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ON NON-HYPERBOLIC QUASI-CONVEX SPACES

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Abstract. We show that if the universal covering of a compact Riemannian manifold with no conjugate points is a quasi-convex metric space then the following assertion holds: Either the universal covering of the manifold is a hyperbolic geodesic space or it contains a quasi-isometric immersion of $Z \times Z$.

Introduction

The existence of quasi-isometric immersions of the plane in complete, simply connected geodesic spaces has played an important role in the study of the geometry and topology of manifolds in recent years. From the work of Eberlein [5] in the early 70's we know for instance that the universal covering of a compact, non-positively curved Riemannian manifold is not a visibility manifold if and only if it contains a flat, totally geodesic plane. Further development of the theory of visibility manifolds (Eberlein, O'Neil [6]) shows remarkable connections between these manifolds and negatively curved manifolds. Also related to this subject we have the theory of 3-manifolds having incompressible tori. Thurston [14] stated the hyperbolization conjecture for 3-manifolds, which says that closed, irreducible 3-manifolds with infinite fundamental group are covered by hyperbolic space if and only if there are no incompressible tori in the manifold, and proved that this statement holds in a large class of 3-manifolds. Gromov [9] began the development of a geometric group theory and extended the study of such problems to the context of groups and graphs, and asked for instance if a non-hyperbolic (in Gromov's sense) group of isometries of a CAT-0 space contains a subgroup isomorphic to $Z \times Z$. Bangert and Schroeder [2] obtained some positive results in this direction in the case of analytic manifolds of non-positive curvature, but the general question for non-positively curved manifolds remains open. Recently, as a consequence of the works of Mess [10] and Gabai [7] we know that closed, oriented, irreducible 3-manifolds whose fundamental groups contain a subgroup isomorphic to $Z \times Z$ either are Seifert fibered or contain incompressible tori.

In the present work we consider the problem of the existence of quasi-isometric immersions of the plane in the context of manifolds with no conjugate points (i.e., where the exponential map is non-singular). Let us say that a metric space (X, d)

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is K, C-quasi-convex for $K, C \geq 0$ if for any two given geodesic segments $[a, b]$, $[c, d]$ in X we have that

$$sup(d(p, [a, b]), d(q, [c, d])) \leq K sup(d(a, c), d(b, d)) + C,$$

where $p \in [c, d]$, $q \in [a, b]$ and $d(p, [a, b])$ is the infimum of the distances from p to the points of $[a, b]$. For a given Riemannian manifold M, denote by $\tilde{M}$ the universal covering of M endowed with the pull-back of the metric of M by the covering map. The main theorem of the paper is the following:

Theorem 1. *Let M be a compact manifold with no conjugate points such that $\tilde{M}$ is K, C-quasi-convex. Then either $\tilde{M}$ is a hyperbolic geodesic space or there exists a quasi-isometric map $f : Z \times Z \longrightarrow \tilde{M}$, where $Z \times Z$ is endowed with the standard flat metric.*

V. Schroeder proved this result in [13] assuming that M has non-positive curvature, and as in the work of Eberlein the convexity of the metric is essentially used in the arguments. In our case, however, there is no restriction on the curvature, so the standard trigonometric inequalities and comparison theory of non-positive curvature may not hold. For the proof of Theorem 1 we use the following result which is proved in [12]:

Theorem 2. *Let M be a compact manifold with no conjugate points such that $\tilde{M}$ is K, C-quasi-convex. Then either $\tilde{M}$ is a hyperbolic geodesic space or $\forall n \in N$ there exist geodesics $\gamma_n, \beta_n : R \longrightarrow \tilde{M}$ and constants $D(n) > 0$ such that*

1. *$d(\gamma_n(t), \beta_n(t)) \leq D(n)$ $\forall t \in R$,*
2. *$d(\gamma_n(0), \beta_n) \geq n$ $\forall n$,*
3. *β_n is an orbit of the Busemann flow of γ_n.*

Statement 3 in Theorem 2 just means that β_n is the limit of geodesic segments joining $\beta_n(0)$ and $\gamma_n(t)$ letting, $t \to +\infty$ (see section 1). Notice that Theorem 2 implies Theorem 1 when M has no focal points, since from the flat strip theorem–which also holds in the absence of focal points–we get a sequence of flat strips whose widths go to infinity; thus a co-compactness argument allows us to obtain a convergent subsequence approaching a flat, totally geodesic plane in the universal covering. This result already generalizes the results of Eberlein and Schroeder mentioned above, since the absence of focal points does not imply the convexity of the metric. In the quasi-convex case we use Theorem 2 to obtain positive constants $\hat{K}, \hat{C}$, an element $\theta = (p, v) \in T_1\tilde{M}$ (the unit tangent bundle of $\tilde{M}$), a geodesic γ with $\gamma(0) = p$, $\gamma\prime(0) = v$ and a sequence of points in $\tilde{M}$, $\{ a_i \} \subset H_{(p,v)}$, $i \in Z$, such that

$$\hat{K}^{-1} \mid i - j \mid -\hat{C} \leq d(a_i, a_j) \leq \hat{K} \mid i - j \mid +\hat{C}$$

for every i, j. Here $H_{(p,v)}$ denotes the stable horosphere of the geodesic $\gamma(t)$ containing the point p. In other words, we obtain a quasi-isometric immersion of Z contained in $H_{(p,v)}$. This statement clearly generalizes the no focal points case, where such an intersection would typically contain a real geodesic of $\tilde{M}$. Notice that in the absence of a flat strip theorem it is not clear that we would find something like a plane bounded by the geodesics γ_n, β_n in $\tilde{M}$.

The paper contains 4 sections. The first section is devoted to the introduction of some basic notions of the theory of manifolds without conjugate points. In section 2 we shall construct a sequence of good, quasi-isometric copies of $Z(m_n) =$

$\{k \in N, \mid k \mid \leq m_n\}$, with $m_n \longrightarrow \infty$, as $n \longrightarrow \infty$ satisfying $Z(m_n) \subset H_{(p_n,v_n)}$, where $\gamma_n(0) = p_n$ and $\gamma_n'(0) = v_n$. A result about the existence of quasi-geodesic subcurves in curves having some minimizing properties (Proposition 2.1) plays a key role in this construction. The proof of this proposition will be done in section 4. Then we shall obtain v, γ and the quasi-isometric immersion of Z as desired by means of convergent subsequences of (p_n, v_n) and the quasi-isometric copies of $Z(m_n)$. Finally, in section 3 we shall show that this last fact together with analogous co-compactness arguments yield the existence of a vector $\psi = (q, u) \in T_1\tilde{M}$ (perhaps different from θ) such that the saturation by the Busemann flow of ψ of $H_{(q,u)}$ contains a quasi-isometric copy of $Z \times Z$.

We could expect that $H_{(q,u)} \cap H_{(q,-u)}$ would contain a whole geodesic as in the non-positive curvature case. However it is not even known if such an intersection contains a rectifiable curve (see [4] for instance) when one drops curvature restrictions. The most difficult part of the paper is the proof of Proposition 2.1, which gives information about the existence of quasi-geodesics in curves whose lengths are comparable with the distance between their endpoints.

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1. Horospheres and asymptotic properties of geodesics in $\tilde{M}$

We start by fixing some notations. We shall always consider C^∞ Riemannian manifolds. Given $\theta = (p, v) \in T_1M$, we shall often use the notation $\gamma_\theta(t)$ for a geodesic having initial conditions $\gamma_\theta'(0) = v$, $\gamma_\theta(0) = p$. All the geodesics will be parametrized by arc length. A very special property of manifolds with no conjugate points is the existence of the so-called *Busemann functions*: given $\theta = (p, v) \in T_1\tilde{M}$, the *Busemann function* $b_\theta : \tilde{M} \longrightarrow R$ associated to θ is defined by

$$b_\theta(x) = lim_{t\to+\infty}(d(x, \gamma_\theta(t)) - t)$$

The level sets of b_θ are the *horospheres* $H_\theta(t)$, where the parameter t means that $\gamma_\theta(t) \in H_\theta(t)$ (notice that $\gamma_\theta(t)$ intersects each level set of b_θ perpendicularly at only one point). In the introduction we used the notation H_θ for $H_\theta(0)$.

Lemma 1.1. 1. *b_θ is a C^1 function for every θ.*
2. *The gradient ∇b_θ has norm equal to one at every point.*

Therefore, horospheres are C^1 imbedded submanifolds of $\tilde{M}$ of dimension $n-1$, where n is the dimension of M, and there is a flow $\psi_t^\theta : \tilde{M} \longrightarrow \tilde{M}$ which we shall call the Busemann flow, whose orbits are geodesics which are everywhere tangent to the vector field $-\nabla b_\theta$. In particular the geodesic γ_θ is an orbit of this flow, and we have

$$\psi_t^\theta(H_\theta(s)) = H_\theta(s + t)$$

for any real numbers t, s. So this flow acts in $\tilde{M}$ perpendicular to the corresponding horosphere foliation and always preserving it. Next, we sketch an elementary construction of horospheres which gives a more precise geometrical description of them:

For $s > t$, let $B(s - t)$ be the sphere of radius $s - t$ and center at $\gamma_\theta(s)$. Clearly, $\gamma_\theta(t) \in B(s - t)$ for every s. Now, letting $s \to +\infty$ we obtain the horosphere $H_\theta(t)$

as a limit of $B(s-t)$ uniformly on compact sets in the C^1 topology. A classical reference for the subject is Busemann's work [3]; more recent expositions can be found in [11] and [4].

Although it is clear that $H_\theta(t)$ depends continuously on $t \in R$, it is not known whether $H_\theta(t)$ depends continuously on θ or not. In [11], Pesin observes that there is a large class of manifolds with no conjugate points satisfying this property.

Definition 1.1. Let M be a Riemannian manifold with no conjugate points. Two geodesics $\gamma(t)$, $\beta(t)$ in $\tilde{\mathrm{M}}$ are called *asymptotic* if there exists $C > 0$ such that

$$d(\gamma(t), \beta(t)) \leq C$$

for every $t \geq 0$.

If γ, β are asymptotic we shall often call β an asymptote of γ (or vice versa). If $\gamma(t), \beta(t)$ are asymptotic and $\bar{\gamma}(t) = \gamma(-t), \bar{\beta}(t) = \beta(-t)$ are also asymptotic, then we say that γ, β are *bi-asymptotic* .

Definition 1.2. Let M be a Riemannian manifold with no conjugate points. We say that $\tilde{\mathrm{M}}$ satisfies the *Axiom of Asymptoticity* if for every $\theta = (p, v)$ in $T_1\tilde{M}$ the following statement holds:

Let $q_n \to q$ be a sequence of points in $\tilde{M}$ and $\theta_n \to \theta$ be a sequence in $T_1\tilde{M}$. Then for every sequence of real numbers $t_n \to +\infty$ the sequence β_n of geodesic segments joining q_n to $\gamma_{\theta_n}(t_n)$ converges to an asymptote of γ_θ.

Manifolds with no focal points satisfy this axiom, and in particular manifolds of non-positive curvature. Now from [11] we have

Lemma 1.2. *Let M be a compact Riemannian manifold with no conjugate points. If $\tilde{M}$ satisfies the Axiom of Asymptoticity, then the vector fields ∇b_θ depend continuously on θ in the C^0 topology and the horospheres $H_\theta(t)$ depend continuously on $\theta \in T_1\tilde{M}$ in the compact open topology of $\tilde{M}$. Moreover, the orbits of the Busemann flow ψ_t^θ are all asymptotic to γ_θ.*

So the orbits of ψ_t^θ can be viewed as the "rays" of the horosphere $H_\theta(0)$ (and therefore as rays of every horosphere $H_\theta(t)$). Of course, in our case we have

Lemma 1.3. *Let M be a Riemannian manifold with no conjugate points. If $\tilde{\mathrm{M}}$ is quasi-convex then it satisfies the Axiom of Asymptoticity.*

Indeed, the quasi-convexity condition allows us to control the convergence of sequences of geodesics rays. Lemma 1.3 follows easily from

Lemma 1.4. *Let M be compact without conjugate points and suppose that $\tilde{\mathrm{M}}$ is a K, C-quasi-convex space. Then the orbits of the Busemann flow ψ_t^θ are asymptotic to γ_θ $\forall \theta \in T_1\tilde{M}$. Moreover, given $p \in \tilde{M}$, the geodesic $\beta(t) = \psi_t^\theta(p)$ satisfies*

$$sup_{t\geq 0} d(\beta(t), \gamma_\theta) \leq K d(\gamma_\theta(0), p) + C.$$

Proof. Although the argument of this proof is quite standard in global geometry, we shall write it down here for the sake of completeness. Let us abbreviate $\gamma_\theta = \gamma$. Given $s > 0$, let $[p, \gamma(s)]$ be the geodesic segment joining p to $\gamma(s)$. From the quasi-convexity we get

$$d([p, \gamma(s)], [\gamma(0), \gamma(s)]) \leq K d(\gamma(0), p) + C.$$

for every $s > 0$, where $d(,)$ in the left hand side of the inequality means Hausdorff distance. Since this inequality is preserved by limits, by letting $s \to +\infty$ we obtain a convergent subsequence of the segments $[p, \gamma(s)]$ whose limit is a geodesic $\beta(t)$ satisfying

1. $\beta(0) = p$,
2. $d(\beta(t), \gamma) \leq Kd(\gamma(0), p) + C$ for every positive t.

Claim. $\beta(t)$ is an orbit of the Busemann flow ψ_t^θ.

A sketch of the proof of this fact goes as follows (see [11] for instance): Notice that $[p, \gamma(s)]$ is always perpendicular to the spheres $B(d(p, \gamma(s)))$ centered at $\gamma(s)$ with radius $d(p, \gamma(s))$. By the construction of horospheres, the spheres $B(d(p, \gamma(s)))$ converge uniformly on compact sets in the C^1 topology to a certain horosphere of $\gamma(t)$ which is clearly perpendicular to $\beta(t)$ at p. Now, by the uniqueness of geodesics in terms of their initial conditions, $\beta(t)$ must be an orbit of ψ_t^θ.

2. On horospheres containing quasi-isometries of Z

We start by giving some basic definitions and notations. A metric space (X, d) is called a *geodesic space* if every pair of points can be joined by a geodesic segment.

Definition 2.1. A map $f : X \longrightarrow Y$ between two geodesic spaces (X, d), (Y, h) is a *quasi-isometry* if there exist positive constants L, B and a map $g : Y \longrightarrow X$ satisfying the following conditions:

1. $d(f(p), f(q)) \leq Ld(p, q) + B \ \forall p, q \in X$,
2. $d(g(p), g(q)) \leq Ld(p, q) + B \ \forall p, q \in Y$,
3. $d(f(g(p)), p) \leq B \ \forall p \in Y$,
4. $d(g(f(p)), p) \leq B \ \forall p \in X$.

The spaces X and Y are called *quasi-isometric*, and we shall refer to the image $f(X)$ either as a quasi-isometric immersion of X or as a quasi-isometric copy of X.

Definition 2.2. A geodesic space (X, d) is called δ-*hyperbolic* (or simply *hyperbolic*) if every geodesic triangle is δ-thin, i.e., every point of a given side of the triangle is at distance at most δ from the union of the other two sides.

The above definitions are taken from Gromov [9].

Definition 2.3. Let (X, d) be a metric space. A rectifiable curve C parametrized by arc length $c : [a, b] \longrightarrow X$ is called L, B-*quasi-geodesic* (or just *quasi-geodesic*) if the metric spaces $(c, \hat{d})$ and $([a, b], ||)$ are L, B-quasi-isometric, where $\hat{d}$ is the metric induced by d on the curve c and $||$ is the metric given by the absolute value in the interval $[a, b]$.

In other words, a rectifiable curve is quasi-geodesic in X if the length of any connected component of the parametrization is comparable with the distance in X between the endpoints of the component. For every continuous parametrization $c : [a, b] \longrightarrow C$ of a curve C we shall consider what we call the *length function* λ_c of the curve C, which is defined as follows:

$$\lambda_c((c(x), x), (c(y), y)) = \lambda_c(c(x), c(y)) = | x - y |$$

for every $x, y \in [a, b]$. The map λ_c is in general a function defined not in $C \times C$, but in the *parametrized* curve product $\{(c(t), t), t \in [a, b]\} \times \{(c(t), t), t \in [a, b]\}$, simply because a point in C may be equal to $c(t)$ for different values of t. However, $(c(t), t)$ is formally different from $(c(s), s)$ if $t \neq s$, even if $c(t) = c(s)$. In particular, if c is 1-1 then λ_c is a function in $C \times C$. We shall write $\lambda_c(c(x), c(y))$ instead of $\lambda_c((c(x), x), (c(y), y))$ as above, to shorten notation. The length function λ_c determines a flat, intrinsic metric in the curve $f(t) = (c(t), t)$, and it is clear that

$$\lambda_c(c(x), c(y)) + \lambda_c(c(y), c(z)) = \lambda_c(c(x), c(z))$$

for every $a \leq x \leq y \leq z \leq b$. We shall look for the quasi-geodesic properties of the curve C by comparing these length functions with the distance of the ambient space restricted to the curve C. Throughout the forthcoming sections we shall refer to λ_c simply as λ.

Definition 2.4. Let (X, d) be a metric space, let $c : [0, l(C)] \longrightarrow X$ be a continuous parametrization of a curve C and let λ be its length function. We say that (C, λ) is A, B-*quasi-convex* in X if

$$\mid t - s \mid = \lambda(c(t), c(s)) \leq Ad(c(t), c(s)) + B$$

$\forall t, s \in [a, b]$. We say that (C, λ) is A, B-*almost-quasi-convex* if

$$\lambda(c(a), c(b)) \leq Ad(c(a), c(b)) + B.$$

It is clear that if (C, λ) is A, B quasi-convex in X then (C, σ) is A', B'-quasi-convex in X for every length σ in c equivalent to λ (i.e. there is a constant $A > 0$ such that $\frac{\sigma(c(s),c(t))}{A} \leq \lambda(c(s), c(t)) \leq A\sigma(c(s), c(t))$). If we take $\lambda =$ arclength of C, then (C, λ) is quasi-convex in X if and only if C is a quasi-geodesic of X. It is also clear that quasi-convexity implies almost quasi-convexity, and although the converse statement is false in general one could ask whether almost quasi-convex curves contain quasi-convex parts. One of the key results used to prove the main theorem is the following:

Proposition 2.1. *Let (X,d) be a metric space. Given $A, B > 0$, there exist $A', B' > 0$ and a sequence m_n of positive integers with $lim_{n \to +\infty} m_n = +\infty$ such that the following statement holds:*

Given any curve $c : [0, l(C)] \longrightarrow C$ with length function λ satisfying the three conditions

1. $\lambda \geq d$,
2. (C, λ) *is A, B-almost-quasi-convex,*
3. $\lambda(c[0, l(C)]) \geq n$,

there exists a subcurve $c[x, y]$ of C which is A', B'-quasi-convex and such that

$$\lambda(c(x), c(y)) \geq m_n.$$

We shall prove Proposition 2.1 in section 4. The purpose of this section is to show that the lack of hyperbolicity of a quasi-convex manifold with no conjugate points is connected with the existence of quasi-geodesics in the horospheres. This fact clearly generalizes the existence of geodesics in horospheres determined by the lack of hyperbolicity in manifolds of non-positive curvature ([5]). More precisely, we prove

Proposition 2.2. *Let M be a compact manifold with no conjugate points such that $\tilde{M}$ is a K, C-quasi-convex metric space. Then either $\tilde{M}$ is a hyperbolic geodesic space or there exist a unit tangent vector (p, v), $p \in \tilde{M}$, and a quasi-isometry $f : Z \longrightarrow \tilde{M}$ with the following properties:*

1. *$f(i) \in H_{(p,v)}$ for every $i \in Z$.*
2. *The orbits of the Busemann flow $\psi_t^{(p,v)}(f(i))$ are bi-asymptotic to $\gamma_{(p,v)}$. Moreover, there exist $K', C' > 0$ such that for all $t \in R$*

$$d(\psi_t^{(p,v)}(f(i)), \psi_t^{(p,v)}(f(j))) \leq K' \mid i - j \mid +C'$$

We shall subdivide the proof of Proposition 2.2 into several steps. Throughout the paper, all metric spaces will be assumed to be complete. Let us consider the set $Z(n) = \{k \in N, \mid k \mid \leq n\}$. First we show

Lemma 2.3. *Let (X, d) be a metric space admitting a co-compact action of isometries and let $f_i : Z(n_i) \longrightarrow X$, $n_i \longrightarrow \infty$, be a sequence of K, C-quasi-isometries of $Z(n_i)$. Then, up to isometries of X, there exists a convergent subsequence of the f_i whose limit is a K, C-quasi-isometry of Z.*

Proof. We shall obtain a convergent subsequence by means of a diagonal process. By the hypotheses we can assume, up to isometries of X, that there exists a compact subset V of X such that $f_i(0) \in$ V for every integer i. So by induction, assume that for a given $n \in Z$ we have a subsequence f_{n_j} of the collection of quasi-isometries satisfying $f_{n_j}(k) \longrightarrow f(k)\ \forall \mid k \mid \leq$ n, where f is a K, C-quasi-isometry of $Z(n)$. Notice that

1. There exists a subsequence $f_{n_{j_i}}$ of the sequence f_{n_j} such that $f_{n_{j_i}}(n+1)$ and $f_{n_{j_i}}(-(n+1))$ are convergent, since by hypotheses we have that

$$\begin{aligned} d(f_{n_j}(n), f_{n_j}(n+1)) &\leq Kd(n, n+1) + C, \\ d(f_{n_j}(-n), f_{n_j}(-(n+1))) &\leq Kd(-n, -(n+1)) + C \\ &\leq K + C, \end{aligned}$$

so we obtain convergent subsequences by compactness. Let us call $f(n+1)$, $f(-(n+1))$ their limit points. This contruction extends the map f to a map from $Z(n+1)$ to $\tilde{M}$.
2. The inequalities of K, C-quasi-convexity are clearly satisfied by the points $f_{n_{j_i}}(s)\ \ \forall$ s $\in Z(n+1)$, since the fact of preserving such inequalities is itself preserved under limit operations.

This means that $f(n+1)$, $f(-(n+1))$ extend the map f as a K, C-quasi-isometry, which clearly implies the lemma. □

Next, we prove a general property of asymptotic geodesics in manifolds with no conjugate points, which allows us to compare the distance between two asymptotes with the distance between a pair of points in them belonging to the same horosphere. Recall that from Lemmas 1.1 and 1.2, given a geodesic $\gamma_\theta(t) = \gamma(t)$ in $\tilde{M}$ we can parametrize every asymptote $\beta(t)$ in such a way that $\beta(t) \in H_\theta(t)$. The idea behind the next two results is to show that, modulo a multiplying factor, the point $\beta(t)$ is the 'nearest' point of β to $\gamma(t)$. This fact is well-known in manifolds of non-positive curvature (and easy to check; see [1] for instance), where $\beta(t)$ is actually the point of β nearest to $\gamma(t)$. Although the arguments are standard in the theory of horospheres, since we do not know any written version of it, we include its proof for the sake of completeness.

Lemma 2.4. *Let M be a compact Riemannian manifold with no conjugate points. Let $\gamma_\theta(t) = \gamma(t)$ be a geodesic of $\tilde{M}$ and let β be any asymptote of γ with $\beta(0) \in H_\theta(0)$. Then*

$$d(\gamma(0), \beta(t)) \geq \frac{1}{3} d(\gamma(0), \beta(0)) \quad \textit{for every } t \in R.$$

Proof. Suppose that there exists t_n such that

$$d(\gamma(0), \beta(t_n)) \leq \frac{1}{n} d(\gamma(0), \beta(0)).$$

We know that the distance between two horospheres $H_\theta(t)$ and $H_\theta(s)$ is precisely $\mid t - s \mid$. Moreover, the distance from $\gamma(t)$ (and in general from $\alpha(t)$, where α is an asymptote of γ) to $H_\theta(s)$ is $\mid t - s \mid$. So on the one hand we have

$$\begin{aligned} t_n = d(\gamma(0), H_\theta(t_n)) &\leq d(\gamma(0), \beta(t_n)) \\ &\leq \frac{1}{n} d(\gamma(0), \beta(0)) \end{aligned}$$

since $\beta(t_n) \in H_\theta(t_n)$. On the other hand, we get

$$\begin{aligned} t_n &= d(\beta(0), H_\theta(t_n)) \\ &= d(\beta(0), H_{(\beta(0), \beta'(0))}(t_n)) \\ &= d(\beta(0), \beta(t_n)) \\ &\geq d(\beta(0), \gamma(0)) - d(\gamma(0), \beta(t_n)) \\ &\geq d(\beta(0), \gamma(0)) - \frac{1}{n} d(\beta(0), \gamma(0)) \\ &= (1 - \frac{1}{n}) d(\gamma(0), \beta(0)). \end{aligned}$$

Combining these two estimates of t_n we conclude that the inequality

$$(1 - \frac{1}{n}) d(\gamma(0), \beta(0)) \leq t_n \leq \frac{1}{n} d(\gamma(0), \beta(0))$$

implies

$$1 - \frac{1}{n} \leq \frac{1}{n},$$

which in turn implies

$$n \leq 2.$$

This means for instance that $d(\gamma(0), \beta(t)) \geq \frac{1}{3} d(\gamma(0), \beta(0))$ for every real t, which proves the lemma. □

Corollary 2.5. *Let M be a compact manifold without conjugate points. Suppose that $\tilde{M}$ is K, C-quasi-convex. Then, given two asymptotic geodesics $\gamma_\theta(t) = \gamma(t)$, $\beta(t)$ in $\tilde{M}$ with $\beta(0) \in H_\theta(0)$, at least one of the following two assertions hold:*

1. $d(\gamma(t), \beta(t)) \geq \frac{1}{3K} d(\gamma(0), \beta(0)) - \frac{C}{K}$ $\forall t \geq 0$;
2. $d(\gamma(t), \beta(t)) \geq \frac{1}{3K} d(\gamma(0), \beta(0)) - \frac{C}{K}$ $\forall t \leq 0$.

Proof. Let us denote by $[p, q]$ the geodesic segment joining p, $q \in \tilde{M}$, and let us denote by $d(z, [p, q])$ the infimum of the distances between z and the points of $[p, q]$ as usual. From the quasi-convexity we get that

$$d(\gamma(0), [\beta(s), \beta(t)]) \leq K sup\{d(\gamma(s), \beta(s)), d(\gamma(t), \beta(t))\} + C$$

for every $s \leq 0$ and $t \geq 0$. So if we let $d(\gamma(t), \beta(t)) < \frac{1}{K} d(\gamma(0), [\beta(s), \beta(t)]) - \frac{C}{K}$ for some positive t, then automatically $d(\gamma(s), \beta(s)) \geq \frac{1}{K} d(\gamma(0), [\beta(s), \beta(t)]) - \frac{C}{K}$ for every negative s. But from Lemma 2.4 this implies

$$\begin{aligned} d(\gamma(s), \beta(s)) &\geq \frac{1}{K} d(\gamma(0), [\beta(t), \beta(s)]) - \frac{C}{K} \\ &\geq \frac{1}{K} d(\gamma(0), \beta) - \frac{C}{K} \\ &\geq \frac{1}{3K} d(\gamma(0), \beta(0)) - \frac{C}{K} \end{aligned}$$

for every negative s. This proves the corollary. □

Lemma 2.6. *Let M be a compact manifold with no conjugate points such that $\tilde{M}$ is a K, C-quasi-convex metric space which is not Gromov hyperbolic. Then there exist a sequence of unit tangent vectors $\theta_n = (p_n, v_n)$ with $p_n \in \tilde{M}$, a sequence of maps $f_n : Z(N_n) \longrightarrow \tilde{M}$ with $N_n \longrightarrow \infty$, and constants K', C' not depending on n such that:*

1. *$f_n(i) \in H_{\theta_n}$ and*

$$d(\psi_t^{\theta_n}(f_n(i)), \psi_t^{\theta_n}(f_n(j))) \leq 3(K + C) \mid i - j \mid$$

   $\forall\, i, j \in Z(N_n)$ and $\forall t \in R$.
2. *There exist parametrizations c_n of the curves*

$$C_n = \bigcup_{i=-N_n}^{N_n - 1} [f_n(i), f_n(i+1)]$$

   such that
   - *$(K+C)\lambda_{c_n} \geq d$, where d is the distance in $\tilde{M}$,*
   - *(C_n, λ_{c_n}) is K',C' almost quasi-convex in $\tilde{M}$.*

Proof. Let d be the distance in $\tilde{M}$ induced by the pullback of the metric of M. By Theorem 2 in the introduction to the paper we have that if $\tilde{M}$ is not a hyperbolic space then there exist geodesics γ_n, β_n in $\tilde{M}$ such that

1. $d(\gamma_n(0), \beta_n(t)) \geq n \ \forall t \in R$,
2. $d(\gamma_n(t), \beta_n(t)) \leq D(n) \ \forall t \in R$,
3. β_n is an orbit of the Busemann flow $\psi_t^{\theta_n}$, where $\theta_n = (\gamma_n(0), \gamma_n'(0))$.

Here, t is the arc length parameter of the geodesics. We can suppose without loss of generality that $\beta_n(0) \in H_{\theta_n}(0)$.

Claim 1. Given $\delta > 0$, there exist n_0 and geodesics $\hat{\gamma_n}$, $\hat{\beta}_n$ such that for every $n \geq n_0$

1. $d(\hat{\gamma_n}(t), \hat{\beta}_n(t)) \geq \frac{1}{4K} n \ \forall t \in R$,
2. $d(\hat{\gamma_n}(t), \hat{\beta}_n(t)) \leq d(\hat{\gamma_n}(0), \hat{\beta}_n(0)) + 2\delta \ \forall t \in R$,
3. $\hat{\beta}_n$ is an orbit of the Busemann flow $\psi_t^{\hat{\theta}_n}$, where $\hat{\theta}_n = (\hat{\gamma}_n(0), \hat{\gamma}_n'(0))$.

The proof is as follows: from Corollary 2.5 and Theorem 2 in the introduction we can assume without loss of generality that

$$\begin{aligned} d(\gamma_n(t),\beta_n(t)) &\geq \frac{1}{3K}d(\gamma_n(0),\beta_n(0)) - \frac{C}{K} \\ &\geq \frac{n}{3K} - \frac{C}{K} \\ &\geq \frac{n}{4K} \end{aligned}$$

for every $t \leq 0$ and $n \geq n_0$ large enough, depending on K, C. Notice that, from Lemma 2.4, this implies that $\gamma_n(t)$ is indeed far away from β_n for every $t \leq 0$, since γ_n, β_n are orbits of the same Busemann flow, and this implies

$$d(\gamma_n(t),\beta_n) \geq \frac{1}{3}d(\gamma_n(t),\beta_n(t)) \tag{1}$$

$$\geq \frac{n}{12K} \tag{2}$$

for every $t \leq 0$. Next, since the distance between $\gamma_n(t)$ and $\beta_n(t)$ is uniformly bounded in t, there exists a sequence $t_m \to -\infty$ such that

$$limsup_{t\to-\infty} d(\gamma_n(t),\beta_n(t)) \leq d(\gamma_n(t_m),\beta_n(t_m)) + \delta$$

by the definition of *limsup*. Consider the geodesics $\gamma_n^m(t) = \gamma_n(t+t_m)$, $\beta_n^m(t) = \beta_n(t+t_m)$. By the choice of the t_m there exists $N > 0$ such that for all $t + t_m \leq -N$ we get

$$d(\gamma_n^m(t), \beta^m n(t)) \leq d(\gamma_n(t_m),\beta_n(t_m)) + 2\delta \tag{3}$$

$$= d(\gamma_n^m(0),\beta_n^m(0)) + 2\delta \tag{4}$$

and also, for every $t \leq -t_m$ we get

$$d(\gamma_n^m(t),\beta_n^m(t)) \geq \frac{n}{4K} \tag{5}$$

Let $\theta_n^m = (\gamma_n^m(0), \gamma_n^{m\prime}(0))$, $\eta_n^m = (\beta_n^m(0), \beta_n^{m\prime}(0))$. Letting $m \to +\infty$, we get, up to covering translations of $\tilde{M}$, convergent subsequences $\theta_n^{m_k} \to \hat{\theta}_n$, $\eta_n^{m_k} \to \hat{\eta}_n$ such that the geodesics $\hat{\gamma}_n$ and $\hat{\beta}_n$ defined by $(\hat{\gamma}_n(0), \hat{\gamma}_n'(0)) = \hat{\theta}_n$, $(\hat{\beta}_n(0), \hat{\beta}_n'(0)) = \hat{\eta}_n$ clearly satisfy conditions 1 and 2 in the statement of Claim 1 as a consequence of inequalities (3), (4) and (5). Assertion 3 of Claim 1 follows from the continuity of horospheres (Lemma 1.2): since γ_n^m, β_n^m are orbits of the Busemann flow $\psi^{\theta_n^m}$, we have that $H_{\theta_n^m}(0) = H_{\eta_n^m}(0)$; on the other hand we have that $H_{\theta_n^{m_k}}(0) \to H_{\hat{\theta}_n}(0)$ and $H_{\eta_n^{m_k}}(0) \to H_{\hat{\eta}_n}(0)$ as $k \to +\infty$. This implies that $H_{\hat{\theta}_n}(0) = H_{\hat{\eta}_n}(0)$, or equivalently, $\hat{\gamma}_n$ and $\hat{\beta}_n$ are orbits of the same Busemann flow, thus finishing the proof of Claim 1.

To begin the construction of the curves in the statement let us fix an integer n. Recall that $[p, q]$ is the geodesic segment of $\tilde{M}$ joining $p, q \in \tilde{M}$. For simplicity we shall use the notation γ_n, β_n, θ_n instead of $\hat{\gamma}_n$, $\hat{\beta}_n$, $\hat{\theta}_n$.

Claim 2. There exists a collection of geodesics $\sigma_i(t) = \psi_t^{\theta_n}(q_i)$, where $q_i \in H_{\theta_n}(0)$ for $0 \leq i \leq l+1$, (l is the integer part of $d(\gamma_n(0), \beta_n(0))$), such that

1. $\sigma_0(t) = \gamma_n(t)$ and $\sigma_{l+1}(t) = \beta_n(t)$,
2. $d(\sigma_i, \sigma_j) \leq K \mid i-j \mid + C$ for every i, j,
3. $\frac{1}{3}d(\sigma_i(0), \sigma_j(0)) \leq K \mid i-j \mid + C$,
4. the number l_n of distinct σ_i's is at least $\frac{n}{12K(K+C)}$.

To see this, let I_m be the integer part of $d(\gamma_n(m), \beta_n(m))$. Let $m < 0$ and make a partition

$$[\gamma_n(m), \beta_n(m)] = \bigcup_{i=0}^{I_m} [q_i^m, q_{i+1}^m],$$

where:

- $q_0^m = \gamma_n(m)$, $q_{I_m+1}^m = \beta_n(m)$.
- The length of $[q_i^m, q_{i+1}^m]$ is 1 for every $0 \leq i \leq I_m$, and the length of $[q_{I_m}^m, q_{I_m+1}^m]$ is less than or equal than 1.
- $[q_{i-1}^m, q_i^m] \cap [q_i^m, q_{i+1}^m] = q_i^m$ for every i.

Let σ_i^m be the orbit of the Busemann flow $\psi_t^{\theta_n}$ passing through q_i^m, and parametrize σ_i^m so that have

$$\sigma_i^m(t) = \psi_t^{\theta_n}(\sigma_i^m \cap H_{\theta_n}(0)) = \psi_t^{\theta_n}(Q_i^m).$$

In this way we have that $q_i^m = \sigma_i^m(s_i^m)$ for appropriate sequences $\{s_i^m\}$ satisfying $lim_{m\to\infty} s_i^m = -\infty$ for every $0 \leq i \leq I_m + 1$. Then from Lemma 1.4 we deduce that

$$\begin{aligned} d(\sigma_i^m(t), \sigma_j^m) &\leq K d(\sigma_i^m(0), \sigma_j^m(0)) + C \\ &= K \mid i - j \mid + C \end{aligned}$$

for every $t \geq s_i^m$ and $0 \leq i, j \leq I_m + 1$. Moreover, from Lemma 2.4 we get

$$\begin{aligned} d(Q_i^m, Q_j^m) &= d(\sigma_i^m(0), \sigma_j^m(0)) \\ &\leq 3d(\sigma_i^m(0), \sigma_j^m) \\ &\leq 3K \mid i - j \mid + 3C \end{aligned}$$

for every i, j, m. Since $Q_0^m = \gamma_n(0)$, all the points Q_i^m are contained in a ball of radius $3K(I_m + 1) + 3C$ centered at $\gamma_n(0)$. So letting $m \to -\infty$ we obtain, by a diagonal process, subsequences $Q_i^{m_k} \to q_i$ for $0 \leq i \leq sup_m I_m + 1$ (recall that $I_m \leq d(\gamma_n(0), \beta_n(0)) + 2\delta$ $\forall m < 0$ by Claim 1) which determine geodesics $\sigma_i(t) = \psi_t^{\theta_n}(q_i)$, all bi-asymptotic to γ_n. Furthermore, from the above inequalities we have that

1. $d(\sigma_i, \sigma_j) \leq K \mid i - j \mid + C$
2. $d(q_i, q_j) \leq 3K \mid i - j \mid + 3C$

for every i, j. In this way we get items 1, 2 and 3 in the statement of Claim 2. To show item 4 we argue by contradiction: the number l_n must be at least $\frac{d(\gamma_n(0), \beta_n)}{K+C}$. Otherwise we would have a path from $\gamma_n(0)$ to β_n of length less than $(K + C)\frac{d(\gamma_n(0), \beta_n)}{K+C} = d(\gamma_n(0), \beta_n)$, which is clearly a contradiction (recall that $d(\sigma_i, \sigma_{i+1}) \leq K + C$). On the other hand, by inequalities (1), (2) in the proof of Claim 1 we have that $d(\gamma_n(0), \beta_n) \geq \frac{n}{12K}$, from which we easily deduce item 4, thus concluding the proof of the claim.

Notice that some of the σ_i may coincide, which implies that some of the segments $[\sigma_i(0), \sigma_{i+1}(0)]$ may be just points. So now we proceed to throw away some of the redundant indices by choosing a certain collection $\{i_s\}$ in the following way:

Let $i_0 = 0$; now look at the segments $[\sigma_{i_0}(0), \sigma_{i_0+j}(0)]$, and let j_0 be the first positive j such that the above segment is not a point. Define $i_1 = i_0 + j_0$. Continuing with this process, we obtain a collection of bi-asymptotes of γ_n:

$$\gamma_n = \sigma_0, \sigma_{i_1}, \cdots, \sigma_{i_{l'}} = \beta_n$$

whose indices satisfy $i_r < i_s$ $\forall 0 < r < s \leq l'$, where $\frac{n}{12K(K+C)} \leq l' \leq l+1$, and such that every segment $[\sigma_{i_r}(0), \sigma_{i_{r+1}}(0)]$ is not a point. Of course, these segments may still coincide on subsegments, and from the process of their selection we get immediately

Claim 3.

$$\frac{1}{3} d(\sigma_{i_r}(0), \sigma_{i_{r+1}}(0)) \leq K + C$$

for every $0 \leq r \leq l'$.

In fact, since $\sigma_{i_{r+1}}$ is the first geodesic not coinciding with σ_{i_r}, this means that all the geodesics σ_j with $j \leq i_{r+1} - 1$ coincide with σ_{i_r}. In particular,

$$\begin{aligned} d(\sigma_{i_{r+1}}, \sigma_{i_r}) &= d(\sigma_{i_{r+1}}, \sigma_{i_{r+1}-1}) \\ &\leq K + C, \end{aligned}$$

where $d(\alpha, \beta)$ denotes the Hausdorff distance between α and β, and K, C are the quasi-convexity constants of the metric in $\tilde{M}$. Now we can apply Lemma 2.4 to get Claim 3.

Finally, define

$$C_n = \bigcup_{r=0}^{l'-1} [\sigma_{i_r}(0), \sigma_{i_{r+1}}(0)]$$

and let $F_n : \{0, 1, \ldots, l'\} \longrightarrow \tilde{M}$ be defined by $F_n(r) = \sigma_{i_r}(0)$. The next remarks follow from the construction:

1. $arclength(C_n) \leq 3(K+C)l' \leq 3(K+C)l$
$$\begin{aligned} &\leq 3(K+C)(d(\gamma_n(0), \beta_n(0)) + 2\delta) \\ &\leq 3(K+C)(d(C_n(0), C_n(l')) + 2\delta). \end{aligned}$$
2. $arclength(C_n) \geq d(\gamma_n(0), \beta_n(0))$.

Let us parametrize C_n in such a way that all the segments $[\sigma_{i_r}(0), \sigma_{i_{r+1}}(0)]$ have λ-length equal to 1 (simply by dividing the arc length parametrization in each geodesic segment by its length with respect to the metric in $\tilde{M}$). In that way we obtain a continuous monotone reparametrization $c_n : [0, l'] \longrightarrow C_n$ such that

$$\bigcup_{r}^{s-1} [\sigma_{i_r}(0), \sigma_{i_{r+1}}(0)] = c_n[r, s]$$

for every $r \leq s$. In particular, if $r = s$ then $F_n(r) = \sigma_{i_r}(0) = c_n(r)$. From the construction and the remarks above it is straightforward to conclude:

1. $F_n(i) \in H_{\theta_n}(0)$;
2. $3(K+C)\lambda_{c_n} \geq d$, where d is the distance in $\tilde{M}$ restricted to c_n;
3. $\lambda_{c_n}(C_n) = l' \leq d(c_n(0), c_n(l')) + 2\delta$.

Furthermore, from Claim 3 we have that

$$arclength(c_n[i,j]) \leq 3(K+C) \mid i - j \mid \tag{6}$$

for every $0 \leq i, j \leq l'$. Letting $N_n = \frac{l'}{2}$ if l' is even, or $N_n = \frac{l'-1}{2}$ if l' is odd (recall that l' depends on n), we obtain maps $f_n : Z_{N_n} \longrightarrow \tilde{M}$ and the curves defined by $f_n(i) = F_n(N_n + i)$ in the statement of Lemma 2.6 satisfying all the required properties, where $K' = 1$, $C' = \delta$. To finish the proof of Lemma 2.6 we just notice that N_n goes to ∞ when n increases, since Claim 2, item 4 gives $l' \geq l_n \geq \frac{n}{12K(C+K)}$. $\square$

Proof of Proposition 2.2. Let M , $\tilde{M}$ be as in the hypotheses of Proposition 2.2. Note that, in the case of surfaces, if $\tilde{M}$ is not hyperbolic, then M is the flat torus and $\tilde{M}$ is the flat plane. In general, we obtain the sequence of curves (C_n, λ_{c_n}) with length functions λ_{c_n} which are K', C'-almost quasi-convex in $\tilde{M}$ according to Lemma 2.6. From Proposition 2.1 there exists a sequence $\tilde{C_n} \subset C_n$ of connected curves such that:

1. $\lambda_{c_n}(\tilde{C_n}) \longrightarrow \infty$.
2. For some constants A, B and $(\tilde{C_n}, \lambda_{c_n})$ is A, B-quasi-convex for every n.

Here of course, $\tilde{C_n}$ inherits the parametrization c_n of C_n. Without loss of generality we can suppose that the curves $\tilde{C_n}$ are unions of segments

$$\tilde{C_n} = \bigcup_{i=p_n}^{j_n} [c_n(i), c_n(i+1)].$$

But this implies that

$$\mid i - j \mid = \lambda_{c_n}[c_n(i), c_n(j)] \leq Ad(c_n(i), c_n(j)) + B$$

for every $p_n \leq i, j \leq j_n$. So we get

$$\frac{1}{A} \mid i - j \mid - \frac{B}{A} \quad \leq \quad d(c_n(i), c_n(j)) \tag{7}$$

$\forall p_n \leq i, j \leq j_n$. Since from Claim 3 in the proof of Lemma 2.6 we have that $d(c_n(i), c_n(j)) \leq 3(K + C) \mid i - j \mid$, we get that the curves $\tilde{C_n}$ are quasi-geodesics for constants $K_1 = sup(A, 3(K + C)), C_1 = \frac{B}{A}$ which do not depend on n, and the maps $f_n : \{p_n, p_n + 1, ..., j_n\} \longrightarrow \tilde{M}$ given by $f_n(i) = c_n(i)$ are quasi-isometries of $\{p_n, p_n + 1, ..., j_n\}$–endowed with the Euclidean metric–into their images. Assume without loss of generality that $\mid p_n - j_n \mid$ is even for every n. Then we obtain a sequence of quasi-isometries of $Z(\frac{p_n - j_n}{2})$ defined by

$$g_n : Z(\frac{p_n - j_n}{2}) \longrightarrow \tilde{M},$$

$$g_n(i) = f_n(\frac{p_n - j_n}{2} + i),$$

whose quasi-isometric constants do not depend on n. By Lemma 2.6 there is a sequence $\theta_n \in T_1\tilde{M}$ such that $g_n(i) \in H_{\theta_n}(0)$ and such that the orbits $\psi_t^{\theta_n}(g_n(i))$ satisfy the inequalities in Lemma 2.6 (1). We can assume for simplicity that $g_n(0) = q_n$. Then, up to covering translations of $\tilde{M}$, by Lemmas 2.3, 2.6, and the continuity of horospheres (Lemma 1.2 (1)) there exist subsequences $g_{n_k} : Z \longrightarrow \tilde{M}$ converging to $g : Z \longrightarrow \tilde{M}$ and $\theta_{n_k} \longrightarrow \theta = (q, v)$ with $g(0) = q$ satisfying

- $g(i) \in H_\theta(0)$ $\forall i \in Z$,
- the map g is a K_1, C_1-quasi-isometry of Z, and
- $d(\psi_t^\theta(g(i)), \psi_t^\theta(g(j))) \leq K_1 \mid i - j \mid + C_1$ for every $t \in R$,

thus proving Proposition 2.2.

3. The quasi-isometry of $Z \times Z$

In this section we shall show how to obtain the quasi-isometry of $Z \times Z$ in $\tilde{M}$ given that it is non-hyperbolic, without conjugate points and K, C-quasi-convex. In this section all geodesics will again be parametrized by arc length. We start with the following result:

Lemma 3.1. *Let $\tilde{M}$ be as before, and let $f : Z \times Z \longrightarrow \tilde{M}$ satisfy the following conditions:*

1. *There exists a geodesic $\gamma = \gamma_\theta$ in $\tilde{M}$ such that:*
   1.i $f(i,j) \in H_\theta(i)\ \forall\ j \in Z$.
   1.ii *For each j there is an asymptote γ_j of γ such that $f(i,j) \in \gamma_j$ for every $i \in Z$, i.e., $f(i,j) = H_\theta(i) \cap \gamma_j$.*
2. *The maps $f_i : Z \longrightarrow \tilde{M}$ defined by $f_i(j) = f(i,j)$ are A, B-quasi-isometries for every $i \in Z$.*

Then f is a A_0, B_0-quasi-isometry, where A_0, B_0 depend on A, B, K, C.

Proof. Recall that the distance between horospheres is constant, i.e., the distance between any point $p \in H_\theta(t)$ and the submanifold $H_\theta(s)$ is $\mid t - s \mid$ and the geodesic of minimum distance from p to $H_\theta(s)$ is the asymptote of γ which contains the point p. Now we proceed to estimate the distance $d(f(i,j), f(k,s))$, where, as always, d is the distance induced by the metric of $\tilde{M}$. The upper inequalities hold easily from the hypotheses:

$$\begin{aligned} d(f(i,j), f(k,s)) &\leq d(f(i,j), f(i,s)) + d(f(i,s), f(k,s)) \\ &\leq A \mid j - s \mid + B + \mid i - k \mid \\ &\leq sup(1, A)(\mid j - s \mid + \mid i - k \mid) + B \\ &\leq sup(1, A)\sqrt{2}\sqrt{\mid j - s \mid^2 + \mid i - k \mid^2} + B, \end{aligned}$$

where in the second inequality we used items 1 and 2 in the hypotheses of the lemma. It remains to analyze the lower inequalities. We have two cases:

1. $\mid j - s \mid \geq \mid k - i \mid$. In this case, since $f(i,j)$ and $f(i,s)$ are in the same horosphere $H_\theta(i)$ and $f(k,s), f(i,s) \in \gamma_s$, we get from Corollary 2.5 that

   $$\begin{aligned} d(f(i,j), f(k,s)) &\geq inf_{t \in R} d(f(i,j), \gamma_s(t)) \\ &\geq \frac{1}{3K} d(f(i,j), f(i,s)) - \frac{C}{K} \\ &\geq \frac{1}{3KA} \mid j - s \mid - \frac{C}{K} - \frac{B}{3K}, \end{aligned}$$

   where in the last inequality we used item 2 of the hypotheses. So we have

   $$\begin{aligned} \sqrt{2} d(f(i,j), f(k,s)) &\geq \frac{1}{3KA}\sqrt{2} \mid j - s \mid - \sqrt{2}(\frac{C}{K} - \frac{B}{3K}) \\ &\geq \frac{1}{3KA}\sqrt{\mid j - s \mid^2 + \mid k - i \mid^2} - \sqrt{2}(\frac{C}{K} - \frac{B}{3K}), \end{aligned}$$

   which combined with the previous estimates gives the desired result.

2. $|j-s| \leq |i-k|$. In this case, since $f(k,s)$, $f(k,j)$ are in the same horosphere $H_\theta(k)$, we get from the remark at the beginning of the proof that the inequality

$$\begin{aligned} d(f(i,j), f(k,s)) &\geq inf_{q \in H_\theta(k)} d(f(i,j), q) \\ &\geq d(f(i,j), f(k,j)) \\ &= |i-k| \end{aligned}$$

implies that

$$\begin{aligned} \sqrt{2} d(f(i,j), f(k,s)) &\geq \sqrt{2} \mid i-k \mid \\ &\geq \sqrt{\mid i-k \mid^2 + \mid j-s \mid^2} \end{aligned}$$

from which we easily conclude the statement of the lemma. □

The next step of the proof will be the construction of a lattice in $\tilde{M}$ of the type described in the statement of Lemma 3.1. To do this, consider the map $f : Z \longrightarrow \tilde{M}$ and the unit vector $\theta = (p,v)$ of Proposition 2.2. Let us extend f to a map from $Z \times Z$ to $\tilde{M}$–which we shall denote by F–in the following way:

$$F : Z \times Z \longrightarrow \tilde{M},$$

$$F(i,j) = \psi_i^\theta(f(j)),$$

where $\psi_t^\theta : \tilde{M} \longrightarrow \tilde{M}$ is the Busemann flow associated to the geodesic $\gamma(t)$ defined by $\gamma(0) = p$, $\gamma'(0) = v$. The integral curves of this flow are the asymptotes of γ and the flow preserves horospheres, i.e., $\psi_t^\theta(H_\theta(s)) = H_\theta(t+s)$ for every $s,t \in R$. In this way we get a lattice in $\tilde{M}$ such that each "column" of the lattice is an iterate of a quasi-isometric image of Z by the time 1 diffeomorphism of the Busemann flow of γ. So it is clear that the n^{th} column $F_n : Z \longrightarrow \tilde{M}$, $F_n(i) = F(n,i)$, of the lattice is a subset of

$$H_\theta(n) = \psi_n(H_\theta(0))$$

for every $n \in N$. However, it is not clear a priori that the columns are also quasi-isometries of Z, which according to Lemma 3.1 would suffice to finish the proof of the main theorem. The idea now is to apply the argument at the end of the proof Proposition 2.2 recursively to obtain quasi-isometries of Z in each horosphere $H_\eta(i)$ for $i \in Z$ and some $\eta \in T_1\tilde{M}$.

Proof of Theorem 1. Let us consider the map F defined before. Let K, C be the quasi-convexity constants of the metric in $\tilde{M}$. According to the construction of this map in the previous section we know the following facts:

1. There exist geodesics γ_j asymptotic to $\gamma_0 = \gamma_\theta$, for some $\theta = (p,v) \in T_1\tilde{M}$, such that

$$F(i,j) = \gamma_j(i) = H_\theta(i) \cap \gamma_j$$

and

$$d(\gamma_j, \gamma_k) \leq K \mid j-k \mid + C$$

for every i, j, k, where d is the Hausdorff distance.

2. Moreover, from Corollary 2.5 we can assume without loss of generality that there exist sequences $\{a_n\}$, $\{b_n\}$, with $lim_{n\to+\infty}(b_n - a_n) = +\infty$, such that for every $t \geq 0$ we have
$$\begin{aligned} d(\gamma_{a_n}(t), \gamma_{b_n}(t)) &\geq \frac{1}{3}d(\gamma_{a_n}(0), \gamma_{b_n}(0)) - \frac{C}{K} \\ &\geq \frac{1}{3}d(f(b_n), f(a_n)) - \frac{C}{K} \\ &\geq \frac{1}{3}(\frac{\mid b_n - a_n \mid}{K_1} - C_1) - \frac{C}{K}, \end{aligned}$$
where K_1, C_1 are the quasi-isometry constants of f.
3. By item 1 and Lemma 2.6 we have
$$d(F_i(j), F_i(k)) \leq 3K \mid j - k \mid +3C$$
for every i, j, k.

The problem here is, as it was in Proposition 2.2, the verification of the inequalities bounding from below in the definition of quasi-isometry (i.e., the inequalities $K' \mid i - j \mid -C' \leq d(F_i(j), F_i(k))$ for some constants $K', C' > 0$). However, assertions 2, 3 above allow us to apply Proposition 2.1 to the curves
$$\Gamma_n(i) = \bigcup_{s=a_n}^{b_n-1} [F(i,s), F(i,s+1)]$$
for $i \in N$, from which we get that for each $i \in N$ there exists sequences $S_k(i) = \{j_k^i, \cdots, j_k^i + N_k^i\}$ with $N_k^i \to \infty$ if $k \to \infty$ such that the maps
$$F_i : [j_k^i, j_k^i + N_k^i] \longrightarrow \tilde{M},$$
$$F_i(j) = F(i,j),$$
are A, B quasi-isometries for A, B depending on K and C. Now, by induction on i we shall obtain a map $\hat{F}$ which is a A, B-quasi-isometry of $Z \times Z$ into $\tilde{M}$. By Proposition 2.2 we have that F_0 is already a A, B-quasi-isometry. So let us assume, as induction hypotheses, that we have a geodesic $\gamma = \gamma_\theta$, and a map that we shall still call F from $Z \times Z$ into $\tilde{M}$ satisfying the following conditions:

1. F_i is an A, B-quasi-isometry of Z for every $0 \leq i \leq n$.
2. For every $i, j \in Z$ we have
$$F(i,j) = H_\theta(i) \cap \gamma_j.$$

The construction in the previous section and Proposition 2.1 give us sequences $S_k(n+1)$ as described above. Assuming without loss of generality that the numbers N_k^{n+1} are even $\forall k$, let us take a sequence of covering translations Φ_k of $\tilde{M}$ such that $\Phi_k(F_{n+1}(j_k^{n+1} + \frac{N_k^{n+1}}{2}))$ belongs to a certain compact fundamental domain V of M in $\tilde{M}$ $\forall k$. Define $g_{n+1}^k : Z(\frac{N_k^{n+1}}{2}) \longrightarrow \tilde{M}$ by
$$g_{n+1}^k(i) = \Phi_k(F_{n+1}(j_k^{n+1} + \frac{N_k^{n+1}}{2} + i)).$$

Let $p_k = j_k^{n+1} + \frac{N_k^{n+1}}{2}$ and let $\alpha_i^k(t) = \Phi_k(\gamma_{p_k+i}(t))$. Again, by the previous construction we obtain that
$$d(\alpha_i^k, \alpha_j^k) \leq K \mid i - j \mid +C$$

for every $i, j \in Z(\frac{N_k^{n+1}}{2})$, because the geodesics γ_j already had this property. Moreover, from the induction hypotheses we get

$$\frac{1}{A} \mid i-m \mid -B \leq d(\Phi_k(\gamma_{p_k+i}(j)), \Phi_k(\gamma_{p_k+m}(j))) \tag{8}$$

$$= d(\alpha_i^k(j), \alpha_m^k(j)) \tag{9}$$

for every $0 \leq j \leq n$ and every $i, m \in Z$. Notice that

1. g_{n+1}^k is a A, B-quasi-isometry and $g_{n+1}^k(0) = \alpha_0^k(n+1) \in V \ \forall k$,
2. $g_{n+1}^k(i) = \alpha_i^k(n+1)$ by condition 2 in the induction hypotheses,
3. $g_{n+1}^k(i) \in H_{(\alpha_0^k(0), \alpha_0^{k\prime}(0))}(n+1)$ for every i, k, and

$$d(g_{n+1}^k(i), g_{n+1}^k(j)) \leq 3K \mid i-j \mid +3C$$

because the corresponding inequalities for the F_i are preserved by isometries.

Letting $k \to +\infty$, we obtain

- a subsequence of the $\alpha_0^k \longrightarrow \alpha_0 = \gamma_{\theta_0}$ for some $\theta_0 \in T_1\tilde{M}$,
- a subsequence of the g_{n+1}^k converging to an A, B-quasi-isometry g_{n+1} of Z (by Lemma 2.3) satisfying
  1. $g_{n+1}(0) = \alpha_0(n+1)$,
  2. $g_{n+1}(i) \in H_{\theta_0}(n+1)$ for every i, by the continuity of horospheres (Lemma 1.2),
- subsequences of the geodesics α_j^k (determined by the subsequences of the g_{n+1}^k) converging to geodesics α_j satisfying

  $$g_{n+1}(j) = \alpha_j(n+1);$$

  these geodesics α_j are clearly bi-asymptotic to α_0, and from equation (1) above we deduce that

  $$\frac{1}{A} \mid j-m \mid -B \leq d(\alpha_j(i), \alpha_m(i))$$

  for every $j, m \in Z$ and every $0 \leq i \leq n$.

Now, define $G_i : Z \longrightarrow \tilde{M}$ for $i \in Z$ by

$$G_i(j) = \alpha_j(i).$$

Thus, the map $G : Z \times Z \longrightarrow \tilde{M}$, $G(i, j) = G_i(j)$, satisfies the induction hypotheses for $0 \leq i \leq n+1$.

In this way, we get a sequence of maps $G^n : 0, 1, \ldots, n \longrightarrow \tilde{M}$ for every integer n, and a sequence of geodesics $\gamma_n \subset \tilde{M}$ such that $G_i^n : Z \longrightarrow \tilde{M}$, $G_i^n(j) = G^n(i, j)$, is an A, B-quasi-isometry of Z $\forall 0 \leq i \leq n$ and

$$G^n(i, j) = \phi_i^n(G^n(0, j))$$

for every i, j, where ϕ_t^n is the Busemann flow associated to γ_n. Finally, for n even define $F^n(i, j) = G^n(\frac{n}{2} + i, j) \ \forall \mid i \mid \leq \frac{n}{2}$. F_i^n is an A, B-quasi-isometry of Z for every $i \in Z(n)$, and therefore, up to covering translations, by Lemmas 1.2 and 3.1 we obtain a convergent subsequence of the F^n whose limit is an A, B-quasi-isometry of $Z \times Z$ into $\tilde{M}$ satisfying the conditions of the main theorem. $\square$

4. Almost convex curves have convex parts

In this section we shall prove Proposition 2.1. The results in this section hold for metric spaces, and they do not depend on the existence of differentiable or Riemannian structures in the space. We shall subdivide the argument into several lemmas.

Lemma 4.1. *Let (X, d) be a metric space. Let $c : [0, l(C)] \longrightarrow C$ be a parametrized curve in X with length function λ. Let $c_1 = c[x, y]$, $c_2 = c[z, t]$ with $x \leq y \leq z \leq t$ be two connected subsets of c such that (c_1, λ) and (c_2, λ) are not A, B-almost quasi-convex. Then $(c[x, t], \lambda)$ is not A, B'-almost quasi-convex, where $B' = 2B - Ad(c(y), c(z))$.*

Proof. This is a straightforward consequence of the triangle inequality. The hypotheses imply the inequalities

$$Ad(c(x), c(y)) + B < \lambda(c(x), c(y)),$$
$$Ad(c(z), c(t)) + B < \lambda(c(z), c(y)),$$

which in turn inply that

$$d(c(x), c(y)) < \frac{1}{A}\lambda(c(x), c(y)) - \frac{B}{A},$$
$$d(c(z), c(t)) < \frac{1}{A}\lambda(c(z), c(t)) - \frac{B}{A},$$

and finally

$$\begin{aligned} d(c(x), c(t)) &< \frac{1}{A}[\lambda(c(x), c(y)) + \lambda(c(z), c(t))] - \frac{2B}{A} + d(c(y), c(z)) \\ &< \lambda(c(x), c(t)) - \frac{2B}{A} + d(c(y), c(z)) \end{aligned}$$

where in the last inequality we used the fact that λ is a length function. This clearly implies the lemma. ☐

Lemma 4.2. *Let (X, d) be a metric space and let $c : [0, l(C)] \longrightarrow C, \lambda$ be a curve and a length function respectively with $\lambda\ (c(x), c(y)) \geq d(c(x), c(y))\ \forall\ x, y$ in $[0, l(C)]$. Assume also that there exist positive constants A,B,D such that $\forall\ x, y$ in $[0, l(C)]$ there are x', y' in $[0, l(C)]$ with*

1. $\lambda(c(x), c(x')) \leq D$, $\lambda(c(y), c(y')) \leq D$,
2. $\lambda(c(x'), c(y')) \leq Ad(c(x'), c(y')) + B$.

Then (C, λ) is A, B'-quasi-convex in X, where $B' = 2D(1 + A) + B$.

Proof. By the hypotheses we have that for every $x, y \in [0, l(C)]$

$$\begin{aligned} \lambda(c(x), c(y)) &\leq \lambda(c(x), c(x')) + \lambda(c(x'), c(y')) + \lambda(c(y'), c(y)) \\ &\leq 2D + Ad(c(x'), c(y')) + B \\ &\leq 2D + A[d(c(x), c(y)) + d(c(x'), c(y')) + d(c(y'), c(y))] + B \\ &\leq 2D + 2AD + B + Ad(c(x), c(y)), \end{aligned}$$

where in the last inequality we used the fact that $\lambda \geq d$. This concludes the proof of the lemma. ☐

Now we begin to study the non-convex parts of a given curve endowed with a length function in (X, d). So let $c : [0, l(C)] \longrightarrow C$ be, as before, a continuous curve and let λ be the length function of c, which in addition satisfies $\lambda \geq d$.

Definition 4.1. Given a subcurve $c[x,y] \subset C = c[0,l(C)]$, let $P_n[x,y]$ be the maximal connected component $c(z,t)$ of c containing c[x,y] which is not $n,0$-almost quasi-convex. Let $P_n \subset C$ be the union over subcurves $c[x,y]$ as above of C of $P_n[x,y]$.

Some elementary remarks about P_n:

1. $P_n[x,y] = c(z,t)$ is always open in C (and, of course, it may be empty if C is $n,0$-quasi-convex). The points $c(z)$, $c(t)$ satisfy
$$d(c(z),c(t)) \leq \frac{1}{n}\lambda(c(z),c(t)).$$
2. Let $I_1 = (x_1,y_1)$, $I_2 = (x_2,y_2)$ be two different maximal components of the preimage of P_n by c. Then (x_1,y_1), (x_2,y_2) are disjoint. This is a consequence of Lemma 4.1: If they were not disjoint then their union would be both connected in $[0,l(C)]$ and not $n,0$-quasi-convex by Lemma 4.1, contradicting their maximality.
3. Let $I = c[t,z]$ be a connected component of the complement of P_n. Then I is $n,0$-quasi-convex. Otherwise there would exist $c[x,y]$ in I which is not $n,0$-quasi-convex, and it would be included in its maximal non-$n,0$-quasi-component.

The following result shows that there could exist quasi-convex subintervals of c not in the complement of P_n.

Lemma 4.3. *Let (X,d) be a metric space and $c : [0,l(C)] \longrightarrow C$, $\lambda \geq d$, $P_n = \bigcup_{i=1}^{m} C_i$ as above, where $C_i = c(t_i,s_i)$, $t_i < s_i < t_{i+1}$ $\forall i = 1,\cdots,m$, are the connected components of P_n. Assume that there exist a constant $L > 0$ and a chain of C_i's, $i = i_1, i_1+1,\cdots,i_1+q = i_k$, such that*
$$\lambda(c_j) = \lambda[c(t_j),c(s_j)] \leq L$$
for every $j = i_1,\cdots,i_k$. Then the subcurve $I = c[t_{i_1},s_{i_k}]$ is n,B'-quasi-convex for B' depending on n, B and L.

Proof. The idea here is to apply Lemma 4.2, since from the hypotheses we have control over the size of non-convex parts of (C,λ). Notice that if $c(x),c(y) \in I$ and neither $c(x)$ nor $c(y)$ is in P_n, then we must have
$$\lambda(c(x),c(y)) \leq nd(c(x),c(y)).$$
Otherwise, the subcurve $c[x,y]$ would not be $n,0$-almost quasi-convex, which would imply that $c[x,y]$ is included in some $c_i \subset P_n$, contradicting the choice of x,y. Now it is clear that we can apply Lemma 4.2 to I, where $A = n$, $B = 0$ and $D = L$ are the constants defined in the statement of Lemma 4.2. This concludes the proof of Lemma 4.3. ◻

From the proof of Lemma 4.3 we see that outside of a component of P_n there are a lot of 'good' inequalities ($n,0$-quasi-convexity inequalities) which are satisfied. The next lemma shows that if a certain collection of good inequalities are satisfied in (C,λ), then we have a sort of control over the λ-length of the components of P_n.

Lemma 4.4. *Let (X,d) be a metric space, $c : [0,l(C)] \longrightarrow C$, $\lambda \geq d$, and P_n as above. Suppose that for all $x \in [0,l(C)]$*
$$\lambda(c(0),c(x)) \leq Ed(c(0),c(x)) + F$$

for certain positive constants E, F. Then, if $z \le y$ is such that $c[z,y]$ is a subset of any component of P_n for $n > 2E$, we have

$$\lambda(c(z), c(y)) \le (2E-1)\lambda(c(0), c(z)) + 2F.$$

Proof. Since $c[z,y]$ is a component of P_n, we have

$$d(c(z), c(y)) \le \frac{1}{n}\lambda(c(z), c(y)).$$

By the hypotheses on c the inequalities

$$d(c(0), c(z)) \ge \frac{1}{E}\lambda(c(0), c(z)) - \frac{F}{E},$$
$$d(c(0), c(y)) \ge \frac{1}{E}\lambda(c(0), c(y)) - \frac{F}{E}$$

imply

$$\begin{aligned}\frac{1}{E}\lambda(c(0), c(y)) - \frac{F}{E} &\le d(c(0), c(y)) \\ &\le d(c(0), c(z)) + d(c(z), c(y)) \\ &\le \lambda(c(0), c(z)) + \frac{1}{n}\lambda(c(z), c(y)),\end{aligned}$$

and so

$$\frac{1}{E}\lambda(c(0), c(y)) - \lambda(c(0), c(z)) \le \frac{1}{n}\lambda(c(z), c(y)) + \frac{F}{E}.$$

Using the additivity of λ and the fact that $n > 2E$, we get

$$\frac{1}{E}\lambda(c(0), c(z)) + \frac{1}{E}\lambda(c(z), c(y)) - \lambda(c(0), c(z)) \le \frac{1}{n}\lambda(c(z), c(y)) + \frac{F}{E},$$

which implies

$$\lambda(c(0), c(z))(\frac{1}{E} - 1) + \frac{1}{E}\lambda(c(z), c(y)) \le \frac{1}{n}\lambda(c(z), c(y)) + \frac{F}{E},$$

and, in turn

$$(\frac{1}{E} - \frac{1}{n})\lambda(c(z), c(y)) \le (1 - \frac{1}{E})\lambda(c(0), c(z)) + \frac{F}{E}.$$

Hence,

$$\lambda(c(z), c(y)) \le \frac{(1-\frac{1}{E})}{(\frac{1}{E} - \frac{1}{n})}\lambda(c(0), c(z)) + \frac{F}{E(\frac{1}{E} - \frac{1}{n})}.$$

We can assume without loss of generality that $E > 1$, so if $n > 2E$ we get $\frac{1}{E} - \frac{1}{n} > \frac{1}{2E}$, which implies

$$\begin{aligned}\lambda(c(z), c(y)) &\le 2E(1 - \frac{1}{E})\lambda(c(0), c(z)) + 2F \\ &= (2E-1)\lambda(c(0), c(z)) + 2F,\end{aligned}$$

thus proving Lemma 4.4. □

Now we are ready to prove the main result of the section.

Proof of Proposition 2.1. Let P_n be defined as above for $n \in N$. □

Claim 1. There exist n_0 and $\alpha < 1$ such that for every connected component S of P_n, $n \geq n_0$,

$$\lambda(S) \leq \alpha\lambda(C).$$

Here of course, $\lambda(S) = \lambda(c(t), c(s))$, where $S = c(t, s)$. The proof is as follows. Let $S = c[t, s]$ with $t < s$. Then $d(c(t), c(s)) \leq \frac{1}{n}\lambda(c(t), c(s))$. From this and the hypotheses on C we get

$$\begin{aligned}\frac{1}{A}\lambda(C) - B = \frac{1}{A}&\lambda(c(0), c(l(C))) - B \\ \leq &\;\; d(c(0), c(l(C)) \\ \leq &\;\; d(c(0), c(t)) + d(c(t), c(s)) + d(c(s), c(l(C))) \\ \leq &\;\; d(c(0), c(t)) + d(c(s), c(l(C))) + \frac{1}{n}\lambda(S),\end{aligned}$$

whence

$$\begin{aligned}&\frac{1}{A}(\lambda(c(0), c(t)) + \lambda(c(t), c(s)) + \lambda(c(s), c(l(C))) \\ &\qquad\qquad \leq \;\; d(c(0), c(t)) + d(c(s), c(l(C))) + \frac{1}{n}\lambda(S) + B,\end{aligned}$$

and so,

$$\begin{aligned}\lambda(S)(\frac{1}{A} - \frac{1}{n}) &\leq \{\lambda(c(0), c(t)) + \lambda(c(s), c(l(C)))\}(1 - \frac{1}{A}) + B \\ &= (\lambda(C) - \lambda(S))(1 - \frac{1}{A}) + B,\end{aligned}$$

where in the last inequality we used the additivity of λ. This implies that

$$\begin{aligned}\lambda(S)(\frac{1}{A} - \frac{1}{n} + 1 - \frac{1}{A}) = \lambda(S)&(1 - \frac{1}{n}) \\ &\leq \lambda(C)(1 - \frac{1}{A}) + B,\end{aligned}$$

and so

$$\lambda(S) \leq \lambda(C)\frac{(1 - \frac{1}{A})}{(1 - \frac{1}{n})} + \frac{B}{(1 - \frac{1}{n})}.$$

Thus it is clear that for $A > 1$ and n suitably large we obtain $\alpha' < 1$ with

$$\frac{\lambda(S)}{\lambda(C)} \leq \alpha' + \frac{2B}{\lambda(C)}$$

So, letting $\lambda(C)$ be arbitrarily large (since otherwise the proposition is trivially true), we deduce that the constant α of the claim is very close to α', thus proving the claim.

From the above claim we deduce that, given any component S in P_n, there exists a connected component I of $C - S$ such that

$$\frac{\lambda(I)}{\lambda(C)} \geq \frac{1}{2}(1 - \alpha).$$

And it is clear that I satisfies the hypotheses of Lemma 4.4, i.e., if $I = c[p, q]$ then either $c(p)$ or $c(q)$ must be an endpoint of S which is a maximal component of non-$n, 0$-almost quasi-convex subintervals of C, so by the argument of Lemma 4.3 we must have that either

$$\lambda[c(p), c(x)] \leq nd(c(p), c(x))$$

for every $c(x)$ in I if $c(p)$ belongs to S, or

$$\lambda[c(q), c(x)] \leq nd(c(q), c(x))$$

for every $c(x)$ in I if $c(q)$ belongs to S. Let us suppose without loss of generality that $c(p) \in S$. In this case we have estimates for the λ-length of the components of P_n which are included in I. Indeed, according to Lemma 4.4 they depend linearly on the distance from their endpoints to $c(p)$. We get two alternatives:

1. There exists a connected component of the complement of P_n in I whose λ-length goes to infinity if $\lambda(C)$ goes to infinity, and in this case the proposition holds.
2. There exists $m > 0$ such that every component of the complement of P_n in I has λ-length bounded by m independently of the length of c.

In this case we have the following assertion:

Claim 2. The λ-length of every component of P_n in I is bounded by a constant independent of the length of C.

Indeed, let $c_i = c[t_i, s_i]$, $t_i < s_i < t_{i+1}, i = i_1, \cdots, i_k$, be the connected components of P_n in I. Between c_i and c_{i+1} there is a connected component of the complement of P_n, namely $c[s_i, t_{i+1}]$, with $\lambda(c(s_i), c(t_{i+1})) \leq m$ by the hypotheses in alternative 2. On the other hand, we have that every interval of the type $c[s_i, q]$ must satisfy

$$\lambda(c(s_i), c(x)) \leq nd(c(s_i), c(x))$$

for every $x \in [s_i, q]$, again by the maximality of $c[t_i, s_i]$. So from Lemma 4.4 we can estimate the λ-length of c_{i+1} by

$$\begin{aligned} \lambda(c_{i+1}) = \lambda[c(t_{i+1}), c(s_{i+1})] &\leq (2n-1)\lambda[c(s_i), c(t_{i+1})] \\ &\leq (2n-1)m, \end{aligned}$$

proving the statement of alternative 2.

Now, notice that the curve I satisfies the hypotheses of Lemma 4.3, so it is quasi-convex for certain constants depending on n and m. This finishes the proof of Proposition 2.1.

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