

CLASSES OF SINGULAR INTEGRALS ALONG CURVES AND SURFACES

ANDREAS SEEGER, STEPHEN WAINGER, JAMES WRIGHT, AND SARAH ZIESLER

ABSTRACT. This paper is concerned with singular convolution operators in \mathbb{R}^d , $d \geq 2$, with convolution kernels supported on radial surfaces $y_d = \Gamma(|y'|)$. We show that if $\Gamma(s) = \log s$, then L^p boundedness holds if and only if $p = 2$. This statement can be reduced to a similar statement about the multiplier $m(\tau, \eta) = |\tau|^{-i\eta}$ in \mathbb{R}^2 . We also construct smooth Γ for which the corresponding operators are bounded for $p_0 < p \leq 2$ but unbounded for $p \leq p_0$, for given $p_0 \in [1, 2)$. Finally we discuss some examples of singular integrals along convex curves in the plane, with odd extensions.

1. INTRODUCTION

This paper is primarily concerned with singular integral operators T in dimensions $d \geq 2$ defined for $f \in C_0^\infty(\mathbb{R}^d)$ by

$$(1.1) \quad Tf(x', x_d) = \text{p.v.} \int f(x' - y', x_d - \Gamma(|y'|)) \frac{\Omega(y')}{|y'|^{d-1}} dy'$$

where $x' \in \mathbb{R}^{d-1}$. We assume that $\Gamma : (0, \infty) \rightarrow \mathbb{R}$ is a smooth function, $\Omega \in L^q(S^{d-2})$ for some $q > 1$ and

$$(1.2) \quad \int_{S^{d-2}} \Omega(\theta) d\sigma(\theta) = 0.$$

We include the case $d = 2$ with the interpretation of $S^0 = \{-1, 1\}$ and the surface measure being counting measure.

It is easy to see using (1.2) that the principal value integral (1.1) exists everywhere for $f \in C_0^\infty$. The question is for which $p \in (1, \infty)$ the operator T extends to a bounded operator on $L^p(\mathbb{R}^d)$. If we consider the case of convex Γ it is known that, then L^2 boundedness implies L^p boundedness for $1 < p < \infty$ (see [10], [2] for the case $d = 2$ and [8] for the case $d \geq 3$, at least in the case of smooth Ω). Moreover it was shown in [8] (again assuming that Ω is smooth and Γ is C^1 in $(0, \infty)$) that in dimension $d \geq 3$ the operators T are bounded in $L^2(\mathbb{R}^d)$, without any convexity assumption on Γ . Our primary concern here is whether T extends to a bounded operator on L^p without any further restriction on Γ . Our first theorem shows that this is not the case, in fact in our example Γ is chosen to be *concave*.

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Theorem 1.1. *Suppose that $\Omega \in L^q(S^{d-2})$ where $q > 1$ and suppose that the cancellation property (1.2) holds. Suppose $\Gamma(t) = \log t$. Then T extends to a bounded operator on $L^p(\mathbb{R}^d)$ if and only if $p = 2$ or $\Omega = 0$ almost everywhere.*

Remark. The analogous maximal operator M_γ defined as the pointwise supremum of averages over $\{(x + y', \log(|x + y'|)) : |y'| \leq h\}$, $h > 0$, is unbounded on all L^p spaces, see the argument in [14, p. 1291]. Moreover the L^2 estimate may fail if the standard homogeneous Calderón-Zygmund kernels $\Omega(y'/|y'|)|y'|^{1-d}$ are replaced by other (standard) singular kernels, such as the kernel for fractional integration of imaginary order, see Remark 2.3 below.

We shall see that the unboundedness of T for $p \neq 2$ follows from a negative result for a Fourier multiplier on \mathbb{R}^2 . In what follows M^p denotes the class of Fourier multipliers of L^p and $\|m\|_{M^p}$ is the L^p operator norm of the convolution operator with Fourier multiplier m .

Proposition 1.2. *Let χ be a bounded function in $C^1(\mathbb{R})$ and define*

$$(1.3) \quad h(\tau, \eta) = \chi(\eta)|\tau|^{-i\eta}.$$

Then $h \in M^p(\mathbb{R}^2)$ if and only if $p = 2$ or $\chi \equiv 0$.

If χ_+ denotes the characteristic function of $(0, \infty)$, then the same statement holds with $h(\tau, \eta)$ replaced by $h_\pm(\tau, \eta) = h(\tau, \eta)\chi_+(\pm\tau)$.

Remark. This result should be compared with the fact that for every η the multiplier $\tau \mapsto |\tau|^{-i\eta}$ is a multiplier in $M^p(\mathbb{R})$ for $1 < p < \infty$ (it is the multiplier corresponding to fractional integration of imaginary order; the L^p boundedness follows from the Marcinkiewicz multiplier theorem).

In our second theorem we exhibit operators T with a prescribed range of L^p boundedness.

Theorem 1.3. *Suppose $1 < r \leq 2$. There is a function Γ defined on $[0, \infty)$ with $\Gamma(0) = 0$, such that the symmetric extension $\Gamma(|x'|)$ to \mathbb{R}^{d-1} is smooth and such that the following holds.*

Let $d \geq 2$ and T be as in (1.1), where $\Omega \in L^q(S^{d-2})$ for some $q > 1$ and the cancellation property (1.2) is assumed. Then T extends to a bounded operator on $L^p(\mathbb{R}^d)$ if and only if $r \leq p \leq r/(r-1)$ or $\Omega = 0$ almost everywhere.

Remarks. (i) Let $1 \leq r < 2$. A slight modification of our construction yields Γ such that T is bounded on $L^p(\mathbb{R}^d)$ if and only if $r < p < r/(r-1)$ or $\Omega = 0$ a.e.

(ii) Examples where the maximal operator associated to the curve is bounded on some L^p spaces but not on others have been constructed by M. Christ [4], see also Vance, Wright and Wainger [15] and unpublished work by Wierdl. Examples of this kind for singular integral operators seem to be new; however in [3] an example of a convex Γ was constructed, so that the Hilbert transform associated to the odd extension was bounded only on $L^2(\mathbb{R}^2)$.

(iii) In an appendix (§5) we include some observations related to the examples in [3] and [4], dealing with singular integrals with convolution kernels supported on curves $\{(t, \gamma(t))\}$ in the plane; here γ is the odd extension of a convex function on $(0, \infty)$.

2. L^2 -ESTIMATES

We shall now consider the case

$$\Gamma(t) = \log t$$

and show that T is bounded on L^2 (provided that $\Omega \in L^q, q > 1$). This is achieved by showing that

$$\begin{aligned} m_R(\xi) &= \int_{|x'| \leq R} e^{-i\langle x', \xi' \rangle + \xi_d \log |x'|} \frac{\Omega(x'/|x'|)}{|x'|^{d-1}} dx' \\ (2.1) \quad &= \int_0^R e^{-i\xi_d \log r} \int_{S^{d-2}} e^{-i\langle r\theta, \xi' \rangle} \Omega(\theta) d\sigma(\theta) \frac{dr}{r} \end{aligned}$$

is bounded uniformly in ξ and R and converges to a bounded function as $R \rightarrow \infty$. By changing variables $r \mapsto r|\xi'|$ and using the cancellation of Ω we see that

$$(2.2) \quad m_R(\xi) = e^{i\xi_d \log |\xi'|} M_{R|\xi'|}(\xi'/|\xi'|, \xi_d)$$

with

$$(2.3) \quad M_R(\vartheta, \xi_d) = \int_0^R e^{-i\xi_d \log r} \int_{S^{d-2}} (e^{-i\langle r\theta, \vartheta \rangle} - 1) \Omega(\theta) d\sigma(\theta) \frac{dr}{r}$$

for $\vartheta \in S^{d-2}$.

We split $M_R = \sum_{i=1}^3 \mathcal{E}_i^R$ where

$$\begin{aligned} \mathcal{E}_1^R(\vartheta, \xi_d) &= \int_0^R e^{-i\xi_d \log r} \int_{\theta: r|\langle \theta, \vartheta \rangle| \leq 1} (e^{-i\langle r\theta, \vartheta \rangle} - 1) \Omega(\theta) d\sigma(\theta) \frac{dr}{r}, \\ (2.4) \quad \mathcal{E}_2^R(\vartheta, \xi_d) &= \int_0^R e^{-i\xi_d \log r} \int_{\theta: r|\langle \theta, \vartheta \rangle| \geq 1} e^{-i\langle r\theta, \vartheta \rangle} \Omega(\theta) d\sigma(\theta) \frac{dr}{r}, \\ \mathcal{E}_3^R(\vartheta, \xi_d) &= - \int_0^R e^{-i\xi_d \log r} \int_{\theta: r|\langle \theta, \vartheta \rangle| \geq 1} \Omega(\theta) d\sigma(\theta) \frac{dr}{r}. \end{aligned}$$

First observe that

$$|\mathcal{E}_1^R(\vartheta, \xi_d)| \leq \int |\Omega(\theta)| \int_0^{\min\{|\langle \theta, \vartheta \rangle|^{-1}, R\}} |e^{-i\langle r\theta, \vartheta \rangle} - 1| \frac{dr}{r} d\sigma(\theta) \leq C.$$

To estimate \mathcal{E}_2^R interchange the order of the integration and observe that after a change of variables $s = r|\langle \theta, \vartheta \rangle|$ in the inner integral we have

$$\begin{aligned} \mathcal{E}_2^R(\vartheta, \xi_d) &= \int_{\langle \theta, \vartheta \rangle \geq R^{-1}} \Omega(\theta) e^{i\xi_d \log |\langle \theta, \vartheta \rangle|} u_+(\xi_d, R|\langle \theta, \vartheta \rangle|) d\sigma(\theta) \\ &\quad + \int_{\langle \theta, \vartheta \rangle \leq -R^{-1}} \Omega(\theta) e^{i\xi_d \log |\langle \theta, \vartheta \rangle|} u_-(\xi_d, R|\langle \theta, \vartheta \rangle|) d\sigma(\theta) \end{aligned}$$

where

$$(2.5) \quad u_{\pm}(\gamma, N) = \int_1^N \exp(-i(\pm s + \gamma \log s)) \frac{ds}{s}.$$

We show that u is uniformly bounded in γ and $N \geq 1$.

Assume first that $|\gamma| > 1/2$. Then we split the integral (2.5) into three parts depending on whether $|\gamma| \geq 5s$ or $s < |\gamma|/5$ or $|\gamma|/5 < s < 5|\gamma|$. The integral over $s \in [|\gamma|/5, 5|\gamma|]$ is trivially bounded.

If $N > 5|\gamma|$, then we integrate by parts to get

$$\begin{aligned} \int_{5|\gamma|}^N e^{-i(\pm s + \gamma \log s)} \frac{ds}{s} &= \int_{5|\gamma|}^N \frac{d(e^{i(\mp s + \gamma \log s)})}{\mp i s - i \gamma} \\ &= i \left(\frac{e^{-i(\pm N + \gamma \log N)}}{\gamma \mp N} - \frac{e^{-i(\pm 5\gamma + \gamma \log 5\gamma)}}{\gamma \mp 5|\gamma|} \right) \\ &\mp i \int_{5|\gamma|}^N e^{-i(\pm s + \gamma \log s)} \frac{ds}{(\gamma \pm s)^2} \end{aligned}$$

and this is bounded (since $|\gamma| \geq 1/2$).

We treat the integral $\int_1^{|\gamma|^{1/5}} e^{-i(\pm s + \gamma \log s)} \frac{ds}{s}$ similarly. If $|\gamma| < 1/2$ and $N \geq 1$, then

$$(2.6) \quad \int_1^N e^{-i(\pm s + \gamma \log s)} \frac{ds}{s} = \pm i (e^{\mp i N} N^{-i\gamma-1} - e^{\mp i}) \pm (i\gamma + 1) \int_1^N e^{\mp i s} s^{-i\gamma-2} ds$$

which is bounded. This shows that $|\mathcal{E}_2^R(\vartheta, \xi_d)| = O(1)$, uniformly in R .

Finally to estimate $\mathcal{E}_3^R(\vartheta, \xi_d)$ we observe that

$$\begin{aligned} \mathcal{E}_3^R(\vartheta, \xi_d) &= - \int_{|\langle \theta, \vartheta \rangle| \geq 1/R} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{dr}{r} d\sigma(\theta) \\ &= -\mathcal{E}_{3,1}^R(\vartheta, \xi_d) + \mathcal{E}_{3,2}^R(\vartheta, \xi_d) \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}_{3,1}^R(\vartheta, \xi_d) &= \int_{S^{d-2}} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{dr}{r} d\sigma(\theta), \\ \mathcal{E}_{3,2}^R(\vartheta, \xi_d) &= \int_{|\langle \theta, \vartheta \rangle| \leq 1/R} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{dr}{r} d\sigma(\theta). \end{aligned}$$

Now

$$\mathcal{E}_{3,1}^R(\vartheta, \xi_d) = - \int_{S^{d-2}} \Omega(\theta) \frac{R^{-i\xi_d} - |\langle \theta, \vartheta \rangle|^{i\xi_d}}{-i\xi_d} d\sigma(\theta) = - \int_{S^{d-2}} \Omega(\theta) \frac{1 - |\langle \theta, \vartheta \rangle|^{i\xi_d}}{-i\xi_d} d\sigma(\theta)$$

where we have used the cancellation of Ω again. We see that

$$\begin{aligned} |\mathcal{E}_{3,1}^R(\vartheta, \xi_d)| &\leq \int_{S^{d-2}} |\Omega(\theta)| \frac{|e^{-i\xi_d \log |\langle \theta, \vartheta \rangle|} - 1|}{|\xi_d|} d\sigma(\theta) \\ &\leq \int_{S^{d-2}} |\Omega(\theta)| \log |\langle \theta, \vartheta \rangle|^{-1} d\sigma(\theta) \end{aligned}$$

and the last integral is bounded uniformly in ϑ because of our assumption $\Omega \in L^q$. Moreover by a straightforward estimate

$$\begin{aligned} \mathcal{E}_{3,2}^R(\vartheta, \xi_d) &\leq \int_{|\langle \theta, \vartheta \rangle| \leq 1/R} |\Omega(\theta)| [\log R + \log |\langle \theta, \vartheta \rangle|^{-1}] d\sigma(\theta) \\ &\leq 2 \int_{S^{d-2}} |\Omega(\theta)| \log |\langle \theta, \vartheta \rangle|^{-1} d\sigma(\theta). \end{aligned}$$

We have shown that M_R is bounded uniformly in (ϑ, ξ_d) . An examination of the above argument also shows that if $|\xi_d| \leq J$ and $J \geq 1$, then for $J \leq R \leq R'$

$$|M_R(\vartheta, \xi_d) - M_{R'}(\vartheta, \xi_d)| \leq C_J \left[\int_{|\langle \theta, \vartheta \rangle| \leq 10JR^{-1}} |\Omega(\theta)|(1 + \log |\langle \theta, \vartheta \rangle|^{-1}) d\sigma(\theta) + \int_{|\langle \theta, \vartheta \rangle| \geq R^{-1}} |\Omega(\theta)|(R|\langle \theta, \vartheta \rangle|)^{-1} d\sigma(\theta) \right]$$

which is $O(R^{-1+1/q}(1 + \log R))$. Therefore $\lim_{R \rightarrow \infty} M_{R|\xi'|}(\xi'/|\xi'|, \xi_d)$ exists and the convergence is uniform with respect to (ξ', ξ_d) in compact subsets of $(\mathbb{R}^{d-1} \setminus \{0\}) \times \mathbb{R}$. Since each M_R is easily seen to be a smooth function on $S^{d-1} \times \mathbb{R}$ we have proved

Proposition 2.1. *Suppose that $\Gamma(t) = \log t$, $\Omega \in L^q(S^{d-2})$, $q > 1$, and that (1.2) holds. Then T is bounded on $L^2(\mathbb{R}^d)$ and the Fourier transform of its convolution kernel is given by*

$$m(\xi) = e^{i\xi_d \log(|\xi'|)} M(\xi'/|\xi'|, \xi_d) f$$

where M is a bounded continuous function on $S^{d-2} \times \mathbb{R}$.

Remark 2.2. If Ω is odd, then T is L^2 bounded if (1.2) holds and Ω is merely in $L^1(S^{d-2})$. To see this one uses the method of rotations (see [1]). Define

$$H_\theta f(x) = \text{p.v.} \int f(x' - t\theta, x_d - \log|t|) \frac{dt}{t};$$

then one can see by transferring our result in two dimensions to d dimensions that H_θ is bounded on $L^2(\mathbb{R}^d)$ with operator norm independent of θ . If Ω is odd, then $T = c \int_{S^{d-2}} \Omega(\theta) H_\theta d\sigma(\theta)$ and the L^2 boundedness of T follows. For general Ω satisfying (1.2) the assumption $\Omega \in L \log L(S^{d-2})$ yields L^2 boundedness of T .

Remark 2.3. For $\alpha \neq 0$ let $m_\alpha(\tau) = |\tau|^{i\alpha}$ and $k_\alpha = \mathcal{F}^{-1}[m_\alpha]$, then k_α is a standard singular integral kernel on \mathbb{R}^{d-1} (although not homogeneous of degree $1 - d$). For $f \in C_0^\infty(\mathbb{R}^d)$ define

$$\mathcal{H}_\alpha f(x) = \int f(x' - t, x_d - \log|t|) k_\alpha(t) dt.$$

Then \mathcal{H}_α is unbounded on $L^2(\mathbb{R}^d)$. To see this observe that the associated multiplier

$$c_\alpha \int_{\mathbb{R}^{d-1}} e^{-i(\langle \xi', x' \rangle) + (\xi_d + \alpha) \log|x'|} |x'|^{1-d} dx'$$

is unbounded as $\xi_d \rightarrow -\alpha$.

For later use we shall now show that for $\xi_d \neq 0$ the function M is actually differentiable as a function of ξ_d ; in particular we shall show that

$$(2.7) \quad \left| \xi_d \frac{\partial M(\vartheta, \xi_d)}{\partial \xi_d} \right| \leq C \quad \text{if } 0 < |\xi_d| \leq 1/2.$$

The proof of (2.7) follows the lines above. Differentiation with respect to ξ_d gives another factor of $-i \log r$ in the formulas (2.4). In the estimation of $\mathcal{E}_1^R(\vartheta, \xi_d)$ this yields an additional factor of $\log |\langle \theta, \vartheta \rangle|^{-1}$ which is harmless in view of our assumption $\Omega \in L^q(S^{d-2})$. In the estimation of $\mathcal{E}_2^R(\vartheta, \xi_d)$ we shall only need to consider the term corresponding to (2.6) since we assume that $|\xi_d| \leq 1/2$, and we get boundedness of the derivative (again the calculation yields an additional factor

of $\log |\langle \theta, \vartheta \rangle|^{-1}$). The term corresponding to $\mathcal{E}_3^R(\vartheta, \xi_d)$ has to be handled with some care; it is a difference of $\tilde{\mathcal{E}}_{3,2}^R(\vartheta, \xi_d)$ and $\tilde{\mathcal{E}}_{3,1}^R(\vartheta, \xi_d)$ given by

$$\begin{aligned} \tilde{\mathcal{E}}_{3,1}^R(\vartheta, \xi_d) &= -i \int_{S^{d-2}} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{\log r}{r} dr d\sigma(\theta), \\ \tilde{\mathcal{E}}_{3,2}^R(\vartheta, \xi_d) &= -i \int_{|\langle \theta, \vartheta \rangle| \leq 1/R} \Omega(\theta) \int_{r=|\langle \theta, \vartheta \rangle|^{-1}}^R e^{-i\xi_d \log r} \frac{\log r}{r} dr d\sigma(\theta). \end{aligned}$$

Now for $\xi_d \neq 0$

$$\int_{r=a}^R e^{-i\xi_d \log r} \frac{\log r}{r} dr = i\xi_d^{-1} R^{-i\xi_d} (\log R - i\xi_d^{-1}) - i\xi_d^{-1} a^{-i\xi_d} (\log a - i\xi_d^{-1}).$$

Using this for $a = |\langle \theta, \vartheta \rangle|^{-1}$ we may copy the argument for $\mathcal{E}_{3,1}^R(\vartheta, \xi_d)$, $\mathcal{E}_{3,2}^R(\vartheta, \xi_d)$ above, producing an additional factor of ξ_d^{-1} . Moreover the limiting argument above can be carried over as long as we stay away from $\xi_d = 0$. This yields (2.7).

3.1. The model multiplier in two dimensions. We now give a proof of Proposition 1.2. Clearly $h \in M_2$ since h is bounded. Let $1 < p < 2$ and assume that χ is not identically zero. We argue by contradiction and assume that $h \in M^p$. Our proof is related to an argument by Littman, McCarthy and Rivière [9].

We may choose an interval $I = (\alpha_0, \alpha_1)$ so that $\chi(\eta) \neq 0$ if η belongs to the closure of I . Let $\Phi \in \mathcal{S}(\mathbb{R})$ so that the Fourier transform $\widehat{\Phi}$ is compactly supported in I but does not identically vanish. Let β be a C^∞ function so that β is supported in $\{\tau : |\tau| \leq 1\}$, $\beta(\tau) = 1$ if $|\tau| \leq 1/2$.

Let

$$g_N(\tau, \eta) = \sum_{k=10}^N \frac{\widehat{\Phi}(\eta)}{\chi(\eta)} \beta(\tau - e^{2^k}) e^{-i\eta(2^k - \log \tau)}.$$

Then it is easy to see by the sharp form of the Marcinkiewicz multiplier theorem ([13, p. 109]) that

$$\|g_N\|_{M^p} \leq C_p \text{ for } 1 < p < \infty.$$

Let

$$h_N(\tau, \eta) = \sum_{k=10}^N \widehat{\Phi}(\eta) \beta(\tau - e^{2^k}) e^{-i\eta 2^k},$$

then $h_N = g_N h$ and therefore

$$\|h_N\|_{M^p} \leq C_p \|h\|_{M^p}.$$

However we shall show that

$$(3.1) \quad \|h_N\|_{M^p} \geq cN^{1/p-1/2}$$

so h cannot be in M^p .

Define f_N by

$$\widehat{f}_N(\tau, \eta) = \sum_{k=10}^N \beta(\tau - e^{2^k}) \widehat{\Psi}(\eta)$$

where $\widehat{\Psi}$ is compactly supported but equals 1 on the support of $\widehat{\Phi}$, so $\Phi = \Phi * \Psi$.

Then by Littlewood-Paley theory

$$\|f_N\|_p \approx \left\| \left(\sum_{k=10}^N |\mathcal{F}^{-1}[\beta]|^2 \right)^{1/2} \right\|_p \approx N^{1/2}.$$

But

$$\mathcal{F}^{-1}[h_N \widehat{f_N}](x) = \sum_{k=10}^N \mathcal{F}^{-1}[\beta^2](x_1) e^{ix_1 e^{2^k}} \Phi(x_2 - 2^k)$$

and since $\Phi \neq 0$ is a Schwartz function it is easy to see that

$$\|\mathcal{F}^{-1}[h_N \widehat{f_N}]\|_p \geq cN^{1/p}.$$

This yields (3.1) and therefore the desired contradiction. The above argument also proves the corresponding statement for the multiplier h_+ and then also for h_- .

3.2. Failure of L^p -boundedness in Theorem 1.1. We now show that if $\Gamma(t) = \log t$ and if T is bounded on $L^p(\mathbb{R}^d)$, then $p = 2$, assuming that Ω is not identically 0. By the Riesz-Thorin theorem we may assume that $1 < p < \infty$. Let χ_+ be the characteristic function of $(0, \infty)$. If m is the corresponding multiplier, then we know by de Leeuw’s theorem [7] that for almost all $\vartheta \in S^{d-2}$ the function $(\tau, \eta) \rightarrow \chi_+(\tau)m(\tau\vartheta, \eta)$ is a Fourier multiplier on $L^p(\mathbb{R}^2)$.

Now $m(\tau\vartheta, \eta) = |\tau|^{i\eta} M(\vartheta, \eta)$ for $\tau > 0$, by Proposition 2.1. Let K_Ω be the kernel $\Omega(x'/|x'|)|x'|^{-d}$ on \mathbb{R}^{d-1} . Then its Fourier transform in \mathbb{R}^{d-1} is homogeneous of degree zero and equals $M(\xi'/|\xi'|, 0)$. The latter cannot be zero almost everywhere by uniqueness of Fourier transforms. Therefore there is $\vartheta \in S^{d-2}$ such that $m(\tau\vartheta, \eta)$ is a Fourier multiplier on $L^p(\mathbb{R}^2)$ and such that $M(\vartheta, 0) \neq 0$. Since M is continuous in η there is $0 < \epsilon < 1/2$ and $c > 0$ so that $|M(\vartheta, \eta)| \geq c$ for $\epsilon/2 \leq \eta \leq \epsilon$. Let χ be a C^∞ function supported in $(\epsilon/2, \epsilon)$, not identically zero.

From (2.7) we see that $\eta \mapsto \chi(\eta)[M(\vartheta, \eta)]^{-1}$ is a Fourier multiplier on L^p , with bounds uniform in ϑ . Therefore $\chi(\eta)\chi_+(\tau)|\tau|^{i\eta}$ is a Fourier multiplier on $L^p(\mathbb{R}^2)$ and by Proposition 1.2 this implies that $p = 2$.

4. EXAMPLES FOR SPECIFIC L^p SPACES

In this section we give a proof of Theorem 1.3. For each p_0 , with $1 < p_0 \leq 2$, we construct an even function $\Gamma \in C^\infty(\mathbb{R})$ such that $\Gamma(0) = 0$ and $\Gamma(t) = 0$ for $t \geq 1$, and such that the operator T as in (1.1) is bounded on $L^p(\mathbb{R}^d)$ if and only if $p_0 \leq p \leq p'_0$ or $\Omega = 0$ a.e.

Let $\zeta \in C^\infty(\mathbb{R})$ so that $\zeta(t) = 1$ if $t > 1/4$ and $\zeta(t) = 0$ if $t < -1/4$. Let $\delta = \{\delta_n\}$ be a sequence of positive numbers, so that $|\delta_n| \leq 1$ and $\lim_{n \rightarrow \infty} \delta_n = 0$.

Let $\{\gamma_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\gamma_{n+1} \leq \gamma_n/10$ for all $n \geq 1$. Our function Γ is then defined by

$$(4.1) \quad \Gamma(t) = \sum_{n=1}^\infty \gamma_n \zeta(2^{n^2+n} \delta_n^{-1} (|t| - 2^{-n^2} (1 - \delta_n))) \zeta(2^{n^2+n} \delta_n^{-1} (2^{-n^2} (1 + \delta_n) - |t|)).$$

Then for $n \geq 1$

$$\Gamma(t) = \begin{cases} \gamma_n & \text{if } 2^{-n^2} (1 - \delta_n + \delta_n 2^{-n-2}) \leq |t| \leq 2^{-n^2} (1 + \delta_n - \delta_n 2^{-n-2}), \\ 0 & \text{if } 2^{-(n+1)^2} (1 + \delta_{n+1} + \delta_{n+1} 2^{-n-3}) \leq |t| \leq 2^{-n^2} (1 - \delta_n - \delta_n 2^{-n-2}) \end{cases}$$

and $\Gamma(t) = 0$ for $|t| \geq 2$.

Theorem 4.1. *Let Γ be as in (4.1), T and Ω as in §1, $1 < p < \infty$ and let $s(p) = |1/p - 1/2|^{-1}$. Then T is bounded on L^p if and only if $\delta \in \ell^{s(p)}$ or $\Omega = 0$ almost everywhere.*

Theorem 1.3 is an immediate consequence, except for the fact that the even function Γ may not be smooth at the origin. This however can be achieved by an appropriate choice of γ_n , for example, $\gamma_n \leq \gamma_{n-1} \exp(-2^n \delta_n^{-1})$ for all $n \geq 2$.

Proof of Theorem 4.1. Let $I_n = [2^{-n^2}(1 - \delta_n), 2^{-n^2}(1 + \delta_n)]$ and

$$T_n f(x) = \int_{|y'| \in I_n} f(x' - y', x_d - \gamma_n) \frac{\Omega(y')}{|y'|^{d-1}} dy'.$$

It is easy to see that $T = \sum_{n=1}^\infty T_n + \mathcal{H} + \sum_{n=1}^\infty K_n$ where the L^p operator norm of K_n is $O(2^{-n})$, for $1 \leq p \leq \infty$ and where \mathcal{H} is the extension to $L^p(\mathbb{R}^d)$ of a variant of a Calderón-Zygmund operator acting in the x' variables; the L^p boundedness for $1 < p < \infty$ follows from [1]. It therefore suffices to examine the operator $\sum_n T_n$.

Let L_k denote the standard Littlewood-Paley operator on \mathbb{R}^{d-1} , i.e.,

$$\widehat{L_k f}(\xi) = \phi(2^{-k}|\xi'|)\hat{f}(\xi)$$

where ϕ is a C_0^∞ function supported on $\frac{1}{2} \leq t \leq 2$ such that $\sum_{k=-\infty}^\infty \phi(2^{-k}|t|) = 1$ for $t \neq 0$.

Then for some $\epsilon > 0$, depending on $p > 1$ and $q > 1$

$$(4.2) \quad \|L_{k+n} T_n\|_{L^p} \leq A \min\{2^{-\epsilon|k|}, \delta_n\};$$

see e.g. [6].

Define $\Delta_n = \sum_{j=n^2-n+1}^{n^2+n} L_j$, $\tilde{\Delta}_n = \sum_{j=n^2-n-1}^{n^2+n+2} L_j$, so that $\Delta_n \tilde{\Delta}_n = \Delta_n$. Observe by (4.2) that

$$\sum_{n=1}^\infty \|T_n - T_n \Delta_n\|_{L^p \rightarrow L^p} < \infty$$

for all $p \in (1, \infty)$. The L^p boundedness of T , under the assumption $\delta \in \ell^s$, follows by a well known argument using Littlewood-Paley theory (see [12] and [5]). For convenience we include the short proof. Without loss of generality assume $1 < p \leq 2$. By Littlewood-Paley theory (or Calderón-Zygmund theory for vector-valued singular integrals [13, ch. II]) the inequality $\|\{\Delta_n f\}\|_{L^p(\ell^2)} \leq C\|f\|_p$ holds for all $p \in (1, \infty)$, similarly the corresponding inequality involving $\tilde{\Delta}_n$. Since the L^p operator norm of T_n is $O(\delta_n)$ we see that

$$\begin{aligned} \left\| \sum_n \tilde{\Delta}_n T_n \Delta_n f \right\|_p &\leq C_p \|\{T_n \Delta_n f\}\|_{L^p(\ell^2)} \leq C_p \|\{T_n \Delta_n f\}\|_{L^p(\ell^p)} \\ &= C_p \|\{T_n \Delta_n f\}\|_{\ell^p(L^p)} \leq C_p \left(\sum_n \|T_n\|_{L^p \rightarrow L^p}^p \|\Delta_n f\|_p^p \right)^{1/p} \\ &\leq C'_p \|\delta\|_{\ell^s} \|\{\Delta_n f\}\|_{\ell^2(L^p)} \leq C'_p \|\delta\|_{\ell^s} \|\{\Delta_n f\}\|_{L^p(\ell^2)} \\ &\leq C''_p \|\delta\|_{\ell^s} \|f\|_p. \end{aligned}$$

We now turn to the proof of the converse. We fix $p \in (1, 2)$ and assume that T is bounded on L^p and that Ω does not vanish on a set of positive measure; we then have to prove that $\delta \in \ell^s$, $s = s(p)$.

Let

$$m_n(\xi') = \int_{|y'| \in I_n} e^{i\langle \xi', y' \rangle} \Omega(y'/|y'|) |y'|^{1-d} dy'.$$

Since by (4.1) the operator $\sum_n T_n$ is bounded on L^p ,

$$m(\xi', \xi_d) = \sum_n e^{i\xi_d \gamma_n} m_n(\xi')$$

is a bounded multiplier on $L^p(\mathbb{R}^d)$. Since we assume that Ω does not vanish on some set of positive measure, it follows that there is an open set U on which the Fourier transform $\widehat{\Omega d\sigma}$ does not vanish, in fact we may assume that $|\widehat{\Omega d\sigma}(\xi)| \geq A > 0$ for $\xi \in U$. By de Leeuw's theorem [6] there is $\Xi \in U$ so that

$$u(\tau, \eta) = \sum_n e^{i\eta \gamma_n} m_n(\tau \Xi)$$

is a multiplier in $M^p(\mathbb{R}^2)$.

Since we assume that $\lim_{n \rightarrow \infty} \delta_n = 0$ we can choose a positive integer K so that the closed ball of radius δ_ℓ and center Ξ is contained in U for all $\ell \geq K$. Let $\beta \in C^\infty(\mathbb{R})$ with β supported in $[1/2, 2]$ so that $\beta(t) = 1$ in a neighborhood of 1. By the Marcinkiewicz multiplier theorem $\sum_{\ell=K}^N \beta(\tau - 2^{\ell^2})$ is in $M^r(\mathbb{R})$ for every $r, 1 < r < \infty$, uniformly in N (here and in what follows we assume that $N \geq K$). Therefore the norms in $M^p(\mathbb{R}^2)$ of the multipliers $\sum_{\ell=K}^N \sum_n e^{i\eta \gamma_n} m_n(\tau \Xi) \beta(\tau - 2^{\ell^2})$ are uniformly bounded.

It follows from (4.2) that the $M_r(\mathbb{R}^2)$ norm of $m_n(\tau \Xi) \beta(\tau - 2^{\ell^2})$ is $O(2^{-\epsilon|\ell^2 - n^2|})$, where $\epsilon = \epsilon(r, q) > 0$ if $r > 1, q > 1$. Therefore $\sum_{\ell=K}^N \sum_{n \neq \ell} e^{i\eta \gamma_n} m_n(\tau \Xi) \beta(\tau - 2^{\ell^2})$ is a Fourier multiplier of $L^r(\mathbb{R}^2)$ for all $r \in (1, \infty)$ with bound uniformly in N . Consequently, by our assumption

$$v_N(\tau, \eta) = \sum_{\ell=K}^N e^{i\eta \gamma_\ell} m_\ell(\tau \Xi) \beta(\tau - 2^{\ell^2})$$

is a Fourier multiplier of $L^p(\mathbb{R}^2)$.

Now let

$$A_\ell = \int_{1-\delta_\ell}^{1+\delta_\ell} \int_{S^{d-2}} \Omega(\theta) e^{ir\langle \Xi, \theta \rangle} d\theta r^{-1} dr,$$

$$b_\ell(\tau) = \int_{1-\delta_\ell}^{1+\delta_\ell} \int_{S^{d-2}} \Omega(\theta) [e^{ir2^{-\ell^2} \tau \langle \Xi, \theta \rangle} - e^{ir\langle \Xi, \theta \rangle}] d\theta r^{-1} dr,$$

then

$$v_N(\tau, \eta) = \sum_{\ell=1}^N e^{i\eta \gamma_\ell} (A_\ell + b_\ell(\tau)).$$

Observe that for $\ell \geq K$

$$(4.3) \quad |A_\ell| \geq A \log \left(\frac{1 + \delta_\ell}{1 - \delta_\ell} \right) \geq A \delta_\ell.$$

Moreover $\beta(\cdot - 2^{\ell^2}) b_\ell$ is a Fourier multiplier of $L^1(\mathbb{R})$, with bound independent of ℓ . The L^∞ norm of this function is $O(2^{-\ell^2})$ and therefore by interpolation the

multiplier $\sum_{\ell=K}^N \beta(\cdot - 2^{\ell^2})b_\ell$ belongs to $M_r(\mathbb{R})$ for $r \in (1, \infty)$, with norm bounded in N . We conclude that

$$w_N(\tau, \eta) = \sum_{\ell=K}^N e^{i\eta\gamma_\ell} \beta(\tau - 2^{\ell^2})A_\ell$$

belongs to $M^p(\mathbb{R}^2)$ with norm independent of N .

Let ψ be a nonnegative smooth function not identically zero, with support in $[-1/2, 1/2]$ and let $\psi_N(y) = \gamma_{N+1}^{-1/p} \psi(\gamma_{N+1}^{-1}y)$.

Now let $\alpha = \{\alpha_\ell\}$ be a sequence in $\ell^{2/p}$, so that $\|\alpha\|_{\ell^{2/p}} \leq 1$. Note that $2/p = (s/p)'$. We test w_N on f_N with

$$\widehat{f}_N(\tau, \eta) = \sum_{\ell=K}^N |\alpha_\ell|^{1/p} \beta(\tau - 2^{\ell^2})\widehat{\psi}_N(\eta);$$

then by Littlewood-Paley theory

$$\|f_N\|_{L^p} \leq C \left\| \left(\sum_{\ell=K}^N |\alpha_\ell|^{2/p} |\mathcal{F}^{-1}[\beta]^2 \right)^{1/2} \right\|_{L^p} \leq C'$$

where C' is independent of N . On the other hand, for $(x, y) \in \mathbb{R}^2$,

$$\mathcal{F}^{-1}[w_N \widehat{f}_N](x, y) = \sum_{\ell=K}^N A_\ell |\alpha_\ell|^{1/p} \mathcal{F}^{-1}[\beta^2](x) e^{i2^{\ell^2}x} \psi_N(y - \gamma_\ell).$$

Since $\gamma_{N+1} \leq \gamma_\ell/10$, $\ell = K, \dots, N$, the supports of the functions $\psi_N(y - \gamma_\ell)$ are disjoint. Therefore

$$\left(\sum_{\ell=K}^N |A_\ell|^p |\alpha_\ell| \right)^{1/p} \leq C \|\mathcal{F}^{-1}[w_N \widehat{f}_N]\|_p \leq C \|w_N\|_{M^p} \|f_N\|_p \leq C'$$

uniformly in N . This implies by (4.3) that

$$\sup_{\|\alpha\|_{\ell^{(s/p)'}} \leq 1} \sum_{\ell=K}^\infty |\delta_\ell|^p |\alpha_\ell| < \infty.$$

By the converse of Hölder's inequality it follows that $\{\delta_n^p\} \in \ell^{s/p}$ and therefore $\delta \in \ell^s$. □

5. APPENDIX: ODD EXTENSIONS OF CONVEX CURVES IN THE PLANE

Here we include some observations concerning *odd* curves $(t, \gamma(t))$ where γ is convex in $(0, \infty)$. Our examples are modifications of those in [3] and [4]. For $r > 0$, $\epsilon \geq 0$, and $j \geq 1$ set $\alpha_{\epsilon, j} = \tau 4^{-j} j^{\epsilon-1}$ for a small τ to be chosen later and

$$(5.1) \quad \gamma_{r, \epsilon}(t) = (2j)^r 4^j + ((2j + 2)^r + \alpha_{\epsilon, j})(t - 4^j) \text{ for } 4^j \leq t \leq 4^j(1 + j^{-\epsilon}).$$

For $4^j(1 + j^\epsilon) \leq t \leq 4^{j+1}$, extend $\gamma_{r, \epsilon}$ so $\gamma_{r, \epsilon}''(t)$ is constant in this interval, $\gamma_{r, \epsilon}'$ is continuous at $4^j(1 + j^{-\epsilon})$ and $\gamma_{r, \epsilon}(t)$ is continuous for $t \geq 4$. Similarly extend $\gamma_{r, \epsilon}$ to $[0, 4]$ with constant positive curvature so that $\gamma_{r, \epsilon}(0) = 0$. A calculation shows that $\gamma_{r, \epsilon}$ is convex for $t > 0$. Finally extend $\gamma_{r, \epsilon}$ as an odd function. The

perturbation by $\alpha_{\epsilon,j}$ in (5.1) is convenient in order that arguments in [4] to study maximal functions should apply to singular integral operators. We consider

$$H_{r,\epsilon}f(x, y) = \text{p.v.} \int f(x - t, y - \gamma_{r,\epsilon}(t))t^{-1}dt.$$

Proposition 5.1.

- (i) For any $\epsilon \geq 0$ and $r > 0$, $\|H_{r,\epsilon}f\|_{L^2} \leq A\|f\|_{L^2}$.
- (ii) If $p_0 = \frac{2\epsilon+2}{2\epsilon+1}$, then for any $r > 0$, $\|H_{r,\epsilon}f\|_{L^p} \leq A_p\|f\|_{L^p}$ for $p_0 < p < p'_0$.
- (iii) If $r = 1$ and $\frac{4}{3} \leq p < 2$, $H_{r,\epsilon}$ is unbounded on L^p if $\epsilon < \frac{1}{p} - \frac{1}{2}$.
- (iv) If $r = 1$ and $p \leq \frac{4}{3}$, $H_{r,\epsilon}$ is unbounded on L^p if $\epsilon \leq \frac{3}{p} - 2$.
- (v) If r is a positive integer, then $H_{r,\epsilon}$ is unbounded on L^p if $p < \frac{r+2}{r+1+\epsilon}$.

Remarks. Consider the maximal function $\sup_{h>0} h^{-1} \int_0^h |f(x - t, y - \gamma_{r,\epsilon}(t))|dt$. Then the operator M is unbounded on L^p if $p < \frac{r+2}{r+1+\epsilon}$. This is a slight improvement over a result in [4]. More generally if $r = \frac{m}{n}$ with m and n positive integers, then one can show that M is unbounded if $p < \frac{m+2}{m+1+n\epsilon}$. One achieves this by restricting the values of j in the argument below to be n th powers. Obviously many questions remain open.

Proof of Proposition 5.1. Clearly (i) follows from [10] since $h_{r,\epsilon}(t) = t\gamma'_{r,\epsilon}(t) - \gamma_{r,\epsilon}(t)$ is doubling (see also [3], [16] for a more geometric proof of this result). In particular note that if $I_{j,\epsilon} = [4^j, 4^j(1+j^{-\epsilon})]$, then $\gamma_{r,\epsilon}(t) = s_j t - h_j$ where $s_j = (2j+2)^r + \alpha_{\epsilon,j}$ and $h_j = 4^j[(2j+2)^r - (2j)^r + \alpha_{\epsilon,j}]$.

Now set $\mathcal{I}_{j,\epsilon} = \{t : |t| \in I_{j,\epsilon}\}$ and let

$$G_j f(x, y) = \int_{\mathcal{I}_{j,\epsilon}} f(x - t, y - \gamma_{r,\epsilon}(t))t^{-1}dt.$$

Then $H_{r,\epsilon} = \sum_{j=1}^\infty G_j + E$. In view of the curvature properties of $\gamma_{r,\epsilon}$ in the complement of $\bigcup_j \mathcal{I}_{j,\epsilon}$ (where h is “infinitesimally doubling”) the method of [3] may be applied to yield the L^q boundedness of E for all $q \in (1, \infty)$.

For the remaining assertions of the proposition it suffices to consider $G = \sum_j G_j$. To prove (ii) we consider the analytic family $G_z = \sum_j j^z G_j$. If $\text{Re}(z) < -1$, G_z is clearly bounded in L^1 . (ii) follows by analytic interpolation if we can show that G_z is bounded in L^2 for $\text{Re}(z) < \epsilon$. This however follows by Fourier transform estimates following [11] or [16]. One derives the estimate

$$|\widehat{G_j}(\xi)| \leq C_1 \min\{j^{-\epsilon}; 4^j|\xi_1 + \xi_2(s_j - 4^{-j}h_j)|\} + C_2|\xi_2|4^{-j}h_j; 4^{-j}|\xi_1 + \xi_2s_j|^{-1}\}.$$

The first estimate is obvious, the second estimate uses the oddness of γ and the estimate $|\sin a| \leq |a|$ and the third uses an integration by parts. It is now straightforward to bound the sum $\sum_{j=1}^\infty |j^z \widehat{G_j}(\xi)|$ provided that $\text{Re}(z) < \epsilon$.

To obtain conclusion (v) we follow Christ [4]. We test G on the characteristic function f_N of a union of small rectangles $R_{(a,b)}$ centered at lattice points (a, b) with $0 \leq a \leq 2^N$ and $0 \leq b \leq N^r 2^N$,

$$R_{a,b} = \{(x, y) : a - N^{-r-1}\sigma \leq x \leq a + N^{-r-1}\sigma, b - N^{-1}\sigma \leq y \leq b + N^{-1}\sigma\}.$$

Here σ is small (to be chosen). We let for each pair of positive integers ℓ and j

$$S^{\ell,j} = \{(x, y) : 0 \leq x \leq 2^N, 0 \leq y \leq N^r 2^N, |y - (2j+2)^r x - \ell| \leq N^{-1}\sigma\}.$$

Then $|S^{\ell,j}| \geq \sigma 2^N (2N)^{-1}$ if $j \leq N/4$ and $\ell \leq N^r 2^N / 10$, moreover if $j' \neq j$, $|S^{\ell,j} \cap S^{\ell',j'}| \leq A \sigma^2 N^{-2r-2} |j^{-r} - (j')^{-r}|^{-1} \leq A' \sigma^2 N^{-2} |j^r - (j')^r|^{-1}$.

Fixing ℓ, j , and j' , the number of strips $S^{\ell',j'}$ that intersect $S^{\ell,j}$ is at most $2^N |j^r - (j')^r|$. Since there are at most N values of j' , the measure of the union of all strips intersecting a given $S^{\ell,j}$ is at most $A \sigma |S^{\ell,j}|$, with A an absolute constant not depending on σ . We are going to restrict j to $N/5 \leq j \leq N/4$. We estimate Gf_N for points (x, y) in $S^{\ell,j}$ such that (x, y) is in no $S^{\ell',j'}$ with $j' \neq j$ and such that the vertical distance from (x, y) to the top of $S^{\ell,j}$ is between $10^{-5} \tau / N$ and $10^{-6} \tau / N$. If we first choose σ sufficiently small and then $\tau = \sigma / 100$, we will be estimating Gf_N on a positive fraction of $S^{\ell,j}$. In evaluating Gf_N at such points (x, y) the contribution to Gf_N from pieces of $\gamma_{r,\epsilon}$ with slopes other than $(2j+2)^r$ is zero. The contribution Gf_N at such points comes from two strips

$$S^{\ell+(2j)^r 2^{2j},j} \quad \text{and} \quad S^{\ell-(2j)^r 2^{2j},j}.$$

The contribution from $S^{\ell-(2j)^r 2^{2j},j}$ is at least $10^{-2} j^{-\epsilon} N^{-r-1}$. The absolute value of the contribution from $S^{\ell+(2j)^r 2^{2j},j}$ is at most $10^{-3} j^{-\epsilon} N^{-r-1}$. Thus if G is bounded in L^p ,

$$N^{-(r+1)p} j^{-p\epsilon} \left| \bigcup_{\ell,j} S^{\ell,j} \right| \leq A |\text{supp}(f_N)|.$$

Therefore $N^{-(r+1)p} j^{-p\epsilon} N N^r 2^N (2^N / N) \leq A N^r 2^{2N} N^{-r-2}$ which implies for $N \rightarrow \infty$ the necessary condition $p \geq \frac{r+2}{\epsilon+r+1}$.

Note that (iv) is a special case of (v). Finally (iii) follows along the same lines as in §7 of [3]. Let

$$\begin{aligned} b_\eta(k) &= \int_{4^k}^{4^{k(1+k^{-\epsilon})}} \sin\{\eta[\alpha_{\epsilon,k}(t-4^k) - 4^{k+1}]\} \frac{dt}{t} \\ &= -(\log 2) k^{-\epsilon} \sin(4^{k+1} \eta) + O(k^{-1}). \end{aligned}$$

It then suffices to show that the sequence b_η does not belong to $M^p(\mathbb{Z})$ (the class of Fourier multipliers for Fourier series in $L^p(\mathbb{T})$) uniformly for $\pi \leq \eta \leq 3\pi$. The error $O(k^{-1})$ represents the Fourier coefficients of an L^2 function and belongs to $M_r(\mathbb{Z})$ for all $r \in [1, \infty]$. Now the argument in [3] shows $b_\eta \notin M^p(\mathbb{Z})$ if $\{k^{-\epsilon-1/p'} \log^{-1} k\} \notin \ell^2(\mathbb{Z})$ which is true if $\epsilon < 1/p - 1/2$. \square

REFERENCES

1. A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309. MR **18**:894a
2. H. Carlsson, M. Christ, A. Córdoba, J. Duoandikoetxea, J.L. Rubio de Francia, J. Vance, S. Wainger and D. Weinberg, *L^p estimates for maximal functions and Hilbert transforms along flat convex curves in \mathbb{R}^2* , Bull. Amer. Math. Soc. **14** (1986), 263–267. MR **87f**:42044
3. A. Carbery, M. Christ, J. Vance, S. Wainger and D. Watson, *Operators associated to flat plane curves: L^p estimates via dilation methods*, Duke Math. J. **59** (1989), 675–700. MR **91m**:42017
4. M. Christ, *Examples of singular maximal functions unbounded on L^p* , Publicacions Matemàtiques **35** (1991), 269–279. MR **92f**:42024
5. M. Cowling, G. Fendler and J. Fournier, *Variants of Littlewood-Paley theory*, Math. Ann. **285** (1989), 333–342. MR **90m**:42016
6. J. Duoandikoetxea and J.-L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561. MR **87f**:42046

7. M. Jodeit, *A note on Fourier multipliers*, Proc. Amer. Math. Soc. **27** (1971), 423–424. MR **42**:4965
8. W. Kim, S. Wainger, J. Wright and S. Ziesler, *Singular integrals and maximal functions associated to surfaces of revolution*, Bull. London Math. Soc. **28** (1996), 291–296. MR **97b**:42029
9. W. Littman, C. McCarthy and N. Rivière, *L^p multiplier theorems*, Studia Math. **30** (1968), 193–217. MR **37**:6681
10. A. Nagel, J. Vance, S. Wainger and D. Weinberg, *Hilbert transforms for convex curves*, Duke Math. J. **50** (1983), 735–744. MR **85a**:42025
11. A. Nagel, J. Vance, S. Wainger and D. Weinberg, *Maximal functions for convex curves*, Duke Math. J. **52** (1985), 715–722. MR **87k**:42017
12. A. Seeger, *Some inequalities for singular convolution operators in L^p -spaces*, Trans. Amer. Math. Soc. **308** (1988), 259–272. MR **89j**:42015
13. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970. MR **44**:7280
14. E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295. MR **80k**:42023
15. J. Vance, S. Wainger and J. Wright, *The Hilbert transform and maximal function along non-convex curves in the plane*, Revista Math. Iberoamericana **10** (1994), 93–120. MR **95a**:42025
16. S. Ziesler, *L^2 boundedness of maximal functions associated to convex curves in \mathbb{R}^n* , London Math. Soc. **51** (1995), 331–341. MR **96a**:42018

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: `seeger@math.wisc.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706
E-mail address: `wainger@math.wisc.edu`

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY 2052, AUSTRALIA
E-mail address: `jimw@maths.unsw.edu.au`

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND
Current address: Department of Mathematics, Dominican University, River Forest, Illinois 60305
E-mail address: `ziessara@email.dom.edu`