $L_1 \nrightarrow L_2$). Since $f(0) \notin L_3$, by Remark 11.9, L_3 is f -covered by some basic interval of T different from itself. Notice that if L_2 f -covers M for some $M \in T$, then $L_3 \rightarrow M$. Set $R = T \setminus \{L_2\} = S \setminus \{L_2\}$. Therefore each basic interval of R is f -covered by some basic interval of R different from itself. Furthermore $L_1 \rightarrow L_1 \leftarrow J$ and $\{0, x_0\} \cap J = \emptyset$. So, by Lemma 11.4 interchanging T by R , and Proposition 7.2 it follows that $B(k) \subset \text{Per}(f)$.

Subcase (iii). Suppose that $L_2 \in T$ and L_2 is f -covered by some basic interval different from itself. Since $f(0) \notin L_3$, from Remark 11.9 it follows that each basic interval of T is f -covered by some different basic interval of T . Notice that $T = S$ or $T = S \setminus \{L_3\}$. If there is a basic interval of T f -covering itself, then by Lemma 14.1 the result follows. Hence we can assume that each basic interval of T does not f -cover itself. Then from the graph of T it follows that f has no fixed points in \mathbf{O} . Since $f(0) \in \mathbf{I}$, there exists a closed subinterval $K \subset [p_e, x_0]$ such that $K \rightarrow \mathbf{O}$ and K has endpoints elements of $P \cup \{x_0\}$. From the facts that $f^{k-1}(p_e) \in \mathbf{O}$, and $f(0) \in A$, we get that there are two minimal closed subintervals $M, N \subset \mathbf{O}$ such that M and N have disjoint pairwise interiors and $M, N \rightarrow K$. Moreover $x_0 \notin M \cup N$. By Lemma 11.6 the result follows.

Case 2: $f(0) \in \text{Int}(B)$ and $x_0 = x_b$. By the monotonicity of f on B we have that $f(B) = [f(0), x_0] \subset B$, otherwise B contains a fixed point $y \neq x_0$ in contradiction with the definition of x_0 in Section 6. Notice that B does not f -cover any basic interval. Now we will study the two basic intervals containing $f(0)$ which are L_1 and L_3 .

Subcase (i). Suppose that $L_1 \notin T$. Then by Proposition 10.8, $L_2, L_3 \in T$. We claim that L_3 is f -covered by some $J_1 \in T \setminus \{L_3\}$. Otherwise suppose that the only basic interval of T f -covering L_3 is itself. Therefore, since $f(B) \subset B$ and $B \cup C \rightarrow B \cup C$, we have $C \rightarrow [f(0), 0] \subset B$ and $C \rightarrow C$. So since $L_2 = A \cup C \in T$ and $L_2 \nrightarrow L_3$ it follows that $A \nrightarrow [f(0), x_0] \subset B$. Then either $A \rightarrow [f(0), 0] \cup A \subset B \cup A$ or $A \rightarrow [f(0), 0] \cup C \subset B \cup C$. Since $L_1 \notin T$, we have $A \rightarrow [f(0), 0] \cup C$ and so $L_1 \rightarrow L_3$. Then, since $I_1 \cap B = \{x_0\}$, by the monotonicity of f on each basic interval of S' we have that each basic interval of $\{A, B, C\}$ does not f -cover I_1 . Moreover, by Remark 11.9, I_1 is f -covered by some different basic interval of S' . So we can assume that there exists $J_0 \in S \cap S' \setminus \{I_1\}$ such that $J_0 \rightarrow I_1$. Since $I_1 \cap B = \{x_0\}$ and $L_1 \notin T$, it follows that $J_0 \rightarrow I_1 \cup B \cup C$. In particular $J_0 \rightarrow L_3$, a contradiction with the assumptions. So the claim is proved.

Since $f(0) \notin L_2$ and there is $J_1 \in T \setminus \{L_3\}$ such that $J_1 \rightarrow L_3$, by Remark 11.9 each $J \in T$ is f -covered by some basic interval of $T \setminus \{J\}$. By Lemma 14.1 we can suppose that for each $J \in T$, $J \nrightarrow J$. In particular $I_1 \nrightarrow I_1$ and so $I_1 \rightarrow B$. Hence either $I_1 \rightarrow B \cup A = L_1$ or $I_1 \rightarrow B \cup C = L_3$. By hypotheses $L_1 \notin T$, so we have that $I_1 \rightarrow L_3$ and thus $f(x_1) \in \mathbf{O}$ (by the monotonicity of f on I_1). On the other hand $L_3 \nrightarrow L_3$ and $L_3 \nrightarrow L_1$, which implies that $L_3 \nrightarrow B$. So we have that $L_3 \rightarrow I_1$. More concretely, $C \rightarrow [f(0), x_0] \subset B$ and $C \rightarrow I_1$. So $f(x_c) \in \mathbf{O}$ (otherwise $C \rightarrow A \cup C$ and consequently $L_2 \rightarrow L_2$, a contradiction). Then, from the facts that $f(x_1) \in \mathbf{O}$, $f(x_c) \in \mathbf{O}$, $I_1 \cap B = \{x_0\}$, each basic interval of T does not f -cover itself, $L_1 \notin T$ and by the continuity of f , we obtain that there exists $J \in S \cap S' \setminus \{I_1\}$,

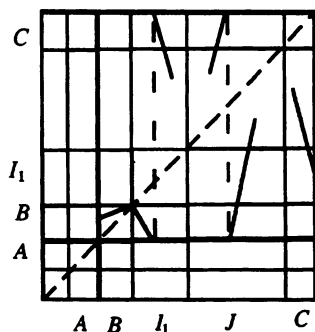


FIGURE 14.1. There exists $J \in S \cap S' \setminus \{I_1\}$, $J \subset \mathbf{O}$, f -covering I_1 and L_3 .

$J \subset \mathbf{O}$ such that $I_1 \leftarrow J \rightarrow L_3$ (see Figure 14.1). Notice that $\{x_0, 0\} \cap J = \emptyset$. Since $I_1 \rightleftharpoons L_3$ and $I_1 \cap L_3 = \{x_0\}$, from Lemma 11.5 and Proposition 7.2 we obtain $B(k) \subset \text{Per}(f)$.

Subcase (ii). Suppose that $L_1 \in T$ and that L_1 is the unique interval of T such that f -covers L_1 . Then, since $f(B) = [f(0), x_0] \subset B$, it follows that $A \rightarrow [f(0), 0] \cup A \subset B \cup A$. Hence $f(x_a) \in \mathbf{I}$. Since $f(P \cap \mathbf{I}) \not\subseteq P \cap \mathbf{I}$, there exists $M \in S \cap S'$, $M \subset \mathbf{I}$ such that $M \rightarrow A$. So either $M \rightarrow A \cup B = L_1$ or $M \rightarrow A \cup C = L_2$. By the assumptions we have that $M \rightarrow L_2$. In particular $L_2 \in T$ and L_2 is f -covered by $M \neq L_2$. Consequently $L_2 \nrightarrow L_1$. Since $L_2 = A \cup C$, $A \rightarrow [f(0), 0] \cup A \subset B \cup A$ and $B \cap I_1 = \{x_0\}$ we get that $L_2 \nrightarrow I_1$. Notice that each basic interval of $\{A, B, C\}$ does not f -cover I_1 . So, by Lemma 11.12 there exists $N \in S \cap S' \setminus \{I_1\}$ such that $N \rightarrow I_1$. Furthermore either $N \rightarrow I_1 \cup B \cup A$ or $N \rightarrow I_1 \cup B \cup C$. By the hypotheses we have that $N \rightarrow I_1 \cup L_3$. In particular $L_3 \in T$ and L_3 is f -covered by $N \neq L_3$. Consider $R = T \setminus \{L_1\} = S \setminus \{L_1\}$. We note that if L_1 f -covers some $J \in R$, then also L_2 f -covers J . Hence, by Remark 11.9 each basic interval of R is f -covered by some different basic interval of R . Since $f(x_a) \in \mathbf{I}$, by the proof of Corollary 11.11(b), interchanging T by R , the result holds.

Subcase (iii). Suppose that L_1 is f -covered by some basic interval of T different from itself. Now we will study the following three possibilities for L_3 .

First suppose that $L_3 \notin T$. So by the definition of T we have $L_2 \in T$. Since $f(0) \notin L_2$ by Remark 11.9 each basic interval of T is f -covered by some different basic interval of T . From Lemma 14.1 we can assume that each basic interval of T does not f -cover itself. By Corollary 11.11(b) we can suppose that $f(x_a) \in \mathbf{O}$. Moreover, since $I_1 \nrightarrow I_1$ we have $I_1 \rightarrow B$ and so either $I_1 \rightarrow B \cup A = L_1$ or $I_1 \rightarrow B \cup C = L_3$. From the assumption that $L_3 \notin T$ we get that $I_1 \rightarrow L_1$. Furthermore $A \rightarrow [x_0, f(0)] \subset B$ and $A \rightarrow I_1$ because $f(0) \in \text{Int}(B)$, $f(x_a) \in \mathbf{O}$ and $L_3 \notin T$. By Remark 10.2 we get that $C \rightarrow [f(0), 0] \subset B$. Hence $L_2 = A \cup C \rightarrow I_1 \cup B$, and so either $L_2 \rightarrow B \cup A = L_1$ or $L_2 \rightarrow B \cup C = L_3$. By the assumptions, we have that $L_2 \rightarrow L_1$. Therefore we have the paths $I_1 \leftarrow L_2 \rightarrow L_1$ and $I_1 \rightleftharpoons L_1$. Since $I_1 \cap L_1 = \{x_0\}$, by Lemma 11.5 there is a path of length $l \leq k - 1$ starting at one of the intervals I_1 or L_1 and ending at L_2 . If the path starts at L_1 , then we consider the loops $L_1 \rightarrow \cdots \rightarrow L_2 \rightarrow L_1 \rightarrow I_1 \rightarrow L_1$, $L_1 \rightarrow \cdots \rightarrow L_2 \rightarrow I_1 \rightarrow L_1$ and $L_1 \rightleftharpoons I_1$ of

lengths $l+3 \leq k+2$, $l+2 \leq k+1$ and 2 respectively. If the path starts at I_1 , then we consider the loops $I_1 \rightarrow \dots \rightarrow L_2 \rightarrow I_1$, $I_1 \rightarrow \dots \rightarrow L_2 \rightarrow L_1 \rightarrow I_1$ and $I_1 \rightarrow L_1$ of lengths $l+1 \leq k$, $l+2 \leq k+1$ and 2 respectively. Clearly $0 \notin I_1$ and $x_0 \notin L_2$. Notice that all the above loops contain I_1 and L_2 . Then by the construction of the loops and by Proposition 7.2 the result holds.

Now suppose that $L_3 \in T$ and L_3 is the unique basic interval of T f -covering L_3 . Then, since $B \rightarrow [f(0), x_0] \subset B$, we have that $C \rightarrow [f(0), 0] \subset B$ and $C \rightarrow C$. By Lemma 11.12 we can assume that C is f -covered by some $M \in S'$, $M \neq C$. Moreover, by the hypotheses we have that $L_1 \in T$ and so $L_1 \nrightarrow L_3$. We claim that $M \neq A$. Otherwise, suppose $M = A$. Therefore, since $L_1 \nrightarrow L_3$ it follows that $A \rightarrow \mathbf{O} \setminus (f(0), 0)$ and $A \rightarrow A$. In particular $L_2 \rightarrow L_2$ and $L_2 \rightarrow L_3$, in contradiction with the assumptions. So the claim is proved. Notice that $M \neq B$, because $f(B) \subset B$. Then we have $M \notin \{A, B, C\}$, and so $M \in S \cap S'$. Since $M \rightarrow C$ and $M \nrightarrow C \cup B = L_3$, we get that $M \rightarrow C \cup A = L_2$. In particular $L_2 \in T$ and L_2 is f -covered by $M \neq L_2$. Notice that if $L_3 \rightarrow N$, for some $N \in T \setminus \{L_3\}$, then $C \rightarrow N$ and consequently $L_2 \rightarrow N$. Set $R = T \setminus \{L_3\} = S \setminus \{L_3\}$. Therefore each basic interval of R is f -covered by some different basic interval of R . From the facts that $L_1 \nrightarrow L_3$, $L_2 \nrightarrow L_3$, $C \rightarrow [0, f(0)] \cup C \subset B \cup C$ and $B \rightarrow [f(0), x_0] \subset B$, it follows that $L_1 \rightarrow L_1$. Then from the proof of Lemma 14.1, interchanging T by R the result holds.

Finally suppose that $L_3 \in T$ and L_3 is f -covered by some different basic interval of T . Since $f(0) \notin L_2$, by Remark 11.9 we have that each $J \in T$ is f -covered by some interval of $T \setminus \{J\}$. From Lemma 14.1 we can assume that for each $J \in T$, $J \nrightarrow J$. In particular $I_1 \nrightarrow I_1$ and so either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_3$. We claim that $C \nrightarrow C$. Otherwise suppose that $C \rightarrow C$. Therefore either $C \rightarrow [0, f(0)] \subset B$ and so $L_3 \rightarrow L_3$; or $C \rightarrow C \cup A$ and $L_2 \rightarrow L_2$ in contradiction with the fact that for each $J \in T$, $J \nrightarrow J$. Thus the claim is proved. From Corollary 11.11(a) we can assume that $f^{k-1}(p_e) \in \mathbf{O}$. By Remark 9.6 there is a closed subinterval $K \subset \sigma$ such that $K \rightarrow \mathbf{O}$. From the fact that $f(B) = [f(0), x_0] \subset B$, $J \nrightarrow J$ for each $J \in T$ and $C \nrightarrow C$, it follows that $K \not\subset \mathbf{O}$. Then there exists $K' \subset I$ such that $K' \rightarrow \mathbf{O} \setminus \text{Int}(B)$ (see Figure 14.2). From the fact that $\{f^{k-1}(p_e), x_0, f(0)\} \subset \mathbf{O}$ it follows that there are two minimal closed subintervals $M, N \subset \mathbf{O} \setminus \text{Int}(B)$ f -covering K' . Then

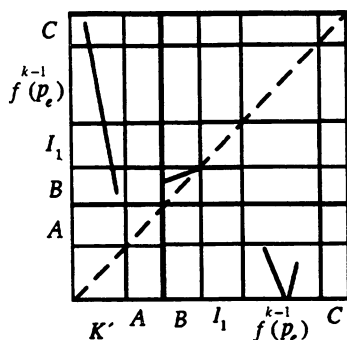


FIGURE 14.2. There is $K' \subset K \cap I$ such that $K' \rightarrow \mathbf{O} \setminus \text{Int}(B)$.

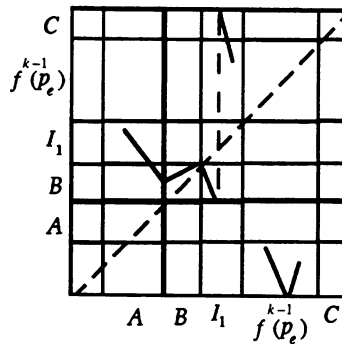


FIGURE 14.3. There is a closed subinterval $M \subset \mathbf{O}$ f -covering L_2 or $L_1 \cup I_1$.

we get $M \rightleftharpoons K' \rightleftharpoons N$. Since $x_0 \notin K'$ we obtain $2N \subset \text{Per}(f)$. If k is even, then $S(k) \subset \text{Per}(f)$ and we are done. Hence we can assume that k is odd. Denote by D_c and D_b the closures of the two components of $\mathbf{O} \setminus \{0, f^{k-1}(p_e)\}$ such that $B \subset D_b$ and $C \subset D_c$.

First assume that $A \rightarrow I_1$. Then $L_1 \rightarrow I_1 \leftarrow L_2$. From Remark 10.2 and since $C \rightarrow C$, we have that $C \rightarrow [f(0), 0] \cup A \subset B \cup A$. Consequently $L_2 \rightarrow L_1 \leftarrow L_3$. On the other hand, either $I_1 \rightarrow L_1$ or $I_1 \rightarrow L_3$. Suppose that $I_1 \rightarrow L_1$. If $f(P \cap \mathbf{I}) = P \cap \mathbf{O}$ and $f(P \cap \mathbf{O}) = P \cap \mathbf{I}$, then k must be even, in contradiction with the assumptions. Therefore there is $J \in S \cap S'$ f -covering either $L_1 \cup I_1$ or L_2 . So we consider the loop $I_1 \rightleftharpoons L_1$ and the path either $L_1 \leftarrow J \rightarrow I_1$ or $L_1 \leftarrow L_2 \leftarrow J \rightarrow L_2 \rightarrow I_1$. From Lemma 11.5 we obtain a non-repetitive loop of length n for each $n \geq k+1$ containing J . Since $\{x_0, 0\} \cap J = \emptyset$ from Proposition 7.2 the result follows. Therefore we can assume that $I_1 \rightarrow L_1$. Hence $I_1 \rightarrow L_3$ and by the monotonicity, $f(x_1) \in \mathbf{O}$. Since $\{f^{k-1}(p_e), f(0)\} \subset \mathbf{O}$ it follows that there exists a closed subinterval $N \subset D_c$ f -covering \mathbf{I} . Moreover, since $\{f(x_1), f^{k-1}(p_e)\} \subset \mathbf{O}$, by the continuity it follows that there exists a closed subinterval $M \subset D_b$ such that $x_0 \notin M$ and either $M \rightarrow L_2$ or $M \rightarrow L_1 \cup I_1$ (see Figure 14.3). If $f(x_a) \in D_b$, then there exists a closed subinterval $K' \subset [p_e, x_a] \subset \mathbf{I}$ with endpoints elements of P such that $K' \rightarrow D_c$. Therefore we have $K' \rightleftharpoons N$ and $x_0 \notin K'$. From Lemma 11.15 the result holds. So suppose that $f(x_a) \in D_c$. Then $A \rightarrow D_b$. In particular $L_1 \rightarrow M \leftarrow L_2$. We consider the loops either $M \rightleftharpoons L_1 \rightarrow I_1 \rightarrow L_3 \rightarrow L_1$; or $M \rightleftharpoons L_2 \rightarrow L_1 \rightarrow M$. Thus we obtain a non-repetitive loop of length n for each $n > k$ odd containing M . Since $x_0 \notin M$ we have $S(k) \subset \text{Per}(f)$.

Finally assume that $A \rightarrow I_1$. Since $J \rightarrow J$ for each $J \in T$, it follows that $A \rightarrow [f(0), 0] \cup C \subset B \cup C$ and so $L_1 \rightarrow L_3$. In particular $f(x_a) \in \mathbf{O}$. Notice that the closed subinterval K f -covering \mathbf{O} is contained in \mathbf{I} . If $f(x_a) \in D_c$, then there is $K' \subset \mathbf{I}$, K' with endpoints elements of P such that $K' \rightarrow D_b$. From the fact that $\{x_0, f^{k-1}(p_e)\} \subset \mathbf{O}$, there is a closed subinterval $M \subset D_c$ such that $M \rightarrow K'$. Then we have $K' \rightleftharpoons M$ and $x_0 \notin K'$. By Lemma 11.15 the result holds. Otherwise $f(x_a) \in D_b$. Hence $A \rightarrow D_c$ and consequently $L_1 \rightarrow D_c \leftarrow L_2$. Now we have two possibilities: $f(x_c) \in \mathbf{O}$ or $f(x_c) \in \mathbf{I}$.

First suppose that $f(x_c) \in \mathbf{O}$. Then we get that $C \rightarrow I_1$ (because $L_3 \rightarrow L_3$)

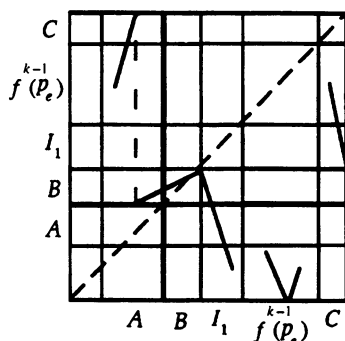


FIGURE 14.4. There is a basic interval of D_c f -covering L_1 or L_2 .

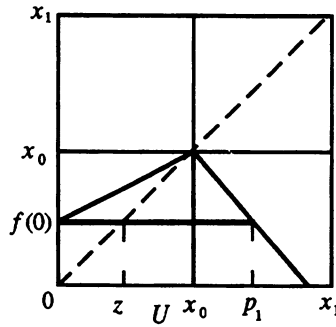
and so $L_3 \rightarrow I_1 \leftarrow L_2 \rightarrow L_3$. On the other hand either $I_1 \rightarrow L_3$ or $I_1 \rightarrow L_1$. If $I_1 \rightarrow L_3$, then $f(x_1) \in \mathbf{O}$. Hence there is a basic interval $M \in S \cap S'$, $M \subset \mathbf{O}$ such that either $M \rightarrow L_3 \cup I_1$ or $M \rightarrow L_2$. Therefore we consider the loop $L_3 \rightrightarrows I_1$ and either the paths $L_3 \leftarrow M \rightarrow I_1$; or $L_3 \leftarrow L_2 \leftarrow M \rightarrow L_2 \rightarrow I_1$. So from Lemma 11.5 and Proposition 7.2 the result holds. Otherwise, $I_1 \rightarrow L_1$. Since $\{f(x_c), f^{k-1}(p_e)\} \subset \mathbf{O}$, there exists $M \in S \cap S'$, $M \subset D_c$, such that either $M \rightarrow L_1$ or $M \rightarrow L_2$ (see Figure 14.4). From the fact that $L_1 \rightarrow D_c \leftarrow L_2$ we obtain either $M \rightrightarrows L_1$ or $L_1 \rightarrow M \rightrightarrows L_2$. So we consider either the loops $L_1 \rightarrow L_3 \rightarrow I_1 \rightarrow L_1$ and $L_1 \rightrightarrows M$; or the loops $L_2 \rightarrow L_3 \rightarrow I_1 \rightarrow L_1 \rightarrow M \rightarrow L_2$ and $L_2 \rightrightarrows M$. So we obtain a non-repetitive loop of length n for each $n \geq 5$ odd containing M . Since $x_0 \notin M$ the result follows.

Finally suppose that $f(x_c) \in \mathbf{I}$. Let $p_1 \in I_1$ be such that $f(p_1) = f(0)$. Set $U = [0, p_1] \subset B \cup I_1$. We define the map $g : \sigma \rightarrow \sigma$ as $g|_{\sigma \setminus U} = f|_{\sigma \setminus U}$ and $g|_U = \{f(0)\}$. Clearly $g(0) = f(0) \in \mathbf{O}$ and g has a fixed point $z \in B$ (see Figure 14.5). So g is a σ map. Moreover P is a periodic orbit of period k for g . Of course $\text{Per}(g) \subset \text{Per}(f)$. We remark that g is monotone on each basic interval associated to $P \cup \{z\} \cup \{0\}$. If we denote by A' , B' and C' the three basic intervals associated to $P \cup \{z\} \cup \{0\}$ with 0 as endpoint, then we have $B' = [0, z]$ and $g(0) = g(z)$. From Remark 12.6 the result follows.

Case 3: $f(0) \in \text{Int}(B)$ and $x_0 = x_c$. We will deal with two basic intervals containing $f(0)$ which are L_1 and L_3 . Notice that by definition of x_0 in Section 6, B does not contain fixed points of f . So $B \rightarrow [f(0), x_b] \subset B$.

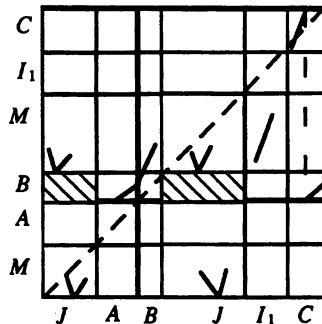
Subcase (i). Suppose that $L_3 \notin T$. So by definition of T we have that $L_1, L_2 \in T$. Notice that C does not f -cover C (otherwise $L_3 = B \cup C \rightarrow B \cup C$). Therefore $C \rightarrow [f(0), x_b] \subset B$ and $C \rightarrow I_1$. So $L_2 \rightarrow I_1$. Since $f(0) \in \text{Int}(B)$ and $B \rightarrow [f(0), x_b] \leftarrow C$, by Remark 10.2 we get that $A \rightarrow [f(0), 0] \subset B$. Furthermore, from the fact that $L_3 \notin T$ we obtain that $f(x_a) \in \mathbf{I}$ (otherwise L_1 and L_2 f -cover L_3) and $A \rightarrow A$. Thus $L_1 \rightarrow L_1 \leftarrow L_2$. Therefore L_1 is f -covered by a basic interval of T different from itself. Since $f(0) \notin L_2$, from Remark 11.9 we have that each basic interval of T is f -covered by some basic interval different from itself. Since $L_1 \rightarrow L_1$ by Lemma 14.1 the result holds.

Subcase (ii). Suppose that $L_3 \in T$ and there are no basic intervals of T different from L_3 f -covering L_3 . Since $B \rightarrow [f(0), x_b] \subset B$ and $L_3 \rightarrow L_3$ it follows that $C \rightarrow [0, f(0)] \cup C \subset B \cup C$.

FIGURE 14.5. The new σ map g in U .

First assume that $L_1 \notin T$. Then by the definition of T we have that $L_2 \in T$. Consequently $L_2 \rightarrow L_3$ and so $A \rightarrow [f(0), x_b] \subset B$. Therefore $A \rightarrow [0, f(0)] \subset B$. Moreover either $A \rightarrow A$ and $L_1 \rightarrow L_1$, or $A \rightarrow C$ and $L_1 \rightarrow L_3$. By the assumption $L_1 \notin T$ we get $A \rightarrow C$ and $L_1 \rightarrow L_3$. Thus $f(x_a) \in \mathbf{O}$ and $A \rightarrow I_1$. In particular $L_2 \rightarrow I_1$. Notice that each basic interval $J_0 \in S \cap S'$ does not f -cover B ; otherwise either $J_0 \rightarrow B \cup A = L_1$ or $J_0 \rightarrow B \cup C = L_3$ in contradiction with the assumptions. Set $R = T \setminus \{L_3\} = S \setminus \{L_1, L_3\}$. Notice that $L_1 \cap L_3 = B$. We claim that if L_3 f -covers M for some $M \in R$, then there is $J \in R$, $J \neq M$ such that $J \rightarrow M$. Now we prove the claim. Since $f(C) \subset L_3$, $B \rightarrow M$. Suppose that $M \subset \mathbf{O}$. So from the facts that $J \rightarrow B$ for each $J \in S \cap S'$, $x_b \in P$ and by the continuity of f , it follows that there is $J \in R$, $J \neq M$ such that $J \rightarrow M$ (see Figure 14.6). Suppose that $M \subset \mathbf{I}$. Then $f(x_b) \in \mathbf{I}$. Hence from the facts that $f(x_b) \in \mathbf{I}$, $f(x_c) \in \mathbf{O}$ and by the continuity of f it follows that there is $J \in R$, $J \subset \mathbf{O}$ (and so $J \neq M$) such that $J \rightarrow M$. Finally suppose that $M = L_2$. Since $A \rightarrow C$ and $C \rightarrow C$, clearly $L_2 \rightarrow L_2$. Since $f(x_a) \in \mathbf{O}$, $f(x_c) \in \mathbf{O}$, $f(f^{k-1}(p_e)) \in \mathbf{I}$, each basic interval of $S \cap S'$ does not f -cover B and by the continuity of f , it follows that there is $J \in S \cap S'$ (and so $J \neq L_2$) such that $J \rightarrow L_2$ (see again Figure 14.6). So the claim is proved.

Hence from Remark 11.9 each basic interval of R is f -covered by some different basic interval of R . If there is $J \in R$ such that $J \rightarrow J$, from the

FIGURE 14.6. There is $J \in T$, $J \neq M$ such that $J \rightarrow M$.

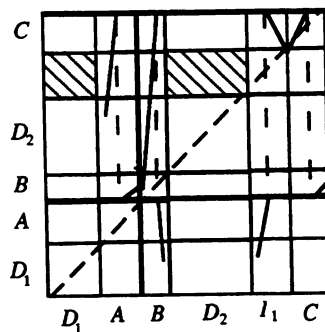
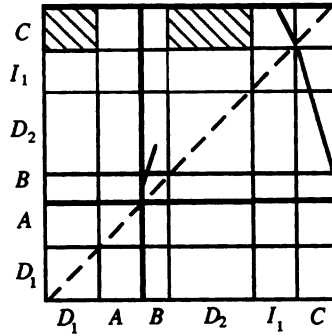


FIGURE 14.7. The graph of f with $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$.

proof of Lemma 14.1, replacing T by R , the result follows. So we can assume that for each $J \in R$ $J \rightarrow J$. By the proof of Corollary 11.11(a) we can suppose that $f^{k-1}(p_e) \in \mathbf{O}$. Moreover $I_1 \rightarrow I_1$ and so $I_1 \rightarrow L_2$ (since $I_1 \rightarrow L_3$). We have the loop $I_1 \rightleftharpoons L_2$, with $I_1 \cap L_2 = \{x_0\}$. If there is $J \in S \cap S'$ such that $I_1 \leftarrow J \rightarrow L_2$, then by the proof of Lemma 11.5 and Proposition 7.2 (replacing T by R) the result holds. Consequently we can assume that for each $J \in S \cap S' \setminus \{I_1\}$, $J \rightarrow I_1$ and $J \rightarrow L_2$. Set $D_1 = [p_e, x_a] \subset \mathbf{I}$ and $D_2 = \text{Cl}(\mathbf{O} \setminus (B \cup C \cup I_1))$ (see Figure 14.7). Therefore we get that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$. Then by Lemma 11.14 the result holds.

Finally assume that $L_1 \in T$. Thus we have $L_1 \rightarrow L_3$. Suppose that L_1 does not f -cover L_1 . Then since $L_1 \rightarrow L_3$, we get that $A \rightarrow [f(0), x_b]$. Consequently $L_2 \rightarrow L_3$ and so $L_2 \notin T$. Since there are no basic intervals of T f -covering L_2 , L_3 is the unique interval of T f -covering L_3 , and $C = L_2 \cap L_3$, it follows that C is not f -covered by any basic interval of S' different from itself. Hence by Lemma 11.12 the result follows. So in the rest of the proof of this subcase we can assume that $L_1 \rightarrow L_1$. Then $f(x_a) \in \mathbf{I}$. Suppose that L_1 is f -covered by some different basic interval of T . Notice that $L_3 \rightarrow L_1$. Set $R = T \setminus \{L_3\}$. We remark that if $L_3 \rightarrow M$, for $M \in R$, then $B \rightarrow M$ (because $f(x_0) = x_0$ and $f(C) \subset L_3$). Consequently $L_1 \rightarrow M$. So by Remark 11.9 each $J \in T$ is f -covered by some interval $T \setminus \{J\}$. Since $L_1 \rightarrow L_1$, by the proof of Lemma 14.1, replacing T by R the result holds.

Now we can suppose that $L_1 \rightarrow L_1$ and L_1 is not f -covered by any basic interval of $T \setminus \{L_1\}$. Then $A \rightarrow [f(0), 0] \cup A \subset B \cup A$. Since $f(P \cap \mathbf{I}) \not\subset P \cap \mathbf{I}$, there exists $z \in P \cap \mathbf{I}$ such that $f(z) \in P \cap \mathbf{O}$. Then, since $x_a, z \in \mathbf{I}$, $f(x_a) \in \mathbf{I}$ and $f(z) \in \mathbf{O}$, by the continuity of f it follows that there exists $J_0 \in S \cap S'$, $J_0 \subset \mathbf{I}$ such that $J_0 \rightarrow A$. So either $J_0 \rightarrow A \cup B = L_1$ or $J_0 \rightarrow A \cup C = L_2$. By the hypotheses we get that $J_0 \rightarrow L_2$. In particular $L_2 \in T$. Notice that each basic interval $J \in S \cap S'$ does not f -cover B , because otherwise either $J \rightarrow B \cup A = L_1$ or $J \rightarrow B \cup C = L_3$ in contradiction with the assumptions. Set $R = T \setminus \{L_1, L_3\} = S \setminus \{L_1, L_3\}$. We have $L_1 \cap L_3 = B$ and $f(0) \in \text{Int}(B)$. We claim that if L_1 or L_3 f -covers M for some $M \in R$, then there is $J \in R$, $J \neq M$ such that $J \rightarrow M$. Now we prove the claim. If $M = L_2$ then take $J = J_0$ and we are done. Suppose that $M \subset \mathbf{O}$. Hence from the facts that $J \rightarrow B$ for each $J \in S \cap S'$, $x_b \in P$, and by the continuity of f it follows (like in Figure 14.6) that there is $J \in R$, $J \neq M$ such that $J \rightarrow M$. Suppose that

FIGURE 14.8. For each $J \in S \cap S'$, $J \rightarrow C$.

$M \subset I$. If $A \rightarrow M$, then take $J = L_2$ and we are done; otherwise, we have $B \rightarrow M$ (because $f(C) \subset O$). Therefore $f(x_b) \in I$. Since $M \subset [f(x_b), 0]$, $f(x_c) \in O$ and by the continuity of f , there exists $J \in S \cap S'$, $J \subset O$ f -covering M . Notice that $J \neq M$. So the claim is proved. Hence each basic interval of R is f -covered by some different basic interval of R . Since $f(x_a) \in I$, from the proof of Corollary 11.11(b) interchanging T by R the result holds.

Subcase (iii). Suppose that $L_3 \in T$ and L_3 is f -covered by some basic interval of T different from itself. Now we shall deal with L_1 .

First we assume that $L_1 \notin T$, then by the definition of T we have that $L_2 \in T$. Since $f(0) \notin L_2$, by Remark 11.9 we have that each basic interval of T is f -covered by some different basic interval of T . From Lemma 14.1 we can assume that each basic interval of T does not f -cover itself. By Corollary 11.11 we can suppose that $f^{k-1}(p_e) \in O$ and $f(x_a) \in O$. Since $x_c = x_0$, $f(0) \in B$ and $L_3 \rightarrow L_3$ we have that $C \rightarrow I_1$. Then $L_2 \rightarrow I_1$ and $L_3 \rightarrow I_1$. Furthermore, since $I_1 \rightarrow I_1$, we get $I_1 \rightarrow C$. Moreover either $I_1 \rightarrow C \cup A = L_2$ or $I_1 \rightarrow C \cup B = L_3$. Suppose that there is $J \in S \cap S' \setminus \{I_1\}$ such that $J \rightarrow C$. Then, since $f(x_0) = x_0$, $J \rightarrow I_1$. Moreover either $J \rightarrow C \cup A = L_2$ or $J \rightarrow C \cup B = L_3$. Thus one of the following subgraphs is satisfied:

- (a) $J \rightarrow I_1 \rightleftharpoons L_i \leftarrow J$ for some $i \in \{2, 3\}$;
- (b) $J \rightarrow L_j \rightarrow I_1 \leftarrow J$ and $I_1 \rightleftharpoons L_i$ for $i, j \in \{2, 3\}$, $i \neq j$.

By Lemma 11.13 the result follows. So we can assume that for each $J \in S \cap S' \setminus \{I_1\}$, $J \rightarrow C$ (see Figure 14.8). Set $D_1 = [p_e, x_a] \subset I$ and $D_2 = \text{Cl}(O \setminus (B \cup C \cup I_1))$. Therefore we get that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$ (because $L_1 \notin T$). By Lemma 11.14 the result holds.

Assume that $L_1 \in T$ and L_1 is the unique interval of T f -covering L_1 . Since $B \rightarrow [f(0), x_b] \subset B$ and $L_1 \rightarrow L_1$, we have that $A \rightarrow [f(0), 0] \subset B$ and $A \rightarrow A$. So $f(x_a) \in I$. Clearly $f(P \cap I) \not\subset P \cap I$. Hence there is $z \in P \cap I$ such that $f(z) \in P \cap O$. Moreover, since $f(x_a) \in I$ and by the continuity of f it follows that there exists $J \in S \cap S'$, $J \subset I$ such that $J \rightarrow A$. Therefore either $J \rightarrow A \cup B = L_1$ or $J \rightarrow A \cup C \cup I_1$. By the assumptions we get that $J \rightarrow L_2 \cup I_1$. In particular $L_2 \in T$. Set $R = T \setminus \{L_1\} = S \setminus \{L_1\}$. Notice that $L_1 \rightarrow L_3$. Furthermore if $L_1 \rightarrow M$ for some $M \in R$, then $A \rightarrow M$ or $B \rightarrow M$. Consequently $L_2 \rightarrow M$ or $L_3 \rightarrow M$. Hence each basic interval of R is f -covered by some different basic interval of R (see Remark 11.9). Since

$f(x_a) \in \mathbf{I}$, by the proof of Corollary 11.11(b), replacing T by R , the result follows.

Finally assume that $L_1 \in T$ and that L_1 is f -covered by some different basic interval of T . Since $f(0) \notin L_2$, from Remark 11.9 each $J \in T$ is f -covered by some basic interval of $T \setminus \{J\}$. Then by Lemma 14.1 we can suppose that each basic interval of T does not f -cover itself. So we have that $I_1 \nrightarrow I_1$ and $L_3 \nrightarrow L_3$. Therefore $I_1 \rightarrow C$ and $C \rightarrow I_1$ (because $f(x_0) = x_0$). Thus L_2 and L_3 f -cover I_1 . Moreover either $I_1 \rightarrow L_2$ or $I_1 \rightarrow L_3$. Suppose that there is $J \in S \cap S' \setminus \{I_1\}$ such that $J \rightarrow C$. Then either $J \rightarrow L_2 \cup I_1$ or $J \rightarrow L_3 \cup I_1$. Hence one of the following subgraphs is satisfied:

- (a) $J \rightarrow I_1 \rightleftharpoons L_i \leftarrow J$ for some $i \in \{2, 3\}$;
- (b) $J \rightarrow L_j \rightarrow I_1 \leftarrow J$ and $I_1 \rightleftharpoons L_i$ for $i, j \in \{2, 3\}$, $i \neq j$.

By Lemma 11.13 the result holds. Consequently we can assume that for each $J \rightarrow S \cap S' \setminus \{I_1\}$, $J \nrightarrow L_2$ and $J \nrightarrow L_3$. Set $D_1 = [p_e, x_a] \subset \mathbf{I}$ and $D_2 = \mathbf{O} \setminus \text{Int}(B \cup C \cup I_1)$. Therefore we get that $f(D_1) \subset D_2$ and $f(D_2) \subset D_1$. By Lemma 11.14 the result holds. \square

Proposition 14.3. *Let f be the standard σ map. Suppose that $f(0) \in \text{Int}(A \cup B \cup C)$ and $x_0 \notin \{x_a, x_b, x_c\}$. Then Theorem 1.4(a) holds.*

Proof. We denote by I_1, I_2 the two basic intervals with x_0 as endpoint. These basic intervals exist in virtue of Remark 6.4. Since $x_0 \notin \{x_a, x_b, x_c\}$, we have that $I_1, I_2 \in S \cap S'$. By symmetry and Remark 6.2 we only need consider the two following cases.

Case 1: $f(0) \in \text{Int}(A)$ and $x_0 \in \mathbf{I}$. Without loss of generality we can assume that $I_1 \subset [p_e, x_0]$. So by the definition of x_0 in Section 6 we have that $[p_e, x_0)$ does not contain fixed points of f . Consequently $I_1 \nrightarrow I_1$. So $I_1 \rightarrow I_2$. Since $f(P \cap \mathbf{I}) \not\subseteq P \cap \mathbf{I}$, it follows that there is $z \in P \cap \mathbf{I}$ such that $f(z) \in P \cap \mathbf{O}$. By hypotheses $x_0 \in \mathbf{I}$. Hence, by the continuity of f , there exists $J_0 \in S \cap S'$ such that $J_0 \rightarrow A$. Therefore either $J_0 \rightarrow A \cup B = L_1$ or $J_0 \rightarrow A \cup C = L_2$. Without loss of generality we can assume that $J_0 \rightarrow A \cup B = L_1$. Thus $L_1 \in T$ and L_1 is f -covered by $J_0 \in S \cap S'$, $J_0 \neq L_1$. Since $[p_e, x_0)$ does not contain fixed points of f we get that $f^{k-1}(p_e) \notin [p_e, x_0]$. So since $I_1 \cap I_2 = \{x_0\}$, $f(0) \in A$ and $f(f^{k-1}(p_e)) = p_e \in \mathbf{I}$, there is $J'_1 \in S \setminus \{I_1, I_2\}$ such that $I_1 \leftarrow J'_1 \rightarrow I_2$. Therefore, from the definition of T , there is $J_1 \in T \setminus \{I_1, I_2\}$ such that $I_1 \leftarrow J_1 \rightarrow I_2$.

Subcase (i). Suppose that $L_2 \notin T$. Then by the definition of T we have $L_3 \in T$. Since $f(0) \notin \text{Int}(L_3)$ and L_1 is f -covered by $J_0 \in S \cap S'$, from Remark 11.9 it follows that each $J \in T$ is f -covered by some basic interval of $T \setminus \{J\}$. By Lemma 14.1 we can assume that $J \nrightarrow J$ for each $J \in T$. In particular $I_2 \nrightarrow I_2$ and so $I_2 \rightarrow I_1$. Consider the loops $I_1 \rightleftharpoons I_2$ and $I_1 \leftarrow J_1 \rightarrow I_2$. Since $0 \notin I_1$ and $x_0 \notin J_1$, from Lemma 11.5 and Proposition 7.2, we get that $B(k) \subset \text{Per}(f)$. Hence the proposition holds.

Subcase (ii). Suppose that $L_2 \in T$ and there are no basic intervals of $T \setminus \{L_2\}$ f -covering L_2 . Consider $R = T \setminus \{L_2\}$. Since $f(0) \in \text{Int}(A)$ and $L_2 \rightarrow L_2$, L_2 does not f -cover L_1 and L_3 . We claim that if $L_2 \rightarrow M$ for some $M \in R$, then there is $J \in R \setminus \{M\}$ such that $J \rightarrow M$. Now we prove the claim. Notice that since $f(0) \in \text{Int}(A)$, if $L_2 \rightarrow M$, then $A \rightarrow M$ or $C \rightarrow M$. If $A \rightarrow M$, then take $J = L_1$ and we are done. So suppose that $C \rightarrow M$. If $L_3 \in T$, then we take $J = L_3$ and we are done. Therefore suppose that $L_3 \notin T$. From

the facts that $f(0) \in \text{Int}(A)$, $f(f^{k-1}(p_e)) = p_e \in \mathbf{I}$ and by the continuity of f , it follows that each basic interval $N \in T$, $N \subset \mathbf{I}$ is f -covered by some basic interval of S' different from C . Hence we can assume that $M \not\subset \mathbf{I}$. Then $M \subset \mathbf{O}$. Since $C \rightarrow M$, $f(0) \in \text{Int}(A)$, $A \cup C \rightarrow A \cup C$, we have that $A \rightarrow [f(0), x_a] \subset A$ and $C \rightarrow [f(0), 0] \cup C \subset A \cup C$. By Lemma 11.12 we can suppose that there is $J_2 \in S'$, $J_2 \neq C$ such that $J_2 \rightarrow C$. Notice that $J_2 \neq A$ because $f(A) \subset \mathbf{I}$. Moreover $J_2 \neq B$; otherwise $L_1 = A \cup B \rightarrow A \cup C = L_2$ in contradiction with the assumptions. Hence $J_2 \in S \cap S'$. Therefore either $J_2 \rightarrow C \cup A = L_2$ or $J_2 \rightarrow C \cup B = L_3$ a contradiction. So the claim is proved.

Thus each basic interval of R is f -covered by some different basic interval of R . By the proof of Lemma 14.1, interchanging T by R , we can assume that $N \nrightarrow N$ for each $N \in R$. In particular $I_2 \nrightarrow I_2$ and so $I_2 \rightarrow I_1$. Then we consider the paths $I_1 \rightleftharpoons I_2$ and $I_1 \leftarrow J_1 \rightarrow I_2$. Now the proposition follows as in Subcase (i) but thinking the proof of Lemma 11.5 with R instead of T .

Subcase (iii). Suppose that $L_2 \in T$ and L_2 is f -covered by some $M \in T \setminus \{L_2\}$. Since $J_0 \rightarrow L_1$ and $f(0) \notin \text{Int}(L_3)$, from Remark 11.9 we have that each $J \in T$ is f -covered by some $N \in T \setminus \{J\}$. By Lemma 14.1 we can assume that $J \nrightarrow J$ for each $J \in T$. Then we get $I_2 \rightarrow I_1$. Consider the paths $I_1 \leftarrow J_1 \rightarrow I_2$, $I_1 \rightleftharpoons I_2$, and the proposition follows as above.

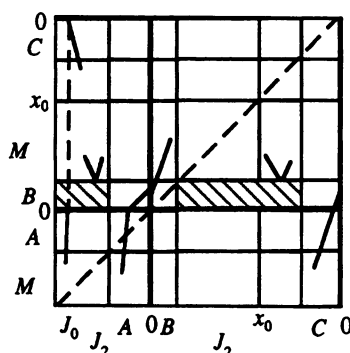
Case 2. $f(0) \in \text{Int}(B)$ and $x_0 \in \mathbf{O}$. By definition of x_0 in Section 6 we have that there are no fixed points of f in $B \cup C$. Therefore $B \rightarrow [f(0), x_b] \subset B$, $C \nrightarrow C$ and so $L_3 \nrightarrow L_3$. Notice that $I_1 \cap I_2 = \{x_0\}$. If there are no basic intervals of T f -covering $I_1 \cup I_2$, then we consider the Y map

$$g: \sigma \setminus \text{Int}(I_1 \cup I_2) \longrightarrow \sigma \setminus \text{Int}(I_1 \cup I_2)$$

defined as f restricted to $\sigma \setminus \text{Int}(I_1 \cup I_2)$. Clearly $k \in \text{Per}(g)$. Hence by the Y Theorem the result follows. So we can assume that there is $J_1 \in T \setminus \{I_1, I_2\}$ such that $I_1 \leftarrow J_1 \rightarrow I_2$. We remark that $x_0 \notin J_1$ and $0 \notin I_1 \cup I_2$.

Subcase (i). Suppose that $L_1 \notin T$. Then by definition of T we have that $L_2, L_3 \in T$. Since $L_3 \nrightarrow L_3$ we get that L_3 is f -covered by some different basic interval of T . Moreover $f(0) \notin \text{Int}(L_2)$, so by Remark 11.9 we have that each $J \in T$ is f -covered by some different basic interval of T . From Lemma 14.1 we can assume that $J \nrightarrow J$ for each $J \in T$. Then we obtain the loop $I_1 \rightleftharpoons I_2$. Therefore consider the above loop together with the path $I_1 \leftarrow J_1 \rightarrow I_2$. Since $x_0 \notin J_1$ and $0 \notin I_1$ again the proposition follows as in the Subcase (i) of Case 1.

Subcase (ii). Suppose that $L_1 \in T$ and there are no basic intervals of $T \setminus \{L_1\}$ f -covering L_1 . Since $B \rightarrow [f(0), x_b] \subset B$, we get that $f(x_a) \in \mathbf{I}$ and $A \rightarrow [f(0), 0] \cup A \subset B \cup A$. The facts that $f(x_a) \in \mathbf{I}$, $f(P \cap \mathbf{I}) \not\subset P \cap \mathbf{I}$ and the continuity of f imply that there is $J_0 \in S \cap S'$, $J_0 \subset \mathbf{I}$ such that $J_0 \rightarrow A$. Moreover either $J_0 \rightarrow A \cup B = L_1$ or $J_0 \rightarrow A \cup C = L_2$. By the hypotheses we have that $J_0 \rightarrow L_2$. Therefore $L_2 \in T$ and L_2 is f -covered by $J_0 \neq L_2$. Furthermore $L_2 \nrightarrow L_1$ and so $C \nrightarrow [f(0), x_b] \subset B$. Since $C \nrightarrow C$, we get $C \rightarrow [f(0), 0] \cup A \subset B \cup A$. Then $L_3 = B \cup C \rightarrow L_1$, and thus $L_3 \notin T$. Set $R = T \setminus \{L_1\} = S \setminus \{L_1, L_3\}$. We claim that if $L_1 \rightarrow M$ for some $M \in R$, then there is $J_2 \in R$, $J_2 \neq M$ such that $J_2 \rightarrow M$. Notice that $J_0 \rightarrow L_2$ and $J_0 \in S \cap S'$. Since $f(0) \in \text{Int}(B)$ and $B = L_1 \cap L_3$, then we can assume that $A \rightarrow M$ or $B \rightarrow M$. The facts that $f(0) \in \text{Int}(B)$, $f(f^{k-1}(p_e)) = p_e \in \mathbf{I}$ and the continuity of f imply that if $M \subset \mathbf{I}$, then M is f -covered by some

FIGURE 14.9. M is f -covered by some $J_2 \in R \setminus \{M\}$.

$J_2 \in R$, $J_2 \neq N$ (see Figure 14.9). So we can assume that $M \subset O$. From the facts that $x_b \in P$, $L_3 \notin T$, L_1 is not f -covered by any basic interval of $T \setminus \{L_1\}$ and by the continuity of f it follows that there exists $J_2 \in R$, $J_2 \neq M$ such that $J_2 \rightarrow M$ (see again Figure 14.9). Hence the claim is proved. Therefore each $J \in R$ is f -covered by some basic interval of $R \setminus \{J\}$. By the proof of Lemma 14.1, interchanging T by R , we can assume that $J \rightarrow J$ for each $J \in R$. Then we obtain $I_1 \rightleftharpoons I_2$. Consider the above loop and the path $I_1 \leftarrow J_1 \rightarrow I_2$. Now the proposition follows as the Subcase (ii) of Case 1.

Subcase (iii). Suppose that $L_1 \in T$ and that L_1 is f -covered by some different basic interval of T . Since $L_3 \rightarrow L_3$ and $f(0) \notin \text{Int}(L_2)$, from Remark 11.9 we have that each $J \in T$ is f -covered by some basic interval of $T \setminus \{J\}$. By Lemma 14.1 we can suppose that $J \rightarrow J$ for each $J \in T$. Then we obtain $I_1 \rightleftharpoons I_2$. Consider the above loop together with the path $I_1 \leftarrow J_1 \rightarrow I_2$. This ends as in the Subcase (i). \square

15. PROOF OF THEOREM 1.4(b)

The goal of this section is to prove Theorem 1.4(b).

Lemma 15.1. *Let $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$. Denote by a and b two endpoints of Y . Then there is an Y map g having the branching point fixed such that $g(a) = a$, $g(b) = b$ and $\text{Per}(g) = S(n_s) \cup G(n_g) \cup R(n_r)$.*

Proof. Take $a_1 \in (a, 0) \subset Y$ and $b_1 \in (b, 0) \subset Y$. We consider the new topological space $\tilde{Y} = Y \setminus ([a, a_1] \cup [b, b_1]) \subset Y$. Of course, \tilde{Y} is homeomorphic to Y . Then by the Y Theorem there is an \tilde{Y} map \tilde{g} having the branching point fixed and such that $\text{Per}(\tilde{g}) = S(n_s) \cup G(n_g) \cup R(n_r)$. Now we define an Y map $g : Y \rightarrow Y$ as follows. Let $g(x) = \tilde{g}(x)$ if $x \in \tilde{Y}$. Let g restricted to $[a, a_1]$ be any homeomorphism between $[a, a_1]$ and $[a, \tilde{g}(a_1)]$ such that $g(a) = a$ and $g(a_1) = \tilde{g}(a_1)$. Finally, let g restricted to $[b, b_1]$ be any homeomorphism between $[b, b_1]$ and $[b, \tilde{g}(b_1)]$ such that $g(b) = b$ and $g(b_1) = \tilde{g}(b_1)$.

By the construction of g we get that g has no periodic points of period $n > 1$ in $Y \setminus \tilde{Y}$. Moreover, since $g|_{\tilde{Y}} = \tilde{g}$ it follows that $g(0) = 0$ and

$\text{Per}(g) = \text{Per}(\tilde{g}) = S(n_s) \cup G(n_g) \cup R(n_r)$. Thus g satisfies the statement of the lemma. \square

Proof of Theorem 1.4(b). Let $n_s \in \mathbb{N}_s$, $n_g \in \mathbb{N}_g$, $n_r \in \mathbb{N}_r$ and $n_b \in \mathbb{N}_b$, we need to construct a σ map f having the branching point fixed such that $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r) \cup B(n_b)$.

Since Y is homeomorphic to $\{(x, y) \in \sigma : y \geq -1\} \subset \sigma$, in the rest of this proof we shall consider $Y = \{(x, y) \in \sigma : y \geq -1\} \subset \sigma$. Let $U = (a, b)$ be the open subinterval of σ defined as $U = \sigma \setminus Y \subset \mathbf{O}$. Clearly a and b are two endpoints of Y . Let g be the Y map given by Lemma 15.1. We will to extend g to a σ map f as follows. Let $f(x) = g(x)$ if $x \in Y$. Then notice that a, b and 0 are fixed points for f . Now we need to define $f|_U$.

Set $n = n_b$. Let $Q = \{q_1, q_2, \dots, q_n\} \subset U$ be such that $q_i \neq q_j$ for $i \neq j$, $(a, q_1) \cap Q = \emptyset$, $(q_n, b) \cap Q = \emptyset$ and for each $i \in \{1, 2, \dots, n-1\}$ $(q_i, q_{i+1}) \cap Q = \emptyset$. Define $f(q_i) = q_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$ and $f(q_n) = q_1$. For each $i \in \{1, 2, \dots, n-2\}$ let f restricted to $[q_i, q_{i+1}]$ be any homeomorphism between $[q_i, q_{i+1}]$ and $[q_{i+1}, q_{i+2}]$. Let f restricted to $[q_{n-1}, q_n]$ be any homeomorphism between $[q_{n-1}, q_n]$ and $[q_n, q_1]$. Let f restricted to $[q_n, b]$ be any homeomorphism between $[q_n, b]$ and $[b, q_1]$. Finally let f restricted to $[a, q_1]$ be any homeomorphism between $[a, q_1]$ and $[a, q_2]$.

Since $f(0) = 0$, f is a σ map. Of course Q is a periodic orbit of period n for f . Consider the set of basic intervals associated to $Q \cup \{a\}$. Therefore we obtain a non-repetitive loop $[a, q_1] \rightarrow [q_1, q_2] \rightarrow [q_2, q_3] \rightarrow \dots \rightarrow [q_{n-1}, q_n] \rightarrow [a, q_1] \rightarrow [a, q_1] \rightarrow \dots \rightarrow [a, q_1]$ of length m for each $m \geq n = n_b$. By Lemma 3.3 there exists a fixed point $x \in [q_1, q_2]$ of f^m for each $m \geq n_b$. By the definition of f x has period m . So $B(n_b) \subset \text{Per}(f)$. By construction of f all periodic points of f on U have period at least n_b . Since $f|_Y = g$, the set of periods of periodic point of $f|_Y$ is $S(n_s) \cup G(n_g) \cup R(n_r)$. Consequently $\text{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r) \cup B(n_b)$ and we are done. \square

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