

DOI-HOPF MODULES, YETTER-DRINFEL'D MODULES AND FROBENIUS TYPE PROPERTIES

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ABSTRACT. We study the following question: when is the right adjoint of the forgetful functor from the category of (H, A, C) -Doi-Hopf modules to the category of A -modules also a left adjoint? We can give some necessary and sufficient conditions; one of the equivalent conditions is that $C \otimes A$ and the smash product $A \# C^*$ are isomorphic as $(A, A \# C^*)$ -bimodules. The isomorphism can be described using a generalized type of integral. Our results may be applied to some specific cases. In particular, we study the case $A = H$, and this leads to the notion of k -Frobenius H -module coalgebra. In the special case of Yetter-Drinfel'd modules over a field, the right adjoint is also a left adjoint of the forgetful functor if and only if H is finite dimensional and unimodular.

0. INTRODUCTION

Let H be a Hopf algebra with bijective antipode over a commutative ring k . Let A be a (left) H -comodule algebra, and C a (right) H -module coalgebra. Doi [6] introduced the notion of unified Hopf module (we will call such a module a Doi-Hopf module). This is a k -module, which is at once a right A -module and a left C -comodule, satisfying the compatibility relation (1.7). Several module structures appear as special cases; let us mention Sweedler's Hopf modules [20], Takeuchi's relative Hopf modules [21], graded modules, and modules graded by a G -set. It was proved recently [2] that Yetter-Drinfel'd modules are also a special case of Doi-Hopf modules.

In [3], induction functors between categories of Doi-Hopf modules are discussed. It turns out that many pairs of adjoint functors are special cases, for example the functor forgetting action or coaction, extension and restriction of scalars and coscalars. In this paper, we focus attention on the functor F from the category of Doi-Hopf modules to the category of right A -modules forgetting the C -coaction. This functor has a right adjoint $G = C \otimes \bullet$. A natural question that arises is the following: when is G also a left adjoint of F ?

In fact, we can view this problem as a Frobenius type problem. It can be shown (cf. [13, Theorem 3.15]) that a ring extension $R \rightarrow S$ is Frobenius (of the first type) if and only if the induction functor (which is a left adjoint of the restriction

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of scalars functor) is isomorphic to the coinduction functor (which is a right adjoint of the restriction of scalars functor). With this example in mind, we can call a functor F a *Frobenius functor* if F has a right adjoint that is also a left adjoint. Then a ring extension $R \rightarrow S$ is Frobenius if and only if the restriction of scalars functor is Frobenius. With this terminology, [4, Proposition 2.5] can be restated as follows: the forgetful functor $F : R\text{-gr} \rightarrow R\text{-mod}$ from the category of G -graded R -modules to R -modules (G a group, R a G -graded ring) is Frobenius if and only if G is a finite group. Our problem can now be restated as follows: when is the forgetful functor from Doi-Hopf modules to right A -modules a Frobenius functor? If C is finitely generated and projective, then this comes down to examining when the extension $A \rightarrow A\#C^*$ is Frobenius in the classical sense.

It turns out that this question is related to the existence of a generalized type of integral. It was observed in [6] and [3] that $C \otimes A$ is a “natural” object of the category of Doi-Hopf modules. In fact, $C \otimes A$ is a two-sided Hopf module, and therefore an A -bimodule. An H -integral of $C \otimes A$ is by definition an element of $(C \otimes A)^A$, and it can be shown that such an integral always exists if G is a left adjoint of F and C is projective as a k -module. Under this hypothesis, we also have that C is finitely generated. We therefore have that C is finitely generated and projective, and this allows us to define another “natural” two-sided Doi-Hopf module, this time isomorphic to $C^* \otimes A$ as a k -module. This new module is in fact isomorphic to the smash product of C^* and A . Our main result, Theorem 2.4, is now the following: if C is projective as a k -module, then G is a left adjoint of F if and only if C is finitely generated and $C^* \otimes A$ and $C \otimes A$ are isomorphic as A -bimodules and left C -comodules (or, equivalently if $A\#C^*$ and $C \otimes A$ are isomorphic as $(A, A\#C^*)$ -bimodules).

In some particular cases, Theorem 2.4 takes an easier form. In the graded case, it follows that the forgetful functor from A -modules graded by a G -set X to A -modules is Frobenius if and only if X is a finite set (Corollary 2.8). A similar result holds for Sweedler’s Hopf modules, at least if we work over a field. Then the functor from Hopf modules to H -modules is Frobenius if and only if H is finite dimensional (Corollary 2.10).

In Section 3, we study the particular situation when $A = H$. We introduce the notion of k -Frobenius H -module coalgebra, generalizing Pareigis’ k -Frobenius Hopf algebras [17]. Several characterizations of k -Frobenius H -module coalgebras are given (Theorem 3.4). If C is projective, then G is a left adjoint of the forgetful functor if and only if C is a k -Frobenius H -module coalgebra. As an easy application, we can give a coalgebra version of a result of Ulbrich [22]: if C is a k -Frobenius H -module coalgebra, then $A\#C^* \cong \text{End}_A^C(C \otimes A)$ as k -algebras.

In Section 4, we apply our results to the category of Yetter-Drinfel’d modules. Here our main result is that the induction functor $H \otimes \bullet$ from right H -modules to Yetter-Drinfel’d modules is a left adjoint of the forgetful functor if and only if H is k -Frobenius and unimodular. We also give characterizations of unimodular Hopf algebras over a field k . Namely, if H is finite dimensional, then H is unimodular if and only if the extension $H \rightarrow D(H)$ is Frobenius. If H is finitely generated and projective, then we can give a new description of the Drinfel’d double. As a k -algebra, $D(H)$ is isomorphic to the endomorphism ring $\text{End}_H^H(H^* \otimes H)$, where $H^* \otimes H$ has a natural structure of Yetter-Drinfel’d module. If H is k -Frobenius, then $D(H) \cong \text{End}_H^H(H \otimes H)$.

Further investigations on Frobenius type properties of functors between categories of Doi-Hopf modules have been carried out recently by Borong Zhou. We refer to the forthcoming [24].

1. PRELIMINARIES

Throughout this paper, k will be a commutative ring with 1. Unless specified otherwise, all modules, algebras, coalgebras and Hopf algebras are over k , and unadorned \otimes and Hom are \otimes_k and Hom_k . In the sequel, H will be a Hopf algebra with invertible antipode S . For a coalgebra $(C, \Delta_C, \varepsilon_C)$ and a left C -comodule (M, ρ_M) we will use Sweedler's \sum -notation $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ and $\rho_M(m) = \sum m_{<-1>} \otimes m_{<0>}$, where $c \in C, m \in M$. ${}^C\mathcal{M}$ will be the category of left C -comodules and C -colinear maps. For a k -algebra A, \mathcal{M}_A (resp. ${}_A\mathcal{M}$) will be used as a notation for the category of right (resp. left) A -modules and A -linear maps.

Recall that a k -algebra A that is also a left H -comodule is called a left H -comodule algebra if the comodule structure map ρ_A is an algebra map. Similarly, a k -coalgebra that is also a right H -module is called a right H -module coalgebra if the module structure map $C \otimes H \rightarrow C, c \otimes h \mapsto (c \cdot h)$ is a coalgebra map.

Suppose that C is a finitely generated projective k -coalgebra, and let $\{c_i^*, c_i | i = 1, \dots, n\}$ be a dual basis for C . Then we can define a left and a right C -coaction on C^* as follows:

$$(1.1) \quad \rho_{C^*}^l(c^*) = \sum c_{<-1>}^* \otimes c_{<0>}^* \iff c^* * d^* = \sum \langle d^*, c_{<-1>}^* \rangle c_{<0>}^*$$

$$(1.2) \quad \rho_{C^*}^r(c^*) = \sum c_{<0>}^* \otimes c_{<1>}^* \iff d^* * c^* = \sum \langle d^*, c_{<1>}^* \rangle c_{<0>}^*$$

for all $d^* \in C^*$. Observe that

$$(1.3) \quad \rho_{C^*}^l(c^*) = \sum_{i=1}^n c_i \otimes (c^* * c_i^*)$$

$$(1.4) \quad \rho_{C^*}^r(c^*) = \sum_{i=1}^n (c_i^* * c^*) \otimes c_i$$

C^* is a C -bicomodule, and

$$(1.5) \quad \sum (c^* * d^*)_{<-1>} \otimes (c^* * d^*)_{<0>} = \sum d_{<-1>}^* \otimes c^* * (d^*)_{<0>}$$

for all $c^*, d^* \in C^*$.

If C is a right H -module coalgebra, then C^* is a left H -module algebra with H -action given by the formula

$$\langle h \cdot c^*, c \rangle = \langle c^*, c \cdot h \rangle$$

and

$$(1.6) \quad \sum (h \cdot c^*)_{<-1>} \otimes (h \cdot c^*)_{<0>} = \sum c_{<-1>}^* (Sh_{(2)}) \otimes h_{(1)} \cdot c_{<0>}^*$$

for all $h \in H$ and $c^* \in C^*$. Indeed,

$$\begin{aligned}
 \sum (h \cdot c^*)_{\langle -1 \rangle} \otimes (h \cdot c^*)_{\langle 0 \rangle} &= \sum_{i=1}^n c_i \otimes (h \cdot c^*) * c_i^* \\
 &= \sum_{i=1}^n c_i \otimes (h_{(1)} \cdot c^*) * (h_{(2)} Sh_{(3)} \cdot c_i^*) \\
 (C^* \text{ is a left } H\text{-module algebra}) &= \sum_{i=1}^n c_i \otimes h_{(1)} \cdot (c^* * (Sh_{(2)} \cdot c_i^*)) \\
 &= \sum_{i,j=1}^n c_i \otimes h_{(1)} \cdot (c^* * \langle Sh_{(2)} \cdot c_i^*, c_j \rangle c_j^*) \\
 &= \sum_{i,j=1}^n c_i \otimes h_{(1)} \cdot (c^* * \langle c_i^*, c_j \cdot Sh_{(2)} \rangle c_j^*) \\
 &= \sum_{j=1}^n c_j \cdot Sh_{(2)} \otimes h_{(1)} \cdot (c^* * c_j^*) \\
 &= \sum c_{\langle -1 \rangle}^* \cdot Sh_{(2)} \otimes h_{(1)} \cdot c_{\langle 0 \rangle}^*
 \end{aligned}$$

1.1. Doi-Hopf modules. Let A be a left H -comodule algebra and C be a right H -module coalgebra. Following [2], we will call the three-tuple (H, A, C) a Doi-Hopf datum. A right-left (H, A, C) -Hopf module is a k -module M which is a right A -module and a left C -comodule via $\rho_M : M \rightarrow C \otimes M$ such that:

$$(1.7) \quad \rho_M^l(ma) = \sum m_{\langle -1 \rangle} \cdot a_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle} a_{\langle 0 \rangle}$$

for all $a \in A, m \in M$. ${}^C\mathcal{M}(H)_A$ will be the category of right-left (H, A, C) -Hopf module and A -linear C -colinear homomorphisms (see [6]).

A left-right (H, A, C) -Hopf module is a left A -module and a right C -comodule such that

$$\sum am_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} = \sum (a_{\langle 0 \rangle} m)_{\langle 0 \rangle} \otimes (a_{\langle 0 \rangle} m)_{\langle 1 \rangle} \cdot a_{\langle -1 \rangle}$$

or

$$(1.8) \quad \rho_M^r(am) = \sum a_{\langle 0 \rangle} m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} \cdot S^{-1}(a_{\langle -1 \rangle})$$

for all $a \in A$ and $m \in M$. The category of left-right (H, A, C) -Doi-Hopf modules will be denoted by ${}_A\mathcal{M}(H)^C$. A two-sided Doi-Hopf module is an object M of the categories ${}^C\mathcal{M}(H)_A$ and ${}_A\mathcal{M}(H)^C$ such that M is an A -bimodule, a C -bicomodule, and

$$\begin{aligned}
 \rho_M^l(am) &= \sum m_{\langle -1 \rangle} \otimes am_{\langle 0 \rangle} \\
 \rho_M^r(ma) &= \sum m_{\langle 0 \rangle} \otimes am_{\langle 1 \rangle}
 \end{aligned}$$

(cf. [3, Def. 2.1]). It is known that $C \otimes A$ is a two-sided Doi-Hopf module, and the structure maps are the following:

$$\begin{aligned} \rho^l(c \otimes a) &= \sum c_{(1)} \otimes c_{(2)} \otimes a \\ (c \otimes a) \cdot b &= \sum c \cdot b_{\langle -1 \rangle} \otimes a_i b_{\langle 0 \rangle} \\ \rho^r(c \otimes a) &= \sum c_{(1)} \otimes a_{\langle 0 \rangle} \otimes (c_2 \cdot S^{-1}(a_{\langle -1 \rangle})) \\ b \cdot (c \otimes a) &= c \otimes ba \end{aligned}$$

Now suppose that C is finitely generated and projective as a k -module. Then C^* is a left H -module algebra, and we can consider the smash product $A\#C^*$. This is $A \otimes C^*$ as a k -module, with multiplication

$$(1.9) \quad (a\#c^*)(b\#d^*) = \sum a_{\langle 0 \rangle} b\#c^* * (a_{\langle -1 \rangle} \cdot d^*)$$

We then have the following isomorphisms of categories:

$${}^C\mathcal{M}(H)_A \cong \mathcal{M}_{A\#C^*} \quad \text{and} \quad {}_A\mathcal{M}(H)^C \cong {}_{A\#C^*}\mathcal{M}$$

(cf. [6, Remark 1.3b]). If $M \in {}^C\mathcal{M}(H)_A$, then the $A\#C^*$ -action on M is given by

$$m \cdot (a\#c^*) = \sum \langle c^*, m_{\langle -1 \rangle} \rangle m_{\langle 0 \rangle} a$$

If $M \in {}_A\mathcal{M}(H)^C$, then the $A\#C^*$ -action on M is given by

$$(a\#c^*) \cdot m = \sum \langle c^*, (am)_{\langle 1 \rangle} \rangle (am)_{\langle 0 \rangle}$$

We also have that the category of two-sided Hopf modules is isomorphic to the category of $A\#C^*$ -bimodules, and this implies that we have another example of a two-sided Hopf bimodule, namely $A\#C^*$. It may be verified easily that the structure maps on $A\#C^*$ are the following ones:

$$\begin{aligned} \rho^l(a\#c^*) &= \sum c^*_{\langle -1 \rangle} \cdot a_{\langle -1 \rangle} \otimes (a_{\langle 0 \rangle} \# c^*_{\langle 0 \rangle}) \\ b \cdot (a\#c^*) &= (b \otimes \varepsilon)(a\#c^*) = \sum b_{\langle 0 \rangle} a\#b_{\langle -1 \rangle} \cdot c^* \\ \rho^r(a\#c^*) &= \sum a\#c^*_{\langle 0 \rangle} \otimes c^*_{\langle 1 \rangle} \\ (a\#c^*) \cdot b &= (a\#c^*)(b \otimes \varepsilon) = ab\#c^* \end{aligned}$$

To make some computations easier, it is convenient to write the action and coaction on $A\#C^*$ in another form. The map $\theta : A\#C^* \rightarrow C^* \otimes A$ defined by

$$\theta(a\#c^*) = \sum S^{-1}(a_{\langle -1 \rangle}) \cdot c^* \otimes a_{\langle 0 \rangle}$$

is a k -module isomorphism, and the inverse map is given by the formula

$$\theta^{-1}(c^* \otimes a) = \sum a_{\langle 0 \rangle} \# a_{\langle -1 \rangle} \cdot c^*$$

for all $a \in A$ and $c^* \in C^*$. The induced A -action and C -coaction take the following form:

$$\begin{aligned} \rho^l(c^* \otimes a) &= \sum c^*_{\langle -1 \rangle} \otimes c^*_{\langle 0 \rangle} \otimes a \\ (c^* \otimes a) \cdot b &= \sum S^{-1}(b_{\langle -1 \rangle}) \cdot c^* \otimes ab_{\langle 0 \rangle} \\ \rho^r(c^* \otimes a) &= \sum c^*_{\langle 0 \rangle} \otimes a_{\langle 0 \rangle} \otimes c^*_{\langle 1 \rangle} \cdot S^{-1}(a_{\langle -1 \rangle}) \\ b \cdot (c^* \otimes a) &= c^* \otimes ba \end{aligned}$$

Observe that $C^* \otimes A \cong A\#C^*$ is a projective generator of the category ${}^C\mathcal{M}(H)_A \cong \mathcal{M}_{A\#C^*}$, and that $C^* \otimes A$ can be viewed as an $A\#C^*$ -bimodule. We also have that

$$A\#C^* \cong \text{End}_{A\#C^*}(A\#C^*) \cong \text{End}_{A\#C^*}(C^* \otimes A) \cong \text{End}_A^C(C^* \otimes A)$$

Let (H, A, C) be a Doi-Hopf datum over k . Then we can define a functor

$$G : \mathcal{M}_A \longrightarrow {}^C\mathcal{M}(H)_A$$

as follows: for $M \in \mathcal{M}_A$, $G(M) := C \otimes M \in {}^C\mathcal{M}(H)_A$ with structure maps

$$(1.11) \quad (c \otimes m) \cdot a = \sum c \cdot a_{<-1>} \otimes ma_{<0>}$$

$$(1.12) \quad \rho_{G(M)}(c \otimes m) = \sum c_{(1)} \otimes c_{(2)} \otimes m_i$$

for all $a \in A$, $c \in C$ and $m \in M$.

The following lemma is a special case of Theorem 1.3 in [3].

Lemma 1.1. *The functor G is a right adjoint of the forgetful functor $F: {}^C\mathcal{M}(H)_A \rightarrow \mathcal{M}_A$.*

1.2. Yetter-Drinfel'd modules. Recall that a right-left Yetter Drinfel'd module is a k -module, which is at once a left H -comodule and a right H -module, such that the following compatibility relation holds:

$$(1.13) \quad \sum m_{<-1>}h_{(1)} \otimes m_{<0>}h_{(2)} = \sum S^{-1}(h_{(3)})m_{<-1>}h_{<1>} \otimes m_{<0>}h_{(2)}$$

for all $h \in H$ and $m \in M$. The category of right-left Yetter-Drinfel'd modules and H -linear H -colinear maps will be denoted by ${}^H\mathcal{YD}_H$. In a similar way, we can introduce left-left, right-right and left-right Yetter-Drinfel'd modules. The corresponding categories are ${}^H_H\mathcal{YD}$, \mathcal{YD}_H^H and ${}_H\mathcal{YD}^H$; we refer to [19] for full details.

In [2] it is shown that there is a category isomorphism

$${}^H\mathcal{YD}_H \cong {}^H\mathcal{M}(H \otimes H^{op})_H$$

The left $H \otimes H^{op}$ -comodule algebra structure on H is defined as follows:

$$(1.14) \quad h \mapsto \sum h_{(1)} \otimes S^{-1}(h_{(3)}) \otimes h_{(2)}$$

and the right $H \otimes H^{op}$ -module coalgebra structure on H is given by the following formula:

$$(1.15) \quad l \cdot (h \otimes k) = klh$$

for all $h, k, l \in H$. In this paper, the categories ${}^H\mathcal{YD}_H$ and ${}^H\mathcal{M}(H \otimes H^{op})_H$ will be identified.

1.2.1. k -Frobenius Hopf algebras. Let H be a Hopf algebra over k . As we have seen above, H^* can be made into a left and right H -module via

$$\langle h \rightharpoonup f, k \rangle = \langle f, kh \rangle \quad \text{and} \quad \langle f \leftarrow h, k \rangle = \langle f, hk \rangle$$

for all $h, k \in H$ and $f \in H^*$.

Definition 1.2. A Hopf algebra H over a commutative ring k is called k -Frobenius if H^* and H are isomorphic as right H -modules.

Lemma 1.3. [17] *Let H be a projective Hopf algebra over a commutative ring k . Then the following assertions are equivalent.*

- 1) H is k -Frobenius;
- 2) there is a right integral $\lambda \in H$ such that $H^* \rightarrow \lambda = H$;
- 3) there is a right integral $\lambda \in H$ such that $\lambda \leftarrow H^* = H$;
- 4) H is finitely generated, and the right integral space of H is free of rank one;
- 5) H is finitely generated, and the left integral space of H is free of rank one;
- 6) there exists an isomorphism $H^* \rightarrow H$ of left H -modules;
- 7) H is finitely generated as a k -module and H^* is k -Frobenius.

Furthermore, if H is k -Frobenius, there exists a right integral λ of H such that the map

$$\varphi : H^* \rightarrow H, \quad h^* \mapsto \lambda \leftarrow h^*, \quad h^* \in H^*$$

is bijective.

Remark 1.4. In [17], Lemma 1.3 is stated only in the case where H is finitely generated over k . It can be shown easily that H is finitely generated if condition 1), 2), 3) or 6) of Lemma 1.3 holds.

It is clear that condition 2) implies that H is finitely generated: every $h \in H$ can be written under the form

$$h = h^* \rightarrow \lambda = \sum \langle h^*, \lambda_2 \rangle \lambda_1$$

and the λ_1 's generate H .

Let us next show that 1) implies 2). Assume that H is k -Frobenius. Let $\varphi : H^* \rightarrow H$ be a right H -module isomorphism. Since $\varepsilon_H \leftarrow h = \varepsilon_H(h)\varepsilon_H$ for any $h \in H$, we have that $\lambda = \varphi(\varepsilon_H)$ is a right integral of H . Now for any $h \in H$,

$$\begin{aligned} (H^* \rightarrow \lambda)h &= \sum \langle H^*, \lambda_{(2)} \rangle \lambda_{(1)}h = \sum \langle H^*, \lambda_{(2)}Sh \rangle \lambda_{(1)} \\ &= (Sh \rightarrow H^*) \rightarrow \lambda \subseteq H^* \rightarrow \lambda \end{aligned}$$

so $I = H^* \rightarrow \lambda$ is a right ideal of H . One can check that I is also a right coideal of H . By the descent theorem for Hopf modules, $I \cong I' \otimes H$. Now for any $x \in I'$, $\Delta x = x \otimes 1_H$, and therefore $x = \varepsilon_H(x)1_H \subseteq k1_H$. Take $J = \{a \in k : a1_H \in I'\}$. Then J is an ideal of k and $I \subseteq I'H \subseteq JH$. Since $\varphi(\varepsilon_H) = \lambda \in I$, we have that $\varphi(\varepsilon_H) \in JH = \varphi(JH^*)$. By the bijectivity of φ we obtain that $\varepsilon_H \in JH^*$. Since $1_k = \varepsilon_H(1_H) \in (JH^*)(1_H) = J$, $J = k$. Therefore $I = I'H = JH = H$, or equivalently $H^* \rightarrow \lambda = H$.

Similar arguments show that 6) implies 3), and that 3) implies that H is finitely generated.

Assume that H is k -Frobenius, and let λ be a free generator of \int_H^r . Then for any $h \in H$, $h\lambda \in \int_H^r$, and therefore $h\lambda = \alpha(h)\lambda$ for a unique $\alpha(h) \in k$. Clearly α is multiplicative, and this implies that α is invertible with $\alpha^{-1} = \alpha \circ S_H$. Following the literature, we call α the distinguished element of H^* . Note that $\alpha \circ S_H^2 = \alpha$. If $\alpha = \varepsilon_H$, or, equivalently, if $\int_H^r = \int_H^l$, then we say that H is unimodular.

2. THE LEFT ADJOINT OF THE FORGETFUL FUNCTOR

Let H be a Hopf algebra with bijective antipode over the commutative ring k . Let A be a left H -comodule algebra and C a right H -module coalgebra. $F : {}^C\mathcal{M}(H)_A \rightarrow \mathcal{M}_A$ is the functor forgetting the C -coaction, and $G = C \otimes \bullet : \mathcal{M}_A \rightarrow {}^C\mathcal{M}(H)_A$ is the right adjoint of F (cf. (1.11) and (1.12)). A natural question that arises is the following: *When is G also a left adjoint of F ?* In this section,

we will give necessary and sufficient conditions. First, we introduce the following generalization of the notion of integral.

Definition 2.1. Let (H, A, C) be a Doi-Hopf datum. An element $x = \sum_{i=1}^n c_i \otimes a_i \in C \otimes A$ is called an H -integral of $C \otimes A$ if

$$(2.1) \quad \sum_{i=1}^n (c_i \otimes a_i) \cdot a = a \cdot \left(\sum_{i=1}^n c_i \otimes a_i \right)$$

or

$$\sum_{i=1}^n c_i \cdot a_{<-1>} \otimes a_i a_{<0>} = \sum_{i=1}^n c_i \otimes a a_i$$

for any $a \in A$.

Remarks 2.2. 1) Let $x = \sum_{i=1}^n c_i \otimes a_i \in C \otimes A$ be an H -integral. Applying $\varepsilon_C \otimes I_A$ to both sides of (2.1), we obtain that $z = \sum_{i=1}^n \varepsilon_C(c_i) a_i$ is a central element of A .

2) Let $C = H$ and suppose that $e \in H$ is a right integral of H . Then $e \otimes 1_A \in H \otimes A$ is an H -integral of $H \otimes A$. Indeed, we have that

$$(e \otimes 1_A) \cdot b = \sum e \cdot b_{<-1>} \otimes b_{<0>} = e \otimes b$$

for all $b \in A$.

3) Let $A = C = H$ and suppose that $x = \sum_{i=1}^n c_i \otimes a_i \in H \otimes H$ is an H -integral of $H \otimes H$. Then $e = \sum_{i=1}^n c_i \varepsilon(a_i)$ and $f = \sum_{i=1}^n c_i S(a_i)$ are right integrals of H . Indeed, if we apply $I_H \otimes \varepsilon_H$ to the identity

$$\sum c_i h_{(1)} \otimes a_i h_{(2)} = \sum c_i \otimes h a_i$$

then we obtain that e is a right integral of H . If we apply $m_H \circ (I_H \otimes S)$ to the same identity, then we obtain that $f \varepsilon(h) = f S(h)$, for all $h \in H$. Now S is bijective, and therefore it follows that $f g = f \varepsilon(g)$, for any $g \in H$, i.e. f is also a right integral of H .

4) We have seen in (1.14) and (1.15) that $(H \otimes H^{op}, H, H)$ is a Doi-Hopf datum. Let $x = \sum_{i=1}^n c_i \otimes a_i \in H \otimes H$ be an $H \otimes H^{op}$ -integral. Then $z = \sum_{i=1}^n \varepsilon(c_i) a_i$ is a central element of H . Observe that in this case $z' = \sum_{i=1}^n c_i \varepsilon(a_i)$ is also a central element of H . Indeed, from (1.14) and (1.15) it follows that $x = \sum_{i=1}^n c_i \otimes a_i$ is an $H \otimes H^{op}$ -integral if and only if

$$\sum S^{-1}(h_{(3)}) c_i h_{(1)} \otimes a_i h_{(2)} = \sum c_i \otimes h a_i$$

for all $h \in H$. Applying $I \otimes \varepsilon$ to this equality, we obtain that

$$\sum S^{-1}(h_{(2)}) z' h_{(1)} = \varepsilon(h) z'$$

for all $h \in H$. Hence z' is a central element of H . Moreover, if we denote $y = \sum_{i=1}^n c_i S(a_i)$, then $y S^2(g) = g y$, for all $g \in H$.

Lemma 2.3. Assume that G is a left adjoint of F . Then for every $M \in \mathcal{M}_A$ and $N \in {}^C \mathcal{M}(H)_A$ we have a natural isomorphism

$$\eta_{M,N} : \text{Hom}_A^C(GM, N) \rightarrow \text{Hom}_A(M, FN)$$

Then

$$(2.2) \quad x = \sum_{i=1}^n c_i \otimes a_i = \eta_{A, C \otimes A}(I_{C \otimes A})(1_A)$$

is an H -integral. Moreover, for any $M \in {}^C\mathcal{M}(H)_A$, $f \in \text{Hom}_A^C(C \otimes M, N)$ and $m \in M$

$$(2.3) \quad \eta_{M,N}(f)(m) = \sum_{i=1}^n f(c_i \otimes ma_i)$$

Proof. G is a left adjoint of F ; hence

$$\eta_{M,GM}(I_{GM}) : M \mapsto F(G(M))$$

is a natural transformation from $\text{Id}_{\mathcal{M}_A}$ to $F \circ G$, cf. [8, p.232]. This means that for any $L, M \in \mathcal{M}_A$ and $f \in \text{Hom}_A(L, M)$, the diagram

$$\begin{CD} L @>\eta_{L,G(L)}(I_{G(L)})>> F(G(L)) \\ @VfVV @VVF(G(f))V \\ M @>\eta_{M,G(M)}(I_{G(M)})>> F(G(M)) \end{CD}$$

is commutative, or, equivalently,

$$(2.4) \quad \eta_{M,G(M)}(I_{G(M)}) \circ f = F(G(f)) \circ \eta_{L,G(L)}(I_{G(L)})$$

for every $f \in \text{Hom}_A(L, M)$. Take $L = M = A$, and let $f = a_l \in \text{Hom}_A(A, A)$ be given by left multiplication by $a \in A$. Then (2.4) takes the following form:

$$\eta_{A,C \otimes A}(I_{C \otimes A}) \circ a_l = (I_C \otimes a_l) \circ \eta_{A,C \otimes A}(I_{C \otimes A})$$

Evaluate both sides of this equation at 1_A . The left hand side is

$$\begin{aligned} \eta_{A,C \otimes A}(I_{C \otimes A}) \circ a_l(1_A) &= \eta_{A,C \otimes A}(I_{C \otimes A})(a) \\ &= (\eta_{A,C \otimes A}(I_{C \otimes A})(1_A)) \cdot a \\ &= \sum_{i=1}^n (c_i \otimes a_i) \cdot a \end{aligned}$$

while the right hand side amounts to

$$\begin{aligned} (I_C \otimes a_l) \circ \eta_{A,C \otimes A}(I_{C \otimes A})(1_A) \\ = (I_C \otimes a_l) \left(\sum_{i=1}^n c_i \otimes a_i \right) = \sum_{i=1}^n c_i \otimes aa_i \end{aligned}$$

Therefore

$$\sum_{i=1}^n (c_i \otimes a_i) \cdot a = \sum_{i=1}^n c_i \otimes aa_i$$

for all $a \in A$, and x is an H -integral of $C \otimes A$.

Let us next show that (2.3) holds if $M = A$. Take $f \in \text{Hom}_A^C(C \otimes A, N)$. η is natural, hence the diagram

$$\begin{CD} \text{Hom}_A^C(C \otimes A, C \otimes A) @>\eta_{A,C \otimes A}>> \text{Hom}_A(A, C \otimes A) \\ @V\text{Hom}_A^C(C \otimes A, f)VV @VV\text{Hom}_A(A, f)V \\ \text{Hom}_A^C(C \otimes A, N) @>\eta_{A,N}>> \text{Hom}_A(A, N) \end{CD}$$

commutes, that is,

$$\eta_{A,N} \circ \text{Hom}_A^C(C \otimes A, f) = \text{Hom}_A(A, f) \circ \eta_{A,C \otimes A}$$

Evaluating this identity at $I_{A \otimes C}$, we obtain

$$\eta_{A,N}(f) = f \circ \eta_{A,C \otimes A}(I_{C \otimes A})$$

Now evaluate both sides of this last equation at $a \in A$. Then it follows that

$$\begin{aligned} \eta_{A,N}(f)(a) &= f \circ \eta_{A,C \otimes A}(I_{C \otimes A})(a) \\ &= f[\eta_{A,C \otimes A}(I_{C \otimes A})(1_A) \cdot a] \\ (2.5) \qquad &= f\left[\sum_{i=1}^n (c_i \otimes a_i) \cdot a\right] \\ &= \sum_{i=1}^n f(c_i \otimes aa_i) \end{aligned}$$

and this proves (2.3) in the case where $M = A$.

Now take an arbitrary right A -module M . For $m \in M$, we write m_l for the right A -linear map $m_l : A \rightarrow M, a \mapsto ma$. From the naturality of η , it now follows that the diagram

$$\begin{array}{ccc} \text{Hom}_A^C(C \otimes A, N) & \xrightarrow{\eta_{A,N}} & \text{Hom}_A(A, N) \\ \uparrow \text{Hom}_A^C(I_C \otimes m_l, N) & & \uparrow \text{Hom}_A(m_l, N) \\ \text{Hom}_A^C(C \otimes M, N) & \xrightarrow{\eta_{M,N}} & \text{Hom}_A(M, N) \end{array}$$

is commutative. Therefore

$$\text{Hom}_A(m_l, N) \circ \eta_{M,N} = \eta_{A,N} \circ \text{Hom}_A^C(I_C \otimes m_l, N)$$

Apply both sides to $g \in \text{Hom}_A^C(C \otimes M, N)$. Then it follows that

$$\text{Hom}_A(m_l, N)(\eta_{M,N}(g)) = \eta_{A,N}(\text{Hom}_A^C(I_C \otimes m_l, N)(g))$$

Now evaluating both sides at $m \in M$, we obtain that

$$\eta_{M,N}(g) \circ m_l = \eta_{A,N}(g \circ (I_C \otimes m_l))$$

and

$$\begin{aligned} \eta_{M,N}(g)(m) &= \eta_{M,N}(g) \circ m_l(1_A) \\ &= \eta_{A,N}(g \circ (I_C \otimes m_l))(1_A) \\ \text{(by (2.5))} \qquad &= \sum_{i=1}^n g \circ (I_C \otimes m_l)(c_i \otimes a_i) \\ &= \sum_{i=1}^n g(c_i \otimes ma_i) \end{aligned}$$

for all $m \in M$. This finishes the proof of the lemma. □

We can now state necessary and sufficient conditions for $G = C \otimes \bullet$ to be a left adjoint of F .

Theorem 2.4. *Let (H, A, C) be a Doi-Hopf datum. Assume that A is faithfully flat as a k -module, and that C is projective as a k -module. Then the following statements are equivalent:*

- 1) *The functor $G = C \otimes \bullet : \mathcal{M}_A \rightarrow {}^C\mathcal{M}(H)_A$ is a left adjoint of the forgetful functor $F : {}^C\mathcal{M}(H)_A \rightarrow \mathcal{M}_A$.*

- 2) C is finitely generated as a k -module, and there exists an H -integral $x = \sum_{i=1}^n c_i \otimes a_i \in C \otimes A$ such that the map

$$\varphi : C^* \otimes A \rightarrow C \otimes A, \quad c^* \otimes a \mapsto \sum_{i=1}^n c_i \leftarrow c^* \otimes a a_i$$

is bijective.

- 3) C is finitely generated as a k -module, and there exists a bijective map $\varphi : C^* \otimes A \rightarrow C \otimes A$ such that φ is at once an A - A -bimodule map and a left C -comodule map, where the left action of A on $C \otimes A$ is given by

$$b \cdot (c \otimes a) = c \otimes ba$$

for all $a, b \in A$ and $c \in C$;

- 4) C is finitely generated as a k -module, and $A \# C^*$ and $C \otimes A$ are isomorphic as $(A, A \# C^*)$ -bimodules;
- 5) C is finitely generated and the extension $A \rightarrow A \# C^*$ is Frobenius.

Proof. 1) \Rightarrow 2). We write $\eta_{M,N} : \text{Hom}_A^C(G(M), N) \rightarrow \text{Hom}_A(M, F(N))$ for the natural isomorphism. Take

$$x = \sum_{i=1}^n c_i \otimes a_i = \eta_{A, G(A)}(I_{G(A)})(1_A)$$

We have seen in Lemma 2.3 that x is an H -integral of $C \otimes A$. For any $i = 1, \dots, n$ we will write

$$\Delta_C(c_i) = \sum_{j=1}^{n_i} c_{ij} \otimes d_{ij} \in C \otimes C$$

Take $c \in C$. $c_l \in \text{Hom}_A(A, C \otimes A)$ will be the right A -module map given by $a \mapsto (c \otimes 1_A) \cdot a$, for all $a \in A$, and let $f = \eta_{A, C \otimes A}^{-1}(c_l) \in \text{Hom}_A^C(C \otimes A, C \otimes A)$. From Lemma 2.3, it follows that

$$(c \otimes 1) = c_l(1_A) = \eta_{A, C \otimes A}(f)(1_A) = f\left(\sum_{i=1}^n c_i \otimes a_i\right)$$

Note that f is C -colinear, and

$$\begin{aligned} \sum c_{(1)} \otimes c_{(2)} \otimes 1_A &= \sum_{i=1}^n c_{i(1)} \otimes f(c_{i(2)} \otimes a_i) \\ &= \sum_{i=1}^n \sum_{j=1}^{n_i} c_{ij} \otimes f(d_{ij} \otimes a_i) \in \left(\sum_{i=1}^n \sum_{j=1}^{n_i} c_{ij} k\right) \otimes C \otimes A. \end{aligned}$$

Applying ε_C to the second factor, we obtain that

$$c \otimes 1_A \in \left(\sum_{i=1}^n \sum_{j=1}^{n_i} c_{ij} k\right) \otimes A$$

From the fact that A is faithfully flat over k , it follows that the map $C \rightarrow C \otimes A$, $c \mapsto c \otimes 1_A$ is injective, cf. e.g. [11, Prop. II.2.1]. It therefore follows that $c \in \sum_{i=1}^n \sum_{j=1}^{n_i} c_{ij} k$, and therefore $C = \sum_{i=1}^n \sum_{j=1}^{n_i} c_{ij} k$ is finitely generated.

Let us now show that the map φ is bijective. From Section 1.1, we know that $C^* \otimes A \in {}^C \mathcal{M}(H)_A$. Consider the map

$$(\varepsilon_C \otimes 1_A)_l : A \rightarrow C^* \otimes A, \quad a \mapsto (\varepsilon_C \otimes 1_A) \cdot a, \quad a \in A$$

Then $(\varepsilon_C \otimes 1_A)_l \in \text{Hom}_A(A, C^* \otimes A)$. By the adjointness property, there exists

$$\psi \in \text{Hom}_A^C(C \otimes A, C^* \otimes A)$$

such that

$$\eta_{A, C^* \otimes A}(\psi) = (\varepsilon_C \otimes 1_A)_l$$

Using (2.3), we obtain, for any $a \in A$,

$$\begin{aligned} \varepsilon_C \otimes a &= \sum S^{-1} a_{\langle -1 \rangle} \cdot \varepsilon_C \otimes a_{\langle 0 \rangle} \\ &= (\varepsilon_C \otimes 1_A) \cdot a = (\varepsilon_C \otimes 1_A)_l(a) \\ (2.6) \quad &= \eta_{A, C^* \otimes A}(\psi)(a) \\ &= \psi\left(\sum_{i=1}^n c_i \otimes aa_i\right) \end{aligned}$$

Note that ψ is colinear. Take a dual basis $\{d_j^*, d_j\}_{j=1}^m$ of C . We then have

$$\begin{aligned} \sum_{j=1}^m d_j \otimes (\varepsilon_C * d_j^*) \otimes a &= \sum (\varepsilon_C \otimes a)_{\langle -1 \rangle} \otimes (\varepsilon_C \otimes a)_{\langle 0 \rangle} \\ &= \sum \psi\left(\sum_{i=1}^n c_i \otimes aa_i\right)_{\langle -1 \rangle} \otimes \psi\left(\sum_{i=1}^n c_i \otimes aa_i\right)_{\langle 0 \rangle} \\ &= \sum_{i=1}^n \left(\sum_{i=1}^n c_i \otimes aa_i\right)_{\langle -1 \rangle} \otimes \psi\left(\left(\sum_{i=1}^n c_i \otimes aa_i\right)_{\langle 0 \rangle}\right) \\ &= \sum c_{i(1)} \otimes \psi(c_{i(2)} \otimes aa_i) \end{aligned}$$

Take $c^* \in C^*$ and apply c^* to the first factor of the above identity. Then we obtain that

$$\begin{aligned} c^* \otimes a &= \sum_{j=1}^m \langle c^*, d_j \rangle d_j^* \otimes a \\ &= \sum_{i=1}^n \sum \langle c^*, c_{i(1)} \rangle \psi(c_{i(2)} \otimes aa_i) \\ &= \sum_{i=1}^n \psi(c_i \leftarrow c^* \otimes aa_i) \\ &= (\psi \circ \varphi)(c^* \otimes a) \end{aligned}$$

and therefore $\psi \circ \varphi = \text{Id}_{C^* \otimes A}$.

Let us now prove that $\varphi \circ \psi = \text{Id}_{C \otimes A}$ or, equivalently,

$$\eta_{A, C \otimes A}(\varphi \circ \psi) = \eta_{A, C \otimes A}(\text{Id}_{C \otimes A}).$$

For every $a \in A$, we have

$$\begin{aligned} \eta_{A,C \otimes A}(\varphi \circ \psi)(a) &= \sum (\varphi \circ \psi)(c_i \otimes aa_i) \quad (\text{by (2.3)}) \\ &= \varphi(\varepsilon_C \otimes a) \quad (\text{by (2.6)}) \\ &= \sum_{i=1}^n c_i \leftarrow \varepsilon_C \otimes aa_i \\ &= \sum_{i=1}^n c_i \otimes aa_i \\ &= \sum_{i=1}^n (c_i \otimes a_i) \cdot a \\ &= \eta_{A,C \otimes A}(\text{Id}_{C \otimes A})(a), \end{aligned}$$

and this proves that φ is bijective.

2) \Rightarrow 3). It suffices to show that the map

$$\varphi : C^* \otimes A \rightarrow C \otimes A, \quad c^* \otimes a \mapsto \sum_{i=1}^n c_i \leftarrow c^* \otimes aa_i, \quad c^* \in C^*, \quad a \in A$$

is left and right A -linear and left C -colinear.

For $c^* \in C^*$ and $a, b, d \in A$, we have that

$$\begin{aligned} \varphi(b \cdot (c^* \otimes a) \cdot d) &= \varphi(S^{-1}d_{\langle -1 \rangle} \cdot c^* \otimes bad_{\langle 0 \rangle}) \\ &= \sum_{i=1}^n \sum c_i \leftarrow (S^{-1}d_{\langle -1 \rangle} \cdot c^*) \otimes ba(d_{\langle 0 \rangle} a_i) \end{aligned}$$

$$\begin{aligned} (x \text{ is } H\text{-integral}) &= \sum_{i=1}^n \sum (c_i \cdot d_{\langle -1 \rangle}) \leftarrow (S^{-1}d_{\langle -2 \rangle} \cdot c^*) \otimes baa_i d_{\langle 0 \rangle} \\ &= \sum_{i=1}^n \sum \langle S^{-1}d_{\langle -2 \rangle} \cdot c^*, (c_i \cdot d_{\langle -1 \rangle})_{(1)} \rangle (c_i \cdot d_{\langle -1 \rangle})_{(2)} \otimes baa_i d_{\langle 0 \rangle} \\ &= \sum_{i=1}^n \sum \langle S^{-1}d_{\langle -3 \rangle} \cdot c^*, c_{i(1)} d_{\langle -2 \rangle} \rangle c_{i(2)} \cdot d_{\langle -1 \rangle} \otimes baa_i d_{\langle 0 \rangle} \\ &= \sum_{i=1}^n \sum \langle c^*, c_{i(1)} \cdot d_{\langle -2 \rangle} S^{-1}d_{\langle -3 \rangle} \rangle c_{i(2)} \cdot d_{\langle -1 \rangle} \otimes baa_i d_{\langle 0 \rangle} \\ &= \sum_{i=1}^n \langle c^*, c_{i(1)} \rangle c_{i(2)} \cdot d_{\langle -1 \rangle} \otimes baa_i d_{\langle 0 \rangle} \\ &= b \cdot \left(\sum_{i=1}^n c_i \leftarrow c^* \otimes aa_i \right) \cdot d \\ &= b \cdot \varphi(c^* \otimes a) \cdot d \end{aligned}$$

and this shows that φ is an A - A -bimodule map. We will now show that φ is left C -colinear. Let $\{d_j^*, d_j\}_{j=1}^m$ be a dual basis of C . For $c^* \in C^*$ and $a \in A$, we have

that

$$\begin{aligned} \sum (c^* \otimes a)_{\langle -1 \rangle} \otimes \varphi((c^* \otimes a)_{\langle 0 \rangle}) &= \sum_{j=1}^m \sum d_j \otimes \varphi(c^* * d_j^* \otimes a) \\ &= \sum_{i=1}^n \sum_{j=1}^m d_j \otimes c_i \leftarrow (c^* * d_j^*) \otimes aa_i \\ &= \sum_{i=1}^n \sum_{j=1}^m \sum d_j \otimes \langle c^* * d_j^*, c_{i(1)} \rangle c_{i(2)} \otimes aa_i \\ &= \sum_{i=1}^n \sum_{j=1}^m d_j \langle d_j^*, c_{i(1)} \leftarrow c^* \rangle \otimes c_{i(2)} \otimes aa_i \\ &= \sum_{i=1}^n \Delta_C(c_i \leftarrow c^*) \otimes aa_i \end{aligned}$$

and φ is C -colinear.

3) \Rightarrow 1). Assume that $\varphi : C^* \otimes A \rightarrow C \otimes A$ is as in 3). Since φ is C -colinear and $\sum \varepsilon_C \langle -1 \rangle \otimes \varepsilon_C \langle 0 \rangle = \sum_{j=1}^m d_j \otimes d_j^*$, where $\{d_j^*, d_j\}_{j=1}^m$ is a dual basis, we have

$$\begin{aligned} &\sum [\varphi(\varepsilon_C \otimes a)]_{\langle -1 \rangle} \otimes [\varphi(\varepsilon_C \otimes a)]_{\langle 0 \rangle} \\ &= \sum (\varepsilon_C \otimes a)_{\langle -1 \rangle} \otimes \varphi((\varepsilon_C \otimes a)_{\langle 0 \rangle}) \\ &= \sum_{j=1}^m d_j \otimes \varphi(d_j^* \otimes a) \end{aligned}$$

for any $a \in A$. Therefore

$$\begin{aligned} &\sum \langle c^*, \varphi(\varepsilon_C \otimes a)_{\langle -1 \rangle} \rangle \varphi(\varepsilon_C \otimes a)_{\langle 0 \rangle} \\ (2.7) \quad &= \varphi\left(\sum_{j=1}^m \langle c^*, d_j \rangle d_j^* \otimes a\right) = \varphi(c^* \otimes a) \end{aligned}$$

for any $c^* \in C^*$.

Consider the map

$$\theta'_M : M \times (C \otimes A) \rightarrow C \otimes M, \quad (m, c \otimes a) \mapsto c \otimes ma, \quad m \in M, \quad a \in A$$

Then

$$\theta'_M(mb, c \otimes a) = c \otimes mba = \theta'_M(m, b \cdot (c \otimes a))$$

for all $m \in M, c \in C$ and $a, b \in A$, and θ'_M induces a map

$$\theta''_M : M \otimes_A (C \otimes A) \rightarrow C \otimes M, \quad m \otimes_A (c \otimes a) \mapsto c \otimes ma, \quad m \in M, \quad c \in C, \quad a \in A$$

It may be verified easily that θ''_M is bijective. The inverse is given by

$$c \otimes m \mapsto m \otimes_A (c \otimes 1_A), \quad c \in C, \quad m \in M$$

Moreover, for any $M' \in \mathcal{M}_A$ and $g \in \text{Hom}_A(M, M')$, $m \in M, c \in C$ and $a \in A$ we have

$$(2.8) \quad (I_C \otimes g)\theta''_M(m \otimes_A (c \otimes a)) = \theta''_{M'}(g(m) \otimes_A (c \otimes a))$$

Let θ_M be the composition of the isomorphisms $I_M \otimes \varphi$ and θ''_M :

$$\theta_M : M \otimes_A (C^* \otimes A) \xrightarrow{I_M \otimes \varphi} M \otimes_A (C \otimes A) \xrightarrow{\theta''_M} C \otimes M$$

Observe that

$$\begin{aligned}
 & \sum \langle c^*, \theta''_M(m \otimes_A (c \otimes a))_{\langle -1 \rangle} \rangle \theta''(m \otimes_A (c \otimes a))_{\langle 0 \rangle} \\
 &= \sum \langle c^*, (c \otimes ma)_{\langle -1 \rangle} \rangle (c \otimes ma)_{\langle 0 \rangle} \\
 &= \sum \langle c^*, c_{(1)} \rangle c_{(2)} \otimes ma \\
 (2.9) \quad &= \sum \theta''_M(m \otimes_A (\langle c^*, (c \otimes a)_{\langle -1 \rangle} \rangle (c \otimes a)_{\langle 0 \rangle}))
 \end{aligned}$$

for any $c^* \in C^*$, $c \in C$, and $a \in A$. Therefore

$$\begin{aligned}
 & \sum \langle c^*, \theta_M(m \otimes_A (\varepsilon_C \otimes a))_{\langle -1 \rangle} \rangle \theta_M(m \otimes_A (\varepsilon_C \otimes a))_{\langle 0 \rangle} \\
 &= \sum \langle c^*, \theta''_M(m \otimes_A \varphi(\varepsilon_C \otimes a))_{\langle -1 \rangle} \rangle \theta''_M(m \otimes_A \varphi(\varepsilon_C \otimes a))_{\langle 0 \rangle} \\
 &= \theta''_M(m \otimes_A \sum \langle c^*, \varphi(\varepsilon_C \otimes a)_{\langle -1 \rangle} \rangle \varphi(\varepsilon_C \otimes a)_{\langle 0 \rangle}) \quad (\text{by (2.9)}) \\
 &= \theta''_M(m \otimes_A \varphi(c^* \otimes a)) \quad (\text{by (2.7)}) \\
 (2.10) \quad &= \theta_M(m \otimes_A (c^* \otimes a))
 \end{aligned}$$

for any $m \in M$, $c^* \in C^*$ and $a \in A$. Take $M \in \mathcal{M}_A$ and $N \in {}^C\mathcal{M}(H)_A$, and define

$$\eta_{M,N} : \text{Hom}_A^C(C \otimes M, N) \rightarrow \text{Hom}_A(M, N)$$

by

$$\eta_{M,N}(f)(m) = f \circ \theta_M(m \otimes_A (\varepsilon_C \otimes 1_A))$$

for $f \in \text{Hom}_A^C(C \otimes M, N)$ and $m \in M$. Then for any $a \in A$,

$$\begin{aligned}
 \eta_{M,N}(f)(ma) &= f \circ \theta_M(ma \otimes_A (\varepsilon_C \otimes 1_A)) \\
 &= f \circ \theta_M(m \otimes_A a \cdot (\varepsilon_C \otimes 1_A)) \\
 &= f \circ \theta_M(m \otimes_A (\varepsilon_C \otimes 1_A) \cdot a) \\
 &= f(\theta_M(m \otimes_A (\varepsilon_C \otimes 1_A))a) \\
 &= f(\theta_M(m \otimes_A (\varepsilon_C \otimes 1_A)))a \\
 &= (\eta_{M,N}(f)(m))a
 \end{aligned}$$

showing that $\eta_{M,N}(f) \in \text{Hom}_A(M, N)$. We have to verify that $\eta_{M,N}$ is the required natural isomorphism.

Take $M' \in \mathcal{M}_A$ and $g \in \text{Hom}_A(M, M')$. Then for any $f \in \text{Hom}_A^C(C \otimes A, N)$, $m \in M$,

$$\begin{aligned}
 \eta_{M,N} \circ \text{Hom}_A^C(I_C \otimes g, N)(f)(m) &= \eta_{M,N}(f \circ (I_C \otimes g))(m) \\
 &= f \circ (I_C \otimes g) \theta_M(m \otimes_A (\varepsilon_C \otimes 1_A)) \quad (\text{by definition}) \\
 &= f[(I_C \otimes g) \circ \theta''_M \circ (I_M \otimes \varphi)(m \otimes_A (\varepsilon_C \otimes 1_A))] \\
 &= f[(I_C \otimes g) \circ \theta''_M(m \otimes_A \varphi(\varepsilon_C \otimes 1_a))] \\
 &= f \circ \theta''_M(g(m) \otimes_A (\varepsilon_C \otimes 1_A)) \quad (\text{by (2.8)}) \\
 &= f \circ \theta_M(g(m) \otimes_A (\varepsilon_C \otimes 1_A)) \\
 &= \eta_{M',N}(f) \circ g(m) \\
 &= \text{Hom}_A(g, N) \circ \eta_{M',N}(f)(m)
 \end{aligned}$$

Thus

$$\eta_{M,N} \circ \text{Hom}_A^C(I_C \otimes g, N) = \text{Hom}_A(g, N) \circ \eta_{M',N}$$

and the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A^C(C \otimes M', N) & \xrightarrow{\eta_{M',N}} & \mathrm{Hom}_A(M', N) \\ \downarrow \mathrm{Hom}_A^C(I_C \otimes g, N) & & \downarrow \mathrm{Hom}_A(g, N) \\ \mathrm{Hom}_A^C(C \otimes M, N) & \xrightarrow{\eta_{M,N}} & \mathrm{Hom}_A(M, N) \end{array}$$

commutes.

Take $N' \in {}^C\mathcal{M}(H)_A$ and $g \in \mathrm{Hom}_A^C(N, N')$. Then for any $f \in \mathrm{Hom}_A^C(C \otimes M, N)$ and $m \in M$,

$$\begin{aligned} \eta_{M,N'} \circ \mathrm{Hom}_A^C(C \otimes M, g)(f)(m) &= \eta_{M,N'}(g \circ f)(m) \\ &= g \circ f \circ \theta_M(m \otimes_A (\varepsilon_C \otimes 1_A)) \\ &= g \circ \eta_{M,N}(f)(m) \\ &= \mathrm{Hom}_A(M, g) \circ \eta_{M,N}(f)(m) \end{aligned}$$

Thus

$$\eta_{M,N'} \circ \mathrm{Hom}_A^C(C \otimes M, g) = \mathrm{Hom}_A(M, g) \circ \eta_{M,N}$$

and the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A^C(C \otimes M, N) & \xrightarrow{\eta_{M,N}} & \mathrm{Hom}_A(M, N) \\ \downarrow \mathrm{Hom}_A^C(C \otimes M, g) & & \downarrow \mathrm{Hom}_A(M, g) \\ \mathrm{Hom}_A^C(C \otimes M, N') & \xrightarrow{\eta_{M,N'}} & \mathrm{Hom}_A(M, N') \end{array}$$

commutes. This proves that η is natural.

The proof will be complete if we can show that $\eta_{M,N}$ is bijective for any $M \in \mathcal{M}_A$ and $N \in {}^C\mathcal{M}(H)_A$. For $g \in \mathrm{Hom}_A(M, N)$, define

$$\tau_g : M \times (C^* \otimes A) \rightarrow N, \quad \tau_g(m, (c^* \otimes a)) = \sum \langle c^*, g(ma)_{\langle -1 \rangle} \rangle g(ma)_{\langle 0 \rangle}$$

$c^* \in C^*$, $m \in M$, $a \in A$. Then for any $b \in A$,

$$\begin{aligned} \tau_g(mb, (c^* \otimes a)) &= \sum \langle c^*, g((mb)a)_{\langle -1 \rangle} \rangle g((mb)a)_{\langle 0 \rangle} \\ &= \sum \langle c^*, g(m(ba))_{\langle -1 \rangle} \rangle g(m(ba))_{\langle 0 \rangle} \\ &= \tau_g(m, (c^* \otimes ba)) = \tau_g(m, b \cdot (c^* \otimes a)) \end{aligned}$$

and therefore τ_g induces a map

$$\tilde{\tau}_g : M \otimes_A (C^* \otimes A) \rightarrow N, \quad m \otimes_A (c^* \otimes a) \mapsto \sum \langle c^*, g(ma)_{\langle -1 \rangle} \rangle g(ma)_{\langle 0 \rangle}$$

where $m \in M$, $c^* \in C^*$, $a \in A$. Now define

$$\xi_{M,N} : \mathrm{Hom}_A(M, N) \rightarrow \mathrm{Hom}_A^C(C \otimes M, N)$$

by

$$\xi_{M,N}(g) = \tilde{\tau}_g \circ \theta_M^{-1} : C \otimes M \rightarrow N.$$

For any $m \in M$, we have that

$$\begin{aligned}
 (\eta_{M,N} \circ \xi_{M,N})(g)(m) &= \eta_{M,N}(\xi_{M,N}(g))(m) \\
 &= \xi_{M,N}(g)(\theta_M(m \otimes_A (\varepsilon_C \otimes 1_A))) \\
 &= \tilde{\tau}_g \circ \theta_M^{-1} \circ \theta_M(m \otimes_A (\varepsilon_C \otimes 1_A)) \\
 &= \tilde{\tau}_g(m \otimes_A (\varepsilon_C \otimes 1_A)) \\
 &= \sum \langle \varepsilon_C, g(m)_{\langle -1 \rangle} \rangle g(m)_{\langle 0 \rangle} \\
 &= g(m)
 \end{aligned}$$

and therefore

$$(2.11) \quad \eta_{M,N} \circ \xi_{M,N} = \text{Id}_{\text{Hom}_A(M,N)}$$

On the other hand, for any $f \in \text{Hom}_A^C(C \otimes M, N)$, $c^* \in C^*$, $m \in M$ and $a \in A$ we have

$$\begin{aligned}
 (\xi_{M,N} \circ \eta_{M,N})(f)(\theta_M(m \otimes_A (c^* \otimes a))) &= \xi_{M,N}(\eta_{M,N}(f))(\theta_M(m \otimes_A (c^* \otimes a))) \\
 &= \tilde{\tau}_{\eta_{M,N}(f)} \theta_M^{-1} \theta_M(m \otimes_A (c^* \otimes a)) \\
 &= \tilde{\tau}_{\eta_{M,N}(f)}(m \otimes_A (c^* \otimes a)) \\
 &= \sum \langle c^*, \eta_{M,N}(f)(ma)_{\langle -1 \rangle} \rangle \eta_{M,N}(f)(ma)_{\langle 0 \rangle} \\
 &= \sum (f\theta_M(ma \otimes_A (\varepsilon_C \otimes 1_A)))_{\langle 0 \rangle} \langle c^*, (f\theta_M(ma \otimes_A (\varepsilon_C \otimes 1_A)))_{\langle -1 \rangle} \rangle \\
 &= \sum f(\theta_M(ma \otimes_A (\varepsilon_C \otimes 1_A)))_{\langle 0 \rangle} \langle c^*, \theta_M(ma \otimes_A (\varepsilon_C \otimes 1_A))_{\langle -1 \rangle} \rangle \\
 &\quad (f \text{ is } C\text{-colinear}) \\
 &= f(\theta_M(ma \otimes_A (c^* \otimes 1_A))) \quad (\text{by (2.10)}) \\
 &= f \circ \theta_M(m \otimes_A (c^* \otimes a))
 \end{aligned}$$

and therefore

$$(\xi_{M,N} \circ \eta_{M,N})(f)\theta_M = f\theta_M$$

Taking into account that θ_M is an isomorphism, we obtain that

$$\xi_{M,N} \circ \eta_{M,N}(f) = f$$

for every $f \in \text{Hom}_A^C(C \otimes M, N)$. Therefore

$$\xi_{M,N} \circ \eta_{M,N} = \text{Id}_{\text{Hom}_A^C(C \otimes M, N)}$$

and $\eta_{M,N}$ is a natural isomorphism. This proves that $G = C \otimes \bullet$ is a left adjoint of the forgetful functor $F : {}^C\mathcal{M}(H)_A \rightarrow \mathcal{M}_A$.

3) \Leftrightarrow 4) follows immediately from the observation made in Section 1.1.

1) \Leftrightarrow 5). We have already seen that 1) implies that C is finitely generated. In this case, the forgetful functor ${}^C\mathcal{M}(H)_A \rightarrow \mathcal{M}_A$ is isomorphic to the restriction of scalars functor $\mathcal{M}_{A\#C^*} \rightarrow \mathcal{M}_A$, and the equivalence then follows immediately from [13, Theorem 3.15], where it is shown that a ring extension $R \rightarrow S$ is Frobenius if and only if the restriction of scalars functor has the same right and left adjoint. \square

Remark 2.5. From the proof of Theorem 2.4, it follows that there exists an H -integral of $C \otimes A$ if and only if there exists an A -bimodule map $\varphi : C^* \otimes A \rightarrow C \otimes A$ which is also a right C^* -module map.

Indeed, if $x = \sum c_i \otimes a_i$ is an H -integral of $C \otimes A$, then $\varphi : C^* \otimes A \rightarrow C \otimes A$, $\varphi(c^* \otimes a) = \sum c_i \leftarrow c^* \otimes aa_i$ is the required map. Conversely, if $\varphi : C^* \otimes A \rightarrow C \otimes A$ is left and right A -linear and right C^* -linear, then $\sum c_i \otimes a_i = \varphi(\varepsilon_C \otimes 1_A)$ is an H -integral of $C \otimes A$, and, moreover $\varphi(c^* \otimes a) = \sum c_i \leftarrow c^* \otimes aa_i$ for all $a \in A$ and $C^* \in C^*$.

Recall ([5]) that a coalgebra C is called right co-Frobenius if there exists a right C^* -monomorphism from C to C^* .

If we take $A = H = k$ in Theorem 2.4, then we obtain the following result.

Corollary 2.6. *Let C be a k -projective coalgebra. The following statements are equivalent:*

- 1) *The functor $G = C \otimes \bullet : \mathcal{M}_k \rightarrow {}^C\mathcal{M}$ is a left adjoint of the forgetful functor $F : {}^C\mathcal{M} \rightarrow \mathcal{M}_k$;*
- 2) *there exist an element $\lambda \in C$ such that the map $\varphi : C^* \rightarrow C$, $\varphi(c^*) = \lambda \leftarrow c^*$ is bijective;*
- 3) *C is finitely generated as a k -module and there exists an isomorphism $\varphi : C^* \rightarrow C$ of right C^* -modules.*

In particular, if the functor $G = C \otimes \bullet : \mathcal{M}_k \rightarrow {}^C\mathcal{M}$ is a left adjoint of the forgetful functor $F : {}^C\mathcal{M} \rightarrow \mathcal{M}_k$, then C is a right co-Frobenius coalgebra.

Proof. This follows immediately from Theorem 2.4 after we identify $C \otimes k \cong C$ and $C^* \otimes k \cong C^*$. Remark that the existence of λ in 2) implies that C is finitely generated, i.e. $C = \sum_{i=1}^n d_i k$, where $\Delta(\lambda) = \sum_{i=1}^n c_i \otimes d_i$. □

Remark 2.7. Obviously, we also have a left-right version of Theorem 2.4. If A is a right H -comodule algebra and C is a left H -module coalgebra, then we can define the category ${}_A\mathcal{M}(H)^C$ (see [6]). The forgetful functor $F : {}_A\mathcal{M}(H)^C \rightarrow {}_A\mathcal{M}$ has a right adjoint $\bullet \otimes C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}(H)^C$. For $M \in {}_A\mathcal{M}$, the A -action and C -coaction on $M \otimes C$ are defined as follows:

$$a \cdot (m \otimes c) := \sum a_{\langle 0 \rangle} m \otimes a_{\langle 1 \rangle} \cdot c$$

and

$$\rho_{M \otimes C}(m \otimes c) = \sum m \otimes c_{(1)} \otimes c_{(2)}$$

If A is faithfully flat as a k -module, and C is projective as a k -module, then the following statements are equivalent.

- 1) *The functor $\bullet \otimes C : {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}(H)^C$ is also a left adjoint of the forgetful functor $F : {}_A\mathcal{M}(H)^C \rightarrow {}_A\mathcal{M}$;*
- 2) *C is finitely generated as a k -module, and there exists $\alpha = \sum a_i \otimes c_i \in A \otimes C$ such that:*
 - a) $\sum a_i b \otimes c_i = \sum b_{\langle 0 \rangle} a_i \otimes b_{\langle 1 \rangle} c_i$, for all $b \in A$;
 - b) *the map $\varphi : A \otimes C^* \rightarrow A \otimes C$, $\varphi(a \otimes c^*) = \sum a_i a \otimes c^* \rightarrow c_i$ is bijective.*

Let us now examine some special situations; first we look at the graded case. Let G be a group, and X a left G -set. Then $H = kG$ is a Hopf algebra, and $C = kX$ is a left H -module coalgebra. A right H -comodule algebra is nothing else then a G -graded k -algebra, cf. e.g. [15]. Doi [6] observed that in this case a Doi-Hopf module is nothing else then a left X -graded A -module in the sense of [16], that is, M can be written as a direct sum of A_e -modules $M = \bigoplus_{x \in X} M_x$ with $A_\sigma M_x \subseteq M_{\sigma x}$ for all $\sigma \in G$ and $x \in X$. We can therefore identify the category ${}_A\mathcal{M}(k[G])^{k[X]}$

and the category of left X -graded A -modules $(G, X, A)\text{-gr}$. It then follows from Lemma 1.1 that the forgetful functor $F : (G, A, X)\text{-gr} \rightarrow {}_A\mathcal{M}$ has a right adjoint $\bullet \otimes kX$. Applying the left-right version of Theorem 2.4, we obtain the following result, generalizing [4, Proposition 2.5], where the case $X = G$ is discussed.

Corollary 2.8. *Let G be a group, X a left G -set and A a faithfully flat G -graded k -algebra. Then the following statements are equivalent:*

- 1) $\bullet \otimes kX$ is also a left adjoint of the forgetful functor $F : (G, A, X)\text{-gr} \rightarrow {}_A\mathcal{M}$;
- 2) X is a finite set.

Proof. If X is a finite set, then $\alpha := \sum_{x \in X} 1 \otimes x$ satisfies condition 2). Indeed, let $a_\sigma \in A_\sigma$ be homogeneous of degree σ . Then we have

$$\sum (a_\sigma)_{\langle 0 \rangle} \cdot 1 \otimes (a_\sigma)_{\langle 1 \rangle} x = \sum_{x \in X} a_\sigma \otimes \sigma x = \sum_{y \in X} 1 \cdot a_\sigma \otimes y$$

and 2a) holds. For every $x \in X$, define $x^* \in kX^*$ by $\langle x^*, y \rangle = \delta_{x,y}$. Then $\{x, x^* | x \in X\}$ is a dual basis for kX . The map $\varphi : A \otimes kX^* \rightarrow A \otimes kX$ is given by

$$\varphi(a \otimes x^*) = \sum_{y \in X} a \otimes x^* \rightarrow y = a \otimes x$$

It is obvious that φ is bijective. □

Another easy consequence of the theorem is the following.

Corollary 2.9. *Let (H, A, C) be a Doi-Hopf datum. If G is a left adjoint of F , then $C \otimes A$ is a projective generator in ${}^C\mathcal{M}(H)_A$.*

Proof. This follows immediately from 3) of Theorem 2.4 and the results in Section 1.1. □

If we apply Theorem 2.4 in the case where $C = H$, then we obtain the following result.

Corollary 2.10. *Let H be a Hopf algebra over a field k and A a left H -comodule algebra. The following statements are equivalent:*

- 1) the functor $G = H \otimes \bullet : \mathcal{M}_A \rightarrow {}^H\mathcal{M}(H)_A$ is a left adjoint of the forgetful functor $F : {}^H\mathcal{M}(H)_A \rightarrow \mathcal{M}_A$;
- 2) H is finite dimensional.

Proof. If F is a right adjoint of G , then by 2) of Theorem 2.4 H is finite dimensional. Conversely, assume that H is finite dimensional. Let $e \in H$ be a nonzero right integral of H . By [20], the map $\iota : H^* \rightarrow H, h^* \rightarrow (e \leftarrow h^*)$ is bijective. Therefore the map $\varphi : H^* \otimes A \rightarrow H \otimes A, h^* \otimes a \mapsto e \leftarrow h^* \otimes a$ is bijective. Obviously $e \otimes 1_A$ is an H -integral of $H \otimes A$, and the result now follows immediately from Theorem 2.4. □

3. k -FROBENIUS H -MODULE COALGEBRAS

We begin this section with a generalization of the notion of k -Frobenius Hopf algebra, cf. Definition 1.2.

Definition 3.1. Let H be a Hopf algebra over a commutative ring k . A right H -module coalgebra C is called a *right k -Frobenius H -module coalgebra* if there exists an element $\lambda \in C$ such that

$$\lambda \leftarrow C^* = C \quad \text{and} \quad \lambda \cdot h = \varepsilon_H(h)\lambda, \quad \forall h \in H$$

We call λ a *right H -integral* of C .

Remark 3.2. If H is k -projective, then it follows from Lemma 1.3 that H is k -Frobenius as a Hopf algebra if and only if H is right k -Frobenius H -module coalgebra.

Example 3.3. Let X be a finite right G -set, and consider $H = k[G]$ and $C = k[X]$. Then C is a right k -Frobenius $k[G]$ -module coalgebra. Indeed, $\lambda = \sum_{x \in X} 1_k x$ is a right $k[G]$ -integral of $k[X]$.

Theorem 3.4. Let H be a Hopf algebra with bijective antipode, and C a k -projective right H -module coalgebra. Then the following statements are equivalent:

- 1) C is a right k -Frobenius H -module coalgebra;
- 2) C is finitely generated as a k -module, and there exists a left C -comodule isomorphism $\varphi : C^* \rightarrow C$ such that

$$\varphi(h \cdot c^*) = \varphi(c^*) \cdot S(h), \quad \forall h \in H, \quad c^* \in C^*$$

If k is a field, then 1) and 2) are also equivalent to

- 3) C is finitely generated and there exists an element $\lambda \in C$ such that

$$C^* \leftarrow \lambda = C \quad \text{and} \quad \lambda \cdot h = \varepsilon_H(h)\lambda \quad \forall h \in H$$

- 4) C is finitely generated, and there exists a nondegenerated bilinear form $[\bullet, \bullet] : C^* \otimes C^* \rightarrow k$ such that

$$[S^{-1}h \cdot c^*, d^*] = [c^*, h \cdot d^*], \quad \text{and} \quad [c^* * d^*, e^*] = [c^*, d^* * e^*]$$

for all $h \in H, c^*, d^*, e^* \in C^*$.

If one the four equivalent conditions is satisfied, then there exists a right H -integral λ of C such that the map $C^* \rightarrow C, c^* \mapsto \lambda \leftarrow c^* (c^* \in C^*)$ is bijective.

Proof. 1) \Rightarrow 2). Assume that λ is as in Definition 3.1. Let $\varphi : C^* \rightarrow C$ be defined by $\varphi(c^*) = \lambda \leftarrow c^*$, for any $c^* \in C^*$. Write $\Delta\lambda = \sum_{i=1}^n c_i \otimes d_i$. Then $C = \lambda \leftarrow C^* \subseteq \sum d_i k \subseteq C$, and C is finitely generated. Now C and C^* are finitely generated and projective, and have the same rank, and the fact that $\varphi : C^* \rightarrow C$ is surjective implies that φ is also injective. Finally,

$$\begin{aligned} \varphi(S^{-1}h \cdot c^*) = \lambda \leftarrow (S^{-1}h \cdot c^*) &= \sum \langle S^{-1}h \cdot c^*, \lambda_{(1)} \rangle \lambda_{(2)} \\ &= \sum \langle c^*, \lambda_{(1)} \cdot S^{-1}h \rangle \lambda_{(2)} \\ &= \sum \langle c^*, \lambda_{(1)} \cdot S^{-1}h_{(3)} \rangle \lambda_{(2)} \cdot S^{-1}(h_{(2)})h_{(1)} \\ &= \sum [(\lambda \cdot S^{-1}h_{(2)}) \leftarrow c^*] \cdot h_{(1)} \\ (\lambda \text{ is a right } H\text{-integral}) &= (\lambda \leftarrow c^*) \cdot h = \varphi(c^*) \cdot h \end{aligned}$$

for all $h \in H$.

- 2) \Rightarrow 1). Let $\lambda = \varphi(\varepsilon_C)$. Using the C -colinearity of φ , we obtain that

$$\sum_{i=1}^n c_i \otimes \varphi(c_i^*) = \Delta(\varphi(\varepsilon_C)) = \Delta(\lambda)$$

Here $\{c_i^*, c_i\}_{i=1}^n$ is a dual basis for C . Therefore

$$\varphi(c^*) = \sum \varphi(\langle c^*, c_i \rangle c_i^*) = \sum \langle c^*, c_i \rangle \varphi(c_i^*) = \sum \langle c^*, \lambda_{(1)} \rangle \lambda_{(2)} = \lambda \leftarrow c^*$$

for all $c^* \in C^*$, and it follows that $\lambda \leftarrow C^* = C$. Now for any $h \in H$,

$$\lambda \cdot h = \varphi(\varepsilon_C) \cdot h = \varphi(S^{-1}h \cdot \varepsilon_C) = \varepsilon_H(h)\varphi(\varepsilon_C) = \varepsilon_H(h)\lambda$$

and this shows that λ is a right H -integral of C .

2) \Rightarrow 4). Assume that 2) holds, and define

$$[\bullet, \bullet] : C^* \otimes C^* \rightarrow k, \quad c^* \otimes d^* \mapsto \langle d^*, \varphi(c^*) \rangle, \quad c^*, d^* \in C^*$$

Then

$$\begin{aligned} [S^{-1}h \cdot c^*, d^*] &= \langle d^*, \varphi(S^{-1}h \cdot c^*) \rangle = \langle d^*, \varphi(c^*) \cdot h \rangle \\ &= \langle h \cdot d^*, \varphi(c^*) \rangle = [h \cdot d^*, c^*] \end{aligned}$$

for all $c^*, d^* \in C^*$ and $h \in H$. Since φ is onto, $d^* = 0$ if and only if $\langle d^*, \varphi(C^*) \rangle = 0$ if and only if $[C^*, d^*] = 0$. If $[c^*, C^*] = 0$, then for any dual basis $\{d_i^*, d_i\}_{i=1}^m$,

$$\varphi(c^*) = \sum \langle d_i^*, \varphi(c^*) \rangle d_i = \sum [c^*, d_i^*] d_i = 0$$

and therefore $c^* = 0$, since φ is bijective. This shows that $[\bullet, \bullet]$ is nondegenerate. On the other hand φ is left C -colinear, and therefore right C^* -linear. Therefore

$$[c^* * d^*, e^*] = \langle e^*, \varphi(c^* * d^*) \rangle = \langle e^*, \varphi(c^*) \leftarrow d^* \rangle = \langle d^* * e^*, \varphi(c^*) \rangle = [c^*, d^* * e^*]$$

Now suppose that k is a field.

4) \Rightarrow 3) Let $\{d_i^*, d_i\}_{i=1}^n$ be a dual basis for C and define

$$\lambda = \sum_{i=1}^n [\varepsilon_C, d_i^*] d_i$$

Then for all $c^*, d^* \in C^*$, we have that

$$\langle c^* * d^*, \lambda \rangle = \sum_{i=1}^n [\varepsilon_C, d_i^*] \langle c^* * d^*, d_i \rangle = [\varepsilon_C, c^* * d^*] = [\varepsilon_C * c^*, d^*] = [c^*, d^*]$$

Suppose that $d^* \rightarrow \lambda = 0$ for some $d^* \in C^*$. Then

$$[C^*, d^*] = \langle C^* * d^*, \lambda \rangle = \langle C^*, d^* \rightarrow \lambda \rangle = 0$$

From the fact that $[\bullet, \bullet]$ is nondegenerate, it follows that $d^* = 0$. Thus the map $\varphi' : C^* \rightarrow C$ defined by

$$\varphi'(c^*) = c^* \rightarrow \lambda$$

for $c^* \in C^*$ is injective. Since C and C^* are finite dimensional vector spaces of the same dimension, it follows that φ' is bijective. Finally, for all $h \in H$, we have that

$$\begin{aligned} \lambda \cdot h &= \sum_{i=1}^n [\varepsilon_C, d_i^*] d_i \cdot h = \sum_{i=1}^n [\varepsilon_C, h \cdot d_i^*] d_i \\ &= \sum_{i=1}^n [S^{-1}h \cdot \varepsilon_C, d_i^*] d_i = \varepsilon_H(h) \sum_{i=1}^n [\varepsilon_C, d_i^*] d_i = \varepsilon_H(h)\lambda \end{aligned}$$

Therefore λ is a right H -integral of C .

3) \Rightarrow 2) Let $\lambda \in C$ be such that $\lambda \cdot h = \varepsilon_H(h)\lambda, \forall h \in H$ and $C^* \rightarrow \lambda = C$. If $\lambda \leftarrow c^* = 0$, then

$$\langle c^*, C \rangle = \langle c^*, C^* \rightarrow \lambda \rangle = \langle C^*, \lambda \leftarrow c^* \rangle = 0$$

and $c^* = 0$. Therefore $\varphi : C^* \rightarrow C, c^* \mapsto (\lambda \leftarrow c^*)(\forall c^* \in C^*)$ is injective, and therefore an isomorphism, since C and C^* are vector spaces of the same dimension. \square

Theorem 3.5. *Let H be a faithfully flat Hopf algebra and C a k -projective right H -module coalgebra. Then $G = C \otimes \bullet : \mathcal{M}_H \rightarrow {}^C\mathcal{M}(H)_H$ is a left adjoint of the forgetful functor $F : {}^C\mathcal{M}(H)_H \rightarrow \mathcal{M}_H$ if and only if C is a right k -Frobenius H -module coalgebra.*

Proof. Assume that G is a left adjoint of F . It follows from Theorem 2.4 that C is finitely generated. Let $x = \sum_{i=1}^n c_i \otimes a_i \in C \otimes H$ be an H -integral as given in 2) of Theorem 2.4, and let $\lambda = \sum_{i=1}^n \varepsilon_H(a_i)c_i$. By Theorem 2.4, we have that

$$\lambda \cdot h = \sum_{i=1}^n \varepsilon_H(a_i)c_i \cdot h = \sum_{i=1}^n \varepsilon_H(a_i h_{(2)})(c_i \cdot h_{(1)}) = \sum_{i=1}^n \varepsilon_H(h a_i)c_i = \varepsilon_H(h)\lambda$$

for any $h \in H$. Take $c \in C$. Then there exists some

$$\sum_{j=1}^m d_j^* \otimes b_j \in C^* \otimes H$$

such that

$$\sum_{i=1}^n \sum_{j=1}^m c_i \leftarrow d_j^* \otimes b_j a_i = c \otimes 1_H$$

Then

$$\begin{aligned} c &= c\varepsilon_H(1_H) = \sum_{i=1}^n \sum_{j=1}^m (c_i \leftarrow d_j^*)\varepsilon_H(b_j a_i) \\ &= \left(\sum_{i=1}^n \varepsilon_H(a_i)c_i \right) \leftarrow \left(\sum_{j=1}^m d_j^* \varepsilon_H(b_j) \right) \\ &= \lambda \leftarrow \left(\sum_{j=1}^m d_j^* \varepsilon_H(b_j) \right) \in \lambda \leftarrow C^* \end{aligned}$$

and $\lambda \leftarrow C^* = C$. Thus C is a k -Frobenius H -module coalgebra.

Conversely, suppose that C is a k -Frobenius H -module coalgebra. From Theorem 3.4 it follows that there exists a left C -comodule isomorphism $\psi : C^* \rightarrow C$ such that $\psi(S^{-1}h \cdot c^*) = \psi(c^*) \cdot h$, for all $c^* \in C^*$ and $h \in H$.

Define

$$\varphi : C^* \otimes H \rightarrow C \otimes H, \quad c^* \otimes g \mapsto \psi(c^*) \otimes g$$

Then φ is a bijective left C -colinear map. For all $h, h' \in H$, we have that

$$\begin{aligned} \varphi(h' \cdot (c^* \otimes g) \cdot h) &= \sum \varphi(S^{-1}h_{(1)} \cdot c^* \otimes h'gh_{(2)}) \\ &= \sum \psi(S^{-1}h_{(1)} \cdot c^*) \otimes h'gh_{(2)} \\ &= \sum \psi(c^*) \cdot h_{(1)} \otimes h'gh_{(2)} \\ &= h' \cdot (\psi(c^*) \otimes g) \cdot h \\ &= h' \cdot \varphi(c^* \otimes g) \cdot h \end{aligned}$$

and therefore φ is an A - A -bimodule map, and we can conclude from Theorem 2.4 that G is a left adjoint of F . \square

Ulbrich [22, theorem 1.3] proved the following duality theorem: if H is a finitely generated, projective Hopf algebra and A is a right H -comodule algebra, then

$$A\#H^* \cong \text{End}_A^H(H \otimes A)$$

as k -algebras. In the next corollary, we give a similar result for coalgebras, and we give a positive answer to a problem in [14], where it was proved that, if C is finitely generated and projective, then the k -algebras $A\#C^*$ and $\text{End}_A^C(C \otimes A)$ are Morita equivalent (see [14, corollary 8]).

Proposition 3.6. *Let (H, A, C) be a Doi-Hopf datum and suppose that C is finitely generated and projective as a k -module. If there exists an isomorphism*

$$\psi : C^* \otimes A \rightarrow C \otimes A$$

in ${}^C\mathcal{M}(H)_A$, then

$$A\#C^* \cong \text{End}_A^C(C \otimes A)$$

as k -algebras. In particular, $A\#C^*$ and $\text{End}_A^C(C \otimes A)$ are isomorphic as k -algebras if C is a k -Frobenius H -module coalgebra.

Proof. The first statement follows immediately from the assumption and (1.10). Suppose that C is a k -Frobenius H -module coalgebra, and let $\varphi : C^* \rightarrow C$ be a left C -comodule isomorphism such that $\varphi(h \cdot c^*) = \varphi(c^*) \cdot Sh$, for all $c^* \in C^*$ and $h \in H$. Define

$$\psi : C^* \otimes A \rightarrow C \otimes A, \quad \psi(c^* \otimes a) = \varphi(c^*) \otimes a$$

Then ψ is a bijective and left C -colinear and

$$\begin{aligned} \psi((c^* \otimes a) \cdot b) &= \sum \varphi(S^{-1}b_{<-1>} \cdot c^* \otimes ab_{<0>}) \\ &= \sum \varphi(c^*) \cdot b_{<-1>} \otimes ab_{<0>} \\ &= (\varphi(c^*) \otimes a) \cdot b \\ &= \psi(c^* \otimes a) \cdot b \end{aligned}$$

and ψ is an isomorphism in ${}^C\mathcal{M}(H)_A$. □

Remark 3.7. The isomorphism

$$\theta : A\#C^* \rightarrow \text{End}_A^C(C \otimes A)$$

can be described explicitly. It is given by the formula

$$\theta(a\#c^*)(\psi(d^* \otimes b)) := \sum \psi(c^*(a_{<-1>} \cdot d^*) \otimes a_{<0>}b)$$

Let us prove that θ is an algebra map. For any $a, b, e \in A$ and $c^*, d^*, f^* \in C^*$ we have

$$\begin{aligned} \theta(a\#c^*)\theta(b\#d^*)\psi(f^* \otimes e) &= \sum \theta(a\#c^*)\psi(d^*(b_{<-1>} \cdot f^*) \otimes b_{<0>}e) \\ &= \sum \psi(c^*(a_{<-1>} \cdot (d^*(b_{<-1>} \cdot f^*))) \otimes a_{<0>}b_{<0>}e) \end{aligned}$$

and

$$\begin{aligned}
 \theta((a\#c^*)(b\#d^*))\psi(f^* \otimes e) &= \sum \theta(a_{<0>}b\#c^*(a_{<-1>} \cdot d^*))\psi(f^* \otimes e) \\
 &= \sum \psi(c^*(a_{<-1>} \cdot d^*)((a_{<0>}b)_{<-1>} \cdot f^*) \otimes (a_{<0>}b)_{<0>}e) \\
 &= \sum \psi(c^*(a_{<-2>} \cdot d^*)(a_{<-1>} \cdot (b_{<-1>} \cdot f^*)) \otimes a_{<0>}b_{<0>}e) \\
 &= \sum \psi(c^*(a_{<-1>} \cdot (d^*(b_{<-1>} \cdot f^*))) \otimes a_{<0>}b_{<0>}e)
 \end{aligned}$$

and it follows that θ is an algebra map. In the last equality we used the fact that C^* is a left H -module algebra.

Corollary 3.8 ([16, Theorem 2.14]). *Let X be a right G -set, and A a G -graded ring. Then*

$$A\#k[X]^* \cong \text{End}_{A\text{-gr}}(U)$$

where $U = \bigoplus_{x \in X} A(X)$.

4. YETTER-DRINFEL'D MODULES AND UNIMODULAR HOPF ALGEBRAS

We recall some basic facts about the Drinfel'd double, as introduced in [7]. Our main references are [12] and [18]. Let H be a finitely generated projective Hopf algebra. Then the antipode S_H is bijective (see [17]), and the Drinfel'd double $D(H) = H \bowtie H^{*cop}$ is defined as follows: $D(H) = H \otimes H^*$ as a k -module. The multiplication, comultiplication, counit and antipode are respectively given by the formulas

- $(h \bowtie f)(h' \bowtie f') = \sum h_{(2)}h' \bowtie f * (f', S^{-1}h_{(3)}?h_{(1)});$
- $\Delta_{D(H)}(h \bowtie f) = \sum (h_{(1)} \bowtie f_{(2)}) \otimes (h_{(2)} \bowtie f_{(1)});$
- $\varepsilon_{D(H)} = \varepsilon_H \otimes \varepsilon_{H^{*cop}};$
- $S_{D(H)}(h \bowtie f) = \sum f_{(2)} \leftarrow (S_H h_{(1)}) \bowtie (S(h_{(2)}) \leftarrow (S_{H^*} f_{(1)}),$

for $h, h' \in H$ and $f, f' \in H^*$. Radford [18] showed that, over a field k , $D(H)$ is unimodular with integral $\lambda \bowtie \Lambda$ with nonzero $\lambda \in \int_H^r$ and $\Lambda \in \int_{H^*}^l$. The following lemma is a generalization of Radford's result.

Lemma 4.1. *If H be is a projective k -Frobenius Hopf algebra, then $D(H)$ is a unimodular k -Frobenius Hopf algebra.*

Proof. Let λ be a free generator for \int_H^r and let α be the distinguished element of H^* . Take $\Lambda \in H^*$ such that $\lambda \leftarrow S^*(\Lambda) = 1_H$. Then Λ is a left integral in H^* and $\int_{H^*}^l = \Lambda k$ (see [17]). Note that $\langle \Lambda, S\lambda \rangle = \langle \varepsilon_H, \lambda \leftarrow S^*\Lambda \rangle = 1_k$.

Take a dual basis $\{h_i^*, h_i\}_{i=1}^n$ for H and assume that $y = \sum_{i=1}^n h_i \bowtie f_i \in \int_{D(H)}^l$, where $f_i \in H^*$. Then for any $f \in H^*$,

$$\sum_{i=1}^n h_i \bowtie (f * f_i) = (1_H \bowtie f) \left(\sum_{i=1}^n h_i \bowtie f_i \right) = (1_H \bowtie f)y = f(1_H) \sum_{i=1}^n h_i \bowtie f_i$$

For any $1 \leq j \leq n$, and $f \in H^*$, we therefore have

$$f * \left(\sum_{i=1}^n \langle h_j^*, h_i \rangle f_i \right) = \sum_{i=1}^n \langle h_j^*, h_i \rangle f * f_i = f(1_H) \left(\sum_{i=1}^n \langle h_j^*, h_i \rangle f_i \right)$$

and $(\sum_{i=1}^n \langle h_j^*, h_i \rangle f_i) \in \int_{H^*}^l = \Lambda k$. Write $\sum_{i=1}^n \langle h_j^*, h_i \rangle f_i = k_j \Lambda$, with $k_j \in k$ and $\sum_{j=1}^n k_j h_j = x$. Then

$$y = \sum_{i,j=1}^n h_j \langle h_j^*, h_i \rangle \bowtie f_i = \left(\sum_{i=1}^n k_i h_i \right) \bowtie \Lambda = x \bowtie \Lambda$$

Let us show that $x \in \int_H^r$. By assumption $x \bowtie \Lambda = y \in \int_{D(H)}^l$, so for any $h \in H$,

$$\varepsilon_H(h)x \bowtie \Lambda = (h \bowtie \varepsilon_H)(x \bowtie \Lambda) = \sum h_{(2)}x \bowtie \langle \Lambda, S^{-1}h_{(3)}?h_{(1)} \rangle$$

Note that $S(\lambda) \in \int_H^l$. Applying $\langle \bullet, S(\lambda) \rangle$ to the above formula, we obtain

$$\begin{aligned} \varepsilon_H(h)x &= \varepsilon(h)x \langle \Lambda, S\lambda \rangle = \sum h_{(2)}x \langle \Lambda, S^{-1}h_{(3)}S\lambda h_{(1)} \rangle \\ &= \sum h_{(2)}x \langle \Lambda, (S\lambda)h_{(1)} \rangle = \sum h_{(2)}x \langle \Lambda, S[(S^{-1}h_{(1)})\lambda] \rangle \\ &= \sum h_{(2)}x \langle \alpha, S^{-1}h_{(1)} \rangle \langle \Lambda, S\lambda \rangle = \sum h_{(2)}x \langle \alpha^{-1}, h_{(1)} \rangle \end{aligned}$$

and

$$(4.1) \quad hx = \sum \langle \alpha, h_{(1)} \rangle \langle \alpha^{-1}, h_{(2)} \rangle h_{(3)}x = \alpha(h)x$$

for all $h \in H$. Now assume that $x = \lambda \leftarrow f = \sum \langle f, \lambda_{(1)} \rangle \lambda_{(2)}$ for some $f \in H^*$. Then by (4.1),

$$\begin{aligned} \lambda \leftarrow (\langle \alpha, h \rangle f) &= \langle \alpha, h \rangle x = hx = \sum h \langle f, \lambda_{(1)} \rangle \lambda_{(2)} \\ &= \sum \langle f, Sh_{(1)}h_{(2)}\lambda_{(1)} \rangle h_{(3)}\lambda_{(2)} = \lambda \leftarrow \left(\sum \langle \alpha, h_{(2)} \rangle (Sh_{(1)} \rightarrow f) \right) \end{aligned}$$

Using the fact that the map $H^* \rightarrow H$, $h^* \mapsto (\lambda \leftarrow h^*)$ is bijective, we obtain

$$\langle \alpha, h \rangle f = \sum \langle \alpha, h_{(2)} \rangle (Sh_{(1)} \rightarrow f)$$

for all $h \in H$. Thus

$$\langle \alpha, h \rangle f(1_H) = \sum \langle \alpha, h_{(2)} \rangle (Sh_{(1)} \rightarrow f)(1_H) = \sum \langle \alpha, h_{(2)} \rangle \langle f, Sh_{(1)} \rangle$$

for any $h \in H$, that is, $(S^*(f)) * \alpha = f(1_H)\alpha$. Hence $f = f(1_H)\varepsilon_H$. We can now conclude that $x = \lambda \leftarrow f = f(1_H)\lambda$ and $y = f(1_H)\lambda \bowtie \Lambda$. Therefore

$$(4.2) \quad \int_{D(H)}^l \subseteq (\lambda \bowtie \Lambda)k$$

Let $I = \{k \in k \mid k(\lambda \bowtie \Lambda) \in \int_{D(H)}^l\}$. Then I is an ideal of k and $\int_{D(H)}^l = I(\lambda \bowtie \Lambda)$.

By the descent theorem for Hopf modules, $D(H) \cong \int_{D(H)}^l \otimes (D(H))^*$. We therefore have that $1_{D(H)} = \sum_{j=1}^m y_j \leftarrow S^*y_j^*$, with $y_j \in \int_{D(H)}^l$ and $y_j^* \in (D(H))^*$. By (4.2), we have $k_j \in I$ such that $y_j = k_j(\lambda \bowtie \Lambda)$. Thus

$$\begin{aligned} 1_k &= \varepsilon_{D(H)}(1_{D(H)}) = \sum_{j=1}^m \varepsilon_{D(H)}(y_j \leftarrow S^*y_j^*) \\ &= \sum_{j=1}^m k_j(\varepsilon_{D(H)}((\lambda \bowtie \Lambda) \leftarrow S^*y_j^*)) \in I \end{aligned}$$

and it follows that $I = k$. This implies that

$$\lambda \bowtie \Lambda \in \int_{D(H)}^l \quad \text{and} \quad \int_{D(H)}^l = (\lambda \bowtie \Lambda)k$$

Similarly, we have that

$$\lambda \bowtie \Lambda \in \int_{D(H)}^r \quad \text{and} \quad \int_{D(H)}^r = (\lambda \bowtie \Lambda)k$$

and this finishes our proof. □

We will now apply the foregoing results to categories of Yetter-Drinfel'd modules. Using Lemma 1.1 and the identification ${}^H\mathcal{YD}_H = {}^H\mathcal{M}(H \otimes H^{\text{op}})_H$ given by (1.14) and (1.15), the forgetful functor $F : {}^H\mathcal{YD}_H \rightarrow \mathcal{M}_H$ has a right adjoint $H \otimes \bullet : \mathcal{M}_H \rightarrow {}^H\mathcal{YD}_H$. For $M \in \mathcal{M}_H$, the structure maps on $H \otimes M$ are given by the formulas

$$\begin{aligned} (h \otimes m) \cdot k &= \sum S^{-1}(k_{(3)})hk_{(1)} \otimes mk_{(2)}, \\ \rho_{H \otimes M}(h \otimes m) &= \sum h_{(1)} \otimes h_{(2)} \otimes m \end{aligned}$$

for all $h, k \in H$ and $m \in M$. In fact this is the right-left version of [2, Cor. 2.6].

Theorem 4.2. *For a k -projective Hopf algebra H , the following statements are equivalent:*

- 1) $H \otimes \bullet : \mathcal{M}_H \rightarrow {}^H\mathcal{YD}_H$ is a left adjoint of the forgetful functor $F : {}^H\mathcal{YD}_H \rightarrow \mathcal{M}_H$;
- 2) H is k -Frobenius and unimodular;
- 3) there exists an element $c \in H$ such that $c \leftarrow H^* = H$ and

$$(4.3) \quad (c \otimes 1)\Delta(h) = \Delta^{\text{cop}}(h)(c \otimes 1)$$

for all $h \in H$.

Proof. 1) \Rightarrow 2) By Theorem 2.4, H is finitely generated, and $D(H)$ is defined. We have a category isomorphism

$$(4.4) \quad \Phi : {}^H\mathcal{YD}_H \rightarrow \mathcal{M}_{D(H)}$$

$\Phi(M) = M$ as a k -module, and the right $D(H)$ -action is given by the formula

$$m \cdot (h \bowtie h^*) = (m \cdot h^*) \cdot h$$

for $h^* \in H^*$ and $h \in H$. We can check easily that

$$F \circ \Phi^{-1} \cong \text{Hom}_{D(H)}(D(H), \bullet) : \mathcal{M}_{D(H)} \rightarrow \mathcal{M}_H$$

is the restriction of scalars functor. In the sequel we will identify the categories ${}^H\mathcal{YD}_H$ and $\mathcal{M}_{D(H)}$ using the functor Φ .

We have an algebra map $H \rightarrow D(H)$, $h \mapsto (h \bowtie \varepsilon_H)$, and it is well-known that $\bullet \otimes_H D(H)$ is a left adjoint of $\text{Hom}_{D(H)}(D(H), \bullet)$ (see [1]). By assumption, the functors

$$H \otimes \bullet \quad \text{and} \quad \bullet \otimes_H D(H) : \mathcal{M}_H \rightarrow \mathcal{M}_{D(H)}$$

are naturally isomorphic. Let $\xi : H \otimes \bullet \rightarrow \bullet \otimes_H D(H)$ be the natural isomorphism. Then for the trivial right H -module k , we have that

$$\xi_k : H \cong H \otimes k \rightarrow k \otimes_H D(H)$$

is an isomorphism in $\mathcal{M}_{D(H)}$. Now for any $h \in H$ and $h^* \in H^*$, we have that

$$\begin{aligned} 1_k \otimes_H (h \bowtie h^*) &= \sum 1_k \otimes_H (h_{(2)} \bowtie \varepsilon_H)(1_H \bowtie \langle h^*, h_{(3)}?S^{-1}h_{(1)} \rangle) \\ &= \sum 1_k \varepsilon(h_{(2)}) \otimes_H (1_H \bowtie \langle h^*, h_{(3)}?S^{-1}h_{(1)} \rangle) \\ &= \sum 1_k \otimes_H (1_H \bowtie \langle h^*, h_{(2)}?S^{-1}h_{(1)} \rangle) \end{aligned}$$

so any element x in $k \otimes_H D(H)$ has the form

$$x = 1_k \otimes_H (1_H \bowtie h_x^*)$$

for a unique $h_x^* \in H^*$, and it is clear that $h_x^*_{x(1_H \bowtie h^*)} = h_x^* * h^*$, for any $x \in k \otimes_H D(H)$ and $h^* \in H^*$.

Define $\varphi : H \rightarrow H^*$ by $\varphi(h) = h_{\xi_k(h)}^*$, for $h \in H$. Then φ is additive and bijective. Since ξ_k is an isomorphism in $\mathcal{M}_{D(H)}$, we have that

$$\varphi(h \leftarrow h^*) = h_{\xi_k(h-h^*)}^* = h_{\xi_k(h)(1_H \bowtie h^*)}^* = h_{\xi_k(h)}^* * h^* = \varphi(h)h^*$$

is k -Frobenius, and therefore H is k -Frobenius.

Let λ be a free generator of \int_H^r and let Λ be a free generator in $\int_{H^*}^l$. By Lemma 4.1, $\lambda \bowtie \Lambda$ is an integral of $D(H)$ and $\int_{D(H)} = (\lambda \bowtie \Lambda)k$. Now $D(H)$ is unimodular and S_H is invertible, so $S_{D(H)}(\lambda \bowtie \Lambda)$ is an integral of $D(H)$ and, for any $x \in k$, $xS_{D(H)}(\lambda \bowtie \Lambda) = 0$ if and only if $x = 0$.

Write $\lambda' = S(\lambda)$, $\Lambda' = S^*(\Lambda)$. Then λ' is a left integral of H and Λ' is a right integral of H^* . Since $h\lambda = \alpha(h)\lambda$ (α is the distinguished element of H^*), $\lambda'h = S[(S^{-1}h)\lambda] = \alpha(S^{-1}h)\lambda' \in \lambda'k$ for any $h \in H$. Denote the right H -module $\lambda'k$ by M . Then

$$\xi_M : H \otimes M \rightarrow M \otimes_H D(H)$$

is an isomorphism in $\mathcal{M}_{D(H)}$.

It is a routine computation to check that the map $\gamma : H \otimes H^* \rightarrow H \bowtie H^*$ defined by

$$\gamma(h \otimes h^*) = (h \bowtie \varepsilon_H)(1_H \bowtie h^*)$$

is a k -linear isomorphism with inverse given by the formula

$$\gamma^{-1}(h \bowtie h^*) = \sum h_{(2)} \otimes \langle h^*, h_{(3)}?S^{-1}h_{(1)} \rangle$$

Now $(\lambda' \bowtie \varepsilon_H)(1_H \bowtie \Lambda') = S_{D(H)}((1_H \bowtie \Lambda)(\lambda \bowtie \varepsilon_H)) = S_{D(H)}(\lambda \bowtie \Lambda)$ is an integral of $D(H)$, and therefore, for any $h \in H$, we have

$$\begin{aligned} \varepsilon_H(h)(\lambda' \bowtie \varepsilon_H)(1_H \bowtie \Lambda') &= (\lambda' \bowtie \varepsilon_H)(1_H \bowtie \Lambda')(h \bowtie \varepsilon_H) \\ &= (\lambda' \bowtie \varepsilon_H)(h \bowtie \Lambda') \\ &= \sum (\lambda' \bowtie \varepsilon_H)(h_{(2)} \bowtie \varepsilon_H)(1_H \bowtie \langle \Lambda', (h_{(3)})?(S^{-1}h_{(1)}) \rangle) \\ &= \sum (\lambda'h_{(2)} \bowtie \varepsilon_H)(1_H \bowtie \langle \Lambda', (h_{(3)})?(S^{-1}h_{(1)}) \rangle) \end{aligned}$$

Applying γ^{-1} to this formula, we obtain

$$(4.5) \quad \sum \lambda'h_{(2)} \otimes \langle \Lambda', (h_{(3)})?(S^{-1}h_{(1)}) \rangle = \varepsilon_H(h)(\lambda' \otimes \Lambda')$$

for any $h \in H$, $h^* \in H^*$.

Write $h_0 \otimes \lambda' = \xi_M^{-1}(\lambda' \otimes_H 1_H \bowtie \Lambda') \in H \otimes M$. Now

$$\begin{aligned} (\lambda' \otimes_H 1_H \bowtie \Lambda')(h \bowtie h^*) &= (\lambda' \otimes_H (h \bowtie \Lambda' * h^*)) \quad (\Lambda' \text{ is right integral}) \\ &= h^*(1_H)(\lambda' \otimes_H (h \bowtie \Lambda')) \\ &= h^*(1_H) \sum \lambda' \otimes_H \left((h_{(2)} \bowtie \varepsilon_H)(1_H \bowtie \langle \Lambda', (h_{(3)})?S^{-1}h_{(1)} \rangle) \right) \\ &= h^*(1_H) \sum \lambda' h_{(2)} \otimes_H 1_H \bowtie \langle \Lambda', (h_{(3)})?(S^{-1}h_{(1)}) \rangle \\ &= \varepsilon_H(h)h^*(1_H)(\lambda' \otimes_H 1_H \bowtie \Lambda') \quad (\text{by (4.5)}) \end{aligned}$$

for all $h \in H$, $h^* \in H^*$. Furthermore ξ_M^{-1} is a $D(H)$ -module map, and therefore

$$(h_0 \otimes \lambda')(h \bowtie h^*) = \varepsilon_H(h)h^*(1_H)(h_0 \otimes \lambda')$$

that is,

$$(4.6) \quad [(h_0 \leftarrow h^*) \otimes \lambda'] \cdot h = \varepsilon_H(h)h^*(1_H)(h_0 \otimes \lambda')$$

for any $h \in H$, $h^* \in H^*$.

In particular, for $h = 1_H$, we obtain that

$$\langle h^*, h_0 \rangle \lambda' = \langle \varepsilon_H, h_0 \leftarrow h^* \rangle \lambda' = \varepsilon_H(h_0)h^*(1_H)\lambda'$$

for any $h^* \in H^*$, and therefore $h_0 = \varepsilon_H(h_0)1_H \in k1_H$. Write $k_0 = \varepsilon(h_0) \in k$. Take $x \in k$ and observe that $k_0x = 0$ if and only if $x(h_0 \otimes \lambda') = 0$, or, equivalently, $x(\lambda' \otimes_H 1_H \otimes \Lambda') = 0$, or $x = 0$.

From (4.6), it follows that

$$\sum k_0(S^{-1}h_{(3)})h_{(1)} \otimes \lambda'h_{(2)} = k_0\varepsilon_H(h)1_H \otimes \lambda'$$

for all $h \in H$. Applying $\varepsilon_H \otimes I_H$ to this equation, we obtain that

$$k_0\lambda'h = k_0\lambda'\varepsilon_H(h)$$

for all $h \in H$. Recall that $\lambda'h = \alpha(S^{-1}h)\lambda'$, so $(k_0\alpha(S^{-1}h) - k_0\varepsilon_H(h))\lambda' = 0$. λ' is a k -free generator, and therefore $k_0\alpha(S^{-1}h) - k_0\varepsilon_H(h) = 0$. Thus $\alpha(S^{-1}h) - \varepsilon_H(h) = 0$ for any $h \in H$, and this implies that

$$\alpha \circ S^{-1} = \varepsilon_H \quad \text{or} \quad \alpha = \varepsilon_H \circ S_H = \varepsilon_H$$

From the definition of α we know that λ is also a left integral of H . Hence H is unimodular.

2) \Rightarrow 3) Assume that H is k -Frobenius and unimodular, and write $\int_H = \lambda k$. It is clear that $c = \lambda$ satisfies the required properties.

3) \Rightarrow 1). Suppose $c \in H$ is such that $c \leftarrow H^* = H$ and (4.3) holds. Then the element $c \otimes 1_H$ is a $H \otimes H^{op}$ -integral of $H \otimes H$ (we identify ${}^H\mathcal{YD}_H$ and ${}^H\mathcal{M}(H \otimes H^{op}_H)$). Now for any $h \in H$,

$$\begin{aligned} (c \otimes 1_H) \cdot h &= \sum (S^{-1}h_{(3)})ch_{(1)} \otimes h_{(2)} \\ &= \sum (S^{-1}h_{(3)})h_{(2)}c \otimes h_{(1)} = c \otimes h \end{aligned}$$

and it follows from Theorem 2.4 that $H \otimes \bullet$ is a left adjoint of the forgetful functor. This finishes the proof of the theorem. \square

As an immediate consequence of Theorem 4.2, we can now give the following functorial characterization of unimodular Hopf algebras.

Corollary 4.3. *Let H be a Hopf algebra over a field k . Then the functor $H \otimes \bullet : \mathcal{M}_H \rightarrow {}^H\mathcal{YD}_H$ is a left adjoint of the forgetful functor $F : {}^H\mathcal{YD}_H \rightarrow \mathcal{M}_H$ if and only if H is finite dimensional and unimodular.*

Let H be a finitely generated projective Hopf algebra. As we have seen in Section 1.1, $H^* \otimes H$ is a natural object in ${}^H\mathcal{YD}_H$ and a left H -module as follows:

- $(h^* \otimes h) \cdot g = \sum (S^{-1}(g_{(1)}) \otimes g_{(3)}) \cdot h^* \otimes hg_{(2)} = \sum \langle h^*, g_{(3)} \rangle S^{-1}(g_{(1)}) \otimes hg_{(2)}$
- $\rho_{H^* \otimes H}(h^* \otimes h) = \sum h^*_{\langle -1 \rangle} \otimes h^*_{\langle 0 \rangle} \otimes h$
- $g \cdot (h^* \otimes h) = h^* \otimes gh$ for all $h, g \in H, h^* \in H^*$.

We can view $H \otimes H$ as a natural object in ${}^H\mathcal{YD}_H$ and as a left H -module as follows:

- $(h \otimes k) \cdot l = \sum S^{-1}(l_{(3)})hl_{(1)} \otimes kl_{(2)}$
- $\rho_{H \otimes H}(h \otimes k) = \sum h_{(1)} \otimes h_{(2)} \otimes k$
- $l \cdot (h \otimes k) = h \otimes lk$

for all $h, k, l \in H$ (see [3, 2]).

Remark 4.4. It was proved in [13, Theorem 3.15] that a ring extension $R \rightarrow S$ is Frobenius if and only if the induction functor $\bullet \otimes_R S$ is isomorphic to the coinduction functor $\text{Hom}_R(S, \bullet)$. Now, let H be a finite dimensional Hopf algebra over a field k . Then it is known [23] that ${}^H\mathcal{YD}_H \cong \mathcal{M}_{D(H)}$, and therefore the forgetful functor $\mathcal{M}_{D(H)} \rightarrow \mathcal{M}_H$ is isomorphic to the restriction of scalars functor $F : \mathcal{M}_{D(H)} \rightarrow \mathcal{M}_H$. The coinduction functor $\text{Hom}_H(D(H)_H, \bullet) : \mathcal{M}_H \rightarrow \mathcal{M}_{D(H)}$ is always a right adjoint of F and the induction functor $\bullet \otimes_H D(H) : \mathcal{M}_H \rightarrow \mathcal{M}_{D(H)}$ is always a left adjoint of F . From the above corollary, it follows that the induction functor is isomorphic to the coinduction functor if and only if H is unimodular. That means, the algebra extension $H \rightarrow D(H)$ is Frobenius if and only if H is unimodular.

Using again the fact that the categories ${}^H\mathcal{YD}_H$ and ${}^H\mathcal{M}(H \otimes H^{op})_H$ are isomorphic, it follows from Theorem 2.4 and Theorem 4.2 that we have the following remarkable characterization of unimodular Hopf algebras.

Corollary 4.5. *Let H be a finite dimensional Hopf algebra over a field k . The following statements are equivalent:*

- 1) H is unimodular.
- 2) The extension $H \rightarrow D(H), h \mapsto h \bowtie \varepsilon_H$ is Frobenius.
- 3) There exists an element $c \in H$ such that $c \leftarrow H^* = H$ and

$$(c \otimes 1)\Delta(h) = \Delta^{cop}(h)(c \otimes 1)$$

for all $h \in H$.

- 4) There exists an $H \otimes H^{op}$ -integral $x = \sum_{i=1}^n c_i \otimes a_i \in H \otimes H$ such that the map

$$\varphi : H^* \otimes H \rightarrow H \otimes H, \quad \varphi(h^* \otimes h) = \sum_{i=1}^n c_i \leftarrow h^* \otimes ha_i$$

is bijective.

- 5) There exists an isomorphism $H^* \otimes H \cong H \otimes H$ in ${}^H\mathcal{YD}_H$ which is also a left H -module map.

Remark 4.6. S. Montgomery pointed out to us that the equivalence of 1) and 2) in Corollary 4.5 can also be proved using [9, Theorem 3.7] and the fact that $D(H)$ is unimodular. Another proof of the same equivalence can be found in [10].

Proposition 4.7. *Let H be a k -projective k -Frobenius Hopf algebra. Then $H^* \otimes H$ and $H \otimes H$ are isomorphic in ${}^H\mathcal{YD}_H$.*

Proof. Since H is k -Frobenius, we can find a right H -integral $\lambda \in H$ such that $\lambda \leftarrow H^* = H$. For any $h \in H$, $h\lambda$ is in \int_H^r , so $h\lambda = \alpha_h\lambda$ for a unique $\alpha_h \in k$. Define $\alpha \in H^*$ by $\alpha(h) = \alpha_h$, for every $h \in H$. It is clear that α is multiplicative and invertible. The inverse of α is $\alpha \circ S$.

Define

$$\varphi : H^* \otimes H \rightarrow H \otimes H, \quad h^* \otimes h \mapsto (\lambda \leftarrow h^* \otimes \alpha^{-1} \rightarrow h)$$

for $h^* \in H^*$ and $h \in H$. φ is bijective, because the maps $H^* \rightarrow H, \quad h^* \mapsto (\lambda \leftarrow h^*)$ and $H \rightarrow H, \quad h \mapsto \alpha^{-1} \rightarrow h$ are bijective. The first of these two maps is left H -colinear, and this implies that φ is left H -colinear. We still have to show that φ is right H -linear. For any $h^* \in H^*$ and $h, g \in H$, we have that

$$\begin{aligned} \varphi((h^* \otimes h) \cdot g) &= \sum \varphi(S_{(H \otimes H^{op})}^{-1} g_{<-1>} \cdot h^* \otimes hg_{<0>}) \\ \text{(by (1.14))} \quad &= \sum \varphi\left(S_{(H \otimes H^{op})}^{-1}(g_{(1)} \otimes S_H^{-1}g_{(3)}) \cdot h^* \otimes hg_{(2)}\right) \\ &= \sum \varphi\left((S_H^{-1} \otimes S_H)(g_{(1)} \otimes S_H^{-1}g_{(3)}) \cdot h^* \otimes hg_{(2)}\right) \\ &= \sum \varphi\left((S^{-1}g_{(1)} \otimes g_{(3)}) \cdot h^* \otimes hg_{(2)}\right) \\ &= \sum \lambda \leftarrow [(S_H^{-1}g_{(1)} \otimes g_{(3)}) \cdot h^*] \otimes \alpha^{-1} \rightarrow (hg_{(2)}) \\ &= \sum \langle (S_H^{-1}g_{(1)} \otimes g_{(3)}) \cdot h^*, \lambda_{(1)} \rangle \lambda_{(2)} \otimes \alpha^{-1} \rightarrow (hg_{(2)}) \\ \text{(by (1.15))} \quad &= \sum \langle h^*, g_{(3)} \lambda_{(1)} S^{-1}g_{(1)} \rangle \lambda_{(2)} \otimes \alpha^{-1} \rightarrow (hg_{(2)}) \\ &= \sum \langle h^*, g_{(3)} \lambda_{(1)} \rangle \lambda_{(2)} g_{(1)} \otimes \alpha^{-1} \rightarrow (hg_{(2)}) \\ &= \sum \langle h^*, \lambda_{(1)} \rangle \alpha(g_{(4)}) (S^{-1}g_{(5)}) \lambda_{(2)} g_{(1)} \otimes \alpha^{-1} (h_{(2)} g_{(3)}) h_{(1)} g_{(2)} \\ &= \sum (S^{-1}g_{(3)}) (\lambda \leftarrow h^*) g_{(1)} \otimes (\alpha^{-1} \rightarrow h) g_{(2)} \\ &= \sum (\lambda \leftarrow h^*) \cdot (g_{(1)} \otimes S^{-1}g_{(3)}) \otimes (\alpha^{-1} \rightarrow h) g_{(2)} \\ &= ((\lambda \leftarrow h^*) \otimes (\alpha^{-1} \rightarrow h)) \cdot g \\ &= \varphi(h^* \otimes h) \cdot g \end{aligned}$$

and this shows that φ is right H -linear. □

It was proved in [2] that H has a left $H \otimes H^{op}$ -comodule algebra structure as follows:

$$h \mapsto \sum h_{(1)} \otimes S^{-1}(h_{(3)}) \otimes h_{(2)}$$

and H^* has a left $H \otimes H^{op}$ -module algebra structure via

$$\langle (h \otimes k) \triangleright h^*, l \rangle = \langle h^*, k l h \rangle$$

for all $h, k, l \in H$ and $h^* \in H^*$. With these structures, the Drinfel'd double is a smash product, $D(H) = H \# H^*$ (see [2]).

Corollary 4.8. *Let H be a finitely generated projective Hopf algebra. Then*

$$D(H) \cong \text{End}_H^H(H^* \otimes H)$$

as k -algebras. If H is k -Frobenius, then

$$D(H) \cong \text{End}_H^H(H \otimes H)$$

as k -algebras.

Proof. This follows immediately (1.10) and Proposition 4.7. \square

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