REPRESENTATIONS OVER PID'S WITH THREE DISTINGUISHED SUBMODULES

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ABSTRACT. Let R be a principal ideal domain. The R-representations with one distinguished submodule are classified by a theorem of Gauß in the case of finite rank, and by the "Stacked Bases Theorem" of Cohen and Gluck in the case of infinite rank. Results of Hill and Megibben carry this classification even further. The R-representations with two distinguished pure submodules have recently been classified by Arnold and Dugas in the finite-rank case, and by the authors for countable rank. Although wild representation type prevails for R-representations with three distinguished pure submodules, an extensive category of such objects was recently classified by Arnold and Dugas. We carry their groundbreaking work further, simplifying the proofs of their main results and applying new machinery to study the structure of finite- and infinite-rank representations with two, three, and four distinguished submodules. We also apply these results to the classification of Butler groups, a class of torsion-free abelian groups that has been the focus of many investigations over the last sixteen years.

1. INTRODUCTION

The abundance of torsion-free abelian groups [18, 19, 20, 29] indicates that it is hopeless to classify them; any sort of classification of torsion-free groups must inevitably be carried out under rather severe restrictions.

Well-known examples are limitations to finitely generated groups or to groups of rank 1, that is to say to subgroups of $\mathbb{Q}$, see [29]. Classification theorems follow the usual strategy. First we find a result showing that the groups of the class under consideration can be decomposed into direct sums of indecomposables. This is possible for many rings R and R-modules of finite rank.

Assuming that the "building blocks"–indecomposable modules–determine the module uniquely (which is often the case), it then remains to classify the indecomposable modules, which is possible if there are not too many of them. A good candidate for such objects are Butler groups with small typesets. Fortunately these torsion-free groups of finite rank can be seen as images under a functor, called the Butler functor, taking representations $\mathbf{F} = (F, F_0, F_1, ..., F_{n-1}) \in Rep_n R$ of free

R-modules F over principal ideal domains R with n distinguished submodules F_i to certain torsion-free abelian groups, see [4, 5, 8, 11, 13, 14, 15, 16]. The crucial point is that this Butler functor preserves direct sums. Hence the indicated decomposition problem of Butler groups can be transported to $Rep_n R$, which is studied independently.

Earlier results [5, 8, 12, 23, 33, 34, 48] and the discussion in [33], for instance, show that classification in $Rep_n R$ is possible only for $n \leq 4$. If R is a field and $n \leq 4$ this is now well-known, see [48]. However if R is a PID and not a field new complications arise and classification may be hopeless even for $n = 2$, witness [23]. The key observation is the fact that classification strongly depends on the kinds of embeddings of the F_i's in F. It helps substantially if the F_i's are pure submodules of F for $\mathbf{F} = (F, F_0, F_1) \in Rep_2 R$, a condition which trivially holds for fields. However, we can still find many R_2-modules $\mathbf{F}$ (elements in $Rep_2 R$) of this sort, of rank $\aleph_0$, with $F_0 \cap F_1 = 0$ and $End\ \mathbf{F} = \mathbb{Z}$; see [28] and [23]. Such an abundance of indecomposable R_2-modules of infinite rank shows that a classification theorem needs further restrictions. In [28], when we impose that $F/F_0 \oplus F_1$ is a bounded R-module, we are able to write R_2-modules of at most countable rank as direct sums of (a list of) indecomposable R_2-modules of rank ≤ 2.

In order to deal with R_3-modules, we observe that their structure is even more complex, and classification needs further restrictions.

We say that $\mathbf{F} = (F, F_0, F_1, ..., F_{n-1}) \in Rep_n R$ is a *complemented representation* if F is a free R-module, all the F_i's are pure in F, $F_0 + F_1 = F$, and $F_i \cap F_j = 0$ for all $i \neq j$. Of course, we say that an object $\mathbf{F} \in Rep_n R$ has finite rank if the free R-module F has finite rank.

Later in the paper (Example 4.8), we will note the existence of a complemented representation $\mathbf{F} = (F, F_0, F_1, F_2)$ of countably infinite rank which is not a direct sum of finite rank R_3-modules, but which is such that all factors $F/F_i \oplus F_j$ $(i \neq j)$ are nevertheless direct sums of cyclic p-modules, one for each prime p in R. In order to avoid the obvious difficulties encountered in this example, we will require that $\mathbf{F}$ has finite rank. Now we are successful in obtaining the desired decomposition result (Theorem 2.6):

Let $\mathbf{F} = (F, F_0, F_1, F_2) \in Rep_3 R$ be a complemented representation of finite rank. Then $\mathbf{F}$ is a direct sum of complemented R_3-modules of rank ≤ 2.

The indecomposable complemented R_3-modules are listed as Remark 2.7.

From this, our main classification result concerning $Rep_3 R$, we derive a number of results about representations with two, three and four submodules, as well as certain Butler groups. The following corollary is due to Arnold and Dugas [5]; an alternate proof is given in [28].

Corollary 4.6. *Suppose $\mathbf{F} = (F, F_0, F_1) \in Rep_2 R$ has finite rank and F_0, F_1 are pure in F. Then $\mathbf{F}$ is a direct sum of R_2-modules of ranks 1 and 2.*

We also observe that our decomposition theorem enables one to turn decomposable R_2-modules into indecomposable, complemented R_4-modules:

Corollary 4.7. *Suppose $\mathbf{F} = (F, F_0, F_1) \in Rep_2 R$ has finite rank and F_0, F_1 are pure in F. If $\mathbf{F}$ has no rank 1 summands, then there exist pure submodules $F_2, F_3 \subseteq$*

F such that (F, F_0, F_2, F_1, F_3) is a complemented, indecomposable representation in Rep_4R.

Additional consequences of the main decomposition theorem will be given in the sections to come, in both the global and local cases (Sections 2 and 3, respectively). We remark that the existence of indecomposable, complemented R_3-modules of *infinite* rank was, for a while, a matter of considerable doubt. These objects indeed exist, and we detail their construction in Section 4. The heart of the matter is Proposition 4.3, an old result in abelian group theory due to Reid [47].

In Section 5, we apply our results to Butler groups. Despite the fact that Butler groups are a special class of torsion-free abelian groups, they turn out to be as complex as the totality of torsion-free abelian groups [21, 30, 31]. Every ring with free additive group is realizable as the endomorphism ring of many Butler groups from a proper class, for instance; see [21]. Hence we will place a number of restrictions on the Butler groups we study, most notably restrictions on their ranks and typesets. (Recall that a group of finite rank is a Butler group if it is a pure subgroup of a completely decomposable torsion-free group of finite rank; equivalent conditions are discussed in the forthcoming book [44].) Butler groups with a critical typeset of the form T_n (an antichain of length n with a bottom element) for $n = 2$ are classified in [43] as direct sums of indecomposable T_2-Butler groups of rank ≤ 2. This also follows from Rep_2R-considerations in [5, 28]. We will bring our machinery in Rep_nR for $n = 2, 3$ and 4 to bear on T_n-groups in the final section, obtaining group-theoretical results about Butler groups of various torsion-free ranks, both finite and infinite.

2. The Global Case of the Main Theorem

Throughout this section R will denote any principal ideal domain that is not a field. If M is an R-module, $p \in R$ and $e \in M \setminus pM$, we say e is *p-pure* in M. If e is p-pure for all primes p of R, or equivalently for every non-unit p of R, then e is said to be *pure* in R. We will constantly use the fact that pure elements e of free R-modules M are basic, i.e. $M = eR \oplus M'$.

We will work with special objects of Rep_nR described by the following:

Definition 2.1. A representation $\mathbf{F} = (F, F_i : i < n)$ with F free is a *complemented representation* if $F_0 + F_1 = F$, F_i is pure in F for all i, and $F_i \cap F_j = 0$ for all $i \neq j$.

Lemma 2.2. *Suppose* $\mathbf{F} = (F, F_0, F_1, F_2) \in Rep_3R$ *is such that* $F = F_0 \oplus F_1$. *Let* $\pi_i : F \longrightarrow F_i$ *for* $i = 0, 1$ *be the canonical projections and assume*
$F_2\pi_0 = \bigoplus_{i<\lambda} Rt_ia_i$, *where* $F_0 = \bigoplus_{i<\lambda} Ra_i$ *and* $t_i = t_i' \cdot t_0$ *in* R *for all* $i < \lambda$.
Choose $c_0 \in F_1$ *with* $d_0 = t_0a_0 + c_0 \in F_2$, *and write* $c_0 = s_0e$ *for some pure element* e *of* F_1. *If* $F_2\pi_1 \subseteq s_0F_1$, *then* $(Ra_0 \oplus Re, Ra_0, Re, Rd_0)$ *is a summand of* $\mathbf{F}$.

Proof. By hypothesis we have

$$F/(F_1 \oplus F_2) = (F_1 \oplus F_0)/(F_1 \oplus F_2) \cong F_0/F_1\pi_0$$
$$= \bigoplus_{i<\lambda} Ra_i / \bigoplus_{i<\lambda} Rt_ia_i \cong \bigoplus_{i<\lambda}(R/t_iR).$$

Hence $t_i a_i \in F_1 \oplus F_2$ and there are unique elements $c_i \in F_1, d_i \in F_2$ with $t_i a_i = d_i - c_i$. Clearly $\bigoplus_{i<\lambda} Rd_i = \bigoplus_{i<\lambda} R(t_i a_i + c_i) \subseteq F_2$, and it is easy to check

$$(*) \qquad\qquad F_2 = \bigoplus_{i<\lambda} Rd_i \ \text{ with } \ d_i = t_i a_i + c_i \qquad (i < \lambda).$$

Since $c_0 = s_0 e$ and e is pure in the free module F_1, we get a decomposition $F_1 = Re \oplus F_1'$; let $\pi : F_1 \longrightarrow Re$ be the canonical projection. Denote $c_i \pi = s_i e$ for all $0 \neq i < \lambda$. Since $F_2 \pi_1 \subseteq s_0 F_1$, $d_i \pi_1 \pi = s_i e \in s_0 F_1$, and e is pure in F_1, s_0 must divide all s_i for $i < \lambda$. Put $d_0' = d_0$; $s_i' = \frac{s_i}{s_0}$ and $d_i' = d_i - s_i' d_0$ for $i > 0$. Then clearly $F_2 = Rd_0' \oplus F_2'$, where $F_2' = \bigoplus_{i>0} Rd_i'$ and $F_2' \pi = 0$. We also have $F_0 = Ra_0 \oplus F_0'$, where $F_0' = \bigoplus_{0 \neq i < \lambda} Rt_i a_i'$ with $a_i' = a_i - s_i' t_i' a_0$. Clearly $F_2' \pi_0 \subseteq F_0'$, and it follows that $\mathbf{F} = (Ra_0 \oplus Re, Ra_0, Re, Rd_0) \oplus (F_0' \oplus F_1', F_0', F_1', F_2')$, as desired. $\qquad\square$

Note that although a sum $e_1 + e_2$ of two pure elements e_1, e_2 need not be pure, $p_1 e_1 + p_2 e_2$ is at least p_i-pure for distinct primes p_i $(i = 1, 2)$. Except for trivial cases it will not be pure; take $e_1 = -e_2$, for instance. The following trick to arrange primes was used in Cohen and Gluck's classic paper [17], and generalizes the above remark. If $\{s_0, \dots, s_{n-1}\}$ is a set of elements in R, let $(s_i : i \in n)$ denote its g.c.d., which is unique up to units in R.

Lemma 2.3. *Let $\mathcal{P}$ be a finite set of primes of R, and $e_0, \dots, e_{n-1}$ be p-pure elements of an R-module M for all $p \in \mathcal{P}$. Assume that $s_0, \dots, s_{n-1} \in R$ are given such that $(s_i : i \in n) = 1$. Then we can find $t_0, \dots, t_{n-1} \in R$ such that $\sum_{i \in n} t_i s_i e_i$ is p-pure in M for all $p \in \mathcal{P}$, and $(t_i s_i : i \in n) = 1$ as well.*

Proof. Enumerate $\mathcal{P} = \{p_0, \dots, p_{m-1}\}$ without repetition for some $m \geq 1$. For each $i \in m$, we may choose $\varphi(i) \in n$ such that p_i does not divide $s_{\varphi(i)}$ because $(s_i : i \in n) = 1$. This determines a function $\varphi : m \longrightarrow n$. For $i \in n$, put $t_i = \prod\{p_k \in \mathcal{P} : \varphi(k) \neq i\}$.

Suppose p divides $t_i s_i$ for all $i \in n$. Since $(s_i : i \in n) = 1$, there is an $i \in n$ with $p|t_i$, hence p has a name, say $p = p_k$ for some $k \in m$. If $\varphi(k) = j$, then clearly $p_k \nmid t_j$ by definition of t_j, hence $p_k | s_j$ since $p_k | t_j s_j$. This is impossible because $p_k \nmid s_{\varphi(k)} = s_j$ by definition of φ. The condition $(t_i s_i : i \in n) = 1$ is therefore met.

Now suppose $p_k \in \mathcal{P}$ and p_k divides $\sum_{i \in n} t_i s_i e_i$. If $\varphi(k) = j$, then p_k also divides t_i for all $i \neq j$, hence p_k divides $\sum_{i \in n} t_i s_i e_i - \sum_{j \neq i \in n} t_i s_i e_i = t_j s_j e_j$. But since e_j is p_k-pure, p_k divides $t_j s_j$, and we obtain $p_k | (s_i t_i : i \in n) = 1$, a contradiction. $\qquad\square$

Next we consider a fixed divisibility chain $t_{n-1}|t_{n-2}|\dots|t_0$ in R. We abbreviate divisibility by $n-1|n-2|\dots|0$, and the corresponding quotients $\frac{t_i}{t_k}$ in R by i/k for $i < k$. The divisibility chain gives rise to a class of $(n-1) \times n$ matrices with entry $i/k \cdot *$ in the row labeled by k and the column labeled by $i < k$ ($* = $ any element of R), and an arbitrary entry $*$ for $i \geq k$. Note the the rows are labeled by $1, \dots, n-1$, and the columns by $0, \dots, n-1$. The matrices have the following

form, where we suppress the $*$ in $i/k \cdot *$.

$$(+) \qquad \begin{pmatrix} 0/1 & * & * & * & * \\ 0/2 & 1/2 & * & * & * \\ 0/3 & 1/3 & 2/3 & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0/n-1 & 1/n-1 & 2/n-1 & 3/n-1 & * \end{pmatrix}$$

We also fix $j \in n$ and have a *forbidden* j-th column (the column labeled by j). Let Π be any product of $n-1$ factors which are entries from exactly $n-1$ rows and $n-1$ columns, but not from the forbidden j-th column. We claim that $0/j$ divides Π in R.

The claim follows by an induction on $r < j$, as follows: If the columns and rows determined by the factors in Π indexed by (a, b) with $b \leq r$ are removed from $(+)$, then we claim $\Pi = 0/r' \cdot \Pi'$ where Π' is the product of all factors coming from rows labeled by (c, d) with $c > r'$, where r' is the index of the largest row we have removed.

Note that if $r = 0$ and we remove column 0 along with the appropriate row labeled by ℓ, the claim is ensured by the entry $0/\ell$.

In the induction step we remove column r and some row, say k, according to the entry (k, r) used in Π. If $k < r'$, the claim $\Pi = 0/r' \cdot \Pi'$ holds trivially. The case $k = r'$ is impossible, because row r' was removed earlier. If $k > r'$, we must show that $\Pi = 0/k \cdot \Pi''$. Clearly the shrunken matrix has only $*$'s on the diagonal above k, hence the corresponding rows have done their duty and just wait to be removed.

By hypothesis $\Pi = 0/r' \cdot \Pi'$ and Π' has a factor r/k from the new entry (k, r) used for Π. Since r rows are removed, we have $r' \geq r$, hence $r'|r$ from our divisibility chain, and $(0/r') \cdot (r/k) = (0/k)(r/r')$ is a product in R. Now it is clear that $\Pi = (0/k) \cdot \Pi''$ with Π'' a product that still has all factors of Π coming from entries of rows after k. The induction is complete.

If $k \geq j$, the claim will follow, and this must happen after at most $j - 1$ steps, which saves us from case distinction due to the forbidden column j.

Corollary 2.4. *If $t_{n-1}|t_{n-2}|\ldots|t_0$ is a given divisibility chain in R and if $|\Omega_j|$ denotes the determinant of the $(n-1) \times (n-1)$ matrix $(+)$ above after removing its j-th column for some $j < n$, then $\frac{t_0}{t_j}$ divides $|\Omega_j|$ in R.*

Proof. Recall that the determinant $|\Omega_j|$ is a sum of all products of the factors considered in the discussion above. Each of them is divisible by $\frac{t_0}{t_j}$, hence $|\Omega_j|$ is, and the corollary follows. $\square$

Remark. We will need only a special case of Corollary 2.4, where entries labeled by (k, j) of the matrices $(+)$ are 0 for all $k > j$. In this case the corollary can also be deduced by means of row-by-row Laplace developments, beginning with the top row of $(+)$.

Corollary 2.4 is used to obtain

Proposition 2.5. *Let $t_{n-1}|t_{n-2}|\ldots|t_0$ be a given divisibility chain in R, and $d_{0k} \in R$ ($k \in n$) be given such that $(d_{0k}\frac{t_0}{t_k} : k \in n) = 1$. Then it is possible to choose additional elements $d_{ij} \in R$ ($1 \leq i < n, j \in n$) such that the $n \times n$ matrix*

$\Delta = (s_{ij})_{i,j \in n}$ *defined by*

$$s_{ij} = \begin{cases} d_{ij} & if \quad i \le j; \\ d_{ij}\frac{t_j}{t_i} & if \quad i > j \end{cases}$$

has determinant $|\Delta| = 1$.

Proof. Let Δ_{ij} denote the minors of Δ obtained by removing the row of Δ indexed by i and the column indexed by j. By induction on n, we may assume $d_{ij} \in R$ $(i,j \in n-1)$ have been chosen so that $|\Delta_{n-1,n-1}| = (d_{0k}\frac{t_0}{t_k} : k \in n-1)$. [Note that we also could assume $d_{ij} = 0$ for $i < j$ in accordance with the remark after Corollary 2.4.] Then since $|\Delta_{n-1,n-1}|$ and $d_{0,n-1}\frac{t_0}{t_{n-1}}$ are coprime in R, we may choose $d,t \in R$ with

$$1 = d \cdot |\Delta_{n-1,n-1}| + td_{0,n-1}\frac{t_0}{t_{n-1}}.$$

If we put $d_{i,n-1} = 0$ for $0 < i < n$, and $d_{n-1,n-1} = (-1)^{n-1}d$, then up to sign we have

$$|\Delta| = d \cdot |\Delta_{n-1,n-1}| + d_{0,n-1}|\Delta_{0,n-1}|$$

by Laplace applied to the last column. If we can choose $d_{n-1,j} \in R$ $(j \in n-1)$ so that $|\Delta_{0,n-1}| = t\frac{t_0}{t_{n-1}}$, then $|\Delta|$ is ± 1 and the proof is complete. Let Ω_j be the matrix obtained from Δ by deleting the rows of Δ indexed by 0 and $n-1$ (the first and last rows), together with the columns of Δ indexed by j and $n-1$, where $j \in n-1$. Then Ω_j has the form described in Corollary 2.4, hence $|\Omega_j| = \frac{t_0}{t_j} \cdot r_j$, where $r_j \in R$. We now carry out Laplace expansion across the top row of $\Delta_{n-1,n-1}$ to see that

$$|\Delta_{n-1,n-1}| = \sum_{j \in n-1}(-1)^j d_{0j}|\Delta_j| = \sum_{j \in n-1}(-1)^j d_{0j}\frac{t_0}{t_j}r_j = (d_{0k}\frac{t_0}{t_k} : k \in n).$$

Therefore $(r_j : j \in n-1) = 1$. At the same time, we see by expanding about the bottom row of $\Delta_{0,n-1}$ that, up to sign,

$$|\Delta_{0,n-1}| = \sum_{j \in n-1}(-1)^j d_{n-1,j}\frac{t_j}{t_{n-1}}|\Delta_j| = \frac{t_0}{t_{n-1}}\sum_{j \in n-1}(-1)^j d_{n-1,j}r_j.$$

Then because $(r_j : j \in n-1) = 1$, the elements $d_{n-1,j}$ $(j \in n-1)$ can be chosen so that the last expression in brackets is ± 1, as desired. This completes the proof. $\square$

Theorem 2.6. *Let* $\mathbf{F} = (F, F_0, F_1, F_2) \in Rep_3 R$ *be a (nonzero) complemented representation of finite rank. Then* $\mathbf{F}$ *is a direct sum of complemented* R_3-*modules of ranks 1 and 2.*

Proof. Since F is finite rank free, F_i is also free of finite rank. If F_2 has rank $rkF_2 < rkF_0$, then we can write $F_2 = \bigoplus_{i \in n} R(a_i + b_i)$ with $a_i \in F_0$ and $b_i \in F_1$, where $\langle a_i : i \in n \rangle$ is contained in a direct summand F_0' of F_0 of rank $\le rkF_2$. Then $F_0 = F_0' \oplus C$ and $\mathbf{F} = (C, C, 0, 0) \oplus (F_0' \oplus F_1, F_0', F_1, F_2)$. Since $(C, C, 0, 0) \ne 0$ decomposes into rank 1 summands and $rkF_2 = rkF_0'$, we may assume $rkF_2 \ge rkF_0$ (and by symmetry $rkF_2 \ge rkF_1$) for our original R_3-module $\mathbf{F}$. Then since $F_2 \oplus F_i \subseteq F_0 \oplus F_1$, also $rkF_2 \le rkF_i$ for $i = 0, 1$, hence $rkF_0 = rkF_1 = rkF_2 = n$ for some n.

We now show that our (adjusted) representation $\mathbf{F}$ can be brought into the special form required by Lemma 2.2. To that end, let $\pi_i : F \longrightarrow F_i$ $(i = 0, 1)$ be

the projections relative to $F = F_0 \oplus F_1$. Clearly $F_0 \supseteq F_2\pi_0$ is finitely generated, and Gauss' theorem applies. We get stacked bases for F_0 and this submodule, namely

(1)
$$\begin{cases} F_0 = \bigoplus_{i \in n} Ra_i \\ \text{There is a divisibility chain } t_{n-1}|t_{n-2}|\ldots|t_0 \text{ such that} \\ F_2\pi_0 = \bigoplus_{i \in n} Rt_ia_i \end{cases}$$

and similarly for $F_1 \supset F_2\pi_1$ we obtain

(2)
$$\begin{cases} F_1 = \bigoplus_{i \in n} Rb_i \\ \text{There is a divisibility chain } s_0|s_1|\ldots|s_{n-1} \text{ such that} \\ F_2\pi_1 = \bigoplus_{i \in n} Rs_ib_i. \end{cases}$$

Note that none of the s_i, t_i are 0, and divisibility is expressed in opposite orders in (1) and (2). Since $s_0|s_i$ for all $i \in n$ we have $F_2\pi_1 \subseteq s_0F_1$, a condition required in Lemma 2.2.

As in the proof of Lemma 2.2 we may choose unique $c_i \in F_1$ such that $d_i = t_ia_i + c_i$ and $F_2 = \bigoplus_{i \in n} Rd_i$. Write $c_i = g_ie_i$ with $g_i \in R$ and c_i pure in F_1 for all $i < n$. Because $F_2\pi_1$ has rank n, the sum

$$F_2\pi_1 = \bigoplus_{i \in n} Rg_ie_i = \bigoplus_{i \in n} Rs_ib_i \subseteq s_0F_1$$

is indeed direct. Since e_i is pure in F_1, s_0 must divide $(g_i : i \in n)$, and since b_0 is pure in F_1, the converse holds as well. Therefore $s_0 = (g_i : i \in n)$. Clearly, d_0 is pure in F_2, which is pure in F, hence d_0 is pure in F as well. The element $d_0 = t_0a_0 + g_0e_0$ must satisfy $(t_0, g_0) = 1$. We deduce $\left(\frac{g_i \, t_0}{s_0 \, t_i} : i \in n\right) = 1$.

If $\mathcal{P}$ denotes all primes dividing g_n, then by Lemma 2.3, there are elements $d_{0i} \in R$ $(i \in n)$ such that

(3)
$$e = \sum_{i \in n} d_{0i}\left(\frac{g_i \, t_0}{s_0 \, t_i}\right)e_i$$

is p-pure in F_1 for all $p \in \mathcal{P}$, and such that

(4)
$$\left(d_{0i}\frac{g_i \, t_0}{s_0 \, t_i} : i < n\right) = 1.$$

It follows that $s_0e = \sum_{i \in n} d_{0i}\frac{t_0}{t_i}c_i$ is pure in $\bigoplus_{i \in n} Rc_i$, hence s_0e is p-pure in F_1 for all primes $p \notin \mathcal{P}$ because $F_1/F_2\pi_2$ is p-torsion-free for these primes by (2). Therefore, e is pure in F_1. Because of (4) we may apply Proposition 2.5 and find an $n \times n$ matrix Δ, with $|\Delta| = 1$, of the form $\Delta = (s_{ij})$ with $s_{ij} = d_{ij}\frac{t_i}{t_j}$ for $i > j$, where d_{0i}, t_i are given above. Since $|\Delta| = 1$, Δ represents an automorphism of the free module $F_2 = \bigoplus_{i \in n} Rd_i$.

We denote $\Delta d_i = d'_i = \sum_{j \in n} s_{ij}d_j$ for $i \in n$. Using $d_i = t_ia_i + c_i$ we get $d'_i = t_ia'_i + c'_i$, where similarly we denote $a'_i = \sum_{j \in n} s_{ij}a_j$ and $c'_i = \sum_{j \in n} s_{ij}c_j$. In particular, $d'_0 = t_0a'_0 + s_0e$ because of (3). Clearly $a'_i \in F_0$, and $F_0 = \bigoplus_{i \in n} Ra_i = \bigoplus_{i \in n} Ra'_i$ since $|\Delta| = 1$. Note $F'_2 := \bigoplus_{i \in n} Rd'_i \subseteq F_2$. In order to show equality, we observe that $F_0/F_2\pi_0 \cong (F_0/F'_2\pi_0)/(F_2\pi_0/F'_2\pi_0)$, hence $F_2\pi_0 = F'_2\pi_0$ because $F_0/F_2\pi_0 \cong$

$\bigoplus_{i \in n} R/Rt_i \cong F_0/F_2'\pi_0$ is finitely generated. Now $F_0 = \bigoplus_{i \in n} Ra_i' \supset F_2 = \bigoplus_{i \in n} Rd_i'$ is of the form required by Lemma 2.2 to split off the rank 2 summand $(Ra_0' \oplus Re, Ra_0', Re, Rd_0')$ from $\mathbf{F}$. An induction on the rank of $\mathbf{F}$ now completes the proof of the theorem. $\square$

Remark 2.7. The nonzero indecomposable, complemented R-representations (F, F_0, F_1, F_2) of finite rank have the following forms:

(1) $(R, R, 0, 0)$ and $(R, 0, R, 0)$;

(2) $(Re \oplus Rf, Re, Rf, R(ae + bf))$, where a and b are nonzero and coprime in R.

3. The Local Case of the Main Theorem

In this section, we prove our main theorem in the much easier case where the base ring R is a discrete valuation domain. Let pR be the maximal ideal of R, and let $\mathbf{F} = (F, F_i : i < 3) \in Rep_3R$ be a nonzero, complemented representation of finite rank. We claim that $\mathbf{F}$ has a direct summand of rank 1 or 2. In what follows, $\pi_i : F = F_0 \oplus F_1 \to F_i$ $(i = 0, 1)$ denotes projection.

We first show that if $F/(F_1 + F_2)$ is not torsion, then $\mathbf{F}$ has a rank 1 direct summand. An easy check verifies $F/(F_1 + F_2) = (F_0 \oplus F_1)/(F_1 + F_2) \cong F_0/F_2\pi_0$. By assumption, the (finitely generated) quotient $F/(F_1 + F_2)$ has a direct summand isomorphic to R. Therefore we obtain a decomposition $F = F_0' \oplus F_0''$ with the first summand isomorphic to R and $F_2\pi_0 \subseteq F_0''$. Since clearly $F_2 \subseteq F_2\pi_0 \oplus F_1$, it follows that $\mathbf{F} = (F_0', F_0', 0, 0) \oplus (F_0'' \oplus F_1, F_0'', F_1, F_2)$, as desired.

Now we assume that $F_0/F_2\pi_0$ is torsion. Since F_2 is pure in F, it follows that $F_0 \neq 0$. By Gauss' theorem, we get stacked bases $F_0 = \bigoplus_{1 \leq i \leq m} Re_i$ and $F_2\pi_0 = \bigoplus_{1 \leq i \leq m} Rp^{n_i}e_i$, where $n_1 \geq \cdots \geq n_m \geq 0$. Since $F_2 \cap F_1 = 0$, there are unique $f_i \in F_2$ such that $f_i\pi_0 = p^{n_i}e_i$ for $i \leq i \leq m$. It is easy to check that the f_i's are independent and span F_2, whence $F_2 = \bigoplus_{1 \leq i \leq m} Rf_i$. Since F_2 is p-pure in F and $F_2 \cap F_0 = 0$, $f_1\pi_1$ is a nonzero, pure element of F_1 if $n_1 > 0$. If $n_1 = 0$, we may relabel to put ourselves in the situation where the p-height $h(f_1\pi_1)$ of the nonzero element $f_1\pi_1$ is *minimal* in the set $\{h(f_i\pi_1) : 1 \leq i \leq m\}$. Let Rg be the purification of $Rf_1\pi_1$ in F_1. Then $F_1 = Rg \oplus F_1'$; for each i, write r_ig $(r_i \in R)$ for the projection of $f_i\pi_1$ onto Rg. Then by construction $r_i/r_1 \in R$ for all i. Put $f_i' = f_i - (r_i/r_1)f_1$ and $e_i' = e_1 - p^{n_1-n_i}(r_i/r_1)e_i$ for $1 < i \leq m$. Clearly $F_0 = Re_0 \oplus F_0'$ and $F_2 = Rf_1 \oplus F_2'$, where $F_0' = \bigoplus_{1 < i \leq m} Re_i'$ and $F_2' = \bigoplus_{1 < i \leq m} Rf_i'$. Observe $F_2'\pi_0 = \bigoplus_{1 < i \leq m} Rp^{n_i}e_i' \subseteq F_0'$, and $F_2'\pi_1 \subseteq F_1'$ because we have arranged by our choice of bases that $F_2'\pi_1$ has trivial projection onto Rg. Thus $F_2' \subseteq F_0' \oplus F_1'$. Evidently $\mathbf{F} = (Re_0 \oplus Rg, Re_0, Rg, Rf_1) \oplus (F_0' \oplus F_1', F_0', F_1', F_2')$, as desired.

An induction on the rank of $\mathbf{F}$ now completes the proof of the decomposition theorem:

$\mathbf{F}$ *is a direct sum of (complemented) indecomposable representations of ranks 1 and 2.*

Remark 3.1. Up to isomorphism, the indecomposable, complemented representations $\neq 0$ in Rep_3R have the following forms:

(1) $(R, R, 0, 0)$ and $(R, 0, R, 0)$;

(2) $(Re \oplus Rf, Re, Rf, R(p^ne + p^mf))$, where $0 \in \{n, m\}$.

We conclude with some remarks about various subcategories of Rep_3R.

For $0 \leq i \leq j$, let $C_3(i,j)$ denote the category of finite-rank objects $(F, F_i : i < 3)$ subject to the following conditions:

(i) F_i is pure in F for $i < 3$;

(ii) $F_i \cap F_j = 0$ if $i \neq j$;

(iii) the torsion parts of the factors $F/(F_0 \oplus F_1)$ and $F/(F_1 \oplus F_2)$ are bounded by p^j; and

(iv) at least one of the factors in (iii) is bounded by p^i.

We are thankful to D. Arnold for discussions about the representation types of $C_3(i,j)$ for different values of the parameters. If $1 \leq i < j$, the category $C_3(i,j)$ contains indecomposables of arbitrarily large rank, hence has infinite representation type. The category $C_3(i,j)$ has so-called "endo-wild" representation type if $2 \leq i \leq j$. The latter notion is a generalization of wild representation type for representations of posets over fields, and indicates that a meaningful classification of all indecomposables in $C_3(i,j)$ is unfeasible when $2 \leq i \leq j$.

Note that after a permutation of the distinguished submodules, each representation in $C_3(0,j)$ is a finite-rank, complemented representation over R. Conversely, every finite-rank complemented representation in Rep_3R lies in $C_3(0,j)$ for some value of j. Remark 3.1 can be used to deduce that $C_3(0,j)$ contains exactly $3j + 4$ nonzero indecomposables, hence is of finite representation type.

Finally, it is relatively straightforward to verify that the following representation $(F, F_i : i < 3)$ of rank $2m$ is an indecomposable member of $C_3(1,j)$ if $j \geq 1$.

Let e_{ij} $(1 \leq i \leq m, j = 0, 1)$ be a set of generators for a free R-module of rank $2m$, and define:

$F_j = \bigoplus_{1 \leq i \leq m} Rpe_{ij}$ for $j = 0, 1$;

$F = F_0 \oplus F_1 + \bigoplus_{1 \leq i \leq m} R(e_{i-1,0} + e_{i1})$, where $e_{00} = 0$;

$F_2 = \bigoplus_{1 \leq i \leq m} Rp(e_{i0} + e_{i1})$.

4. Applications of the Main Theorem

In this section, we show how Theorem 2.6 can be applied to obtain new facts about modules with *two* distinguished submodules.

Let Σ_2R denote the full subcategory of Rep_2R consisting of all finite-rank representations (F, F_0, F_1) such that $F = F_0 + F_1$ (we make no assumption about the purity of the submodules here). We thank D. Arnold for pointing out the following category equivalence.

Proposition 4.1. *There is an additive equivalence from Σ_2R onto the category of all finite-rank, complemented representations in Rep_3R.*

We briefly indicate the (covariant) functors involved in the proof of this result. Define

$$f : \Sigma_2R \longrightarrow Rep_3R$$

by putting

$$f(\mathbf{F}) = (F_0 \times F_1, F_0, F_1, F_2),$$

where $\mathbf{F} = (F, F_0, F_1) \in \Sigma_2R$ and F_2 is the kernel of the homomorphism $(x, y) \mapsto x + y$ from the external direct sum $F_0 \times F_1$ onto F.

It is straightforward to verify that $f(\mathbf{F})$ is complemented, and that f commutes with direct sums in the category.

Conversely, if $\mathbf{U} =(U, U_0, U_1, U_2) \in Rep_3 R$ is complemented and of finite rank, then it is easy to verify that

$$g(\mathbf{U}) = (U/U_2, (U_0 \oplus U_2)/U_2, (U_1 \oplus U_2)/U_2))$$

lies in $\Sigma_2 R$. Morphisms $\mathbf{F} \longrightarrow \mathbf{F}'$ induce morphisms $f(\mathbf{F}) \longrightarrow f(\mathbf{F}')$ in a natural way, and similarly for the other functor g.

Observe that if $(F, F_0, F_1) \in \Sigma_2 R$, then

$$gf(F, F_0, F_1) = ((F_0 \times F_1)/F_2, (F_0 \oplus F_2)/F_2, (F_1 \oplus F_2)/F_2) \cong (F, F_0, F_1)$$

because the canonical isomorphism

$$(F_0 \times F_1)/F_2 \longrightarrow F \ ((x, y) + F_2 \mapsto x + y)$$

carries the submodule $(F_0 \oplus F_2)/F_2$ onto F_0 and $(F_1 \oplus F_2)/F_2$ onto F_1. Thus $gf = 1_{\Sigma_2 R}$. A similar check reveals that fg is the other identity.

In view of Remark 2.7 and the discussion above, we have the following classification of objects in $\Sigma_2 R$:

Corollary 4.2. *Nonzero objects in $\Sigma_2 R$ are direct sums of indecomposables of rank ≤ 2. Up to isomorphism, the indecomposable objects $\neq 0$ in $\Sigma_2 R$ have the following forms: $(R, R, 0), (R, 0, R)$ and (Re, Rae, Rbe), where a and b are nonzero and coprime in R.*

It seems rather hard to visualize what kinds of techniques in $Rep_2 R$ itself (opposed to $Rep_3 R$) would be suitable to obtain the decompositions of objects $(F_0 + F_1, F_0, F_1)$ guaranteed by Corollary 4.2. On the other hand, there seem to be no methods available in $Rep_3 R$ to produce even a single specimen of an indecomposable, complemented representation of *infinite* rank.

Concerning this last problem, we remark that it is easy to verify (and we will do so below) that if $\mathbf{F} = (F, F_i : i < 3)$ is complemented and F/F_2 is indecomposable, then $\mathbf{F}$ is indecomposable. The problem here lies in arranging somehow that $F_0 \cap F_2 = 0$ and $F_1 \cap F_2 = 0$. Let us concentrate on the case $R = \mathbb{Z}$. Working in the spirit of Corollary 4.2 and using the functors f and g described above, we are able to formulate the problem at hand as one about abelian groups with two special subgroups:

Which (indecomposable) abelian groups can be written as sums of two *free* subgroups?

This fundamental question was answered some time ago by Reid [47]. We supply our own proof.

Proposition 4.3. *(i)   If G is a torsion-free abelian group of finite torsion-free rank which is a sum of two free subgroups, then G is finitely generated and free.*

(ii)   If G is an abelian group of infinite torsion-free rank not less than its cardinality (in particular, any torsion-free group of infinite rank), then G is the sum of two free subgroups.

Proof. (i) If $G = F_0 + F_1$ for some free subgroups F_i, then F_i must have finite rank and consequently must be finitely generated by assumption on the rank of G. Hence G is finitely generated and $G = E \oplus F$ for some finite group E and some finitely generated free group F. Now it is clear that $E = 0$ and (i) follows.

(ii) Let $\kappa \geq \omega$ be the torsion-free rank of G, and let $F_0 = \bigoplus_{i<\kappa} \mathbb{Z}x_i$ be a maximal free subgroup of G. Then we may enumerate the nonzero elements of G/F_0 as $\{g_j + F_0 : j < \kappa\}$. For each j, there is a least positive integer n_j with $n_j g_j \in F_0$; let $[n_j g_j]$ denote its support with respect to the given basis for F_0. Observe that for each $i < \kappa$, the set $\bigcup_{j\leq i}[n_j g_j]$ has cardinality $< \kappa$. Hence we can define (by recursion) a function $\tau : \kappa \longrightarrow \kappa$ such that for each $i < \kappa$,

$$\bigcup_{j\leq i}[n_j g_j] \cap \{\tau(j) : j \leq i\} = \emptyset.$$

Put $F_1 = \sum_{i<\kappa} \mathbb{Z}(g_i + x_{\tau(i)}) \subset G$. Clearly, $G = F_0 + F_1$. Moreover, it follows by our definition of τ that $F_1 = \bigoplus_{i<\kappa} \mathbb{Z}(g_i + x_{\tau(i)})$, hence F_1 is free. $\qquad\square$

Corollary 4.4. *Let G be an indecomposable, torsion-free group of infinite rank κ. Let F_i $(i = 0, 1)$ be free subgroups of G whose sum is G, and let F_2 be the kernel of the map*

$$F_0 \times F_1 \longrightarrow F \ ((x, y) \mapsto x + y).$$

Then $\mathbf{F} = (F_0 \times F_1, F_0, F_1, F_2) \in Rep_3\mathbb{Z}$ is an indecomposable, complemented representation of rank κ. If G has endomorphism ring $\mathbb{Z}$, then so does $\mathbf{F}$.

Proof. Assume $G = F_0 + F_1$ is indecomposable and torsion-free. Note $F/F_2 \cong G$ is torsion-free, hence F_2 is pure in F. It is easy to check $F_2 \cap F_i = 0$ for $i = 0, 1$. Hence $\mathbf{F}$ is a complemented representation of rank κ. Suppose $\sigma^2 = \sigma \in End(\mathbf{F})$. Then σ induces an idempotent in $\mathrm{End}(F/F_2)$, with F/F_2 indecomposable, hence by replacing σ by $1 - \sigma$ if necessary we may assume that σ induces the zero map on F/F_2, that is, $F\sigma \subseteq F_2$. Then $(F_i)\sigma \subseteq F_i \cap F_2 = 0$ for $i = 0, 1$, whence $\sigma = 0$ and $\mathbf{F}$ is indecomposable. If in addition G has endomorphism ring $\mathbb{Z}$, then given any endomorphism σ of $\mathbf{F}$ we obtain an integer z so that $\sigma - z$ is zero on F/F_2, and proceed in the same way to conclude $\sigma - z = 0$, whence $End\ \mathbf{F} \subseteq \mathbb{Z}$. $\qquad\square$

Of course, a similar strategy will work to produce indecomposables as in Corollary 4.4 over many domains besides $\mathbb{Z}$. We chose $\mathbb{Z}$ because Proposition 4.3 seems to be a noteworthy result for abelian groups, and because Corollary 4.4 will serve as a tool for constructing special Butler groups in the final section.

We turn now to a different application of Theorem 2.6. Here, we are interested in generating indecomposables in Rep_4R from objects in Rep_3R and Rep_2R. In the setting of representations $\mathbf{U} = (U, U_i : i < n)$ over a field K, Brenner [12] realizes the trivial algebra K as $End\ \mathbf{U}$ using four subspaces U_i $(i = 0, 1, 2, 3)$, and in fact provides a complete list of all such representations with dim U finite. An inspection of her list shows that when dim $U = 2m > 2$, U and its first three subspaces constitute a finite rank complemented representation in Rep_3K. Moreover, U_2 is the usual "diagonal" subspace of dimension m. The fourth, more "complicated" subspace U_3 has dimension $m+1$, hence $U_i \cap U_3$ must be nontrivial for all $i < 4$. Hence $\mathbf{U}$ is *not* a complemented representation of Rep_4K. In fact, the rigid representations of infinite rank given in Göbel, May [33] fail to be complemented for the same reason.

In the next section we will want to work only with complemented representations in order to apply our results to T_n-Butler groups, a setting in which each pair of distinguished submodules is required to have zero intersection. Brenner's results indicate that we cannot hope to transform a complemented representation

as in Theorem 2.6 into a rigid one by adjoining a fourth (pure) submodule which intersects the others trivially. However, there is enough leeway to produce an indecomposable representation in this fashion. The following corollary of Theorem 2.6 will be used in Section 5 to construct Butler groups in a very special way.

Corollary 4.5. *Let R be a PID with infinitely many primes. Assume that $\mathbf{F} = (F, F_i : i < 3) \in Rep_3R$ is a complemented representation of finite rank which has no direct summand of rank 1. Then there is a pure submodule $F_3 \subseteq F$ (there are in fact infinitely many) such that the following holds:*
$\mathbf{F}' = (F, F_i : i < 4)$ *is a complemented, indecomposable R_4-module.*

Proof. Since $\mathbf{F}$ has no cyclic summands, Theorem 2.6 and Remark 2.7 provide us with the following explicit bases for the first three submodules:

$$F_0 = \bigoplus_{i=1}^{n} Re_i \; , \;\; F_1 = \bigoplus_{i=1}^{n} Rf_i \text{ and } F_2 = \bigoplus_{i=1}^{n} R(a_ie_i + b_if_i),$$

where $a_i, b_i \in R\backslash\{0\}$ and $(a_i, b_i) = 1$ for all i. Denote $a_ie_i + b_if_i$ by d_i. Since R has infinitely many primes, we may choose a prime $p \in R$ which does not divide any of the coefficients a_i and b_i. We define $F_3 \subseteq F$ by

$$F_3 = \bigoplus_{i=1}^{n-1} R(a_ie_i + p^{i+1}b_{i+1}f_{i+1}) \oplus R(a_ne_n + pb_1f_1)$$

and put $\mathbf{F}' = (F, F_i : i < 4) \in Rep_4R$.

Denote the indicated basis elements of F_3 by $g_1, ..., g_n$ respectively. Since $(a_i, pb_i) = 1$ for all i, it follows easily that F_3 is a pure submodule of F, and obviously $F_3 \cap F_i = 0$ for $i = 0, 1$.

Suppose $x = \sum_{i=1}^{n} r_id_i = \sum_{i=1}^{n} s_ig_i \in F_3 \cap F_2$ is a pure element, where $r_i, s_i \in R$. Upon writing these sums in terms of basis elements e_i and f_i, and equating coefficients, we obtain $r_1 = ps_n$ and $r_i = p^is_{i-1}$ for $i > 1$, hence $x \in p(F_3 \cap F_2)$ and $x = 0$ by purity. Therefore $F_3 \cap F_2 = 0$ as well, and $\mathbf{F}'$ is a complemented representation.

We now consider any $\sigma \in End \, \mathbf{F}'$. Since $F_i\sigma \subseteq F_i \; (i = 0, 1)$ we may write

$$\text{(i)} \qquad\qquad e_i\sigma = \sum_{k=1}^{n} r^i_ke_k \text{ and } f_i\sigma = \sum_{k=1}^{n} s^i_kf_k$$

for $k \leq i \leq n$, where $r^i_k, s^i_k \in R$. Since $F_2\sigma \subseteq F_2$, we may also write $d_i\sigma = a_i(e_i\sigma) + b_i(f_i\sigma) = \sum_{k=1}^{n} t^i_k(a_ke_k + b_kf_k)$ for all i, where $t^i_k \in R$. Inserting the expressions (i) and equating coefficients, we obtain

$$\text{(ii)} \qquad\qquad a_ir^i_k = a_kt^i_k \text{ and } b_is^i_k = b_kt^i_k \; (1 \leq i, k \leq n).$$

Set $k = i$ to see

$$\text{(iii)} \qquad\qquad\qquad r^i_i = s^i_i \; (1 \leq i \leq n).$$

At this point we assume that σ is an idempotent. If we can show that $\sigma = 0$ or $\sigma = 1$ then $\mathbf{F}'$ is indecomposable and (4.5) holds.

Using (i) and Fubini's theorem we compute

$$e_i\sigma = \sum_{k=1}^{n} r_k^i e_k = e_i\sigma^2 = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} r_k^j r_j^i\right)e_k$$

for all i, whence

(iv) $$r_j^i = \sum_{k=1}^{n} r_k^i r_j^k \ (1 \leq i, j \leq n).$$

Now we employ the fact that F_3 is invariant under σ. For each i there are elements $v_k^i \in R$ such that

(v) $$q_i\sigma = \sum_{k=1}^{n} v_k^i g_k.$$

Fix $i < n$ and write the g's in (v) in terms of the e's and f's, apply (i) to the left-hand side of (v) and equate coefficients to derive

(vi) $$\begin{cases} a_i r_k^i = v_k^i a_k \ (1 \leq k \leq n, 1 \leq i < n) \text{ and} \\ p^{i+1} b_{i+1} s_k^{i+1} = v_{k-1}^i p^k b_k \ (1 < k \leq n, 1 \leq i < n). \end{cases}$$

Similarly, we set $i = n$ in (v) to deduce

(vii) $$\begin{cases} a_n r_k^n = v_k^n a_k \ (1 \leq k \leq n), \\ b_1 s_k^1 = v_{k-1}^n p^{k-1} b_k \ (1 < k \leq n), \\ s_1^1 = v_n^n. \end{cases}$$

Recall that the elements a_i and b_i are prime to p. By (vii) we see that $s_k^1 \equiv 0 \bmod p$, when $k > 1$, and a shift of indices in the second equation of (vi) yields $s_k^i \equiv 0 \bmod p$, when $1 < i < k \leq n$. Together with (ii), these facts give us

(viii) $$r_k^i \equiv s_k^i \equiv 0 \bmod p, 1 \leq i < k \leq n.$$

Now (iv) reveals that $r_i^i(1 - r_i^i) \equiv 0 \bmod p$ for all i. Replacing σ by $1 - \sigma$ if necessary, we may assume that $r_1^1 \equiv s_1^1 \equiv 0 \bmod p$. By setting $k = i$ and $k = i+1$ in the first and second equations of (vi), respectively, we obtain $r_i^i = v_i^i = s_{i+1}^{i+1}$ for $1 \leq i < n$. Similarly, $r_n^n = v_n^n = s_1^1$ by (vii). Therefore by (ii), we have $r_i^i \equiv s_i^i \equiv 0 \bmod p$ for all i. Setting $i = j + 1$ in (iv) and employing (viii), we deduce $r_j^{j+1} \equiv 0 \bmod p$ for $j < n$. If $i > 1$, we may assume by induction that $r_j^{j+i-1} \equiv 0 \bmod p$ for $j \leq n + 1 - i$. Then once again an application of (iv) and (viii) reveals that $r_j^{j+i} \equiv 0 \bmod p$ for $j \leq n + i$. We have shown

(ix) $$r_i^j \equiv 0 \bmod p, 1 \leq i, j \leq n.$$

Now (iv) implies $r_i^j \in \bigcap_{k<\omega} p^k R = 0$ for all i and j. By (ii), all s_i^j are zero as well. Therefore $\sigma = 0$, as desired.

Last, we observe that there were infinitely many choices of the prime $p \in R$, hence of the fourth submodule $F_3 \subseteq F$. In fact, we get an infinite family of such **F**'s with only the zero homomorphism between distinct members. $\square$

In recent accounts of representation theory and Butler groups (see [5, 28]), R_2-modules (F, F_0, F_1) with pure submodules F_0 and F_1 have played a major role. It is worth pointing out how an important theorem concerning such representations

[5, Theorem A.1], [28, Corollary 3.3] follows with relative ease from our splitting theorem for R_3-modules.

Corollary 4.6. *Suppose* $\mathbf{F} = (F, F_0, F_1) \in Rep_2 R$ *has finite rank and* F_0, F_1 *are pure in* F. *Then* $\mathbf{F}$ *is a direct sum of* R_2-*modules of ranks 1 and 2.*

Proof. Because F/F_0 is finitely-generated and torsion free, it is free. Therefore F_0 splits in F, say $F = F_0 \oplus C$. We first show how Corollary 4.6 follows when $\mathbf{F}$ satisfies the following condition:

$$(*) \qquad\qquad\qquad F_0 \cap F_1 = C \cap F_1 = 0.$$

Denote $\mathbf{F}' = (F, F_0', F_1', F_2') = (F, F_0, C, F_1)$. Clearly, $\mathbf{F}$ is a complemented representation in $Rep_3 R$. Using Theorem 2.6, there exists a decomposition $F = \bigoplus_{j=1}^{n} A_j$ into summands A_j of rank ≤ 2 such that $F_i' = \sum_{j=1}^{n} (F_i' \cap A_j)$ for $i = 0, 1, 2$. Now $(F, F_0, F_1) = \bigoplus_{j=1}^{n} (A_j, F_0' \cap A_j, F_2' \cap A_j)$ is a sum of R_2-modules of rank ≤ 2, as desired.

Finally we consider any R_2-module $\mathbf{F} = (F, F_0, F_1)$ with pure F_0 and F_1. As above we see that $F_0 \cap F_1$ is a summand in F, say $F = F' \oplus (F_0 \cap F_1)$. Clearly,

$$\mathbf{F} = (F', F_0 \cap F', F_1 \cap F') \oplus (F_0 \cap F_1, F_0 \cap F_1, F_0 \cap F_1).$$

Because the second summand is a direct sum of cyclic R_2-modules, and $(F_0 \cap F') \cap (F_1 \cap F') = (F_0 \cap F_1) \cap F' = 0$, we may assume that our original representation $\mathbf{F}$ satisfies $F_0 \cap F_1 = 0$.

Write $F = F_0 \oplus C$ as above. Then $C = C' \oplus (F_1 \cap C)$ for some C', and we have a decomposition

$$\mathbf{F} = (F_0 \oplus C', F_0, (F_0 \oplus C') \cap F_1) \oplus (F_1 \cap C, 0, F_1 \cap C)$$

in which the second summand is a direct sum of cyclic R_2-modules.

Consequently, it suffices to replace $\mathbf{F}$ by the R_2-module $\mathbf{F}' = (F', F_0', F_1') = (F_0 \oplus C', F_0, (F_0 \oplus C') \cap F_1)$ above. The proof is complete provided $\mathbf{F}'$ satisfies condition $(*)$. To this end, we simply note $F_0' \cap F_1' \subseteq F_0 \cap F_1 = 0$ and $C' \cap F_1' \subseteq C' \cap F_1 = 0$. $\square$

Suppose $\mathbf{F} = (F, F_0, F_1)$ is an R_2-module as in Corollary 4.6, and write $F = F_0 \oplus C$. The second paragraph of the proof of Corollary 4.6 shows that $\mathbf{F}$ has a rank 1 direct summand if either intersection $F_0 \cap F_1$ or $C \cap F_1$ is nonzero. We use this fact to help establish an analogue to Corollary 4.5 for R_2-modules. The following result will be applied to Butler groups in Section 5; note that the second submodule F_1 in Corollary 4.7 is transferred to the third place in the R_4-module.

Corollary 4.7. *Suppose* $\mathbf{F} = (F, F_0, F_1) \in Rep_2 R$ *has finite rank and* F_0, F_1 *are pure in* F. *If* $\mathbf{F}$ *has no rank 1 summands, then there exist pure submodules* $F_2, F_3 \subseteq F$ *such that* (F, F_0, F_2, F_1, F_3) *is a complemented, indecomposable representation in* $Rep_4 R$.

Proof. We may decompose $F = F_0 \oplus C$ since F_0 is pure in F. As remarked above, our hypotheses imply $F_0 \cap F_1 = C \cap F_1 = 0$. Put $F_2 = C$. Because decompositions of the R_3-module $\mathbf{F}' = (F, F_0, F_1, F_2)$ induce ones of $\mathbf{F}$, it follows that $\mathbf{F}'$ has no rank 1 summands. By Corollary 4.5, a fourth pure submodule $F_3 \subseteq F$ with

$F_3 \cap F_i = 0$ $(i = 0, 1, 2)$ can be adjoined to $\mathbf{F}'$ to produce an indecomposable R_4-module (F, F_0, F_2, F_1, F_3) of the desired form. $\qquad\square$

Corollary 4.4 shows in particular that indecomposable, complemented R_3-modules $(F, F_i : i < 3)$ of infinite rank need not decompose into direct sums of R_3-modules of finite rank. A rather obvious necessary condition for such a decomposition to exist is that all quotients $F/(F_i \oplus F_j)$ $(i \neq j)$ be direct sums of cyclic R-modules. The following example shows that this condition does not suffice.

Example 4.8. Suppose F_0 and F_1 are free R-modules of rank ω and the PID R contains infinitely many primes $\{p_i : i < \omega\}$. Then there is a pure submodule $F_2 \subseteq F = F_0 \oplus F_1$ such that $\mathbf{F} = (F, F_0, F_1, F_2)$ is a complemented representation in $Rep_3 R$, which is not a direct sum of finite rank R_3-modules. Moreover,

$$F/(F_0 \oplus F_2) \cong F/(F_1 \oplus F_2) \cong \bigoplus_{1 \leq i < \omega} R/p_i R.$$

Proof. Write $F_0 = \bigoplus_{i < \omega} Ra_i$, $F_1 = \bigoplus_{i < \omega} Rb$, and denote $d_i = a_i + b_i \in F = F_0 \oplus F_1$. Define

$$F_2 = Rd_0 \oplus R(b_0 + p_1 d_1) \oplus \bigoplus_{1 \leq i < \omega} R(d_i + p_{i+1} d_{i+1}).$$

We first show that F_2 is pure in F. Let $\pi_{a_i} : F \longrightarrow Ra_i$ and $\pi_{b_i} : F \longrightarrow Rb_i$ denote the obvious projections. Suppose $q \in R$ is prime and

$$x = rd_0 + s(b_0 + p_1 d_1) + \sum_{i \geq 1} r_i(d_i + p_{i+1} d_{i+1})$$

lies in $F_2 \cap qF$, where $r, s, r_i \in R$.

Then $x\pi_{b_0} = (r + s)b_0$, $x\pi_{a_0} = ra_0$ and $x\pi_{a_1} = (r_1 + sp_1)a_1$ are divisible by q in F. Therefore $r, s, r_1 \in qR$. Now $x\pi_{a_i} = (r_{i-1}p_i + r_i)a_i$ $(i \geq 2)$ is in qF as well, hence it follows by induction that $r_i \in qR$ for all $i \geq 2$. Therefore $x \in qF_2$, and the purity of F_2 in F is established.

Also note that $F/F_2 \cong (Ra_0 \oplus Rb_0 \oplus \bigoplus_{1 \leq i < \omega} Rd_i)/F_2 \oplus \bigoplus_{1 \leq i < \omega} Ra_i$ and the first summand is torsion free of rank 1 of type $(1, 1, ...)$.

It is routine to verify that $\mathbf{F}$ is complemented. Let $\pi_i : F \longrightarrow F_i$ denote the canonical projection for $i = 0, 1$. The composition $F_1 \longrightarrow F \longrightarrow F/F_0 \oplus F_2$ of inclusion followed by the natural epimorphism is surjective with kernel $F_2\pi_1$, hence $F/F_0 \oplus F_2 \cong F_1/F_2\pi_1$. Note $F_2\pi_1$ is generated by the set

$$\{b_0, p_1 b_1\} \cup \{b_i + p_{i+1} b_{i+1} : 1 \leq i < \omega\}.$$

Denote $b'_1 = b_1$ and $b'_i = p_1 \cdots p_{i-1} b_i$ for $i \geq 2$. Because the p_k are distinct primes of R it is easy to see that F_1 is generated by $\{b'_i : 1 \leq i < \omega\}$ and $F_2\pi_1$. Moreover b'_i has order p_i modulo $F_2\pi_1$, hence

$$F_1/F_2\pi_1 = \bigoplus_{1 \leq i < \omega} R(b'_i + F_2\pi_1) \cong \bigoplus_{1 \leq i < \omega} R/p_i R.$$

Similarly, it follows that $F/F_1 \oplus F_2 \cong F_0/F_2\pi_0 \cong \bigoplus_{1 \leq i < \omega} R/p_i R$.

Note that if $\mathbf{F} = (F, F_0, F_1, F_2)$ were a direct sum of finite rank R_3-modules, F/F_2 would be a direct sum of finitely generated torsion-free R-modules, hence

free. However a look back at the definition of F_2 shows $b_0 \notin F_2$, while

$$b_0 \equiv -p_1 d_1 \mod F_2 \quad \text{and} \quad d_i \equiv -p_{i+1}d_{i+1} \mod F_2$$

for all $i \geq 1$. Therefore $0 \neq b_0 + F_2 \in \bigcap_{1 \leq i < \omega} p_i(F/F_2)$, and F/F_2 is certainly not a free R-module. This finishes 4.8. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finally we note an additional property of Example 4.8: Choose an idempotent $\sigma \in End\ \mathbf{F}$, acting on generators as $a_i\sigma = b_i\sigma = 0$ for $i > 1$ and $a_i\sigma = a_1 + p_2 a_2$, $b_1\sigma = b_1 + p_2 a_2$.

Then $d_1\sigma = d_1 + p_2 d_2$ and $\sigma \in End\ \mathbf{F}\backslash\{0,1\}$. Actually

$$\mathbf{F} = (\langle a_1 + p_2 a_2, b_1 + p_2 b_2\rangle, \langle a_1 + p_2 a_2\rangle, \langle b_1 + p_2 b_2\rangle)$$
$$\oplus (\langle a_i, b_i : i > 1\rangle, \langle d_i : i > 1\rangle, \langle b_i : i > 1\rangle, \langle d_0, b_0 + p_2 d_2, d_2 + p_3 d_3, ...\rangle)$$

in $Rep_3 R$, where the second summand $\mathbf{C}_0$ is isomorphic to $\mathbf{F}$.

We can find a descending chain of $Rep_3 R$-submodules $\mathbf{C}_i$ with completely decomposable complements (into summands of ranks 1 and 2) in $\mathbf{F}$ such that $\bigcap_{i \in \omega} \mathbf{C}_i = \langle a_0, b_0\rangle$.

5. Applications to Butler Groups

A *Butler group* of finite rank is a torsion-free, homomorphic image of any finite direct sum of additive subgroups of the rationals. More generally, a countable torsion-free abelian group is called a Butler group if it is the union of an ascending chain of pure, finite-rank Butler subgroups. There are a number of alternate definitions of Butler groups, many of them as deceptively simple as the ones we just gave. This special class of groups, dazzlingly complex, has fundamental links to several branches of mathematics: ring theory, abelian group and module theory, representation theory, even number theory and category theory. It is one of the links to representation theory that concerns us in this brief section. The methodology we summarize in the following paragraph is due to Arnold and Dugas [5].

Fix an additive subgroup C and a subring R of $\mathbb{Q}$. Our work will revolve around a set $\mathfrak{A} = \{A_0, A_1, A_2, \dots\}$ of additive subgroups of $\mathbb{Q}$, each containing 1, that satisfy the following conditions whenever $i \neq j < \omega$:

(i) $A_i \cap A_j \cong C$;
(ii) $\text{Hom}(A_i, A_j) = 0$;
(iii) $\text{End}(A_i) = R$.

Let E be any set of rational primes (possibly empty) such that for each $p \in E$, p^{-1} is not contained in any member of $\mathfrak{A}$. Moreover, let $\tau_i = \text{type}(A_i)$ and $\tau = \text{type}(C)$.

With the sets $\mathfrak{A}$ and E in place, we let $B(T_n, E)$ denote the category of finite-rank Butler groups G with typesets contained in $T_n = \{\tau_0, \dots, \tau_{n-1}, \tau\}$ such that G modulo the sum $G(\tau_0) + \cdots + G(\tau_{n-1})$ of type subgroups is a torsion E-group.

The extended category $B(T_n, E, \omega)$ will denote those countable groups which are obtainable as unions of ascending chains of pure subgroups in $B(T_n, E)$. The theory recently developed in [5] states, roughly, that the groups G in $B(T_n, E, \omega)$ are determined up to isomorphism by representations

$$\mathbf{F} = (F, F_i : i < n) \in Rep_n R, \text{ where } A_i F_i = G(\tau_i)$$

in the divisible hull of G, and F is the E-purification of $F_0 + \cdots + F_{n-1}$ in G. The existence and uniqueness of the free R-modules $F_i \subseteq G$ with the stated property are guaranteed by [5, Lemma 1.1], and the situation is such that:

(a) each F_i is pure in the free R-module F;

(b) $F_i \cap F_j = 0$ if $i \neq j$; and

(c) $F/(F_0 + \cdots + F_{n-1})$ is a torsion E-group.

We use

$$L_n : B(T_n, E, \omega) \to Rep_n R$$

to denote the functor just described, whose image is the full subcategory of $Rep_n R$ comprised of countable-rank representations $\mathbf{F}$ which satisfy the conditions (a), (b) and (c). Theorem 3.1 in [5] asserts that L_n is a full, additive, rank-preserving equivalence whose inverse takes

$$\mathbf{F} = (F, F_i : i < n) \in \text{ Im } L_n \text{ to } G = F + \sum_{i < n} A_i F_i \subseteq \mathbb{Q} \otimes F.$$

G is a member of $B(T_n, E, \omega)$. Moreover, $G(\tau_i)$ coincides with the subgroup $A_i F_i$ used to build G.

The following consequence of Corollary 4.6 has been noted in a number of recent papers [5, 9, 28, 22, 46]. It provides, in effect, a complete classification of finite-rank Butler groups with at most two critical types.

Proposition 5.1. *Groups in $B(T_2, E)$ are direct sums of indecomposable groups isomorphic to A_0, A_1 and $A_0 e + A_1 f + R\frac{e + af}{b} \subseteq \mathbb{Q}e \oplus \mathbb{Q}f$, where $a, b \neq 0$ are coprime in R and b is a nonunit whose prime factors lie in E.*

On the other hand, there are indecomposable groups in $B(T_3, E)$ of arbitrarily large finite rank, even when $E = \emptyset$ (see [5, Example A.5]). A tractable subclass of $B(T_3, E)$ consists of those members G whose images $L_3(G)$ are the complemented R_3-modules studied earlier in this paper; by Theorem 2.6, Remark 2.7 and the categorical equivalence discussed above, these special Butler groups are direct sums of indecomposable groups isomorphic to A_0, A_1, A_2 and $A_0 e + A_1 f + A_2(ae + bf) \subseteq \mathbb{Q}e \oplus \mathbb{Q}f$, where $a \neq 0 \neq b$ are coprime in R. We refer the interested reader to [5, Corollary A.4] for details about the *group-theoretical* properties of this particular class of Butler groups with three or fewer critical types.

We know of no result quite like the following anywhere in the literature on Butler groups. It is complementary to Proposition 5.1, and asserts that most Butler groups with two critical types are full-rank subgroups of *indecomposable* Butler groups with only two additional critical types.

Proposition 5.2. *If $G \in B(T_2, E)$ has no rank one direct summands, then G is contained in an indecomposable group $H \in B(T_4, \emptyset)$ with H/G torsion. Moreover, $H = \sum_{i<4} H(\tau_i)$ can be chosen so that $H(\tau_i) = G(\tau_i)$ for $i = 0, 1$, and so that H/G is a homomorphic image of a finite direct sum of copies of A_2/R and A_3/R.*

Proof. The assumptions (i)-(iii) we made about the set $\mathfrak{A}$ of rank one groups imply that the PID R contains infinitely many rational primes. Assume $G \in B(T_2, E)$ has no rank one summands, and denote $\mathbf{F} = L_2(G) = (F, F_0, F_1)$. Then $\mathbf{F}$ satisfies all conditions needed to apply Corollary 4.7; we conclude the existence of pure submodules $F_2, F_3 \subseteq F$ such that $F_i \cap F_j = 0$ if $0 \leq i \neq j < 4$, and $\mathbf{F}' = (F, F_0, F_2, F_1, F_3)$ is a complemented, indecomposable R_4-module. Note $F = \sum_{i<4} F_i$ because $\mathbf{F}'$ is complemented. Applying the inverse of L_4 to the indecomposable representation

(F, F_0, F_1, F_2, F_3), we obtain an indecomposable member $H = F + \sum_{i<4} A_i F_i \subseteq \mathbb{Q} \otimes F$ of $B(T_4, \emptyset)$. If we identify G with $F + A_0 F_0 + A_1 F_1 = L_2^{-1}(\mathbf{F})$, then G is embedded in H with $G(\tau_i) = A_i F_i = H(\tau_i)$ for $i = 0, 1$. Clearly, H/G is torsion because $\mathrm{rank}(H) = \mathrm{rank}(F) = \mathrm{rank}(G)$. In fact, since $H = G + H(\tau_2) \oplus H(\tau_3)$ and $F_2 \oplus F_3$ is contained in $G \cap (H(\tau_2) \oplus H(\tau_3))$, it follows that

$$\frac{H}{G} \cong \frac{H(\tau_2) \oplus H(\tau_3)}{G \cap (H(\tau_2) \oplus H(\tau_3))}$$

is a homomorphic image of

$$\frac{A_2 F_2 \oplus A_3 F_3}{F_2 \oplus F_3},$$

hence has the form we desired. $\qquad\qquad\square$

Finally, we give some examples of Butler groups of infinite rank that arise naturally from the complemented representations of infinite rank constructed in the previous section. For the moment, we assume $R = \mathbb{Z}$, and that $\mathfrak{A}$ has been chosen so that the characteristic of 1 in the group A_2 consists of only zeros and ones.

Example 5.3. There is a group $G \in B(T_3, \emptyset, \omega)$ of infinite rank such that
   (i) $\mathrm{End}(G) = \mathbb{Z}$;
   (ii) G modulo the completely decomposable subgroup $G(\tau_0) \oplus G(\tau_1)$ is a direct sum of elementary p-groups.

To construct G, let $\mathbf{F} = (F, F_0, F_1, F_2)$ be a rigid, complemented $\mathbb{Z}_3$-module, the existence of which was established in Corollary 4.4. Applying the inverse of the (full) functor L_3 to $\mathbf{F}$, we obtain a rigid group $G = \sum_{i<3} G(\tau_i)$ in $B(T_3, \emptyset, \omega)$. Now since $F = F_0 \oplus F_1$, we reason that

$$\frac{G}{G(\tau_0) \oplus G(\tau_1)} \cong \frac{G(\tau_2)}{G(\tau_2) \cap (G(\tau_0) \oplus G(\tau_1))} = \frac{A_2 F_2}{A_2 F_2 \cap (A_0 F_0 \oplus A_1 F_1)}$$

is a homomorphic image of $A_2 F_2 / F_2 \cong \bigoplus_\omega A_2/\mathbb{Z}$. Because of our assumption about the characteristic of 1 in A_2, it follows that the last factor is a direct sum of elementary p-groups, establishing (ii).

We remark that rigid extensions H of completely decomposable groups $D = D(\tau_0) \oplus D(\tau_1)$ by torsion groups were previously constructed in [22], but with the important difference that H/D was *divisible*. The discussion and examples in [28, Section 4] make it clear that different techniques are required to obtain rigid groups H for which H/D is, say, a direct sum of cyclic groups. Our work in Section 4 that led to indecomposable, complemented $\mathbb{Z}_3$-modules of infinite rank enabled us to realize the group in Example 5.3 above with relative ease; compare the construction of a similar kind of rigid group given in [28, Examples 2.2 and 5.2]. Depending on the choice of $\mathbf{F}$, however, it can be the case that G modulo the completely decomposable subgroup $G(\tau_1) \oplus G(\tau_2)$ is divisible.

We want to point out the existence of what are still fairly intractable members G of $B(T_3, \emptyset, \omega)$ for which *all* quotients $G/(G(\tau_i) \oplus G(\tau_j))$ $(i \neq j)$ are direct sums of elementary p-groups. To do this, we continue to take $R = \mathbb{Z}$, and make a slightly more careful choice of $\mathfrak{A}$. Partition the rational primes into two disjoint, infinite sets Π_1 and Π_2, and choose $\mathfrak{A}$ such that for $i = 0, 1, 2$, the characteristic of 1 in A_i consists of zeros and ones, with the latter entries occurring only in places corresponding to primes in Π_1.

Example 5.4. There is a Butler group $G = \sum_{i<3} G(\tau_i) \in B(T_3, \emptyset, \omega)$ of infinite rank such that:

(i) If $0 \leq i \neq j < 3$, then G modulo the completely decomposable subgroup $G(\tau_i) \oplus G(\tau_j)$ is a direct sum of elementary p-groups;

(ii) G is not a direct sum of finite-rank subgroups.

To construct G, we invoke Example 4.8 and obtain a complemented $\mathbb{Z}_3$-module $\mathbf{F} = (F, F_0, F_1, F_2)$ of rank ω such that

$$F/F_0 \oplus F_2 \cong F/F_1 \oplus F_2 \cong \bigoplus_{p \in \Pi_2} \mathbb{Z}/p\mathbb{Z},$$

but $\mathbf{F}$ is not a direct sum of finite-rank $\mathbb{Z}_3$-modules. Let

$$G = \sum_{i<3} A_i F_i \subseteq \mathbb{Q} \otimes F.$$

Since $L_3(G) = \mathbf{F}$, G is not a direct sum of finite-rank subgroups. Because $\mathbf{F}$ is complemented, it follows as in the proof of Example 5.3 that $G/G(\tau_0) \oplus G(\tau_1)$ is a homomorphic image of $\bigoplus_\omega A_2/\mathbb{Z}$, hence is a direct sum of elementary p-groups. Note that our choice of the sets Π_1 and Π_2 (disjoint) and of the characteristic of 1 in A_0 implies that $A_0F/F_1 \oplus F_2$ is a direct sum of elementary p-groups, where p ranges over Π_2 and the places where the characteristic of 1 in A_0 is nonzero. Since $G/G(\tau_1) \oplus G(\tau_2)$ is naturally a homomorphic image of $A_0F/F_1 \oplus F_2$, it is of the desired form. Similarly, the other factor $G/G(\tau_0) \oplus G(\tau_2)$ is an image of $A_1F/F_0 \oplus F_2$ and is therefore a direct sum of elementary p-groups, as well.

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