TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 8, Pages 3243–3265
S 0002-9947(99)02167-4
Article electronically published on April 7, 1999

KNOT INVARIANTS FROM SYMBOLIC DYNAMICAL SYSTEMS

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ABSTRACT. If G is the group of an oriented knot k, then the set $\mathrm{Hom}(K, \Sigma)$ of representations of the commutator subgroup $K = [G, G]$ into any finite group Σ has the structure of a shift of finite type Φ_Σ, a special type of dynamical system completely described by a finite directed graph. Invariants of Φ_Σ, such as its topological entropy or the number of its periodic points of a given period, determine invariants of the knot. When Σ is abelian, Φ_Σ gives information about the infinite cyclic cover and the various branched cyclic covers of k. Similar techniques are applied to oriented links.

1. INTRODUCTION

For almost 70 years useful information about knots has been obtained by examining representations of the associated knot groups. Alexander conducted possibly the first such study in [Al]. Several years later Reidemeister and Seifert demonstrated the value of representations onto Fuchsian groups. For example, Seifert used them in [Se] to exhibit a nontrivial knot with trivial Alexander polynomial. Representations onto finite metacyclic groups were considered by Fox in [Fo]. An advantage of considering representations of a knot group G into a *finite* group Σ is that the set $\mathrm{Hom}(G, \Sigma)$ of representations is itself finite and so can be tabulated. Riley, for example, catalogued representations of knot groups onto the special projective group $PSL(2, p)$, where p is a prime, using a computer [Ri].

This paper explores a new direction in the study of knot group representations, as promised in our earlier paper [SiWi1]. For any knot group G and finite group Σ, we describe the set $\mathrm{Hom}(K, \Sigma)$ of representations of the commutator subgroup $K = [G, G]$. At first sight this new direction is frightening. Often the subgroup K is nonfinitely generated and $\mathrm{Hom}(K, \Sigma)$ is infinite. However, the situation is promising. The commutator subgroup K has a presentation that is *finite* $\mathbf{Z}$*-dynamic* (see section 2), reflecting the fact that G is an infinite cyclic extension of K. Consequently, $\mathrm{Hom}(K, \Sigma)$ has the structure of a compact 0-dimensional dynamical system known as a *shift of finite type* and can be completely described by a finite graph. In order to emphasize the dynamical aspects of $\mathrm{Hom}(K, \Sigma)$, we denote it by Φ_Σ, continuing the notation established in [SiWi1]. The representations of the knot group G appear (by restricting their domain) as certain periodic points in the shift Φ_Σ.

When considering representations of K rather than G, we can gain information from relatively uncomplicated target groups Σ. Using finite abelian groups Σ, we

Received by the editors June 27, 1996 and, in revised form, July 16, 1997.
1991 *Mathematics Subject Classification.* Primary 57Q45; Secondary 54H20, 20E06, 20F05.

discover new results about the homotopy and homology of branched cyclic covers (Theorem 4.2 and Corollary 4.6). Using finite nonabelian groups, we find surprising features in the groups of alternating knots (Theorem 6.3).

We apply our techniques to links as well as knots. We obtain, in particular, information about the total linking number cover (see section 2). It is also possible to associate a finite type $\mathbf{Z}^d$-shift to any link of d components and finite abelian group. We will do this in a future paper.

It is a pleasure to thank Patrick Gilmer and Alex Suciu for their comments and suggestions. We also thank Dong-biao Zheng for many useful computer calculations [Zh].

2. Representation shifts

An *augmented group system* (AGS) is a triple $\mathcal{G} = (G,\ \chi,\ x)$ consisting of a finitely presented group G, an epimorphism $\chi : G \to \mathbf{Z}$, and a distinguished element $x \in G$ such that $\chi(x) = 1$. Such systems were introduced in [Si1] and used in [SiWi1]. A *mapping* from $\mathcal{G}_1 = (G_1,\ \chi_1,\ x_1)$ to $\mathcal{G}_2 = (G_2,\ \chi_2,\ x_2)$ is a homomorphism $h : G_1 \to G_2$ such that $h(x_1) = x_2$ and $\chi_1 = \chi_2 \circ h$. If h is an isomorphism between G_1 and G_2, then we say that $\mathcal{G}_1$ and $\mathcal{G}_2$ are *equivalent*. Equivalent AGS's are regarded as the same.

As usual an *oriented knot* is a smoothly embedded oriented circle k in the 3-sphere S^3. Two oriented knots k_1 and k_2 are *equivalent* if there exists a diffeomorphism $f : S^3 \to S^3$ such that $f(k_1) = k_2$, preserving all orientations. We regard equivalent oriented knots as the same. Given an oriented knot k, we can associate an AGS $\mathcal{G}_k = (G,\ \chi,\ x)$ as follows. Let $N(k) = S^1 \times D^2$ be a tubular neighborhood of k. The closure of $S^3 - N(k)$, the *exterior* of k, is denoted by $X(k)$. Define G to be $\pi_1(X(k), *)$, where the basepoint $*$ lies in the boundary $\partial X(k)$, and define $x \in G$ to be the element represented by a meridian of k with orientation induced by the knot. We complete the definition of $\mathcal{G}_k$ by choosing χ to be the abelianization homomorphism of G that sends x to 1. It is an immediate consequence of the uniqueness of tubular neighborhoods that $\mathcal{G}_k$ is well defined. Moreover, equivalent oriented knots determine equivalent AGS's.

We can also associate an AGS to an oriented link. An *(ordered) oriented link* is a smoothly embedded union $l = l_0 \cup \ \dots \ \cup \ l_\mu$ of oriented circles in S^3. Two oriented links $l = l_0 \cup \ \dots \ \cup \ l_\mu$ and $l' = l'_0 \cup \ \dots \ \cup \ l'_\mu$ are *equivalent* if there exists a diffeomorphism $f : S^3 \to S^3$ such that that $f(l_i) = l'_i$ for each $i = 1, \dots, \mu$, preserving all orientations. Equivalent oriented links are regarded as the same. Given an oriented link $l = l_0 \cup \ \dots \ \cup \ l_\mu$, we associate an AGS $\mathcal{G}_l = (G,\ \chi,\ x)$ in the following way. Let $G = \pi_1(X(l), *)$ where $X(l)$ is the exterior of l, defined the same as for a knot, and $*$ lies in the component of the boundary $\partial X(l)$ corresponding to l_0. Define $x \in G$ to be the element represented by an oriented meridian of l_0. In general there are many possible choices for χ. A natural choice is the homomorphism that sends the classes of all meridians to 1, called the *total linking number homomorphism* for an obvious reason. Once an interpretation for χ is chosen, $\mathcal{G}_l$ depends only on l. Example 2.3 below shows that in general the equivalence class of the AGS might change if the order of the components of l is altered.

When $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS we denote the kernel of χ by K. If $\mathcal{G}$ is the AGS associated to an oriented knot, then K is the commutator subgroup of the knot group G. If $\mathcal{G}$ is associated to an oriented link and χ is the total linking number

homomorphism, then K is known as the augmentation subgroup of the link group G.

Let Σ be any finite group. The associated *representation shift*, denoted by $\Phi_\Sigma(\mathcal{G})$ (or simply Φ_Σ), is the set of representations $\rho : K \to \Sigma$ together with the *shift map* $\sigma_x : \Phi_\Sigma \to \Phi_\Sigma$ defined by

$$\sigma_x \rho(a) = \rho(x^{-1}ax) \ \ \forall\ a \in K.$$

The underlying set Φ_Σ has a natural topology determined by the basis sets

$$\mathcal{N}_{a_1,\ \ldots,\ a_s}(\rho) = \{\rho' \mid \rho'(a_i) = \rho(a_i),\ i = 1,\ \ldots\ ,\ s\},$$

where $\rho \in \Phi_\Sigma,\ a_1,\ \ldots\ ,\ a_s \in K$. It is a straightforward matter to check that σ_x is a homeomorphism. The pair $(\Phi_\Sigma,\ \sigma_x)$ is a *(topological) dynamical system*; that is, a compact topological space together with a homeomorphism.

It is customary to regard two dynamical systems as the same if they are *topologcally conjugate*. Dynamical systems (Φ_1, σ_1) and (Φ_2, σ_2) are topologically conjugate if there exists a homeomorphism $h : \Phi_1 \to \Phi_2$ such that $h \circ \sigma_1 = \sigma_2 \circ h$. In this case, h induces a bijection between Fix $\sigma_1^r = \{\rho : \sigma_1^r \rho = \rho\}$ and Fix $\sigma_2^r = \{\rho : \sigma_2^r \rho = \rho\}$ for each $r \geq 1$; stated briefly, topologically conjugate dynamical systems have the same number of periodic points for every period.

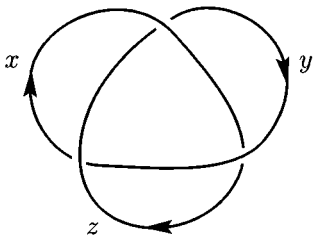
One of the main results of [SiWi1] is that $(\Phi_\Sigma,\ \sigma_x)$ is a special type of symbolic dynamical system called a *shift of finite type* that can be completely described by a finite directed graph. Let Γ be a finite directed graph with edge set E, and give $E^{\mathbf{Z}}$ the product topology. The shift of finite type associated with Γ consists of the set of all $y = (y_i)$ in $E^{\mathbf{Z}}$ that correspond to bi-infinite paths in the graph, together with the shift map σ given by $(\sigma y)_i = y_{i+1}$.

In order to find a graph for the representation shift, we first employ the Reidemeister-Schreier Theorem [LySc] to obtain a presentation for K having the form

$$K = \ \langle a_{i,j} \mid r_{k,j} \rangle,$$

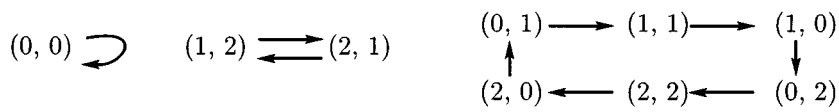
where $1 \leq i \leq n,\ 1 \leq k \leq m,\ j \in \mathbf{Z}$. Each relator $r_{k,j}$ is a word in the $a_{i,j}$ such that $r_{k,q+t}$ is obtained from $r_{k,q}$ by "shifting" the generators, adding t to the second subscript of every symbol in $r_{k,q}$. (Such a presentation is termed *présentation* $\mathbf{Z}$-*dynamique finie* in [HauKe].) We can think of any represention $\rho : K \to \Sigma$ as a homomorphism from the free group on the generators $a_{i,j}$ such that every image $\rho(r_{i,j})$ is trivial. Such functions correspond to bi-infinite paths in the finite directed graph Γ that we now construct.

Assume without loss of generality, that if any word $r_{1,0},\ \ldots\ ,\ r_{n,0}$ contains $a_{i,j}$, then it contains $a_{i,0}$ but no $a_{i,j}$ with $j < 0$ (see [Ra]). Let M_i be the largest value of j such that $a_{i,j}$ occurs in $r_{1,0},\ \ldots\ ,\ r_{n,0}$, or 0 if $a_{i,j}$ does not occur in the relators for any j. Let A_0 denote the set of generators obtained from $A = \{a_{1,0}, \ldots\ , a_{1,M_1}, a_{2,0}, \ldots\ , a_{n,M_n}\}$ by deleting $a_{1,M_1}, \ldots\ , a_{n,M_n}$. The vertex set of Γ consists of all functions $\rho_0 : A_0 \to \Sigma$; that is, vectors in Σ^M, where $M = M_1 + \ldots + M_n$. Construct an edge $\rho_0\rho_0'$ from ρ_0 to ρ_0' if two conditions are satisfied. We require that (1) $\rho_0(a_{i,j+1}) = \rho_0'(a_{i,j})$ for each $a_{i,j} \in A_0$. Regarding ρ_0 as a partial assignment of elements of Σ to the generators $a_{i,j}$, requirement (1) is an "overlap condition" that enables us to use ρ_0' to extend the assignment of elements of Σ to the larger subset A of generators by sending $a_{i,j+1}$ to $\rho_0'(a_{i,j})$. Of course, if there is an edge from ρ_0' to some vertex ρ_0'', then we can extend the assignment



The knot $k = 3_1$

(a)



The graph Γ

(b)

FIGURE 1

further to $A \cup \{a_{1,M_1+1}, \dots, a_{n,M_N+1}\}$. In addition to (1), we require that (2) the images of $r_{1,0}, \dots, r_{n,0}$ be trivial in Σ. Now the extended assignment determined by the two edges $\rho_0\rho_0'$ and $\rho_0'\rho_0''$ would also map $r_{1,1}, \dots, r_{n,1}$ trivially. Continuing this reasoning, it is clear that bi-infinite paths in Γ correspond to homomorphisms of K into Σ.

When constructing the graph Γ, one often discovers that some vertex is "dead" in the sense that no edge leads from it. In such a case, we can safely remove the vertex together with any edge that terminates at it. Likewise, we may remove a vertex with no edges leading to it, together with the edges that originate there. Repeating the process until no dead vertices remain eventually yields a graph in which every edge lies in some bi-infinite path. We will always assume that Γ has been "pruned" in this way.

Example 2.1. Consider the trefoil knot $k = 3_1$ as it appears in Figure 1a with Wirtinger generators indicated. Its group G has presentation

$$\langle x,\ y,\ z \mid xy = zx,\ xy = yz\rangle.$$

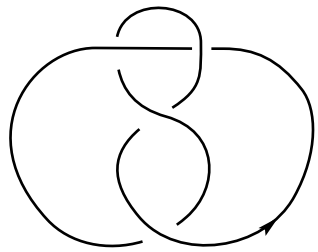
We use the second relation to eliminate z from the presentation, obtaining

$$\langle x,\ y \mid y^{-1}x^{-1}y^{-1}xyx\rangle.$$

We choose x to be the distinguished generator, and we replace y by xa (i.e., we introduce a new generator a and remove y using Tietze moves). The following presentation for G results:

$$G = \langle x,\ a \mid a^{-1} \cdot x^{-2}a^{-1}x^2 \cdot x^{-1}ax\rangle.$$

The Reidemeister-Schreier Theorem enables us to write a presentation for K, the commutator subgroup of G. The generators are the elements $x^{-j}ax^j$, denoted by the symbols a_j. Defining relators are found by conjugating the relation in the last



The knot $k = 4_1$

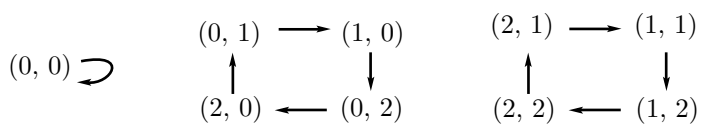


FIGURE 2

presentation by powers of x and rewriting the words in terms of the a_j. We find

$$K = \langle a_j \mid a_j^{-1} a_{j+2}^{-1} a_{j+1}, \quad j \in \mathbf{Z} \rangle.$$

(In examples, we drop the cumbersome double subscripts and replace the $a_{i,j}$ by n families a_j, b_j, etc.; in this example $n = 1$.)

We will let Σ be the cyclic group $\mathbf{Z}/3$. The vertex set of Γ is

$$\{\rho_0 \mid \rho_0 : \{a_0, a_1\} \to \mathbf{Z}/3\} \simeq \mathbf{Z}/3 \times \mathbf{Z}/3$$

while the edge set is

$$\{\rho_0 \rho_0' \mid \rho_0(a_1) = \rho_0'(a_0),\ \rho_0'(a_1) = \rho_0(a_1) - \rho_0(a_0)\}.$$

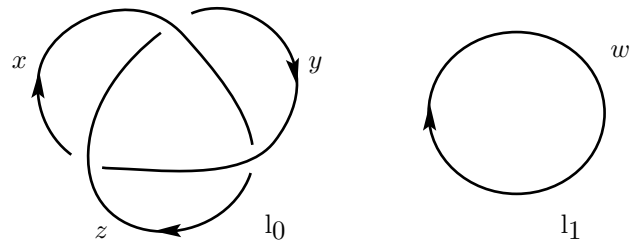
The graph Γ appears in Figure 1b. From it we immediately see that there are exactly nine representations $\rho : K \to \mathbf{Z}/3$. For example, the circuit

$$(1,0) \to (0,2) \to (2,2) \to (2,0) \to (0,1) \to (1,1) \to (1,0),$$

regarded as the unique bi-infinite path in Γ beginning at $(1,0)$, corresponds to the representation ρ such that $\rho(a_j) = 0$ if $j \equiv 1, 4 \pmod 6$; $\rho(a_j) = 1$ if $j \equiv 0, 5 \pmod 6$; $\rho(a_j) = 2$ if $j \equiv 2, 3 \pmod 6$. We can determine the effect of σ_x directly from Γ. For example, since $\sigma_x \rho(a_j) = \rho(x^{-1} a_j x) = \rho(a_{j+1})$, we find that $\sigma_x \rho(a_j) = 0$ if $j \equiv 0, 3 \pmod 6$; $\sigma_x \rho(a_j) = 1$ if $j \equiv 4, 5 \pmod 6$; $\sigma_x \rho(a_j) = 2$ if $j \equiv 1, 2 \pmod 6$.

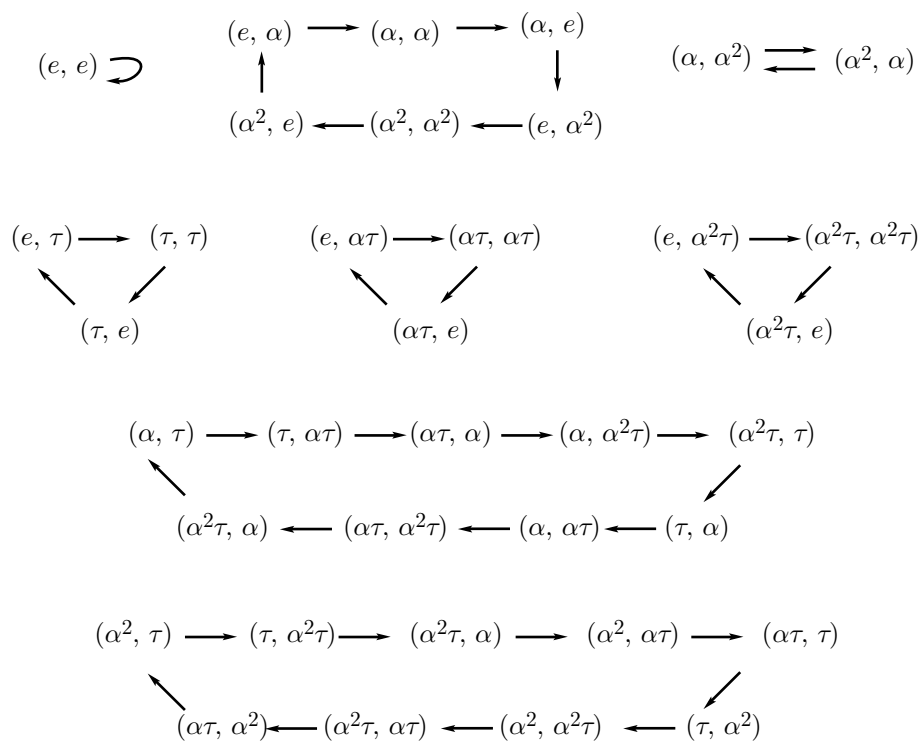
From Γ we see that the representation shift $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$ contains 1 fixed point, 2 points of period 2, and 6 points of period 6.

Example 2.2. Following the steps of the previous example using the figure eight knot $k = 4_1$ produces a very different graph Γ (see Figure 2). The details are left to the reader. We find that $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ contains 1 fixed point and 8 points of period 4. Although $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{4_1})$ has the same number of representations as $\Phi_{\mathbf{Z}/3}(\mathcal{G}_{3_1})$, the two shifts are different.



The link 1

(a)



The graph Γ

(b)

FIGURE 3

Example 2.3. Consider the link $l = l_0 \cup l_1$ in Figure 3a consisting of a right-hand trefoil $l_0 = 3_1$ and a distant unknot l_1 with Wirtinger generators indicated. The group G of the link has presentation

$$G = \langle x, y, w \mid xyx = yxy \rangle.$$

Let $\chi : G \to \mathbf{Z}$ be the total linking number homomorphism. First we choose the trefoil to be the distinguished component. In order to find the appropriate presentation for K, replace generators y, w by xa, xb respectively. The presentation for G becomes

$$G = \langle x, a, b \mid x^{-2}ax^2 = x^{-1}ax \cdot a^{-1} \rangle.$$

An application of the Reidemeister-Schreier Theorem produces the following presentation for K.

$$K \;=\; \langle a_j, b_j \mid a_{j+2} = a_{j+1}a_j^{-1}, \quad j \in \mathbf{Z} \rangle.$$

Notice that K is the free product of the commutator subgroup of the trefoil knot group and the free group on the generators b_j. It follows that for any finite group Σ, the associated representation shift Φ_Σ decomposes as $\Phi_\Sigma(\mathcal{G}_{3_1}) \times \Sigma^{\mathbf{Z}}$, where $\Sigma^{\mathbf{Z}}$ is the *full shift on* Σ described by the complete directed graph on the set Σ. The shift map on this product space is the product of the shift map on $\Phi_\Sigma(\mathcal{G}_{3_1})$ and the mapping $(s_j) \mapsto (s'_j)$, where $s'_j = s_{j+1}$, defined on $\Sigma^{\mathbf{Z}}$. Notice that the full shift and hence the product shift are uncountable whenever Σ is nontrivial. We specialize to the case in which Σ is the dihedral group $D_3 = \langle \tau, \alpha \mid \alpha^3 = e, \tau\alpha\tau^{-1} = \alpha^2 \rangle$ of order 6. The graph Γ describing $\Phi_\Sigma(\mathcal{G}_{3_1})$ has vertices corresponding to pairs of elements of D_3, and it appears in Figure 3b. From Γ we see that $\Phi_{D_3}(\mathcal{G}_{3_1})$ contains exactly one fixed point. Since $\Sigma^{\mathbf{Z}}$ clearly has 6 fixed points, the representation shift $\Phi_{D_3}(\mathcal{G}_l)$ of the link contains exactly 6 fixed points.

If we construct a second link l' by interchanging the components l_0 and l_1, then we must follow somewhat different steps to get a presentation for K. We replace x, y by wc, wd respectively, and obtain

$$K = \; \langle c_j, d_j \mid c_{j+2}d_{j+1}c_j = d_{j+2}c_{j+1}d_j, \quad j \in \mathbf{Z} \rangle.$$

The associated representation shift $\Phi_{D_3}(\mathcal{G}_{l'})$ has 6 obvious fixed points corresponding to $c_j = d_j =$ constant for all j. However, the assignment $c_j = \tau, d_j = \alpha\tau$ also describes a fixed point. (In fact, this shift has exactly 12 fixed points.) Hence $\Phi_{D_3}(\mathcal{G}_{l'})$ is not topologically conjugate to $\Phi_{D_3}(\mathcal{G}_l)$.

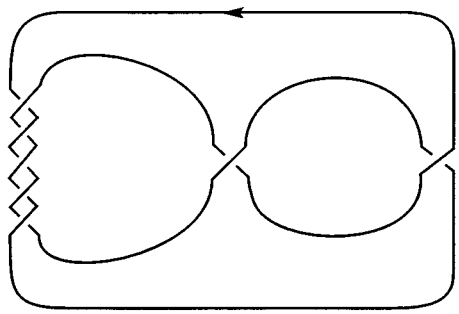
Example 2.3 demonstrates that the order of the components of a link can affect the associated AGS and hence the various representation shifts. However, we will see in section 3 that when Σ is abelian the representation shifts are independent of the order of the components. (See also Theorem 6.3 of [SiWi1].)

In general a nonabelian group Σ can produce a representation shift for a knot that is also uncountable. For example, this is the case for the knot 5_2 when Σ is the symmetric group S_4 [SiWi1]. We offer another example.

Example 2.4. Pretzel knots $k = k(2p+1, 2q+1, 2r+1)$ [BuZi] are attractive examples for demonstrating representation shift techniques. Given integers p, q and r, one can immediately write a finite $\mathbf{Z}$-dynamic presentation for the commutator subgroup of the knot group.

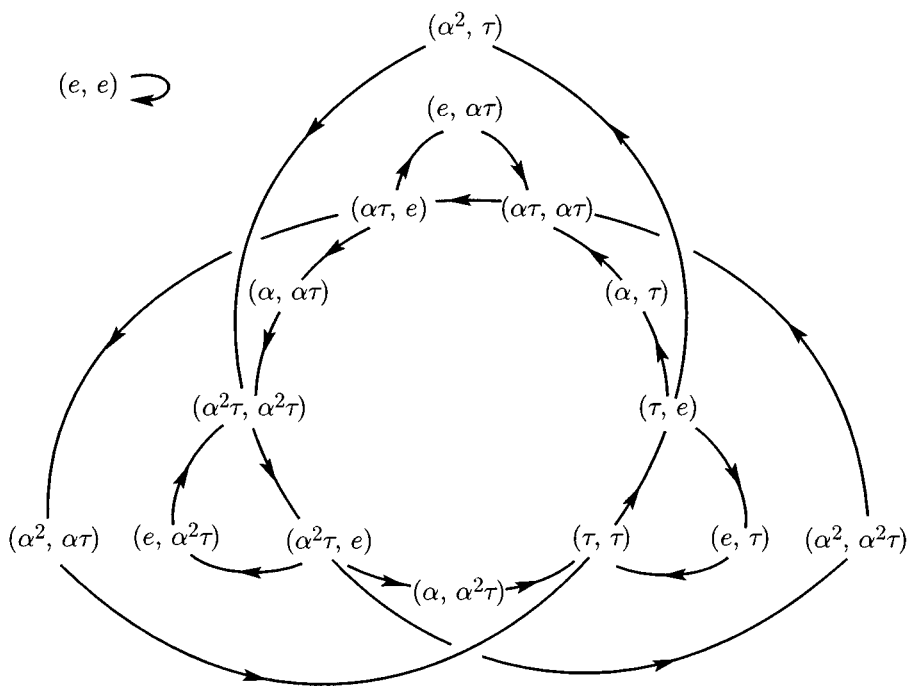
$$\begin{aligned} K \;=\; \langle a_j,\ b_j \mid (b_j a_j^{-1})^{-q} a_j^{p+1} = (b_{j+1}^{-1} a_{j+1})^{q+1} a_{j+1}^{p}, \\ b_j^r (b_j a_j^{-1})^{q+1} = b_{j+1}^{r+1} (b_{j+1}^{-1} a_{j+1})^{-q} \rangle. \end{aligned}$$

(See Equation 1 of [CrTr].) The graph describing $\Phi_\Sigma(\mathcal{G}_k)$ has vertices labeled by ordered pairs of elements in Σ, a pleasant situation that often allows computations by



The pretzel knot $k = k(5,\,1,\,1)$

(a)



(b)

FIGURE 4

hand. For example, consider the knot $k = k(5,1,1)$ in Figure 4a. The commutator subgroup K has presentation

$$K \;=\; \langle a_j, b_j \mid a_j^3 = b_{j+1}^{-1} a_{j+1}^3,\; b_j a_j^{-1} = b_{j+1}, \quad j \in \mathbf{Z}\rangle.$$

The graph Γ describing the shift $\Phi_{D_3}(\mathcal{G}_k)$ appears in Figure 4b. The shift has two components, a component consisting only of the trivial representation and another component that is uncountable. The nontrivial component has a 3-fold

symmetry reflecting the fact that the subgroup generated by α acts nontrivially on these representations by conjugation. In addition, this component can be partitioned into three sets that are permuted cyclically by σ_x: the set of representations for which $(\rho(a_0), \rho(b_0))$ is in $\{(\tau,\tau),(\alpha\tau,\alpha\tau),(\alpha^2\tau,\alpha^2\tau)\}$, those for which it is in $\{(\tau,e),(\alpha\tau,e),(\alpha^2\tau,e)\}$, and the remainder. Each of these sets, under the action of σ_x^3, forms a dynamical system that can be seen to be topologically conjugate to the full 3-shift $\{0,1,2\}^{\mathbf{Z}}$.

In Examples 2.1, 2.2 and 2.4 the only fixed points in the representation shifts correspond to the trivial representation. This is a consequence of the fact that the knot group is the normal closure of the distinguished generator.

Proposition 2.5. *Let $\mathcal{G} = (G,\ \chi,\ x)$ be an AGS such that G is the normal closure of x. Then in any representation shift $\Phi_\Sigma(\mathcal{G})$ the trivial representation is the only fixed point; that is, $\sigma_x\rho = \rho$ implies that ρ is trivial.*

Proof. Assume that $\sigma_x\rho = \rho$ for some $\rho \in \Phi_\Sigma(\mathcal{G})$. Then $\rho(x^{-1}ax) = \rho(a)$ for all $a \in K$; equivalently, $\rho(x^{-1}axa^{-1}) = e$ for all $a \in K$. The hypothesis that G is the normal closure of x implies that the elements $x^{-1}axa^{-1}$, where $a \in K$, generate K (see [HauKe], for example.) Hence ρ is trivial. □

The conclusion of Proposition 2.5 is false without the assumption about the distinguished generator. (See Example 3.5.)

Given any AGS $(G,\ \chi,\ x)$ with K finitely generated, the representation shift Φ_Σ is finite for every Σ. However, there are other AGS's for which such a conclusion holds. For example, the shift Φ_Σ is finite for every Σ whenever G has an ascending HNN decomposition with respect to χ (see [SiWi1].) Such a decomposition consists of a *base group*, a finitely generated subgroup U of K, say with presentation $\langle X \mid R\rangle$, together with a monomorphism $\phi : U \hookrightarrow U$ and an element $t \in \chi^{-1}(1)$ such that $G = \langle t, X \mid R,\ t^{-1}ut = \phi(u),\ \forall u \in U\rangle$. In such a case K is the union of an ascending chain

$$\ldots \subset t^{-1}Ut \subset U \subset tUt^{-1} \subset t^2Ut^{-2} \subset \ldots$$

Problem 2.6. Characterize AGS's $\mathcal{G}$ such that $\Phi_\Sigma(\mathcal{G})$ is finite for every group Σ.

We offer the following result concerning Problem 2.6.

Proposition 2.7. *If $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS such that $\Phi_\Sigma(\mathcal{G})$ is infinite for some group Σ, then K contains a free group of rank 2.*

Proof. We sketch the modification of an argument in [Ba2] (see pages 67-69). Recall that K has a presentation $K = \langle a_{i,j} \mid r_{k,j}\rangle$, where $1 \le i \le n,\ 1 \le k \le m,\ j \in \mathbf{Z}$ and each relator $r_{k,j}$ is a word in the $a_{i,j}$ such that $r_{k,q+t}$ is obtained from $r_{k,q}$ by "shifting" the generators, adding t to the second subscript of every symbol in $r_{k,q}$. We continue to assume that if any word $r_{1,0},\ \ldots,\ r_{n,0}$ contains $a_{i,j}$, then it contains $a_{i,0}$ but no $a_{i,j}$ with $j < 0$, and again we define M_i to be the largest value of j such that $a_{i,j}$ occurs in $r_{1,0},\ \ldots,\ r_{n,0}$. Let ν be the maximum of $M_1, \ldots, M_n$. Define K^+ to be the subgroup of K generated by $a_{i,j},\ 1 \le i \le n,\ j \ge 0$. Define K^- to be the subgroup generated by $a_{i,j},\ 1 \le i \le n,\ j \le \nu$. Finally, define U to be the subgroup generated by $a_{i,j},\ 1 \le i \le n,\ 0 \le j \le \nu$. By a straightforward argument [Ba2] one can show that K is the free product of K^+ and K^- amalgamated over U. We complete the proof by considering three cases: (1) If $U \ne K^+$ and $U \ne K^-$, and if U has index at least 3 in either K^+ or K^-, then U contains a free subgroup

of rank 2 (see Lemma 4 of [Ba2]). (2) If $U \neq K^+$ and $U \neq K^-$, and if U has index 2 in both K^+ or K^-, then both K^+ and K^- are finitely generated. However, this implies that K is finitely generated and hence $\Phi_\Sigma(\mathcal{G})$ is finite, contrary to hypothesis. (3) If U is equal to either K^+ or K^-, then G has an ascending HNN decomposition with respect to χ^{-1} or χ and with base group U. Again, this implies that the shift $\Phi_\Sigma(\mathcal{G})$ is finite. Hence the only possibility is (1), and thus K contains a free group of rank 2. □

Example 2.8. Proposition 2.7 can be used to show that certain finitely generated groups G are not finitely presented. Consider the triple $(G,\ \chi,\ x)$ consisting of the finitely generated group

$$G \ = \ \langle x, a \mid [a, x^{-j} a x^j], \quad j \in \mathbf{Z} \rangle$$

and epimorphism $\chi : G \to \mathbf{Z}$ such that $\chi(x) = 1,\ \chi(a) = 0$. The group G is the reduced wreath product of two infinite cyclic groups, and it is known to be nonfinitely presented [Hal] (see also [Ba1]). The kernel K of χ is the free abelian group with presentation

$$K \ = \ \langle a_j \mid [a_j, a_{j+\nu}], \quad j, \nu \in \mathbf{Z} \rangle.$$

Clearly there exist infinitely many representations $\rho : K \to \mathbf{Z}/2$. If G were finitely presented, then $(G,\ \chi,\ x)$ would be an AGS, and hence by Proposition 2.7 the kernel K would contain a free group of rank 2, which is certainly not the case. We can conclude that G is finitely generated but not finitely presented.

From another perspective, the above argument shows that K is not finite $\mathbf{Z}$-dynamically presented. One might compare this group to the finite $\mathbf{Z}$-dynamically presented group $\langle a_j \mid [a_j, a_{j+1}], \quad j \in \mathbf{Z} \rangle$.

3. Abelian representation shifts

Representation shifts have extra structure and special properties when the target group is abelian.

Proposition 3.1. *Assume that $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS and Σ is a finite abelian group. Let Λ be the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$.*

(1) The representation shift $\Phi_\Sigma(\mathcal{G})$ is an abelian group. Moreover, the operation $\rho \mapsto \sigma_x \rho$ induces a Λ-module structure on $\Phi_\Sigma(\mathcal{G})$.

(2) $\Phi_\Sigma(\mathcal{G})$ is finite if and only if the trivial representation is isolated (i.e., has a neighborhood that contains no other point).

(3) If $\Phi_\Sigma(\mathcal{G})$ is infinite, then it is uncountable.

Proof. Since any two representations into an abelian group can be added together, $\Phi_\Sigma(\mathcal{G})$ is obviously an abelian group. The proof of the second assertion is straightforward.

We will prove the remaining assertions using an idea of Kitchens [Kit]. Recall that elements of the representation shift $\Phi_\Sigma(\mathcal{G})$ are described by bi-infinite paths of a graph Γ, vertices of which are labeled by vectors in Σ^M (see section 2). Given any vertex v, we define the *follower set* $f(v)$ to be the set of vertices v' such that there is an edge in Γ from v to v'. Let v_0 be the trivial vertex $(e, \dots, e)$. One verifies easily that $f(v_0)$ is a subgroup of Σ^M, and for each vertex v, the set $f(v)$ is a coset of $f(v_0)$. We can make the same argument for the *predecessor set* $p(v)$ of vertices v' with an edge in Γ from v' to v. Since we have assumed that Γ is "pruned" (see the remarks before Example 2.1), $\Phi_\Sigma(\mathcal{G})$ is finite if and only if $f(v_0) = \{v_0\} = p(v_0)$. Since this

holds if and only if the trivial representation is isolated in $\Phi_\Sigma(\mathcal{G})$, assertion (2) is proved.

In order to prove assertion (3), assume that $\Phi_\Sigma(\mathcal{G})$ is not finite. Then every vertex in Γ has out-degree greater than 1. In such a case, there exist uncountably many bi-infinite paths in Γ, and hence $\Phi_\Sigma(\mathcal{G})$ is uncountable. $\square$

The assumption in Proposition 3.1(3) that Σ is abelian is needed. K.H. Kim and F. Roush discovered the following example of an AGS $\mathcal{G}$ such that $\Phi_{A_5}(\mathcal{G})$ is countably infinite [KimRou].

Example 3.2. Recall that the alternating group A_5 has presentation $\langle a, b \mid a^2 = b^3 = (ab)^5 = e\rangle$. Let $\mathcal{G}$ be the AGS $(G,\ \chi,\ x)$ such that

$$G \ = \ \langle x, a, b \mid a^2 = b^3 = (ab)^5 = e,\ [ax^{-1}a^{-1}x, x^{-1}ax] = [bx^{-1}b^{-1}x, x^{-1}ax]$$

$$= [ax^{-1}a^{-1}x, x^{-1}bx] = [bx^{-1}b^{-1}x, x^{-1}bx] = e\rangle,$$

where $[u, v]$ is the commutator $u^{-1}v^{-1}uv$, and $\chi : G \to \mathbf{Z}$ is the homomorphism mapping $x \mapsto 1,\ a, b \mapsto 0$. The kernel K has presentation

$$K \ = \ \langle a_j, b_j \mid a_j^2 = b_j^3 = (a_jb_j)^5 = e,\ [a_ja_{j+1}^{-1}, a_{j+1}] = [b_jb_{j+1}^{-1}, a_{j+1}]$$

$$= [a_ja_{j+1}^{-1}, b_{j+1}] = [b_jb_{j+1}^{-1}, b_{j+1}] = e, \quad i \in \mathbf{Z}\rangle.$$

Consider the graph Γ that describes $\Phi_{A_5}(\mathcal{G})$. From the presentation for K, it is clear that the vertices of Γ are all possible pairs (α, β) of elements of A_5 such that $\alpha^2 = \beta^3 = (\alpha\beta)^5 = e$. Also, every vertex has a self-loop (i.e., an edge which begins and ends at the vertex). Since A_5 is simple, the subgroup generated by α and β is either trivial or all of A_5. Clearly, every vertex has an edge leading to (e, e). We claim that there are no other edges in Γ other than the ones we have described. In order to see this, suppose that there exists an edge from (e, e) to some vertex (α, β). From the presentation of K we see that α and β must commute, and hence $(\alpha, \beta) = (e, e)$. Now suppose that there exists an edge from a vertex (α, β) to some vertex (γ, δ) other than (e, e). Again using the presentation for K, we find that $\alpha\gamma^{-1}$ and $\beta\delta^{-1}$ commute with both γ and δ, and since the center of A_5 is trivial, $\alpha = \gamma$ and $\beta = \delta$ and hence the edge is just a self-loop on (α, β).

From the form of Γ, we see that the shift $\Phi_{A_5}(\mathcal{G})$ is countable.

The work of Kitchens mentioned previously provides an attractive structure theorem for representation shifts with abelian Σ. The following is an immediate consequence of Theorem 1 (ii) of [Kit].

Theorem 3.3. *Assume that $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS and Σ is a finite abelian group. The component of the trivial representation in $\Phi_\Sigma(\mathcal{G})$ is a finite-index subgroup Φ_Σ^0 invariant under σ_x and topologically conjugate to a full shift $\{1, 2, \dots, n\}^{\mathbf{Z}}$ for some positive integer n. Moreover, the quotient group $\overline{\Phi}_\Sigma = \Phi_\Sigma(\mathcal{G})/\Phi_\Sigma^0$ is a retract of $\Phi_\Sigma(\mathcal{G})$. Consequently, $\Phi_\Sigma(\mathcal{G}) = \Phi_\Sigma^0 \times \overline{\Phi}_\Sigma$, both as groups and as dynamical systems.*

Example 3.4. Recall from Example 2.3 that if $l = l_0 \cup l_1$ is the 2-component link in Figure 2a and Σ is any finite abelian group, then $\Phi_\Sigma(\mathcal{G}_l) = \Sigma^{\mathbf{Z}} \times \Phi_\Sigma(\mathcal{G}_{3_1})$, the cartesian product of a nontrivial full shift and a finite shift.

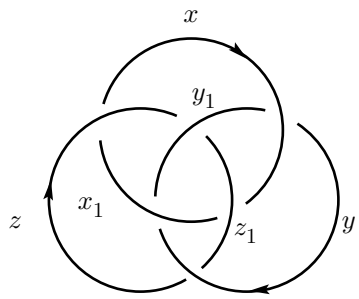


FIGURE 5

Example 3.5. Let $l = l_0 \cup l_1$ be the trivial 2-component link. Its group G is free on meridinal generators x and y corresponding to the two components. Consider the associated AGS $\mathcal{G}_l = (G,\ \chi,\ x)$, where χ is the total linking number homomorphism. Replace y by xa; i.e., eliminate y and introduce the new generator a, using Tietze moves. The kernel K is free on the generators $a_j = x^{-j}ax^j$, where $j \in \mathbf{Z}$. Given any finite abelian group Σ, the representation shift $\Phi_\Sigma(\mathcal{G}_l)$ is the full shift $\Sigma^{\mathbf{Z}}$. (This is true without the assumption that Σ is abelian.) In this example, $\overline{\Phi}_\Sigma$ is trivial.

Example 3.6. Consider the Borromean rings 6^3_2, the 3-component link $l = l_0 \cup l_1 \cup l_2$ that appears in Figure 5 with Wirtinger generators indicated. Let $\mathcal{G}_l = (G,\ \chi,\ x)$ be the associated AGS, where χ is the total linking number homomorphism. The group G has presentation

$$\langle x,\ x_1,\ y,\ y_1,\ z,\ z_1 \mid zx = x_1z,\ xy = y_1x,\ yz = z_1y,\ z_1y_1 = y_1z,\ z_1x = x_1z_1\rangle.$$

Using the first three relators we can eliminate $x_1,\ y_1$ and z_1 from the presentation, obtaining

$$\langle x,\ y,\ z \mid yzy^{-1}xyx^{-1} = xyx^{-1}z,\ zxz^{-1}yzy^{-1} = yzy^{-1}x\rangle.$$

We replace y by xa and z by xb and apply the Reidemeister-Schreier method to produce the following presentation for K.

$$\langle a_j,\ b_j \mid a_{j+2}b_{j+1}a_{j+1}^{-1}a_jb_{j+1}^{-1}a_{j+1}^{-1},\quad b_{j+2}b_{j+1}^{-1}a_{j+1}b_ja_j^{-1}a_{j+1}b_{j+1}^{-1}a_{j+2}^{-1},\quad j \in \mathbf{Z}\rangle.$$

The abelianization $K/[K,K]$ has presentation

$$\langle a_j,\ b_j \mid a_{j+2} = -a_j + 2a_{j+1},\ b_{j+2} = -b_j + 2b_{j+1},\quad j \in \mathbf{Z}\rangle.$$

Since any representation $\rho : K \to \Sigma$ must factor through the quotient map $K \to K/[K,K]$, it follows at once that the representation shift $\Phi_\Sigma(\mathcal{G}_l)$ is finite. The vertex set of the directed graph Γ describing the shift consists of all functions from $\{a_0,\ a_1,\ b_0,\ b_1\ \}$ to Σ, and there is exactly one edge from each vertex. In fact, $\Phi_\Sigma(\mathcal{G}_l)$ is the cartesian product of two identical finite shifts. In this example, Φ^0_Σ is trivial.

For each AGS $\mathcal{G} = (G,\ \chi,\ x)$ associated to an oriented knot that we have seen, the representation shift $\Phi_\Sigma(\mathcal{G})$ is finite whenever Σ is abelian. We show now that finiteness is a consequence of the fact that G has infinite cyclic abelianization.

Proposition 3.7. *Assume that $(G,\ \chi,\ x)$ is an AGS. If the abelianization of G is infinite cyclic, then the representation shift Φ_Σ is finite for every abelian group Σ.*

Proof. Conjugation by x in G induces an automorphism of K, and hence it induces an automorphism s_x of the abelianization $K/[K,K]$. Consider the homomorphism $g = s_x - id$. The hypothesis that $G/[G,G]$ is infinite cyclic implies that K is equal to $[G,G]$, and from this it is not difficult to see that g is surjective. (One way to see this is to describe the abelianization of G as $\langle x \mid \rangle \times K/[K,K]/\langle\langle g(a)\ \forall a \in K/[K,K]\rangle\rangle$. The second factor must vanish.) Consequently, g induces an injection $g^* : \mathrm{Hom}(K/[K,K],\Sigma) \to \mathrm{Hom}(K/[K,K],\Sigma)$ defined by $(g^*\rho)a = \rho(g(a))$, for all $a \in K/[K,K]$. Since Σ is abelian, we can identify $\mathrm{Hom}(K/[K,K],\Sigma)$ with the representation shift Φ_Σ. If Φ_Σ is infinite, then Φ^0_Σ is a nontrivial full shift and hence contains at least one nontrivial constant representation, a representation ρ such that $\rho(s_x(a)) = \rho(a)$ for all $a \in K/[K,K]$. However, g^* maps every constant representation to the trivial one. Since g^* is injective, Φ^0_Σ must be trivial, and hence Φ_Σ is finite by Theorem 3.3. □

One can give an alternative (but longer) proof of Proposition 3.7 using techniques of algebraic topology found in [Mi].

Representation shifts with abelian target groups Σ are determined by the pair $(G,\ \chi)$ and are unaffected by the choice of distinguished element. In order to see this, consider two AGS's $\mathcal{G}_i = (G,\ \chi,\ x_i),\ i = 1,2$, that differ only in the choice of distinguished element. Certainly, $\Phi_\Sigma(\mathcal{G}_1)$ and $\Phi_\Sigma(\mathcal{G}_2)$ are the same as sets. We can write $x_2 = x_1 u$, where u is an element of K. If Σ is an abelian group, then $\sigma_{x_2}\rho(a) = \rho(x_2^{-1}ax_2) = \rho(u^{-1}x_1^{-1}ax_1u) = \rho(u)^{-1}\rho(x_1^{-1}ax_1)\rho(u) = \rho(x_1^{-1}ax_1) = \sigma_{x_1}\rho(a)$ for all $a \in K$. Hence the two shifts are the same. It follows from this that the representation shift $\Phi_\Sigma(\mathcal{G}_l)$ associated to an oriented link l is independent of the order of components of l whenever Σ is abelian.

More delicate is the question of dependence of the representation shift on the choice of epimorphism $\chi : G \to \mathbf{Z}$. Given any AGS $\mathcal{G}_k = (G,\ \chi,\ x)$, we can form a second system $\mathcal{R}\mathcal{G}_k = (G,\ -\chi,\ x^{-1})$, easily seen to be the AGS associated to k with orientation reversed. For any finite group Σ, the representation shift $\Phi_\Sigma(\mathcal{R}\mathcal{G}_k)$ can be described by reversing all arrows in a directed graph Γ that determines $\Phi_\Sigma(\mathcal{G}_k)$. This *inverse shift* is topologically conjugate to $\Phi_\Sigma(\mathcal{G}_k)$ when the latter is finite, since in that case the shift consists of finitely many periodic orbits. In general, a shift of finite type need not be topologically conjugate to its inverse shift (the shift obtained by reversing all arrows in a finite graph that describes it). However, the most tractable shift invariants will not distinguish a shift from its inverse. We ask:

Question 3.8. Does there exist a noninvertible knot k such that $\Phi_\Sigma(\mathcal{G}_k)$ and $\Phi_\Sigma(\mathcal{R}\mathcal{G}_k)$ are different for some group Σ?

If the answer to Question 3.8 is no, then $\Phi_\Sigma(\mathcal{G}_k)$ is an *unoriented* knot invariant. We believe that this is not the case.

4. Periodic representations

Suppose $\mathcal{G} = (G,\ \chi,\ x)$ is an AGS. Recall that the kernel of χ is denoted by K. For any integer $r \geq 1$ we define K_r to be the quotient group

$$K/\langle\langle x^{-r}ax^r = a\ \forall a \in K\rangle\rangle.$$

The projection $p_r : K \to K_r$ induces an embedding $p_r^* : \mathrm{Hom}(K_r,\Sigma) \to \Phi_\Sigma(\mathcal{G})$ defined by $(p_r^*\rho)(a) = \rho(p_r(a))$ for any finite group Σ.

Proposition 4.1. *Let $\mathcal{G} = (G,\ \chi,\ x)$ be an AGS, and let Σ be any finite group. The image of p_r^* is precisely the set* Fix σ_x^r *of points of period r in $\Phi_\Sigma(\mathcal{G})$.*

Proof. Suppose that $\rho \in \mathrm{Hom}(K_r, \Sigma)$. Then $\sigma_x^r(p_r^*\rho)(a) = (p_r^*\rho)(x^{-r}ax^r) = \rho(p_r(x^{-r}ax^r)) = \rho(p_r(a)) = (p_r^*\rho)(a)$ for all $a \in K$. Thus $p_r^*\rho \in \mathrm{Fix}\ \sigma_x^r$. Conversely, any $\rho \in \mathrm{Fix}\ \sigma_x^r$ induces a representation $\tilde{\rho} \in \mathrm{Hom}(K_r, \Sigma)$ such that $p_r^*\tilde{\rho} = \rho$. □

If l is an oriented link, we will denote its r-fold branched cyclic cover by $\hat{X}_r$ [Rol]. This cover is constructed from the ordinary r-fold cyclic cover X_r of the link exterior X (i.e., the cover corresponding to the kernel of $G \xrightarrow{\chi} \mathbf{Z} \to \mathbf{Z}/r$) by attaching solid tori $S^1 \times D^2$, one for each component of l, in such a way that each meridian of l is covered r times by some $S^1 \times 0$.

Theorem 4.2. *Let l be an oriented link and let Σ be any finite group. Then the cardinality $|\mathrm{Hom}(\pi_1\hat{X}_r, \Sigma)|$ satisfies a linear recurrence relation.*

Proof. The proof of Theorem 4.2 in the special case of an oriented knot is relatively easy, and we present it first for pedagogical reasons. Let $\mathcal{G}_k = (G,\ \chi,\ x)$ be the associated AGS. The fundamental group of $\hat{X}_r$ is easily seen to be isomorphic to K_r, and so by Proposition 4.1 the quantity $|\mathrm{Hom}(\pi_1\hat{X}_r, \Sigma)|$ is equal to $|\mathrm{Fix}\ \sigma_x^r|$. This in turn is equal to the trace of $\mathbf{A}^r$, where $\mathbf{A}$ is the adjacency matrix of a directed graph Γ that describes $\Phi_\Sigma(\mathcal{G}_k)$. It is well known that such a quantity satisfies a linear recurrence relation. For the convenience of the reader we recall the argument. First expand the product $\prod(t - \lambda_i)$, where the λ_i are the distinct nonzero eigenvalues of $\mathbf{A}$, as $\alpha_0 + \alpha_1 t + \ldots + \alpha_{m-1}t^{m-1} + t^m$. Since each eigenvalue λ_j satisfies $\alpha_0 + \alpha_1\lambda_j + \ldots + \alpha_{m-1}\lambda_j^{m-1} + \lambda_j^m = 0$, we find that $x_r = \lambda_j^r$ satisfies the linear recurrence relation

$$\alpha_0 x_r + \alpha_1 x_{r+1} + \ldots + \alpha_{r-1}x_{r+m-1} + x_{r+m} = 0.$$

However, the trace of $\mathbf{A}^r$ is expressible as $\sum_j c_j\lambda_j^r$, where the sum is taken over the distinct nonzero eigenvalues λ_j of $\mathbf{A}$ and c_j is the multiplicity of λ_j. By linearity $|\mathrm{Hom}(\pi_1\hat{X}_r, \Sigma)|$ satisfies the same linear recurrence relation.

In order to prove the general result, assume that l is an arbitrary link with associated AGS $\mathcal{G}_l = (G,\ \chi,\ x)$. Here χ is the total linking number homomorphism. Let $\langle a_{1,j}, \ldots, a_{n,j} \mid r_{1,j}, \ldots, r_{m,j},\ \ j \in \mathbf{Z}\rangle$ be a presentation for K obtained directly from a Wirtinger presentation for l and the Reidemeister-Schreier method so that each $xa_{i,0}$ is the class of a meridian. The fundamental group of the r-fold cyclic cover X_r has presentation $\langle y, a_{i,j} \mid r_{k,j},\ y^{-1}a_{i,j}y = a_{i,j+r},\ \ j \in \mathbf{Z}\rangle$, where the element y is equal to x^r in G, a trivial element in $\pi_1(\hat{X}_r)$. Since the elements $xa_{i,0}$, $i = 1, \ldots, n$, represent the meridians of l, $\langle a_{i,j} \mid r_{k,j},\ a_{i,j} = a_{i,j+r},\ (xa_{i,0})^r = 1,\ \ j \in \mathbf{Z}\rangle$ is a presentation of $\pi_1(\hat{X}_r)$. The relation $(xa_{i,0})^r = 1$ is equivalent to $x^{-r}(xa_{i,0})^r = 1$, and the latter is the same as $a_{i,r-1}\cdots a_{i,0} = 1$. Conjugating by powers of x produces the relators $a_{i,j+r-1}\cdots a_{i,j}$ for all $j \in \mathbf{Z}$. Hence

$$\pi_1(\hat{X}_r) = \langle a_{i,j} \mid r_{k,j},\ a_{i,j} = a_{i,j+r},\ a_{i,j+r-1}\cdots a_{i,j} = 1,\ \ j \in \mathbf{Z}\rangle.$$

(When l has just one component, the class of any meridian is conjugate to x and so the relations $a_{i,j+r-1}\cdots a_{i,j} = 1$ are redundant. Thus we have produced a presentation for the group K_r.) As in the proof of Proposition 4.1 the projection $q_r : K \to \pi_1(\hat{X}_r)$ induces an embedding $q_r^* : \mathrm{Hom}(K_r, \Sigma) \to \Phi_\Sigma(\mathcal{G}_l)$. The image consists of all period r points $\rho \in \Phi_\Sigma(\mathcal{G}_l)$ such that $\rho(a_{i,j+r-1})\cdots\rho(a_{i,j})$ vanishes for

every generator $a_{i,j}$. These special periodic points are in $|\Sigma|^n$-to-1 correspondence with the period r points of a certain finite-cover shift $(\tilde{\Phi}, \tilde{\sigma})$ that we describe next. The remainder of the proof is identitical to that for an oriented knot, using the finite-cover shift instead of $(\Phi_\Sigma(\mathcal{G}_l), \sigma_x)$.

Consider the group $\tilde{K}$ with presentation

$$\langle a_{1,j}, \dots, a_{n,j}\ b_{1,j}, \dots, b_{n,j} \mid r_{1,j}, \dots, r_{m,j},\ b_{i,j+1} = a_{i,j+1} b_{i,j}, \quad j \in \mathbf{Z}\rangle.$$

From the form of the relations $b_{i,j+1} = a_{i,j+1} b_{i,j}$ we see that $\tilde{K}$ is the free product of K with the free group on $b_{1,0}, \dots, b_{n,0}$. The finite $\mathbf{Z}$-dynamic presentation of $\tilde{K}$ corresponds to an AGS $\tilde{\mathcal{G}}$; let $(\tilde{\Phi}, \tilde{\sigma})$ denote the corresponding representation shift. Adapting the notation established in section 2, let $\tilde{A}_0$ be the set of generators obtained from $\{a_{1,0}, \dots, a_{1,M_1}, a_{2,0}, \dots, a_{n,M_n}, b_{1,0}, \dots, b_{n,0}\}$ by deleting $a_{1,M_1}, \dots, a_{n,M_n}$. Then $(\tilde{\Phi}, \tilde{\sigma})$ is described by a directed graph Γ with vertex set equal to the set of functions $\rho_0 : \tilde{A}_0 \to \Sigma$. There is an edge from ρ_0 to ρ_0' if (1) $\rho_0(a_{i,j+1}) = \rho_0'(a_{i,j})$ for each $a_{i,j} \in \tilde{A}_0$; (2) the images of $r_{1,0}, \dots, r_{r,0}$ are trivial in Σ; and (3) $\rho_0'(b_{i,0}) = \rho_0'(a_{i,0})\rho_0(b_{i,0})$.

The natural embedding of $K \hookrightarrow \tilde{K}$ induces a projection of $\tilde{\Phi}$ onto $\Phi_\Sigma(\mathcal{G}_l)$. The projection is $|\Sigma|^n$-to-1 and sends any representation $\rho : \tilde{K} \to \Sigma$ to its restriction $\rho|_K$. The reader can easily check that $\rho \in \tilde{\Phi}$ has period r if and only if its projection in $\Phi_\Sigma(\mathcal{G}_l)$ has period r and satisfies $\rho(a_{i,j+r-1}) \cdots \rho(a_{i,j}) = 0$ for every generator $a_{i,j}$. □

Example 4.3. We return to the pretzel knot $k = k(5,1,1)$ in Example 2.4 with $\Sigma = D_3$. A calculation shows that the distinct nonzero eigenvalues of $\mathbf{A}$ are 1 and the cube roots of 3. Hence

$$|\operatorname{Hom}(\pi_1 \hat{X}_{r+4}, D_3)| - |\operatorname{Hom}(\pi_1 \hat{X}_{r+3}, D_3)| - 3|\operatorname{Hom}(\pi_1 \hat{X}_{r+1}, D_3)| + 3|\operatorname{Hom}(\pi_1 \hat{X}_r, D_3)| = 0.$$

The initial conditions

$$|\operatorname{Hom}(\pi_1 \hat{X}_1, D_3)| = |\operatorname{Hom}(\pi_1 \hat{X}_2, D_3)| = 1,$$

$$|\operatorname{Hom}(\pi_1 \hat{X}_3, D_3)| = 10, \quad |\operatorname{Hom}(\pi_1 \hat{X}_4, D_3)| = 1$$

are easily read from Γ. Consequently, $|\operatorname{Hom}(\pi_1 \hat{X}_5, D_3)| = 1, |\operatorname{Hom}(\pi_1 \hat{X}_6, D_3)| = 28$, and in general $|\operatorname{Hom}(\pi_1 \hat{X}_{3m}, D_3)| = 3^{m+1} + 1$, while the other terms in the sequence are 1. The cardinalities $|\operatorname{Hom}(\pi_1 \hat{X}_{3m}, D_3)|$ grow asymptotically by a factor of 3. In fact, the exponential growth rate $\frac{\log 3}{3}$ is an invariant of the shift $\Phi_\Sigma(\mathcal{G}_k)$, its *topological entropy* (see [LiMa]). In general, the topological entropy of a shift of finite type is the log of the spectral radius of the incidence matrix of any directed graph that describes the shift. Hence if $\mathcal{G} = (G,\ \chi,\ X)$ is an AGS, then for any finite group Σ the topological entropy $h_\Sigma(\mathcal{G})$ is an invariant of $\mathcal{G}$. By this means we can obtain numerical invariants of knots and links from their associated representation shifts. This idea was introduced in [SiWi1].

Earlier numerical knot invariants defined using topological entropy were introduced by Los [Lo] and the first author [Si2]. The invariants in [Si2] are most interesting in the case of fibered knots (equivalently, knots for which the commutator subgroup of the knot group is finitely generated [BuZi], [Rol].) The entropy invariants obtained from representation shifts are completely different. When the group of a knot k has finitely generated commutator subgroup, the representation

shift Φ_Σ is finite and hence the entropy $h_\Sigma(\mathcal{G}_k)$ is zero for every finite group Σ. On the other hand, $h_\Sigma(\mathcal{G}_k)$ tends to be positive for nonfibered knots k and sufficiently large nonabelian groups Σ.

Conjecture 4.4. If k is a nonfibered knot and S_n is the symmetric group of degree n, then $h_{S_n}(\mathcal{G}_k) > 0$ for sufficiently large values of n.

Recall that Λ denotes the ring $\mathbf{Z}[t, t^{-1}]$. If l is an oriented link and Σ is a finite abelian group, then the Universal Coefficient Theorem for cohomology [Vi] implies that the underlying Λ-module $\mathrm{Hom}(K, \Sigma)$ of the representation shift $\Phi_\Sigma(\mathcal{G}_l)$ is isomorphic to $H^1(X_\infty; \Sigma)$, where X_∞ is the augmentation cover of X (the infinite cyclic cover if l is a knot) and the Λ-action is induced by covering transformations. Similarly, for any positive integer r, $\mathrm{Hom}(K_r, \Sigma)$ is a Λ_r-module, where $\Lambda_r = \Lambda/(t^r - 1)$, and it is isomorphic to $H^1(\hat{X}_r; \Sigma)$, the Λ_r-action again being induced by covering transformations.

When Σ is a finite nonabelian group, $\Phi_\Sigma(\mathcal{G}_l)$ and $\mathrm{Hom}(\pi_1 \hat{X}_r, \Sigma)$ can be regarded as *cohomology sets* $H^1(X_\infty; \Sigma_c^0)$ and $H^1(\hat{X}_r; \Sigma_c^0)$, respectively. The cohomology set $H^1(X_\infty; \Sigma_c^0)$ is defined in the following way (see [Hir]). For convenience we define only $H^1(X_\infty; \Sigma_c^0)$, the definition of $H^1(\hat{X}_r; \Sigma_c^0)$ being similar.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X_∞ that is *proper* in the sense that distinct indices i, j correspond to distinct sets. For convenience, we will also assume that every U_i and $U_i \cap U_j$, $i, j \in I$, is connected, $U_i \cap U_j$ possibly being empty. A $\mathcal{U}$-cocyle is a function α that assigns to each nonempty $U_i \cap U_j$ an element $\alpha_{i,j} \in \Sigma$ such that $\alpha_{i,j}\alpha_{j,k} = \alpha_{i,k}$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Cocycles α and α' are said to be equivalent if for each $i \in I$ there is an element $g_i \in \Sigma$ such that

$$\alpha'_{i,j} = g_i^{-1} \alpha_{i,j} g_j, \quad \text{whenever } U_i \cap U_j \neq \emptyset.$$

We define $H^1(\mathcal{U}; \Sigma_c)$ to be the set of equivalence classes of $\mathcal{U}$-cocycles. More precisely, we have defined the cohomology set of $\mathcal{U}$ with coefficients in the *constant sheaf* $\Sigma_c : \Sigma \times X_\infty \to X_\infty$, where $(\alpha, x) \mapsto x$ and the group Σ is given the discrete topology. The elements $\alpha_{i,j}$ and g_i should be regarded as sections of the sheaf over $U_i \cap U_j$ and U_i, respectively, constant since Σ is discrete and the sets $U_i \cap U_j, U_i$ are connected. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $H^1(\mathcal{U}; \Sigma_c)$ embeds naturally in $H^1(\mathcal{V}; \Sigma_c)$ (see [Hir], pages 38 and 39). The cohomology set $H^1(X_\infty; \Sigma_c)$ is defined to be the union of the sets $H^1(\mathcal{U}; \Sigma_c)$ as $\mathcal{U}$ runs over all such coverings of X_∞.

The elements of $H^1(X_\infty; \Sigma_c)$ correspond in a natural way to conjugacy classes in $\mathrm{Hom}(K, \Sigma)$. To see this, one must first choose an open set U_{i_0} in $\mathcal{U}$ containing a basepoint $* \in X_\infty$. Let $\alpha = \{\alpha_{i,j}\}$ be a $\mathcal{U}$-cocycle. If γ is any closed curve in X_∞ representing an element of $K = \pi_1(X_\infty, *)$, then we can select a chain $U_{i_0}, U_{i_1}, \ldots, U_{i_n}, U_{i_0}$ of sets in $\mathcal{U}$ that cover γ. We associate to γ the element $\alpha_{i_0,i_1} \cdots \alpha_{i_{n-0},i_n} \alpha_{i_n,i_0} \in \Sigma$. This element depends only on the homotopy class of γ in $\pi_1(X_\infty, *)$, and our procedure determines a representation $\rho : K \to \Sigma$. Moreover, cocycles equivalent to α induce the same representation. If we replace our initial choice of U_{i_0} with another set containing the basepoint, then the new representation ρ' has the form $\rho'(a) = \beta^{-1}\rho(a)\beta$, $\forall a \in K$, for some $\beta \in \Sigma$. In this way we have a bijection between $H^1(X_\infty; \Sigma_c)$ and the set of conjugacy classes of $\mathrm{Hom}(K, \Sigma)$. It is well known that the elements of $H^1(X_\infty; \Sigma_c)$ classify Σ-bundles over X_∞ (see Theorem 3.2.1 of [Hir]).

If we replace Σ_c by the "punctured" sheaf Σ_c^0 obtained by removing every preimage of the basepoint except that corresponding to the identity element of Σ, then

the only section over U_{i_0} is trivial. The reader can verify that the correspondence between the elements of $H^1(X_\infty;\Sigma_c^0)$ and $\mathrm{Hom}(K;\Sigma)$ no longer depends on the initial choice of U_{i_0}. Thus we can interpret $\Phi_\Sigma(\mathcal{G}_l)$ as a cohomology set $H^1(X_\infty;\Sigma_c^0)$. The latter has additional structure, the action of an infinite cyclic group induced by the covering transformations of X_∞, which corresponds to the action of σ_x on $\Phi_\Sigma(\mathcal{G}_l)$. Of course, when Σ is abelian, $H^1(X_\infty;\Sigma_c^0)$ is the same as $H^1(X_\infty;\Sigma_c)$, the ordinary first cohomology group of X_∞ with coefficients in Σ. In the case of a knot, this cohomology group can be identified with the homology group $H_1(X_\infty;\Sigma)$, as Proposition 4.5 shows.

Proposition 4.5. *If k is an oriented knot and Σ is a finite abelian group, then the underlying Λ-module of the representation shift $\Phi_\Sigma(\mathcal{G}_k)$ is isomorphic to $H_1(X_\infty;\Sigma)$. For any integer $r>1$, the Λ_r-module* Fix σ_x^r *of points of period r is isomorphic to $H_1(\hat{X}_r;\Sigma)$.*

Proof. We assume first that Σ is a finite cyclic group. It is well known that $\pi_1(X_\infty)$ is the commutator subgroup K of the group of the knot k. The underlying Λ-module of $\Phi_\Sigma(\mathcal{G}_k)$ is the same as $\mathrm{Hom}(H_1(X_\infty;\mathbf{Z});\Sigma)$, which by properties of tensor product is isomorphic to $\mathrm{Hom}(H_1(X_\infty;\mathbf{Z})\otimes_{\mathbf{Z}}\Sigma;\Sigma)$. It well known and not difficult to prove that $H_1(X_\infty;\mathbf{Z})\otimes_{\mathbf{Z}}\Sigma$ is finite; hence, by properties of Hom and tensor product we have $\mathrm{Hom}(H_1(X_\infty;\mathbf{Z})\otimes_{\mathbf{Z}}\Sigma;\Sigma)\cong(H_1(X_\infty;\mathbf{Z})\otimes_{\mathbf{Z}}\Sigma)\otimes_{\mathbf{Z}}\Sigma$. The last module is isomorphic to $H_1(X_\infty;\mathbf{Z})\otimes_{\mathbf{Z}}\Sigma$, and by the Universal Coefficient Theorem it is isomorphic to $H_1(X_\infty;\Sigma)$.

The proof that $\Phi_\Sigma(\mathcal{G}_k)\cong H_1(X_\infty;\Sigma)$ when Σ is an arbitrary finite abelian group follows from the proof for the case above, since $H_1(X_\infty;\Sigma_1\times\Sigma_2)$ can be decomposed as $H_1(X_\infty;\Sigma_1)\times H_1(X_\infty;\Sigma_2)$.

In order to prove the remaining assertion, assume that $r>1$. We recall that Fix $\sigma_x^r\cong\mathrm{Hom}(H_1(\hat{X}_r;\mathbf{Z});\Sigma)$ by Proposition 4.1 and the remarks that follow its proof. An argument similar to the one above shows that Fix σ_x^r is isomorphic to $H_1(\hat{X}_r;\Sigma)$. □

The following periodicity result has also been proved by W. Stevens [St]. Partial results of this sort were obtained earlier by M. Dellomo [De] and J. Hillman [Hil].

Corollary 4.6. *For any knot k and finite abelian group Σ there is a positive integer $n=n(k,\Sigma)$ such that $H_1(\hat{X}_{r+n};\Sigma)\cong H_1(\hat{X}_r;\Sigma)$.*

Proof. By Proposition 4.5 we can regard $H_1(\hat{X}_\infty;\Sigma)$ as $\Phi_\Sigma(\mathcal{G}_k)$ and $H_1(\hat{X}_r;\Sigma)$ as Fix σ_x^r. Recall from Proposition 3.7 that $\Phi_\Sigma(\mathcal{G}_k)$ is finite whenever Σ is abelian. Let n be the least common multiple of the periods of $\Phi_\Sigma(\mathcal{G}_k)$. Then Fix $\sigma_x^{r+n}=$ Fix σ_x^r, for all r. The assertion about periodicity follows. □

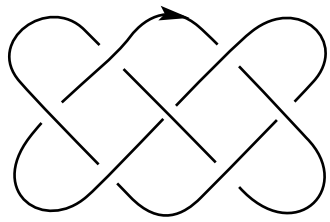
The proof of Corollary 4.6 provides an effective algorithm for determining the period n. The following example illustrates this.

Example 4.7. Consider the knot $k=7_4$ in Figure 6a. The commutator subgroup K of its group has presentation

$$\langle a_j \mid a_{j+2}^{-2}a_{j+1}^2a_j^{-2}a_{j+1}^2a_j^{-2}a_{j+1}a_{j+2}^{-2}a_{j+1}^2,\quad j\in\mathbf{Z}\rangle.$$

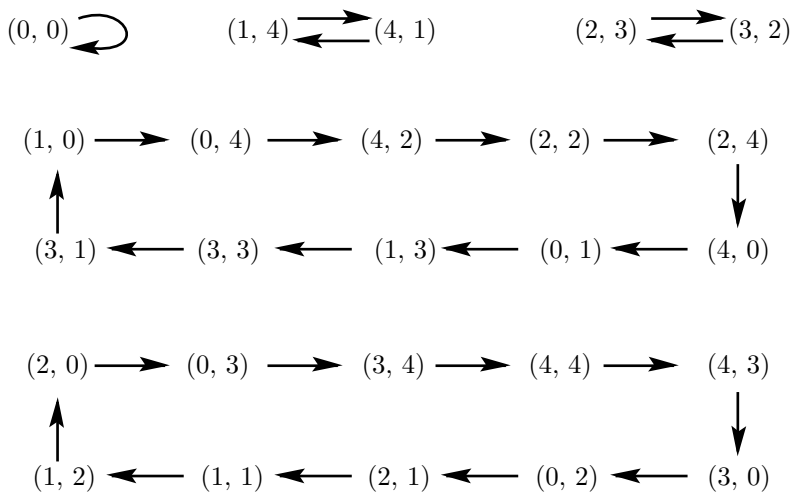
The abelianized group has presentation

$$\langle a_j \mid 4a_{j+2}=7a_{j+1}-4a_j,\quad j\in\mathbf{Z}\rangle.$$



The knot $k = 7_4$

(a)



The graph Γ

(b)

FIGURE 6

A graph Γ describing $\Phi_{\mathbf{Z}/5}(\mathcal{G}_k)$ appears in Figure 6b. Since the least common multiple n of its cycle lengths is 10, we can immediately conclude that $H_1(\hat{X}_{r+10}; \mathbf{Z}/5) = H_1(\hat{X}_r; \mathbf{Z}/5)$ for all $r \geq 1$. The graph also reveals other structure. From it we see, for example, that $H_1(\hat{X}_3; \mathbf{Z}/5),\ H_1(\hat{X}_7; \mathbf{Z}/5),\ H_1(\hat{X}_9; \mathbf{Z}/5),\ \ldots$ are trivial.

The conclusions of Theorem 4.2, Propostion 4.5 and Corollary 4.6 apply to a wider variety of spaces than knot exteriors. Let X be any manifold with finitely presented fundmental group, and suppose that $\chi : \pi_1(X) \to \mathbf{Z}$ is an epimorphism. If c is a simple closed curve in *int* X such that $x = [c]$ is mapped by χ to 1, then we can construct r-fold branched cyclic covering spaces $\hat{X}_r$ with branch set c corresponding to this data. The conclusion of Theorem 4.2 holds. Furthermore, if $\pi_1(X)$ has infinite cyclic abelianization, then it is not difficult to show that the conclusions of Propostion 4.5 and Corollary 4.6 are also valid.

5. Extending representations

Given an AGS $\mathcal{G} = (G,\ \chi,\ x)$ and a finite group Σ, it is natural to ask when a representation $\rho \in \Phi_\Sigma(\mathcal{G})$ extends over G. Proposition 5.1 shows that every periodic representation extends over G in a weak sense.

A representation $\hat{\rho} : K \to \overline{\Sigma}$ is a *lift* of $\rho : K \to \Sigma$ if there exists a homomorphism $\pi : \hat{\rho}(K) \to \Sigma$ such that $\pi\hat{\rho} = \rho$. When $\overline{\Sigma}$ is a finite group, we will call $\hat{\rho}$ a *finite lift* of ρ.

Proposition 5.1. *Let $\mathcal{G} = (G,\ \chi,\ x)$ be an AGS, and assume $\rho \in \Phi_\Sigma(\mathcal{G})$. Then some finite lift of ρ extends over G if and only if the representation ρ is periodic.*

Proof. Assume that some finite lift $\hat{\rho} : K \to \overline{\Sigma}$ of ρ extends to a representation $\overline{\rho} : G \to \overline{\Sigma}$. Then there exists a homomorphism $\pi : \hat{\rho}(K) \to \Sigma$ such that $\pi\hat{\rho} = \rho$. Let r be the order of $\overline{\rho}(x)$ in $\overline{\Sigma}$. Since $\overline{\rho}(a_{i,j+r}) = \overline{\rho}(x^{-r}a_{i,j}x^r) = \overline{\rho}(a_{i,j})$ for all generators $a_{i,j}$ of K, it follows that $\pi\hat{\rho}(a_{i,j+r}) = \pi\hat{\rho}(a_{i,j})$ and so $\rho(a_{i,j+r}) = \rho(a_{i,j})$. Hence $\sigma_x^r\rho = \rho$ and ρ is periodic.

Conversely, assume that $\sigma_x^r\rho = \rho$ for some $r \geq 1$. If $r = 1$, then we can extend $\hat{\rho}$ to a representation $\overline{\rho} : G \to \Sigma$ by defining $\overline{\rho}(x)$ to be the trivial element. Assume that $r \geq 2$. Let $\overline{\Sigma}$ be the semidirect product $\langle \tau \mid \tau^r = e\rangle \times_\theta \Sigma^r$ in which conjugation by τ takes $(c_0, \dots, c_{r-1}) \in \Sigma^r$ to $\theta(c_0, \dots, c_{r-1}) = (c_1, \dots, c_{r-1}, c_0)$. Define $\overline{\rho} : G \to \overline{\Sigma}$ by sending x to τ and any $a \in K$ to $(\rho a, \sigma_x\rho a, \dots, \sigma_x^{r-1}\rho a) \in \Sigma^r$. We conclude the proof by noting that $\hat{\rho} = \overline{\rho}|_K$ is a lift of ρ, since $\pi\hat{\rho} = \rho$, where π is the first-coordinate projection on Σ^r. □

Representations ρ and ρ' in $\Phi_\Sigma(\mathcal{G})$ are *equivalent* if $\rho' = \phi\rho$ for some inner automorphism ϕ of $\rho(K)$. If $\sigma_x\rho = \phi\rho$ for some (not necessarily inner) automorphism ϕ of $\rho(K)$, then by induction $\sigma_x^n\rho = \phi^n\rho$ for any integer n. Since Σ is finite, the automorphism ϕ must have finite order, and it follows that ρ must be periodic.

The next proposition shows that the representations $\rho : K \to \Sigma$ that are restrictions of finite representations of G are special periodic points of the representation shift.

Proposition 5.2. *Let $\mathcal{G} = (G,\ \chi,\ x)$ be an AGS, and assume $\rho \in \Phi_\Sigma(\mathcal{G})$. Then ρ is the restriction of a representation $\overline{\rho} : G \to \overline{\Sigma}$, for some finite overgroup $\overline{\Sigma}$ of Σ, if and only $\sigma_x\rho = \phi\rho$ for some automorphism ϕ of $\rho(K)$. Furthermore, ρ is the restriction of a representation $\overline{\rho} : G \to \Sigma$ if and only if $\sigma_x\rho$ and ρ are equivalent.*

Proof. Assume that $\rho : K \to \Sigma$ is the restriction of $\overline{\rho} : G \to \overline{\Sigma}$. Let $\xi = \overline{\rho}(x)$. Since $\sigma_x\rho(a) = \rho(x^{-1}ax) = \overline{\rho}(x^{-1}ax) = \xi^{-1}\rho(a)\xi$, for all $a \in K$, conjugation in $\overline{\Sigma}$ by ξ restricts to an automorphism ϕ of $\rho(K)$, and $\sigma_x\rho = \phi\rho$. Notice that if $\overline{\Sigma} = \Sigma$, then the element ξ is in $\rho(K)$, and ϕ is, in fact, an inner automorphism of $\rho(K)$.

Conversely, assume that $\sigma_x\rho = \phi\rho$ for some automorphism ϕ of $\rho(K)$. Let r be the order of ϕ. Define $\overline{\Sigma}$ to be the semidirect product $\langle \tau \mid \tau^r = e\rangle \times_\phi \rho(K)$, and extend ρ to a representation $\overline{\rho} : G \to \overline{\Sigma}$ by defining $\overline{\rho}(x) = \tau$. If ϕ is an inner automorphism of $\rho(K)$, say, conjugation by an element $\xi \in \rho(K)$, then we can extend ρ to $\overline{\rho} : G \to \Sigma$ by defining $\overline{\rho}(x)$ to be ξ. □

Proposition 5.2 provides a method for finding all representations $\overline{\rho}$ of the group G of a knot k onto metabelian groups of the form $\overline{\Sigma} = \langle \tau \mid \tau^r = e\rangle \times_\phi \Sigma$, where Σ is a fixed finite abelian group. The method is different from that described at the end of [Har]. As noted in [Har] (see note on page 95), there is no loss of generality in

assuming that $\overline{\rho}$ maps the merdianal generator x of G to τ. We begin by locating the period r representations ρ in $\Phi_\Sigma(\mathcal{G}_k)$ (there can be only finitely many) and selecting those representations that are surjective. We recognize the representations as cycles of length r in the directed graph Γ that describes the representation shift, each cycle corresponding to r distinct representations, any one of which is surjective if and only if all are. Such a cycle has vertices $v_1 \to v_2 \to \ldots \to v_r \to v_1$ labeled by M-tuples $(\alpha^{(\nu)}_{1,0}, \ldots \alpha^{(\nu)}_{1,M_1-1}, \alpha^{(\nu)}_{2,0}, \ldots, \alpha^{(\nu)}_{n,M_n-1})$ of elements of Σ, $1 \leq \nu \leq r$. If the correspondence $\alpha^{(1)}_{i,j} \mapsto \alpha^{(2)}_{i,j}, \ldots, \alpha^{(r)}_{i,j} \mapsto \alpha^{(1)}_{i,j}$ extends to a homomorphism $\phi : \Sigma \to \Sigma$, then ϕ must in fact be an automorphism (reverse the arrows to get its inverse) and ρ extends to a representation $\overline{\rho}$ from G onto $\langle \tau \mid \tau^r = e\rangle \times_\phi \Sigma$ by defining $\overline{\rho}(x)$ to be τ.

Example 5.3. Consider the AGS $\mathcal{G} = (G,\ \chi,\ x)$ associated to the oriented knot 6_1 shown in Figure 7. We will construct all representations of G onto metabelian groups of the form $\langle \tau \mid \tau^r = e\rangle \times_\phi \mathbf{Z}/5$. The commutator subgroup K of the group G has presentation

$$\langle a_j \mid a^2_{j+2}a^{-2}_{j+1}a^2_j a^{-3}_{j+1}, \quad j \in \mathbf{Z}\rangle.$$

The representation shift $\Phi_{\mathbf{Z}/5}(\mathcal{G})$ can be described by a directed graph Γ with a vertex set consisting of pairs of elements of $\mathbf{Z}/5$. There is a directed edge from $(\alpha_0,\ \alpha_1)$ to $(\alpha'_0,\ \alpha'_1)$ if and only if $\alpha_1 = \alpha'_0$ and $\alpha'_1 = -\alpha_0$. Since we are seeking representations ρ such that the correspondence $\rho(a_j) \mapsto \rho(a_{j+1})$ extends to an automorphism of $\mathbf{Z}/5$, it suffices to consider cycles in Γ beginning at vertices of the form $(1, \alpha)$, where $\alpha = 2, 3$ or 4. The cycles we find are

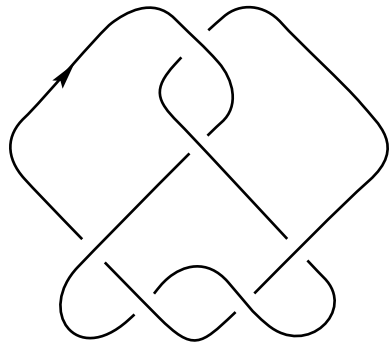
$$(1,2) \mapsto (2,4) \mapsto (4,3) \mapsto (3,1) \mapsto (1,2),$$
$$(1,3) \mapsto (3,4) \mapsto (4,2) \mapsto (2,1) \mapsto (1,3),$$
$$(1,4) \mapsto (4,4) \mapsto (4,1) \mapsto (1,1) \mapsto (1,4).$$

The third cycle does not determine an automorphism of $\mathbf{Z}/5$ since 1 is followed by both 1 and 4, but the first two cycles do. The first cycle detemines the automorphism ϕ_1 mapping 1 to 2, while the second determines ϕ_2 mapping 1 to 3. Consequently, we obtain representations $\overline{\rho}_1$ and $\overline{\rho}_2$ from G onto $\overline{\Sigma}_1 = \langle \tau,\ \alpha \mid \tau^4 = \alpha^5 = e,\ \tau\alpha\tau^{-1} = \tau^2\rangle$ and $\overline{\Sigma}_2 = \langle \tau,\ \alpha \mid \tau^4 = \alpha^5 = e,\ \tau\alpha\tau^{-1} = \tau^3\rangle$, respectively. It is easy to see that the two groups $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ are isomorphic. In fact, the eight metabelian representations of G that we obtain are pairwise equivalent.

Proposition 5.4. *If k is a knot with group G, then for any positive integer n the representations of G onto the dihedral group D_n are in n-to-one correspondence with the elements of $\Phi_{\mathbf{Z}/n}(\mathcal{G}_k)$ having least period 2.*

Proof. Suppose that $\overline{\rho}$ is a representation of G onto $D_n = \langle \tau, \alpha \mid \tau^2 = \alpha^n = e,\ \tau\alpha\tau^{-1} = \alpha^{-1}\rangle$. The image $\overline{\rho}(x)$ must be a reflection, an element of the form $\alpha^i\tau$. The restriction of $\overline{\rho}$ to the commutator subgroup K of G is a representation $\rho \in \Phi_{\mathbf{Z}/n}(\mathcal{G}_k)$, and for any element $a \in K$, we have $\sigma^2_x\rho(a) = \rho(x^{-2}ax^2) = \overline{\rho}(x^{-2})\overline{\rho}(a)\overline{\rho}(x^2) = \rho(a)$. Hence ρ has period 2. In fact, ρ cannot have period 1, since by Proposition 2.5 the trivial representation is the only fixed point in the representation shift.

Conversely, suppose that ρ is a nontrivial representation in $\Phi_{\mathbf{Z}/n}(\mathcal{G}_k)$ with period 2. Since $\rho + \sigma_x\rho$ is evidently a fixed point of the representation shift, it must be trivial; consequently, $\sigma_x\rho = -\rho$. We can extend ρ to a representation $\overline{\rho}$ of G onto



The knot $k = 6_1$

FIGURE 7

D_n by defining $\overline{\rho}(x)$ to be any reflection $\alpha^i\tau$. There are exactly n ways to do this. □

It is well known that dihedral representations of the group of a knot k correspond to the "n-colorings" of a diagram for k (see [CrFo]). We show in [SiWi2] that the representations in $\Phi_{\mathbf{Z}/n}(\mathcal{G}_k)$ of period r greater than 2 correspond to a class of generalized (n, r)-colorings. Like n-colorings, (n, r)-colorings can be determined directly from the knot diagram, and they have many attractive properties.

6. Knots with representations close to the identity

Each representation shift for knots that we have seen so far has the property that the trivial representation is isolated; that is, it has a neighborhood containing no other representation. For any fibered knot this will be the case no matter what target group Σ is used, since its representation shifts will be finite. Example 6.1 shows that there are also nonfibered knots for which the trivial representation will be isolated in every representation shift.

Example 6.1. Consider the AGS $\mathcal{G}_k$ associated to the oriented knot 6_1 (Figure 7). Recall that the commutator subgroup K has presentation $\langle a_j \mid a_{j+2}^2 = a_{j+1}^3 a_j^{-2} a_{j+1}^2, j \in \mathbf{Z}\rangle$. The representation shift $\Phi_\Sigma(\mathcal{G}_k)$ can be described by a finite directed graph Γ with vertices corresponding to elements of $\Sigma \times \Sigma$.

Assume that for some finite group Σ, the trivial representation is not isolated in $\Phi_\Sigma(\mathcal{G}_k)$. We can find an infinite path (e, e), (e, α_1), $(\alpha_1, \alpha_2), \ldots,$ where α_1 is nontrivial. Since Σ is finite, there exist nonnegative integers $r < s$ such that $(\alpha_r, \alpha_{r+1}) = (\alpha_s, \alpha_{s+1})$. Let Σ_0 be the subgroup of Σ generated by $\alpha_1, \ldots, \alpha_s$. It would follow that the trivial representation is not isolated in $\Phi_{\Sigma_0}(\mathcal{G}_k)$. However, from the presentation for K we see that $\alpha_1^2 = e$. Also, $\alpha_2^2 = \alpha_1$. Inductively, we can show that $\alpha_{m+1}^2 = \alpha_m$ for every $m \geq 1$. This implies that the subgroup Σ_0 is generated by α_s, and so it is abelian, in contradiction to Propositions 3.1 and 3.7. Hence the trivial representation is isolated in $\Phi_\Sigma(\mathcal{G}_k)$ for every finite group Σ.

Example 6.1 stands in contrast to the following.

Example 6.2. Consider the AGS $\mathcal{G}_k$ associated to the pretzel knot $k = k(7, 11, 15)$ (notation and knot orientation as in Example 2.4). Let Σ be the alternating group A_5. One can check that the graph Γ describing $\Phi_\Sigma(\mathcal{G}_k)$ contains the loop $v_0 \to v_1 \to \ldots \to v_9 \to v_0$, where $v_0 = (e, e)$; $v_1 = ((345), (354))$; $v_2 = ((13)(24), (345))$; $v_3 = ((253), (12543))$; $v_4 = (e, (13452))$; $v_5 = ((234), (13542))$; $v_6 = (e, (12453))$; $v_7 = ((145), (12435))$; $v_8 = ((23)(45), (15342))$; $v_9 = ((153), e)$. Consequently, $\Phi_\Sigma(\mathcal{G}_k)$ contains nontrivial representations that are arbitrarily close to the trivial representation. This seems surprising since k is an alternating knot. Indeed, the commutator subgroup of the group of any alternating knot is known to be rigid in several respects (see for example Proposition 13.26 and Corollary 13.27 of [BuZi]).

The above example was discovered using a computer search [Zh]; the relevant section of the representation shift that we have exhibited was checked by hand. Since the alternating group A_5 has exponent 30, the knots $k(7 + 30n, 11 + 30n, 15 + 30n)$ provide an infinite number of such examples.

Given a finite group Σ, we can define the *cohomology set with compact supports* $H^1_{\text{cpt}}(X_\infty; \Sigma_c)$ in the following way (see Chapter 6.22 of [Do]). Let K_ν, $\nu \in \mathbf{N}$ be a sequence of compact subsets of X_∞ such that $K_\nu \subset K_{\nu+1}$ and $\bigcup K_\nu = X_\infty$. For each ν, let Σ_c^ν be the constant sheaf Σ_c punctured over each point of $X_\infty - K_\nu$ (see section 4); that is, remove the preimages of each point of $X_\infty - K_\nu$ except those corresponding to the trivial element of Σ. We define $H^1_{\text{cpt}}(X_\infty; \Sigma_c)$ to be the limit of the sets $H^1(X_\infty; \Sigma_c^\nu)$. It is not difficult to check that elements of $H^1_{\text{cpt}}(X_\infty; \Sigma_c)$ correspond to conjugacy classes of representations $\rho : K \to \Sigma$ that are contained in the same component of $\Phi_\Sigma(\mathcal{G}_k)$ as the trivial representation. We summarize the results of this section in the following.

Theorem 6.3. *If X_∞ is the infinite cyclic cover of the pretzel knot $k(7, 11, 15)$ and $\Sigma = A_5$, then $H^1_{\text{cpt}}(X_\infty; \Sigma_c)$ is nontrivial. If X_∞ is the infinite cyclic cover of 6_1, then $H^1_{\text{cpt}}(X_\infty; \Sigma_c)$ is trivial for every finite group Σ.*

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