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LOCATING THE FIRST NODAL LINE IN THE NEUMANN PROBLEM

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Abstract. The location of the nodal line of the first nonconstant Neumann eigenfunction of a convex planar domain is specified to within a distance comparable to the inradius. This is used to prove that the eigenvalue of the partial differential equation is approximated well by the eigenvalue of an ordinary differential equation whose coefficients are expressed solely in terms of the width of the domain. A variant of these estimates is given for domains that are thin strips and satisfy a Lipschitz condition.

1. Introduction

The purpose of this note is to locate the first nodal line for the Neumann problem in a convex planar domain and to estimate the first (smallest) nonzero eigenvalue. The standard physical model for the Neumann problem is a vibrating membrane with "free" ends [CH, p. 299]. The first nonconstant eigenfunction is the lowest mode of vibration, and the frequency of vibration is the first nonzero eigenvalue. This mode of vibration can be photographed using a strobe light set to the frequency. It is the easiest mode to see because it is the one with the most energy (largest amplitude). Its zero set is a curve known as the first nodal line. This curve is easy to see because it is stationary during vibration at the lowest frequency.

In this paper we estimate the location of the first nodal line by showing that it is near the unique zero of the first nonconstant eigenfunction of a certain ordinary differential equation. The ordinary differential equation is defined along an axis that can be taken to be the diameter of the domain, and its coefficients are expressed in a simple way in terms of the width of the domain perpendicular to that axis. By "near" we mean that the nodal set is near the zero (or node) of the eigenfunction of the ordinary differential equation to within a distance comparable to inradius of the domain. This bound is used, in turn, to estimate the difference between the first nonzero eigenvalue for the original partial differential equation and the first nonzero eigenvalue of the ordinary differential equation by a bound proportional to the ratio of the inradius to the diameter of the domain. The point of the estimates is that they are universal, best possible bounds, valid for all convex domains no matter how long and thin. The methods are similar, but much simpler than the methods used in the case of the Dirichlet problem in [J].

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Every convex domain can be written, after translation, in the form

$$\Omega = \{(x,y) : f_0(x) < y < f_1(x),\, 0 < x < N\}$$

where f_0 is a convex function and f_1 is a concave function on $[0,N]$. Denote

$$h(x) = f_1(x) - f_0(x); \qquad \epsilon = \max_{x\in[0,N]} h(x).$$

Rotate so that the projection of Ω on the y-axis has smallest length among all projections of Ω onto a line. It follows that N is comparable to the diameter of Ω and that ϵ is comparable to the inradius of Ω. (See Lemma 2.1.)

Let u be an eigenfunction for Ω associated with the smallest nonzero eigenvalue λ of the Neumann problem for Ω with the normalization above. Denote by ϕ_1 the eigenfunction for the smallest nonzero eigenvalue μ_1 for the Neumann problem on $[0,N]$ given by

$$-(h\phi_1')' = \mu_1 h\phi_1 \text{ on } (0,N); \qquad h(x)\phi_1'(x) \to 0 \text{ as } x\to 0^+ \text{ or } x \to N^-. \tag{1.1}$$

There is a unique $x_1 \in (0,N)$ such that $\phi_1(x_1) = 0$. The main result of the paper is as follows.

Theorem 1.2. *There is an absolute constant C such that*
a) $u(x,y) = 0$ implies $|x - x_1| < C\epsilon$,
b) $(1 - C\epsilon/N)\mu_1 \le \lambda \le \mu_1$.

(It is not hard to show that $\mu_1 = O(1/N^2)$ independent of ϵ. See Lemmas 4.2, 4.3 and 4.4. The eigenfunction u need not be unique, but the theorem only has content in the case in which ϵ/N is smaller than some absolute constant. In that case the eigenfunction is unique up to a scalar multiple.)

Theorem 1.2 is sharp to order of magnitude as will be shown in a joint paper with D. Grieser that is in preparation. A variant of Theorem 1.2 is stated below in Theorem 7.1. The essential hypothesis is that the domain is long and thin, but the uniformity of its shape is captured with an assumption that is different from convexity.

Recall that u minimizes the Dirichlet integral

$$I(v) = \int\int_\Omega (v_x^2 + v_y^2)\,dxdy \tag{1.3}$$

among all functions v on Ω satisfying

$$\int\int_\Omega v^2 dxdy = 1, \quad \int\int_\Omega v\,dxdy = 0. \tag{1.4}$$

The minimum value of I is the eigenvalue λ.

In order to explain the ideas behind the proof and the relationship between ϕ_1 and u, consider functions of x alone. Let $v(x,y) = \phi(x)$, then

$$I(\phi) = \int_0^N \phi'(x)^2 h(x)\,dx \tag{1.5}$$

and the constraints (1.4) are

$$\int_0^N \phi(x)^2 h(x)\,dx = 1, \quad \int_0^N \phi(x)h(x)\,dx = 0. \tag{1.6}$$

The minimizer of (1.5) under the constraints (1.6) is the first nonconstant eigenfunction ϕ_1 for the Sturm-Liouville boundary problem given in (1.1). (For a discussion of Sturm-Liouville problems, see, for instance, [CH, pp. 291–295 and p. 324].) Note that in (1.1) we have stated the so-called natural boundary conditions valid even in the singular case in which h vanishes at one or both of the endpoints.[1] Thus $I(\phi_1) = \mu_1$ and

$$\lambda \le \mu_1. \tag{1.7}$$

This upper bound is the starting point of the proof of part (a) of Theorem 1.2. Consider the nodal domains $\Omega_+ = \{u > 0\}$ and $\Omega_- = \{u < 0\}$. The eigenvalue for the mixed boundary problem on $\Omega_\pm$ with Neumann conditions on $\partial\Omega \cap \partial\Omega_\pm$ and Dirichlet conditions on $\partial\Omega_+ \cap \partial\Omega_-$ is λ, with eigenfunction $u_\pm$. If $\Omega_\pm$ do not resemble $x < x_1$ and $x > x_1$, then one can show that the eigenvalue of Ω_- or Ω_+ is strictly larger than μ_1, which contradicts (1.7). What makes the Neumann problem so much easier than the Dirichlet problem in [J] is that in the Neumann problem the presence of the nodal line $u = 0$ far from $x = x_1$ drives up the value of the Dirichlet integral dramatically. (See Lemma 5.2.) Hence eigenvalue estimates suffice to confine the nodal line. In contrast, the Dirichlet eigenfunction is already zero on the boundary, so that zeros very near the boundary do not increase the Dirichlet integral very much at all. A Hopf-maximum-principle-type argument is needed in [J] to show that there are no thin passages in the nodal domains. All that is required in the Neumann case are certain uniform eigenvalue estimates for pde and ode and Sturm-type comparison theorems for ode. Another key difference between the Dirichlet and Neumann cases is that in the Dirichlet case the eigenfunctions are concentrated in a very short region where Ω is widest. The length of that region (denoted L and called the "length scale" of the eigenfunction in [J]) is the fundamental parameter relative to which one must derive uniform estimates. In the Neumann problem the eigenfunctions are nearly uniformly spread along the whole length N, and the only parameter is the obvious one, ϵ/N, which represents the "eccentricity" of Ω.

The number μ_1 depends on the choice of the rectangular coordinate system. The bound (1.7) is valid for all possible μ_1. But it is only a useful bound if it is close to best possible. This is why one must rotate so that projection onto the y-axis has smallest length. (We do not mean that this orientation is literally required; one can use any orientation to within an angle of $O(\epsilon/N)$ of the one chosen. For example, one could stipulate instead that a diameter of Ω is horizontal.) From the choice of orientation it follows that μ_1 is essentially as small as possible, and one ultimately shows that it is very close to being a lower bound for λ. But that lower bound on λ is proved *post facto*, that is, only after one has shown that the zero set of u is near the line $x = x_1$. In summary, the order of proof is the upper bound in part (b), followed by part (a), followed by the lower bound in part (b). This kind of reasoning was also required in [J].

This paper is a direct descendent of the paper by Payne and Weinberger [PW], which proves a sharp lower bound for the first nonzero eigenvalue of a convex domain in terms of the diameter, d (namely π^2/d^2). The idea of comparison with the ordinary differential equation (1.1) originates in [PW]. The difference between the papers is that the lower bound in [PW] is sharp when compared to the case of a

[1] An alternative to dealing directly with the singular case is to consider an approximate domain that is (slightly) shorter, so that the corresponding function h is strictly positive. Then the boundary conditions at the endpoints are the usual Neumann boundary conditions, $\phi_1'(0) = \phi_1'(N) = 0$.

horizontal strip. The estimate in the present paper is less precise in that it is only asymptotically sharp as the width tends to zero. But it is more precise in the way that it takes into account more detailed information about the width function h. The methods of Payne and Weinberger can be used to give better constants than the ones given, for instance, in Lemma 4.2 below.

There is a large literature concerning the size of the first Neumann and Dirichlet eigenfunctions. We refer the reader to the paper by Smits [S] for further references as well as to the probabilistic interpretation of eigenfunctions and eigenvalues. Our understanding of level sets of eigenfunctions is somewhat limited. For example, the conjecture of J. Rauch that the maximum of the Neumann eigenfunction of a convex domain occurs on the boundary remains open. It was only very recently proved for obtuse triangles by Banuelos and Burdzy [BB].

2. A new coordinate system

For a function F of bounded variation (or BV function) we define $F(a^-)$ and $F(a^+)$ as the left- and right-hand limits of F at a. These limits exist because a function of bounded variation is the sum of a monotone increasing and monotone decreasing function. Throughout this paper inequalities which are stated for $F(a)$ are valid for all numbers between $F(a^\pm)$. In particular, inequalities for $f'(a)$ for a convex or concave function f are meant to be valid for all numbers between $f'(a^\pm)$.

Lemma 2.1 [GJ1, Lemma 3.1]**.** *If the projection of Ω on the y-axis has the smallest length, then f_1 attains its maximum and f_0 its minimum at all points where h attains its maximum. In other words,*

a) $\epsilon = \max_{x\in[0,1]} f_1(x) - \min_{x\in[0,1]} f_0(x)$,

b) $|h'(x^\pm)| = |f_1'(x^\pm)| + |f_0'(x^\pm)|$.

In particular, Ω is contained in an $N\times\epsilon$ rectangle and the diameter and inradius of Ω are comparable to N and ϵ, respectively.

After dilation, we can assume, without loss of generality, that $N = 1$. We will construct a coordinate system on Ω using the functions f_i. The main step is to replace them by linear functions on intervals on which their derivatives exceed 1/2.

Lemma 2.2. *With $\max h = \epsilon$, there exist a_1 and b_1 satisfying $0 \le a_1 \le 2\epsilon$ and $1 - 2\epsilon \le b_1 \le 1$, a convex function $\tilde f_0$ on $\mathbf{R}$ and a concave function $\tilde f_1$ on $\mathbf{R}$ satisfying*

a) $\tilde f_i(x) = f_i(x)$ *for all* $x \in [a_1, b_1]$,

b) $\tilde f_0(x) \le f_0(x) \le f_1(x) \le \tilde f_1(x)$ *for all* $x \in [0, a_1] \cup [b_1, 1]$,

c) $|\tilde f_1'(x^\pm)| + |\tilde f_0'(x^\pm)| = |\tilde h'(x^\pm)| \le 1/2$ *for all* $x \in \mathbf{R}$ *(with* $\tilde h(x) = \tilde f_1(x) - \tilde f_0(x)$*).*

Proof. Define

$$a_1 = \inf\{x \in [0,1] : |h'(x)| \le 1/2\}; \quad b_1 = \sup\{x \in [0,1] : |h'(x)| \le 1/2\}.$$

Because $0 \le h \le \epsilon$ and h is concave, $0 \le a_1 \le 2\epsilon$ and $1-2\epsilon \le b_1 \le 1$. The functions $\tilde f_i$ will be chosen to be linear on the intervals $(-\infty, a_1]$ and $[b_1, \infty)$. The geometric picture is that $\tilde\Omega = \{(x,y) : \tilde f_0(x) < y < \tilde f_1(x)\}$ is the domain formed by taking the convex hull of Ω with the two points where the lines associated to $\tilde f_0$ and $\tilde f_1$ meet. The only task is to specify the slopes of the four lines $\tilde f_i'(a_1^-)$, $\tilde f_i'(b_1^+)$, $i = 0, 1$. If $a_1 > 0$, then $h'(a_1^-) \ge 1/2$ and therefore $a_1 \le x$ for any $x \in [0,1]$ satisfying

$h(x) = \max h$. It follows from Lemma 2.1 that $f_0'(a_1^-) \le 0$ and $f_1'(a_1^-) \ge 0$. Thus one can choose $\tilde f_0'(a_1^-)$ and $\tilde f_1'(a_1^-)$ satisfying

$$f_0'(a_1^-) \le \tilde f_0'(a_1^-) \le \min(f_0'(a_1^+), 0) \quad \text{and} \quad \max(f_1'(a_1^+), 0) \le \tilde f_1'(a_1^-) \le f_1'(a_1^-).$$

The lower bound for $\tilde f_0'(a_1^-)$ and the upper bound for $\tilde f_1'(a_1^-)$ imply the inequalities of (b) in $[0, a_1]$ (and these inequalities are superfluous if $a_1 = 0$). The other inequalities imply that $\tilde f_0$ is convex at a_1 and that $\tilde f_1$ is concave. They also imply that

$$\tilde h'(a_1^-) = |\tilde f_1'(a_1^-)| + |\tilde f_0'(a_1^-)| = \tilde f_1'(a_1^-) - \tilde f_0'(a_1^-) \le f_1'(a_1^+) - f_0'(a_1^+) = h'(a_1^+).$$

And the definition of a_1 implies that $h'(a_1^+) \le 1/2$. The reasoning at b_1 is similar.

Define $\tilde f_t(s) = (1-t)\tilde f_0(s) + t\tilde f_1(s)$. And define coordinates (s, t) by

$$(x, y) = (s, \tilde f_t(s)). \tag{2.3}$$

It follows that

$$dx = ds, \quad dy = \tilde f_t'(s)ds + \tilde h(s)dt; \quad dx \wedge dy = \tilde h(s) ds \wedge dt. \tag{2.4}$$

Denote by $u_x = \partial u/\partial x$, the partial derivative with y held fixed and similarly for u_y. Denote $\partial_s u = \partial u/\partial s$, the derivative with t held fixed, and, similarly, $\partial_t u = \partial u/\partial t$ with s held fixed. Then for any $A > 0$, using Lemma 2.2(c),

$$\begin{aligned}(\partial_s u)^2 &= (u_x + \tilde f_t'(s) u_y)^2 \\ &\le (1 + A)u_x^2 + (1 + A^{-1})\tilde f_t'(s)^2 u_y^2 \\ &\le (1 + A)u_x^2 + (1 + A^{-1})\tilde h'(s)^2 u_y^2.\end{aligned}$$

Put $A = |\tilde h'(s)|$ to obtain

$$\frac{(\partial_s u)^2}{1 + |\tilde h'(s)|} \le u_x^2 + |\tilde h'(s)| u_y^2 \le u_x^2 + u_y^2/2.$$

Note also that $\partial_t u = \tilde h(s) u_y$, so

$$\frac{(\partial_s u)^2}{1 + |\tilde h'(s)|} + \frac{(\partial_t u)^2}{2\tilde h(s)^2} \le u_x^2 + u_y^2. \tag{2.5}$$

3. Dirichlet integral estimates

Next, we use (2.5) to estimate the eigenvalue of a certain Schrödinger equation from above by λ. The eigenfunction u minimizes I and is orthogonal to constants, and hence

$$\int\int_\Omega (u_x v_x + u_y v_y) dxdy = \lambda \int\int_\Omega uv dxdy \tag{3.1}$$

for every v for which $I(v)$ is finite. Denote $\Omega_+ = \{(x, y) \in \Omega : u(x, y) > 0\}$, and similarly for Ω_-. Denote $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. Let $v = u_\pm$ in (3.1) to conclude that

$$\int\int_{\Omega_\pm} (u_x^2 + u_y^2) dxdy = \lambda \int\int_{\Omega_\pm} u^2 dxdy. \tag{3.2}$$

Define $I_+(s) = \{t : (s, \tilde f_t(s)) \in \Omega_+\}$ and define I_- similarly. Denote by $[a_+, b_+]$ the projection of $\overline{\Omega}_+$ on the x-axis and $[a_-, b_-]$ the projection of $\overline{\Omega}_-$ on the x-axis.

These projections are intervals because the Courant nodal domain theorem [CH, p. 452] implies that Ω_+ and Ω_- are connected. Define

$$V_+(s) = \pi^2/8\tilde{h}(s)^2 \text{ for } s \in [a_-, b_-], \text{ and } V_+(s) = 0 \text{ otherwise;}$$
$$V_-(s) = \pi^2/8\tilde{h}(s)^2 \text{ for } s \in [a_+, b_+], \text{ and } V_-(s) = 0 \text{ otherwise.}$$

Recall that if $v(t_0) = 0$ (or $v(t_1) = 0$) and $t_1 - t_0 \le 1$, then

$$\int_{t_0}^{t_1} v'(t)^2 dt \ge \frac{\pi^2}{4} \int_{t_0}^{t_1} v(t)^2 dt.$$

It follows that

$$\frac{1}{2\tilde{h}(s)^2} \int_{I_+(s)} (\partial_t u)^2 dt \ge V_+(s) \int_{I_+(s)} u^2 dt. \tag{3.3}$$

Indeed, the set $I_+(s)$ is a disjoint union of open intervals of length no greater than 1 and on each such interval the zero boundary condition is imposed on u on at least one end because that end also meets $I_-(s)$. Denote

$$w(s) = \frac{1}{1 + |\tilde{h}'(s)|} \tag{3.4}$$

and

$$\mu_+ = \inf \frac{\int_a^b \left(w(s)\phi'(s)^2 + V_+(s)\phi(s)^2 \right) \tilde{h}(s) ds}{\int_a^b \phi(s)^2 \tilde{h}(s) ds} \tag{3.5}$$

where the infimum is taken over all a and b such that $0 \le a \le a_1$, $b_1 \le b \le 1$ and all functions ϕ. Note that ϕ need not be orthogonal to the constant function. The number μ_- is defined similarly.

We wish to prove that

$$\mu_+ \le \lambda \text{ and } \mu_- \le \lambda. \tag{3.6}$$

(These bounds eventually lead to constraints on $V_\pm$ because w is close on average to 1. See (4.14).) The set of (s, t) for which $(s, \tilde{f}_t(s)) \in \Omega$ is given by the inequalities $a(t) \le s \le b(t)$ for some functions a and b satisfying $0 \le a(t) \le a_1$ and $b_1 \le b(t) \le 1$ for all $t \in (0, 1)$. Thus,

$$\begin{aligned}
\lambda \int\int_{\Omega_+} u^2 dxdy &= \int\int_{\Omega_+} (u_x^2 + u_y^2) dxdy \\
&\ge \int_0^1 \int_{I_+(s)} \left(w(s)(\partial_s u_+)^2 + \frac{(\partial_t u_+)^2}{2\tilde{h}(s)^2} \right) \tilde{h}(s) dtds \\
&\ge \int_0^1 \int_{I_+(s)} \left(w(s)(\partial_s u_+)^2 + V_+(s) u_+^2 \right) \tilde{h}(s) dtds \\
&= \int_0^1 \int_{a(t)}^{b(t)} \left(w(s)(\partial_s u_+)^2 + V_+(s) u_+^2 \right) \tilde{h}(s) dsdt \\
&\ge \mu_+ \int_0^1 \int_{a(t)}^{b(t)} (u_+)^2 \tilde{h}(s) dsdt = \mu_+ \int\int_{\Omega_+} u^2 dxdy.
\end{aligned}$$

The proof for μ_- is similar.

4. ODE eigenvalue estimates

Remark 4.1. If h is concave and nonnegative on $[0,1]$, then
a) $(1-a)h(x) \le h(t)$ whenever $0 \le x \le t \le a \le 1$.
b) $ah(t) \le h(x)$ whenever $0 \le a \le x \le t \le 1$.

Proof. The two statements are equivalent by reflection. To prove the second, use concavity on $[0,t]$:

$$h(x) \ge \frac{t-x}{t}h(0) + \frac{x}{t}h(t) \ge \frac{x}{t}h(t) \ge ah(t).$$

Lemma 4.2. *Let h be a concave, nonnegative function on $[0,1]$. Let $a \le 1/2$. Then*

$$\inf_{\{\phi:\phi(a)=0\}} \frac{\int_0^a \phi'(x)^2 h(x)dx}{\int_0^a \phi(x)^2 h(x)dx} \ge \frac{1}{a^2}.$$

Proof. Remark 4.1(a) and $a \le 1/2$ imply that $h(x)/h(t) \le 2$ whenever $0 \le x \le t \le a$. Denote $g(x) = \phi'(x)$, then

$$\begin{aligned}
\int_0^a \left(\int_x^a g(t)dt\right)^2 h(x)dx &\le \int_0^a \left(\int_x^a g(t')^2 h(t')dt'\right)\left(\int_x^a h(t)^{-1}dt\right)h(x)dx \\
&\le \int_0^a \left(\int_x^a g(t')^2 h(t')dt'\right)\int_x^a 2dtdx \\
&\le \int_0^a 2(a-x)dx\left(\int_0^a g(t)^2h(t)dt\right) = a^2\left(\int_0^a g(t)^2 h(t)dt\right).
\end{aligned}$$

Lemma 4.3. *Let h be a concave, nonnegative function on $[a,b]$. Then*

$$\inf_{\{\phi:\phi(a)=0\}} \frac{\int_a^b \phi'(x)^2 h(x)dx}{\int_a^b \phi(x)^2 h(x)dx} \le \frac{6}{(b-a)^2}.$$

Proof. One can dilate and translate so that $a=0$ and $b=1$. Then take the test function $\phi(x) = x$. Then we need to show that

$$\int_0^1 h(x)dx \le 6\int_0^1 x^2 h(x)dx$$

for any nonnegative, concave function h on $[0,1]$. Multiply h by a constant so that $\int_0^1 h = 1$. Define

$$H(x) = \int_x^1 h(t)dt.$$

Then $H'(x) = -h(x)$, $H(1) = 0$, and $H(0) = 1$. We claim that

$$H(x) \ge (1-x)^2 \quad \text{for } 0 \le x \le 1.$$

In fact, let $F(x) = H(x) - (1-x)^2$. Then $F'(x) = -h(x) + 2(1-x)$ is convex. Moreover, $F'(1) = -h(1) \le 0$ and $\int_0^1 F'(x)dx = 0$. Therefore, there exists $x_1 \in [0,1]$ such that $F'(x) \le 0$ for $x_1 \le x \le 1$ and $F'(x) \ge 0$ for $0 \le x \le x_1$. Now $F(0) = 0$ implies that $F(x) \ge 0$ for all $0 \le x \le x_1$. Also, $F(1) = 0$ implies that $F(x) \ge 0$ for all $x_1 \le x \le 1$. In all we have proved the claim.

Now integrate by parts,

$$\int_0^1 x^2 h(x)dx = -\int_0^1 x^2 H'(x)dx = \int_0^1 2xH(x)dx$$

because $H(1) = 0$. Furthermore,

$$\int_0^1 2xH(x)dx \geq \int_0^1 2x(1-x)^2 dx = 1/6.$$

Since we normalized $\int_0^1 h = 1$, we have proved the inequality.

Lemma 4.4. $1/4 < x_1 < 3/4$.

Proof. It is well known that

$$\mu_1 = \inf_{\{\phi:\phi(x_1)=0\}} \frac{\int_0^{x_1} \phi'(x)^2 h(x)dx}{\int_0^{x_1} \phi(x)^2 h(x)dx} = \inf_{\{\phi:\phi(x_1)=0\}} \frac{\int_{x_1}^1 \phi'(x)^2 h(x)dx}{\int_{x_1}^1 \phi(x)^2 h(x)dx}.$$

Lemma 4.2 implies $\mu_1 \geq 1/x_1^2$ if $x_1 \leq 1/2$ and, by symmetry, $\mu_1 \geq 1/(1-x_1)^2$ if $x_1 \geq 1/2$. Hence for all x_1, $0 \leq x_1 \leq 1$, $\mu_1 \geq 1/x_1^2$. But Lemma 4.3 implies $\mu_1 \leq 6/(1-x_1)^2$. Combining these inequalities,

$$\frac{1}{x_1^2} \leq \frac{6}{(1-x_1)^2},$$

which yields $x_1 \geq (-2+\sqrt{24})/10 > 1/4$. By symmetry, $x_1 < 3/4$.

For $[a,b] \subset [0,1]$, define

$$E[a,b] = \inf_{\{\phi:\phi(a)=0\}} \frac{\int_a^b \phi'(x)^2 h(x)w(x)dx}{\int_a^b \phi(x)^2 h(x)dx}$$

where w is given by (3.4). (Note that w is expressed in terms of $\tilde{h}$, not h, so that w has a uniform positive lower bound.)

Lemma 4.5. *Assume that* $1/5 \leq a \leq 4/5$ *and* $b \geq 9/10$. *Suppose that* h *is a concave nonnegative function on* $[0,1]$ *and that* w *is a function of bounded variation satisfying* $1/2 \leq w \leq 1$. *There are absolute constants* C *and* $c > 0$ *such that*

a) $|(\partial/\partial b)E[a,b]| \leq C$,

b) $(\partial/\partial a)E[a,b] \geq c$.

In particular, part (b) of this lemma means that moving the node of these ordinary differential equations changes the eigenvalue by a significant amount, similar to what happens for sines and cosines.

Proof. Let ϕ denote the unique nonnegative minimizer for $E[a,b]$ with the normalization

$$\int_a^b \phi(x)^2 h(x)dx = \max_{[0,1]} h. \tag{4.6}$$

Then

$$-(wh\phi')' = Eh\phi; \quad \phi(a) = 0 \text{ and } (wh\phi')(b^-) = 0 \quad (E = E[a,b]). \tag{4.7}$$

As in [J, Lemma 4.1], one can compute the variational formulas

$$(\partial/\partial b)E[a,b] = -E[a,b]h(b)\phi(b)^2\left(\int_a^b \phi(x)^2h(x)dx\right)^{-1} \tag{4.8}$$

and

$$(\partial/\partial a)E[a^+,b] = w(a^+)h(a)\phi'(a^+)^2\left(\int_a^b \phi(x)^2h(x)dx\right)^{-1}. \tag{4.9}$$

To prove (a) from (4.8), observe first that $b-a \geq 1/10$ and Lemma 4.3 imply

$$E[a,b] \leq 600. \tag{4.10}$$

In light of the normalization (4.6), it suffices to show that ϕ is bounded by an absolute constant. Normalize h by $\max_{[0,1]} h = 1$. Then

$$\begin{aligned}|w(x)h(x)\phi'(x)| &= \left|\int_x^b Eh(t)\phi(t)dt\right|\\ &\leq E\left(\int_a^b \phi(t)^2h(t)dt\right)^{1/2}\left(\int_a^b h(t)dt\right)^{1/2} \leq E.\end{aligned}$$

It follows that $|h\phi'|$ is bounded by $2E$. Since h is concave, nonnegative and $\max h = 1$, $h(1/5) \geq 1/5$, and hence $h(t) \geq (1-t)/4$ for all $t \in [1/5, 1]$. Therefore,

$$\phi(x) = \left|\int_a^x \phi'(t)dt\right| \leq 2E\left|\int_a^x \frac{dt}{h(t)}\right| \leq 2E\left|\int_a^x \frac{4dt}{1-t}\right| \leq 8E\log\left(\frac{1}{1-x}\right).$$

Next, since Remark 4.1 (b) implies $h(t) \leq 5h(x)$ for $a \leq x \leq t \leq 1$,

$$\begin{aligned}|\phi'(x)| &\leq \frac{2}{h(x)}\int_x^b Eh(t)\phi(t)dt \leq 10\int_x^b E\phi(t)dt \leq 80E^2\int_x^b \log\left(\frac{1}{1-t}\right)dt\\ &\leq 80E^2\int_0^1 \log\left(\frac{1}{1-t}\right)dt = 80E^2.\end{aligned}$$

Thus

$$|\phi'(x)| \leq 80(600)^2. \tag{4.11}$$

Finally,

$$\phi(x) = \int_a^x \phi'(x)dx \leq 80E^2(b-a) \leq 80E^2 \leq 80(600)^2. \tag{4.12}$$

This concludes the proof of part (a) of Lemma 4.5.

To prove (b) from (4.9), note that $h(a) \geq 1/5$ and hence $w(a^+)h(a) \geq 1/10$. Therefore it suffices to show that there is an absolute constant $c > 0$ such that

$$\phi'(a^+) > c.$$

In fact, because ϕ is positive, and using (4.10) and (4.12),

$$(wh\phi')(a^+) = \int_a^b Eh(t)\phi(t)dt \geq \frac{E}{\max\phi}\int_a^b h(t)\phi(t)^2dt \geq \frac{E}{80E^2} \geq \frac{1}{(80)(600)}.$$

Define $\tilde{E}[a,b]$ in the same way as $E[a,b]$ with $\tilde{h}$ replacing h. Lemma 4.5 applies to $\tilde{E}$ because the only property of h that we use is that it is nonnegative and concave on $[0,1]$.

Lemma 4.13. *If $w = 1/(1+|\tilde{h}'|)$, then there is an absolute constant C such that for all b, $b_1 \le b \le 1$,*

$$|E[x_1,b]-\mu_1| + |\tilde{E}[x_1,b]-\mu_1| \le C\epsilon.$$

Proof. Because $w \le 1$, $E[x_1,1] \le \mu_1$. Because $1-w \le |\tilde{h}'|$, $\tilde{h}$ is concave, and $0 \le \tilde{h} \le \epsilon$,

$$\int_0^1 (1-w(x))dx \le 2\epsilon. \tag{4.14}$$

This estimate on the integral of $1-w$ is the one for which the assumption that the projection on the y-axis is smallest helps. If the region is rotated too far from the orientation chosen, then this integral is not comparable to the inradius. But any other rotation that gives bounds comparable to (4.14), such as the one in which the diameter is horizontal, gives the same theorem.

Let ϕ be the minimizer for $E[x_1,1]$ with the normalization (4.6). Then

$$\int_{x_1}^1 \phi'(x)^2 h(x)dx = E[x_1,1]\int_{x_1}^1 \phi(x)^2 h(x)dx + \int_{x_1}^1 (1-w(x))\phi'(x)^2 h(x)dx$$

and estimates (4.11) and (4.14) imply that there is an absolute constant C such that

$$\int_{x_1}^1 (1-w(x))\phi'(x)^2 h(x)dx \le C\max h\int_{x_1}^1 (1-w(x))dx \le 2C\epsilon\max h.$$

It follows that

$$\mu_1\int_{x_1}^1 \phi(x)^2 h(x)dx \le \int_{x_1}^1 \phi'(x)^2 h(x)dx \le (E[x_1,1]+C\epsilon)\int_{x_1}^1 \phi(x)^2 h(x)dx.$$

Thus, $|E[x_1,1]-\mu_1| \le C\epsilon$.

Because $1-b_1 \le 2\epsilon$, Lemma 4.5 (a) implies that there is an absolute constant C such that $|E[x_1,b]-E[x_1,1]| \le C\epsilon$. Furthermore, $E[x_1,b_1] = \tilde{E}[x_1,b_1]$, and the estimate $|\tilde{E}[x_1,b]-\tilde{E}[x_1,1]| \le C\epsilon$ is valid because Lemma 4.5 applies to $\tilde{E}$ as well as to E. Putting these estimates together gives Lemma 4.13.

5. ODE comparison theorems

Lemma 5.1. *Suppose that α, β, η_1' and η_2' are functions of bounded variation satisfying*

(i) $\eta_1'(x^\pm) < \alpha(x^\pm)\eta_1(x)^2 + \beta(x^\pm)$ for all $x \in (0,a)$ and at 0^+,
(ii) $\eta_2'(x^\pm) = \alpha(x^\pm)\eta_2(x)^2 + \beta(x^\pm)$ for all $x \in (0,a)$ and at 0^+,
(iii) $\eta_1(0) \le \eta_2(0)$.
Then $\eta_1(x) < \eta_2(x)$ for all $x \in (0,a)$.

This lemma says that solutions to certain ordinary differential inequalities do not cross. As in Sturm type comparison theorems counting the number of oscillations of a solution to a differential equation, this lemma is applied in Lemma 5.2 below to the logarithmic derivative of a solution to second-order equation in order to control its oscillation.

Proof. Since η_i are continuous, if $\eta_1(0) < \eta_2(0)$, then $\eta_1(x) < \eta_2(x)$ for $0 \le x < x_0$ for some $x_0 > 0$. On the other hand, if $\eta_1(0) = \eta_2(0)$, then (i) and (ii) at 0^+ imply that $\eta_1'(0^+) < \eta_2'(0^+)$, so that even in this case $\eta_1(x) < \eta_2(x)$ for $0 < x < x_0$ for some $x_0 > 0$. Denote $x_1 = \inf\{x : 0 < x < a \text{ and } \eta_1(x) = \eta_2(x)\}$. Assume that the set is nonempty, then x_1 exists and $x_1 \geq x_0$. Since $\eta_1(x) < \eta_2(x)$ for all $x < x_1$ and $\eta_1(x_1) = \eta_2(x_1)$, $\eta_1'(x_1^-) \geq \eta_2'(x_1^-)$. But this contradicts (i) and (ii). ☐

The following lemma is stated on a general interval of length N in order to permit a scale change to make another parameter equal to 1. Thus we will change the value of N in the course of the proof of Lemma 5.2, but it will be applied in the case $N = 1$.

Lemma 5.2. *Let h, w and V be functions on $[0, N]$. Suppose that h is concave, nonnegative. Suppose that w is a function of bounded variation, $1/2 \leq w \leq 1$, and V is nonnegative and piecewise continuous. Suppose that there are numbers μ, x_1, T, b, satisfying $0 < T < N/100$, $x_1 + 2T < b \leq N$, $N/4 \leq x_1 \leq 3N/4$, and $0 < \mu < 1/100T^2$, and a function ψ satisfying*

(i) $-(wh\psi')' + Vh\psi = \mu h\psi$ on $[x_1, b]$ and $h(x)\psi'(x) \to 0$ as $x \to b^-$,

(ii) $\psi(x) > 0$ for $x \in [x_1, b]$,

(iii) $\mu < E_{T/2}$ where

$$E_T = \inf_{\{\phi:\phi(x_1+T)=0\}} \frac{\int_{x_1+T}^b \phi'(x)^2 h(x)w(x)dx}{\int_{x_1+T}^b \phi(x)^2 h(x)dx},$$

(iv) $\psi'(x_1^+) \geq 0$.

Then

$$\int_{x_1+T}^{x_1+2T} (V(s) - \mu)h(s)ds \leq (94/T) \max_{[0,N]} h.$$

Proof. The idea of the proof is that the hypotheses on ψ, (notably $\psi'(x_1) \geq 0$) make it like an eigenfunction for Neumann boundary conditions with the potential V. More precisely, if V is too large, then ψ can be modified to be a test function showing that $E_{T/2} \leq \mu$, a contradiction.

To begin the proof, multiply h by a constant so that $\max_{[0,N]} h = 1$. (All the hypotheses and the conclusion of the lemma are unchanged.) With this normalization, $h(x) \geq 1/5$ for $x_1 \leq x \leq x_1 + 2T$. Rescale to the case $T = 1$ as follows. Define $\zeta(x) = \psi(x_1 + Tx)$, then V and μ are replaced by T^2V and $T^2\mu$ and x_1 replaced by 0. Thus it suffices to confirm the case $T = 1$, and we can rewrite the hypotheses as follows:[2]

$$wh \geq 1/10 \text{ on } [0, 2] \text{ and } 0 < \mu < 1/100 \tag{5.3}$$

and ζ satisfies

$$-(wh\zeta')' + Vh\zeta = \mu h\zeta \text{ on } [0, b] \text{ with } (h\zeta)(b^-) = 0 \tag{5.4}$$

and

$$\zeta > 0 \text{ on } [0, b] \text{ and } \zeta'(0^+) \geq 0. \tag{5.5}$$

Finally, $\mu < E_{1/2}$ where

$$E_T = \inf_{\{\phi:\phi(T)=0\}} \frac{\int_T^b \phi'(x)^2 h(x)w(x)dx}{\int_T^b \phi(x)^2 h(x)dx}.$$

[2]The lower bound for wh on $[0, 2]$ follows from the new normalizations, the concavity and nonnegativity of h, and the fact that $T = 1$ implies $N > 100$.

Our goal is to prove that

$$\int_1^2 (V(s)-\mu)h(s)ds \leq 94. \tag{5.6}$$

Denote $z(x) = w(x)h(x)\zeta'(x)/\zeta(x)$. Then

$$z' = (wh\zeta')'/\zeta - wh(\zeta')^2/\zeta^2.$$

In other words,

$$z' = \alpha z^2 + \beta$$

where $\alpha = -1/wh$ and $\beta = (V-\mu)h$. (5.5) implies that $z(0^+) \geq 0$ and (5.3) implies

$$\alpha(x) \geq -10, \quad \beta \geq -\mu > -1/100.$$

We claim that

$$z(x) \geq -1/20 \text{ for all } x \in [0,2].$$

Define $z_1(x) = -(1/32)\tan(x/2)$, then

$$z_1' = -16z_1^2 - 1/64 \text{ and } z_1(0) = 0$$

z_1 is a subsolution, $z_1' < \alpha z_1^2 + \beta$, with an initial condition $z_1(0) = 0 \leq z(0^+)$. Thus by Lemma 5.1,

$$z(x) > z_1(x) \geq \min_{[0,2]} z_1 = z_1(2) = -(1/32)\tan 1 > -1/20$$

for all $x \in (0,2]$. Assume that

$$\int_1^2 (V(x)-\mu)h(x)dx > 94.$$

We will derive a contradiction. First, assume also that $z(x) \leq 3$ for all $x \in [1,2]$, then $z'(x) = \alpha z^2 + (V-\mu)h \geq (V-\mu)h - 90$. Hence

$$z(2) \geq z(1) + \int_1^2 ((V(x)-\mu)h - 90)dx \geq z(1) + 4 > 3.$$

This contradicts the upper bound on z. So there exists $x_2 \in [1,2]$ such that $z(x_2) > 3$. Define

$$\tilde{V}(x) = V(x) \text{ for } x > x_2; \quad \tilde{V}(x) = 0 \text{ for } x < x_2.$$

Let $\tilde{z}$ solve

$$\tilde{z}' = \alpha\tilde{z}^2 + (\tilde{V}-\mu)h; \quad \tilde{z}(x_2) = z(x_2).$$

We claim that there exists s such that $x_2 - 1/3 \leq s < x_2$ and

$$\lim_{x \to s^+} \tilde{z}(x) = \infty.$$

Consider the function

$$z_2(x) = \frac{1}{x - x_2 + 1/3}$$

which satisfies $z_2(x_2) = 3$ and $z_2' = -z_2^2$. This is a subsolution starting at x_2 going to the left because it satisfies $z_2(x_2) = 3$ and $z_2'(x) > \alpha z_2^2 - \mu h$ ($\mu > 0$ and $\alpha \leq -1$). Thus by Lemma 5.1, $z_2(x) < \tilde{z}(x)$ and $z_2(x) \to \infty$ as $x \to (x_2 - 1/3)^+$ and hence s above exists.

Next we wish to find $c > 0$ such that for $x > s$ sufficiently close to s,

$$\tilde{z}(x) \geq c/(x-s). \tag{5.7}$$

In fact, $\tilde{z}' \geq \alpha \tilde{z}^2 - 1 \geq -16\tilde{z}^2 - 1$ implies

$$\int_{s_1}^{x} \frac{\tilde{z}'(t)dt}{16\tilde{z}(t)^2 + 1} \geq -\int_{s_1}^{x} dt.$$

Hence

$$\arctan(4\tilde{z}(x)) - \arctan(4\tilde{z}(s_1)) \geq -4(x - s_1).$$

Taking the limit as $s_1 \to s^+$ one obtains

$$\arctan(4\tilde{z}(x)) \geq \pi/2 - 4(x-s)$$

from which (5.7) follows.

Define

$$\tilde{\zeta}(x) = \exp \int_{x_2}^{x} \frac{\tilde{z}(\sigma)d\sigma}{w(\sigma)h(\sigma)},$$

then (5.7) implies that $\tilde{\zeta}(s^+) = 0$. Furthermore, $\tilde{z}(x) = z(x)$ for $x > x_2$ implies $\tilde{\zeta}(x) = \zeta(x)/\zeta(x_2)$, which implies by (5.4) that $(h\tilde{\zeta}')(b^-) = 0$. Finally,

$$-(wh\tilde{\zeta}')' + \tilde{V}h\tilde{\zeta} = \mu h \tilde{\zeta} \text{ on } [s,b].$$

Multiplying by $\tilde{\zeta}$ and integrating by parts, we find

$$\int_s^b wh\tilde{\zeta}'(x)^2 dx + \int_s^b \tilde{V}(x)h(x)\tilde{\zeta}(x)^2 dx = \mu \int_s^b \tilde{\zeta}(x)^2 dx.$$

But because $\tilde{V} \geq 0$, $\tilde{\zeta}$ is a test function for E_s and $E_s \leq \mu$. Notice that E_s is increasing with s, and $s > 1/2$ implies $E_s \geq E_{1/2}$. This contradicts the assumption that $\mu < E_{1/2}$. Therefore, we have proved (5.6) as desired. $\square$

6. Conclusion of the proof of Theorem 1.2

The idea of the proof of Theorem 1.2 is to use Lemma 5.2 with $\mu = \mu_+$ and $V = V_+$ of (3.5). We will show that $\mu_+ < E_{T/2}$ for any suitable large number T. Hence by Lemma 5.2, V_+ will be small at most points of the interval $[x_1+T, x_1+2T]$. Finally, the nodal line cannot cross the vertical line above any point x at which $V_+(x)$ is small.

To begin the proof of Theorem 1.2(a), recall that we can assume without loss of generality that $N = 1$. (We only briefly switched to general N during the course of the proof of Lemma 5.2.) Define

$$\mu(V;a,b) = \inf_{\psi} \frac{\int_a^b (w(x)\psi'(x)^2 + V(x)\psi(x)^2)\tilde{h}(x)dx}{\int_a^b \tilde{h}(x)\psi(x)^2 dx}. \tag{6.1}$$

Since $V \geq 0$, the Beurling-Deny criterion [RS] implies that the function minimizing $\mu(V;a,b)$ is nonnegative (after multiplication by a suitable constant). Recall that

$$\mu_+ = \inf\{\mu(V_+;a,b) : 0 \leq a \leq a_1,\ b_1 \leq b \leq 1\}. \tag{6.2}$$

Let $C_0 = 4C/c$, where C is from Lemma 4.13 and c is from Lemma 4.5(b). Take $\epsilon_0 = 1/100C_0$. Then if $\epsilon < \epsilon_0$ and $T = C_0\epsilon$, then $\tilde{E}[x_1 + T/2, b] > \mu_1$ for all b, $b_1 \leq b \leq 1$, and $0 < T < 1/100$. (Recall that $\tilde{E}$ was defined above Lemma 4.13.)

From (1.7) and (3.6) one has $\mu_+ \leq \lambda \leq \mu_1$. Therefore there exists $[a,b]$ so that $0 \leq a \leq a_1$, $b_1 \leq b \leq 1$ and $\mu \equiv \mu(V_+;a,b) < \tilde{E}[x_1+T/2,b]$. Let ψ_+ denote the minimizer for $\mu(V_+;a,b)$. Then ψ_+ satisfies the equation of Lemma 5.2(i) with $V = V_+$ on all of $[a,b]$ and $\tilde{h}$ in place of h. Moreover the natural boundary condition is valid at both ends. (We need both ends for symmetry after reflection.) Because $V \geq 0$, uniqueness in the initial value problem for second-order ode's and the Beurling-Deny criterion [RS] imply that we may assume that $\psi_+ > 0$ on $[a,b]$. Lemmas 4.3 and 4.4 imply that $\mu_1 < 100$, so we can also assume that $\mu < 100$. Hence $\mu < 1/100T^2$. Finally, $\mu \geq 0$.

Now assume that $\psi_+'(x_1^+) \geq 0$. The hypotheses of Lemma 5.2 are satisfied (with $\tilde{h}$ in place of h and $N = 1$), and hence

$$\int_{x_1+T}^{x_1+2T} (V_+(s)-\mu)\tilde{h}(s)ds \leq 94\epsilon/T. \tag{6.3}$$

(In the exceptional case $\mu = 0$, (6.3) is trivially valid because $\mu = 0$ implies that V_+ is identically zero on $[a,b]$. We could also have avoided this technicality by proving Lemma 5.2 under the hypothesis $\mu \geq 0$.) Hence

$$\int_{x_1+T}^{x_1+2T} V_+(s)\tilde{h}(s)ds \leq (94/T + 100T)\epsilon < 95\epsilon/T.$$

Since $V_+(s)\tilde{h}(s) \geq 1/\epsilon$ on $[a_-,b_-]$, the length of the intersection

$$|[a_-,b_-] \cap [x_1+T, x_1+2T]| \leq 95\epsilon^2/T < T/2$$

provided $T > 14\epsilon$. Thus $[a_-,b_-]$ contains less than half the interval $[x_1+T, x_1+2T]$. The analogous reasoning with ψ_- associated to V_- shows that if $\psi_-'(x_1^+) \geq 0$, then $[a_+,b_+]$ also covers less than half of $[x_1+T, x_2+2T]$. But this contradicts $[a_-,b_-] \cup [a_+,b_+] = [0,1]$. Therefore it must be that $\psi_-'(x_1^+) \leq 0$. Next, note that $wh\psi_-'$ is continuous, and the factors w and h are strictly positive, so the sign of $\psi_-'(x_1^+)$ is the same as the sign of $\psi_-'(x_1^-)$. Thus $\psi_-'(x_1^-) \leq 0$ and after reflection $x \to x_1 - x$ the same argument as above shows that $[a_+,b_+]$ covers less than half of $[x_1-2T, x_1-T]$. It follows that $[a_-,b_-]$ covers at least some portion of this interval. In particular, $a_- < x_1 - T$. Now let us return to the fact that $[a_-,b_-]$ cannot contain the whole of $[x_1+T, x_1+2T]$. It follows that either $b_- < x_1+2T$ or $a_- > x_1+T$. But the latter possibility has already been ruled out, so it must be that $b_- < x_1+2T$. In all we have shown that $\psi_+'(x_1^+) \geq 0$ implies that $\psi_-'(x_1^-) \leq 0$ and $b_- < x_1+2T$. Thus, $\{(x,y) : x > x_1+2T\} \subset \Omega_+$. Moreover, the assumption $\psi_-'(x_1^-) \leq 0$ implies, similarly that $\{(x,y) : x < x_1-2T\} \subset \Omega_-$. This proves Theorem 1.2(a) with $C\epsilon = 2T$, provided $\epsilon < \epsilon_0$. The case $\psi_+'(x_1^+) \leq 0$ is similar after the reflection $x \to x_1 - x$.

The proof of part (b) is now routine. Define

$$a_+(t) = \inf\{s \in [0,1] : (s, \tilde{f}_t(s)) \in \Omega_+\}.$$

Without loss of generality (choosing between Ω and the reflected domain) we may now assume that $a_+(t) \geq x_1 - 2T$ for some absolute constant T. It follows from Lemmas 4.13 and 4.5 that $\tilde{E}[a_+(t), b(t)] \geq \mu_1 - C\epsilon$, where $b(t)$ was defined after

(3.6). As in the proof of (3.6),

$$\begin{aligned}\lambda \int\int_{\Omega_+} u^2 dxdy &= \int\int_{\Omega_+} (u_x^2 + u_y^2) dxdy \\ &\geq \int_0^1 \int_{a_+(t)}^{b(t)} w(s)(\partial_s u_+)^2 \tilde{h}(s) dsdt \\ &\geq \int_0^1 \tilde{E}[a_+(t), b(t)] \int_{a_+(t)}^{b(t)} (u_+)^2 \tilde{h}(s) dsdt \\ &\geq (\mu_1 - C\epsilon) \int\int_{\Omega_+} u^2 dxdy.\end{aligned}$$

The difference this time is that we are using the extra information that u vanishes at $(a_+(t), \tilde{f}_t(a_+(t)))$. The potential V_+ is superfluous because the zero boundary condition drives up the eigenvalue even more.

7. Final remarks

The assumption of convexity can be replaced by other hypotheses. Here is one example.

Theorem 7.1. *Let $\alpha > 0$ and A be constants. Let f_0 and f_1 satisfy*

$$\max(|f_0'(x)|, |f_1'(x)|) \leq A; \quad \alpha x(1-x) \leq f_1(x) - f_0(x) \leq A \quad \text{for all } x \in [0,1].$$

Denote by u_ϵ the first nonconstant Neumann eigenfunction for

$$\Omega_\epsilon = \{(x,y) : \epsilon f_0(x) < y < \epsilon f_1(x),\ 0 < x < 1\}$$

with eigenvalue λ_ϵ. Let ϕ_1 be the first nonconstant Neumann eigenfunction satisfying (1.1) with $N = 1$ and $h(x) = f_1(x) - f_0(x)$. Let $x_1 \in (0,1)$ be the unique zero of ϕ_1 and let μ_1 be the eigenvalue. There is a constant C depending only on α and A such that

a) $u_\epsilon(x,y) = 0$ implies $|x - x_1| < C\epsilon$,
b) $(1 - C\epsilon)\mu_1 \leq \lambda_\epsilon \leq \mu_1$.

Proof. The proof of this theorem is nearly the same as the proof of Theorem 1.2, so we will only summarize the modifications needed. In all particulars, the proof is simpler. There is no need for the modification of coordinates in Section 2 because when ϵ is small, the slope $A\epsilon << 1/2$. Thus one uses the coordinate system $(s, \epsilon f_t(s)) = (s, \epsilon(1-t)f_0(s) + \epsilon t f_1(s))$. Furthermore, the definition of μ_+ in (3.5) is simpler because one considers only the single interval $(0,1)$ rather than a collection of intervals with initial points a such that $0 \leq a \leq a_1$ and final points b with $b_1 \leq b \leq 1$. The weight factor w is given by

$$w(x) = \frac{1}{1 + \epsilon \max(|f_0'(x)|, |f_1'(x)|)} \tag{7.2}$$

and 2ϵ is replaced by $A\epsilon$ in estimate (4.14). With a suitable change of constants, Remark 4.1 remains true under the new hypotheses on $h(x)$, and hence Lemma 4.2 has the same proof. Lemma 4.3 can also be proved with suitable constants, provided one assumes further that the subinterval under consideration has a length at least some fixed fraction of $(0,1)$ like one tenth. This limitation is of no consequence for the three places where Lemma 4.3 is used, namely to prove Lemma 4.4, (4.10), and the upper bound on μ_1 in the paragraph preceding (6.3). This concludes the list of modifications.

The hypotheses of Theorem 7.1 neither imply nor are implied by those of Theorem 1.2. Evidently there is no hypothesis on second derivatives in Theorem 7.1. Conversely, convexity does not imply any uniform bound on the first derivative. Indeed, the purpose of Section 2 was to take care of this difficulty.

We have not attempted to find the best possible constants in our estimates. Although Theorem 1.2 was proved with a very large value of N/ϵ and a large absolute constant C, we believe that it is true when $N > 3\epsilon$ with the constant C on the order of $1/10$. Because the reasoning here is less complicated than that of [J], there is a realistic chance of improving the constants to reasonably explicit ones that prove the theorem to an accuracy within, say, a factor of 10 of the actual optimal bounds. For example, we expect that in Lemma 4.4 the worst case occurs when $h(x) = x$. The eigenfunction is the Bessel function J_0 suitably scaled to $[0, 1]$, and the zero x_1 is roughly $5/8$. Similarly, the extreme cases in Lemma 4.5 may be given by subintervals $[0, x_1]$ or $[x_1, 1]$ of $[0, 1]$ for the triangles, $h(x) = x$, $h(x) = 1 - x$, or the rectangle $h(x) = 1$.

To pursue the goal of finding better absolute constants, it might be helpful to give a more intrinsic construction of a coordinate system. One way to do so is to mimic the level sets of the eigenfunction u more closely by trying to find a foliation of Ω by curves that meet the boundary at right angles. Presumably one wishes to make these curves as close to circles as possible, and the error would be expressed in terms of the curvature of the boundary. We did not attempt to carry out the construction of such a coordinate system because to do so seemed closer in computational complexity to the problem of computing the eigenfunction u itself. We preferred to make a direct comparison with the simplest function of one rectangular coordinate variable associated to the problem, namely the ODE eigenfunction ϕ_1.

It is our hope that these methods can be extended to the case of higher-dimensional convex domains and to positively curved Riemann surfaces. The main extra ingredient needed in the Riemann surface case is an appropriate coordinate system. In the higher-dimensional case, the maximum principle ideas of [J, J1] should play a role. Finally, it would also be reasonable to try to prove versions for the Neumann problem of the Dirichlet problem results of [GJ, GJ1].

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