TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 9, Pages 4207–4236
S 0002-9947(00)02444-2
Article electronically published on March 21, 2000

ROBIN BOUNDARY VALUE PROBLEMS ON ARBITRARY DOMAINS

DANIEL DANERS

ABSTRACT. We develop a theory of generalised solutions for elliptic boundary value problems subject to Robin boundary conditions on arbitrary domains, which resembles in many ways that of the Dirichlet problem. In particular, we establish L_p-L_q-estimates which turn out to be the best possible in that framework. We also discuss consequences to the spectrum of Robin boundary value problems. Finally, we apply the theory to parabolic equations.

1. INTRODUCTION

It is well known that the equation $-\Delta u = f$ subject to homogeneous Dirichlet or Neumann boundary conditions can be considered on arbitrary bounded domains Ω in $\mathbb{R}^N$. The idea is to introduce a weak formulation and to choose the "right" Hilbert space incorporating the boundary conditions in a generalised sense. One feature of this approach is that if data and domain are smooth enough we are back to classical solutions satisfying the boundary conditions pointwise. Looking at boundary conditions of Robin type such as

$$\frac{\partial}{\partial\nu}u + \beta u = 0 \tag{1.1}$$

(ν being the outer unit normal to the boundary $\partial\Omega$ of Ω, and β a constant) it is well known that Dirichlet and Neumann boundary conditions correspond to two extreme cases, namely $\beta = \infty$ and $\beta = 0$, respectively. However, although introduced quite some time ago by Maz'ja [19, 20, Section 4.11.6], it is not very well known that there is a weak formulation on arbitrary domains if $\beta \in (0, \infty)$. The difficulty is to give sense to the boundary integral appearing in the Dirichlet form

$$a(u,v) = \int_\Omega \nabla u \nabla v\,dx + \int_{\partial\Omega} \beta uv\,d\sigma$$

for $u, v \in W_2^1(\Omega)$ as traces of u, v on $\partial\Omega$ are not well defined for a general domain, and if they are, it is not clear whether they are square integrable over $\partial\Omega$.

We shall use Maz'ja's approach and develop an L_p-theory for Robin boundary value problems on arbitrary domains and general (nonselfadjoint) second order elliptic operators in divergence form with real bounded and measurable coefficients. It is known that the Neumann problem does not have any smoothing properties for a general domain as this is equivalent to embedding theorems for Sobolev spaces (see [6, Corollary 3.4]), and that the Dirichlet problem always has (e.g. [5, Lemma 1]).

Received by the editors April 5, 1996 and, in revised form, May 19, 1998.
2000 *Mathematics Subject Classification.* Primary 35J25; Secondary 35D10, 35B45.
Supported by a grant of the Australian Research Council.

It turns out that the smoothing properties we prove in this paper lie "half way" between the two, and that they are the best possible in that framework (see Theorem 5.11 below). In [7] it is shown that the L_p-L_q-estimates we obtain cannot be improved even if we restrict ourselves to arbitrarily smooth domains if we do not want the constants involved to depend on the geometry of the underlying domain. This follows from domain perturbation results which are proved in the above mentioned paper. The whole theory developed in this paper is in fact motivated by applications to domain perturbation of linear and nonlinear elliptic equations subject to Robin boundary conditions. It provides a priori estimates for solutions not depending on the domain geometry, which allows us to deal with very singular perturbations of the domain such as cutting holes or adding small pieces. The results also allows us to establish a positive lower bound for the first eigenvalue of $\Delta\varphi = \lambda\varphi$ subject to boundary conditions of the form (1.1) with $\beta > 0$, uniformly with respect to all domains of the same volume. It would be interesting to know whether a Faber-Krahn type inequality holds or not.

We further show that the theory carries over to the corresponding parabolic problem, and that the parabolic problem fits into the framework of semigroup theory and abstract parabolic equations on L_p-spaces for $1 \leq p < \infty$. We also get estimates for the semigroup kernel (heat kernel). It turns out that the usual methods to prove kernel estimates such as described in [30] and [10] either apply directly or can be adapted to our situation very easily.

The outline of the paper is as follows. In Section 2 we give the precise assumptions, introduce notation and state some of our main results. Section 3 deals with the L_2-theory of weak solutions on arbitrary domains as introduced by Maz'ja [19, 20]. We discuss some problems arising with this approach and give examples. In Section 4 we provide global bounds for weak solutions of Robin boundary value problems using a version of the well known Moser iteration technique. It is an extension of the results in [6]. These results are the basis to establish an L_p-theory for Robin boundary value problems on arbitrary domains which we develop in Section 5. The final section is concerned with the corresponding parabolic problem. More on the parabolic problem can be found in [8]. The paper concludes with an appendix discussing the operator induced by a bilinear form. In particular, in Appendix A, we discuss maximal restrictions of that operator to Banach spaces, and duality. Then, in Appendix B we establish a priori estimates of solutions of the corresponding abstract elliptic problem employing a version of the well known Moser iteration techique. In Appendix C we consider maximal restrictions to L_p-spaces, and some implications of the a priori estimates. Some of the results are folklore, but as we do not know of an explicit reference, we include the precise statements and give complete proofs.

2. Assumptions and main results

In this section we give the precise assumptions, fix notation and state some of our main results. For more detailed statements, proofs and further comments we refer to later sections. In particular, we refer to Section 5 for the L_p-theory of elliptic problems, and to Section 6 for parabolic problems.

We shall be concerned with the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}u &= f \qquad \text{in } \Omega, \\ \mathcal{B}u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

on arbitrary bounded domains $\Omega \subset \mathbb{R}^N$. (By a domain we mean an open and connected set.) We always assume that $N \geq 2$. For $N = 1$ every bounded domain is smooth and there is nothing to prove. The differential operators $\mathcal{A}$ and $\mathcal{B}$ are supposed to be of the form

$$\mathcal{A}u := -\sum_{i=1}^{N} \partial_i \Big(\sum_{j=1}^{N} a_{ij}(x)\partial_j u + a_i(x)u \Big) + \sum_{i=1}^{N} b_i(x)\partial_i u + c_0(x)u \tag{2.2}$$

and

$$\mathcal{B}u := \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} a_{ij}(x)\partial_j u + a_i(x)u \Big)\nu_i + b_0(x)u\,, \tag{2.3}$$

where $\partial_i := \frac{\partial}{\partial x_i}$ and $\nu := (\nu_1, \dots, \nu_N)$ denotes the outer unit normal on the boundary $\partial\Omega$ of Ω. The coefficients a_{ij}, a_i, b_i and c_0 are real bounded and measurable functions on $\overline{\Omega}$, and b_0 real bounded and measurable on $\partial\Omega$ with respect to the $(N-1)$-dimensional Hausdorff measure. Further, we set $\mathbf{a} := (a_1, \dots, a_N)$ and $\mathbf{b} := (b_1, \dots, b_N)$. We also assume that $\mathcal{A}$ is uniformly strongly elliptic; that is, there exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \tag{2.4}$$

for all $x \in \overline{\Omega}$ and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$. Finally, we assume that

$$\inf_{x \in \partial\Omega} b_0(x) \geq \beta_0 \tag{2.5}$$

for some $\beta_0 > 0$ (take the essential infimum with respect to the $(N-1)$-dimensional Hausdorff measure if b_0 is only measurable).

We denote by $L_p(X)$ the Lebesgue spaces on a measurable subspace of $X \subset \mathbb{R}^N$ and by $\|\cdot\|_{p,X}$ its norm. If no confusion seems likely, we just write L_p and $\|\cdot\|_p$, respectively. Further, we write $L_{p,\mathrm{loc}}(X)$ for the space of all functions u such that $u|_K \in L_p(K)$ for all compact subsets K of X. If $p \in [1, \infty]$, we denote its dual exponent by p'; that is,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover, $W_p^k(\Omega)$ is the Sobolev space, which consists of all functions $u \in L_p(\Omega)$ such that all the distributional derivatives up to the order k lie in $L_p(\Omega)$. Further, $\mathcal{D}(\Omega)$ is the space of smooth functions with compact support in Ω, and $\mathring{W}_2^1(\Omega)$ its closure in $W_2^1(\Omega)$. Finally, $C(\Omega)$ and $C^k(\Omega)$ are the spaces of continuous and k-times differentiable functions, respectively. If E, F are Banach spaces, we write $E \hookrightarrow F$ if $E \subset F$ and the natural injection is continuous, and $E \overset{d}{\hookrightarrow} F$ if in addition E is dense in F. We write E' for the topological dual of E. The domain of definition of a linear operator A on E we denote by $D(A)$. If $A\colon D(A) \to F$ is a densely defined linear operator, we denote its dual operator by A'. Further, $\mathcal{L}(E, F)$ is the Banach space of all bounded linear operators from E to F equipped with the usual operator norm, and $\mathcal{L}(E) := \mathcal{L}(E, E)$. Finally, $\|\cdot\|_{p,q}$ denotes the norm in $\mathcal{L}(L_p, L_q)$.

For the moment let Ω be a bounded Lipschitz domain in the sense that $\partial\Omega$ is locally the graph of a Lipschitz function. Then a function $u \in W_2^1(\Omega)$ is usually called a weak solution of (2.1) if

$$a(u,v) = \langle f, v\rangle \tag{2.6}$$

for all $v \in W_2^1(\Omega)$, where $a(\cdot\,,\cdot)$ is the (generalised) Dirichlet form corresponding to $(\mathcal{A},\mathcal{B})$ defined by

$$\begin{aligned} &a(u,v) \\ &\quad := \int_\Omega \Big(\sum_{i=1}^N \Big(\sum_{j=1}^N a_{ij}\partial_j u + a_i u\Big)\partial_i v + \Big(\sum_{i=1}^N b_i \partial_i u + c_0 u\Big) v\Big)\, dx + \int_{\partial\Omega} b_0 uv\, d\sigma \end{aligned} \tag{2.7}$$

for all $u, v \in W_2^1(\Omega)$. Here, σ is the $(N-1)$-dimensional Hausdorff measure restricted to $\partial\Omega$ which coincides with the usual surface measure if $\partial\Omega$ is smooth (e.g. [12, Theorem 3.2.3]). Further,

$$\langle f, v\rangle := \int_\Omega fv\, dx\,. \tag{2.8}$$

It is well known that if data and domain are smooth any sufficiently smooth weak solution of (2.1) satisfies (2.1) pointwise. Since $W_2^1(\Omega)$-functions do not have well defined traces on $\partial\Omega$ for an arbitrary domain, and if so, it is not clear whether the boundary integral appearing in (2.7) is finite, we cannot use the space $W_2^1(\Omega)$ when dealing with weak solutions of (2.1) on arbitrary domains. The precise definition of a weak solution will be given in Section 3. As in the case of the Dirichlet problem, it turns out that (2.6) makes only sense for $f \in L_p(\Omega)$ if p is larger than some $p_0 > 1$. In order to define solutions for inhomogeneities in "lower" L_p-spaces we "extrapolate" the problem by means of duality. This allows us to define generalised solutions of (2.1) for all $f \in L_p(\Omega)$ and $p \in [1,\infty]$ (see Definition 5.1). The first of our main results concerns the global L_p-regularity of generalised solutions.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain and suppose that $(\mathcal{A},\mathcal{B})$ are as above. Then, any (generalised) solution of* (2.1) *with $f \in L_p(\Omega)$ belongs to $L_{m(p)}(\Omega)$. Here, $1 \le p \le \infty$, and $m(p) := Np(N-p)^{-1}$ if $p \in (1,N)$, $m(p) = \infty$ if $p > N$ and $m(p) < N(N-1)^{-1}$ arbitrary if $p = 1$.*

Second, we establish the Fredholm alternative for solutions of (2.1) on general bounded domains.

Theorem 2.2. *The solutions of* (2.1) *satisfy the Fredholm alternative in L_p ($1 \le p \le \infty$); that is, either* (2.1) *has a unique solution for all $f \in L_p(\Omega)$, or there are infinitely many solutions for some $f \in L_p(\Omega)$ and none for others.*

Finally, we get control over the norm of the resolvent in terms of the coefficients of $(\mathcal{A},\mathcal{B})$ and the measure $|\Omega|$ of the domain Ω. This result is new even in the case of smooth domains, since the known estimates all depend on the geometry of the domain.

Theorem 2.3. *Define the quantities*

$$\gamma := \max\{\alpha_0^{-1}, \beta_0^{-1}\} \qquad \textit{and} \qquad \delta := \alpha_0^{-1}\big(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty\big)^2 + \|c_0^-\|_\infty, \tag{2.9}$$

where c_0^- is the negative part of c_0. Then there exists a constant $C > 0$ depending only on N, p and upper bounds for γ, δ and $|\Omega|$ such that any (generalised) solution *of*

$$\begin{aligned} \mathcal{A}u + \lambda u &= f \qquad \text{in } \Omega, \\ \mathcal{B}u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.10}$$

with $\lambda \geq \delta$ *satisfies the a priori estimate*

$$\|u\|_{m(p)} \leq C\|f\|_p \tag{2.11}$$

for all $f \in L_p(\Omega)$*, where* $m(p)$ *is as in Theorem 2.1. If* $a_i = 0$ *for* $i = 1, \dots, N$*,* $c_0 \geq 0$ *and* $p > N$*, the above estimate holds for* $\delta = 0$*.*

We also show that the eigenvalue problem

$$\begin{aligned} -\mathcal{A}\varphi &= \lambda\varphi \qquad \text{in } \Omega, \\ \mathcal{B}\varphi &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.12}$$

has discrete spectrum, and that the first eigenvalue is algebraically simple with positive eigenfunction. Here, φ is in the Hilbert space used to define weak solutions on general domains introduced later. As a consequence of the above results we show that under some additional assumption, the spectrum has a positive lower bound uniformly with respect to all domains of a given volume.

Corollary 2.4. *Suppose that* $c_0 \geq 0$ *and that* $a_i = 0$ *or* $b_i = 0$ *for all* $i = 1, \dots, N$*. Then, there exists* $\lambda_* > 0$ *depending only on* N *and upper bounds for* γ, δ *and* $|\Omega|$ *such that* $\operatorname{Re}\lambda \geq \lambda_*$ *for all eigenvalues* λ *of* (2.12)*.*

After rewriting the above results in an abstract form, they will be proved in Section 5.

Remark 2.5. (a) If $\mathcal{A} = -\Delta$ and b_0 is a constant, the existence of a positive lower bound λ_* for the first eigenvalue of (2.12) was first observed in Payne and Weinberger [23] for a class of smooth two- and three-dimensional domains Ω lying between two parallel planes. For a related result, see Beale [3, Lemma 4]. The result was rediscovered in [6] using similar ideas, and an extension to arbitrary bounded Lipschitz domains in any dimension was given. Again, for a class of smooth domains, it is shown in [23] that λ_* can be chosen to be the first eigenvalue of the Robin problem on a ball circumscribing the given domain. This result was reproved in Lax and Phillips [16] for all smooth bounded domains using the strong maximum principle. In [23], it is also shown by a counterexample that, unlike in the case of the Dirichlet problem, the first eigenvalue is not a monotone functional of the domain. It is still an open problem whether a Faber-Krahn type inequality holds; that is, whether λ_* can be chosen to be the first eigenvalue of (2.12) on a ball with volume $|\Omega|$. For partial results on that problem see Bossel [4] or Sperb [27, 28]. In all these references it is assumed that the domains satisfy certain geometric conditions such as convexity or restrictions on the curvature of the boundary.

(b) It turns out that the above theorem, its corollary, as well as Theorem 2.6 below remain true if we have Dirichlet boundary conditions on a closed subset of $\partial\Omega$ (see Remark 4.4(b) below).

We finally consider the parabolic problem

$$\begin{aligned} \partial_t u(x,t) + \mathcal{A}u(x,t) &= f(x,t) \qquad \text{in } \Omega \times (0,\infty), \\ \mathcal{B}u(x,t) &= 0 \qquad \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) &= u_0(\cdot) \qquad \text{in } \Omega\,, \end{aligned} \tag{2.13}$$

where $\partial_t := \partial/\partial t$, and we show that the following results hold.

Theorem 2.6. *Suppose that the same assumptions as in Theorem 2.1 hold and let $1 < p \leq q < \infty$ with*

$$N\Big(\frac{1}{p}-\frac{1}{q}\Big) < 1. \tag{2.14}$$

Then, for all $u_0 \in L_q$ and $f \in L_\infty\big((0,\infty), L_p\big)$, equation (2.13) has a unique generalised solution $u \in C\big([0,T], L_q\big)$.

A proof of the above theorem as well as the precise definition of a generalised solution will be given in Section 6. The idea is to reformulate (2.13) as an abstract parabolic equation in L_p-spaces, where standard semigroup theory applies. Clearly, one can then use the same framework to prove existence and uniqueness of solutions of semilinear initial value problems.

3. The L_2-theory

We start this section by recalling Maz'ja's approach to (2.1) on general bounded domains. To do so we define $\tilde{V}$ to be the abstract completion of the space

$$V_0 := V_0(\Omega) := \{u \in W_2^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega) \colon \|u\|_V < \infty\},$$

endowed with the norm $\|\cdot\|_V$ given by

$$\|u\|_V := \big(\|\nabla u\|_2^2 + \|u|_{\partial\Omega}\|_{2,\partial\Omega}^2\big)^{\frac{1}{2}}.$$

Here, $\|\cdot\|_{2,\partial\Omega}$ is the norm in $L_2(\partial\Omega) := L_2(\partial\Omega,\sigma)$, and σ is the restriction of the $(N-1)$-dimensional Hausdorff measure to $\partial\Omega$. (In Maz'ja's notation this is the space $W_{2,2}^1(\Omega,\partial\Omega)$.) The key to the whole theory is the following inequality due to Maz'ja ([18], see [20, Corollary 4.11.1/2]). It asserts that for all $u \in V_0$ the inequality

$$\|u\|_{\frac{2N}{N-1}} \leq c(N,|\Omega|)\|u\|_V \tag{3.1}$$

holds, where $c(N,|\Omega|) > 0$ is a constant depending only on N and an upper bound for $|\Omega|$. This inequality tells us that the natural embedding

$$j_0 \colon V_0 \to L_{\frac{2N}{N-1}}(\Omega)$$

is continuous with norm dominated by $c(N,|\Omega|)$. Hence j_0 has a unique extension $j \in \mathcal{L}(\tilde{V}, L_{\frac{2N}{N-1}})$. Using inequality (3.1) it is easy to check that the form $a(\cdot\,,\cdot)$ defined by (2.7) is continuous on $V_0 \times V_0$. Therefore, it has a unique continuous extension to $\tilde{V} \times \tilde{V}$, which we denote again by $a(\cdot\,,\cdot)$. Moreover, using (3.1) and a standard argument it is easy to see that there exist constants $\lambda_0 \in \mathbb{R}$ and $\alpha > 0$ such that

$$a(u,u) + \lambda_0\|u\|_2^2 \geq \alpha\|u\|_V^2 \tag{3.2}$$

for all $u \in V_0$ and hence for $u \in \tilde{V}$ by continuity and density (see (4.5) with $q = 2$ for explicit values for λ_0, α). Since obviously $\mathcal{D}(\Omega)$ is a subspace of $\tilde{V}$ the image of j is dense in $L_{\frac{2N}{N-1}}$. Therefore, the dual map $j' \in \mathcal{L}(L_{\frac{2N}{N+1}}, \tilde{V}')$ is an injection, which means that each element of $L_{\frac{2N}{N+1}}$ can be identified in a unique way with an element of $\tilde{V}'$. Thus, the following definition makes sense.

Definition 3.1. Let $f \in L_p(\Omega)$ with $2N(N+1)^{-1} \le p \le \infty$. Then, $u \in \tilde{V}$ is said to be a weak solution of (2.1) if and only if

$$a(u,v) = \langle f, j(v)\rangle \quad (= \langle j'(f), v\rangle_V) \tag{3.3}$$

holds for all $v \in \tilde{V}$ (or a dense subset thereof), where $\langle\cdot\,,\cdot\rangle_V$ is the duality pairing between $\tilde{V}$ and $\tilde{V}'$, and $\langle\cdot\,,\cdot\rangle$ is defined by (2.8). In abuse of notation we often write $\langle f, v\rangle$ rather than $\langle f, j(v)\rangle$.

Remark 3.2. (a) If Ω is a Lipschitz domain, then, due to (3.1), the trace inequality $\|u\|_{2,\partial\Omega} \le c\|u\|_{W_2^1(\Omega)}$ (see e.g. [22]), and the fact that $C^\infty(\overline{\Omega})$ is dense in $W_2^1(\Omega)$ (see e.g. [1]) imply that $\tilde{V} = W_2^1(\Omega)$ up to an equivalent norm. Hence, if Ω is Lipschitz, the definition of a weak solution is the same as in the classical theory.

(b) Since $\mathcal{D}(\Omega) \subset \tilde{V}$, it follows that a weak solution of (2.1) is a weak solution of the equation $\mathcal{A}u = f$ in Ω in the usual sense; that is, $a_0(u,v) = \langle f, v\rangle$ for all $v \in \mathcal{D}(\Omega)$ where $a_0(u,v)$ is defined by (2.7) but without the boundary integral. Hence, all results on local properties of weak solutions of elliptic equations such as, for instance, those given in [13, Chapter 8] apply.

(c) If $(\mathcal{A},\mathcal{B})$ is selfadjoint and $c_0 \ge 0$, we can choose $\lambda_0 = 0$ in (3.2) and hence, by the Lax Milgram Theorem, (2.1) has a unique weak solution. This is Maz'ja's result in [19, 20].

(d) There is a difficulty with the space $\tilde{V}$, which is not mentioned in [19, 20]. By definition of the embedding $j \in \mathcal{L}(\tilde{V}, L_{\frac{2N}{N-1}})$ it is not clear whether it is an injection or not. Injectivity of j is equivalent to the fact that for any sequence $(u_n)_{n\in\mathbb{N}}$ in V_0 converging to u in $\tilde{V}$ such that $u_n \to 0$ in $L_{\frac{2N}{N-1}}$ implies that $u = 0$ in $\tilde{V}$. Using (3.1) it is clear that $u_n \to 0$ in $W_2^1(\Omega)$ as n tends to infinity. However, the question which seems to be difficult to answer is whether this implies that $u_n|_{\partial\Omega} \to 0$ in $L_2(\partial\Omega)$ for a general bounded domain Ω. The above arguments also show that $u \in \ker j$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in V_0 such that

$$\lim_{n\to\infty} u_n = 0 \text{ in } W_2^1(\Omega) \quad\text{and}\quad \lim_{n\to\infty} u_n|_{\partial\Omega} = w \text{ in } L_2(\partial\Omega). \tag{3.4}$$

We do not know of an example of a domain such that j is not an injection, and we suspect that it always is. However, we do not have a proof. For further comments concerning this question we refer to Remark 3.5. There is a way to get rid of this difficulty by replacing $\tilde{V}$ by a closed subspace of $\tilde{V}$. To see this, observe that by (3.3) any weak solution of (2.1) is orthogonal to the kernel, $\ker j$, of j with respect to the form $a(\cdot\,,\cdot)$ and, moreover, u is a weak solution of (2.1) if and only if (3.3) holds for v in

$$V_a := \{w \in \tilde{V} \colon a(w,v) = 0 \text{ for all } v \in \ker j\}\,.$$

If it happens that j is not injective, we could replace $\tilde{V}$ by V_a which a priori depends on the operators $(\mathcal{A},\mathcal{B})$. By (3.4) and the definition of $a(\cdot\,,\cdot)$ the function w is in V_a if and only if it is in $\tilde{V}$, and

$$\int_{\partial\Omega} b_0 wv\,d\sigma = 0 \tag{3.5}$$

for all $v \in \ker j$. In particular, this shows that V_a is independent of the coefficients of $\mathcal{A}$. It turns out that V_a is also independent of $\mathcal{B}$. The precise statement is contained in the following proposition.

Proposition 3.3. *For all* $(\mathcal{A}, \mathcal{B})$ *satisfying* (2.2)–(2.5) *we have*

$$V := V_a = (\ker j)^\perp,$$

where $(\ker j)^\perp$ *is the orthogonal complement of* $\ker j$ *in* $\tilde{V}$ *(the complement with respect to the inner product in* $\tilde{V}$*). Furthermore,* $\ker j$ *is isometrically isomorphic to* $L_2(S)$*, where* S *is a measurable subset of* $\partial\Omega$*, and*

$$V = (\ker j)^\perp = \{w \in \tilde{V} : w = 0 \text{ on } S\}. \tag{3.6}$$

Finally, the function u *is a weak solution of* (2.1) *if and only if* $u \in \tilde{V}$ *and* (3.3) *holds for all* $v \in V$.

The proof makes use of a technical lemma which we prove at the end of this section.

Lemma 3.4. *If* $v \in \ker j$*, then* $gv \in \ker j$ *for all* $g \in L_\infty(\partial\Omega)$.

Note that by definition of $\ker j$ we have that $v = 0$ in Ω, and thus it makes sense to multiply v by a function just defined on $\partial\Omega$.

Proof of Proposition 3.3. To prove the first assertion let $w \in \tilde{V}$. We have to show that (3.5) holds for all $v \in \ker j$ if and only if

$$\int_{\partial\Omega} wv \, d\sigma = 0$$

for all $v \in \ker j$. But this is easily seen from the above lemma and (2.5) since $b_0^{-1}v$ and $b_0 v$ are in the kernel of j. Let S be the union of all essential supports of $v \in \ker j$. Then, it follows from the above lemma that all simple functions on S are in $\ker j$, and hence, $\ker j = L_2(S)$. As a consequence of this, we obtain (3.6). The last assertion is clear from the remarks before Proposition 3.3. □

Next we have some more remarks and examples concerning the space $\tilde{V}$.

Remark 3.5. (a) Due to (3.1) it turns out that $\tilde{V}$ is topologically isomorphic to the closure of $\{(u, u|_{\partial\Omega}) : u \in V_0\}$ in $W_2^1(\Omega) \times L_2(\partial\Omega)$. All elements of $\tilde{V}$ have well defined traces on $\partial\Omega$ given by $\lim_{n\to\infty} u_n|_{\partial\Omega}$, where u_n is a sequence in V_0 converging to u in $\tilde{V}$. To keep notation as simple as possible we do not distinguish between u and its trace on $\partial\Omega$ in our notation. However, note that due to the same problem we had with the existence of a kernel for the embedding j we do not know how much the trace can "disconnect" from the function in Ω.

(b) The space

$$V_1 := \{u \in C(\overline{\Omega}) \cap W_2^1(\Omega) : \|u\|_V < \infty\}$$

is dense in $\tilde{V}$. Indeed, a careful analysis of the proof that $W_2^1(\Omega) \cap C^\infty(\Omega)$ is dense in $W_2^1(\Omega)$ such as given in [13, Theorem 7.9] reveals that, if $u \in C(\overline{\Omega}) \cap W_2^1(\Omega)$, the approximating function lies in $C(\overline{\Omega}) \cap C^\infty(\Omega) \cap W_2^1(\Omega)$, and has the same boundary values as u. Therefore, V_1 is dense in V_0 and hence in $\tilde{V}$.

(c) The space $\tilde{V}$ is a lattice, that is, if $u \in \tilde{V}$ then the absolute value $|u|$ also lies in $\tilde{V}$. Moreover, $\||u|\|_V = \|u\|_V$. To see this, note that by [13, Lemma 7.6] the space $W_2^1(\Omega)$ is a lattice, and that $\||u|\|_{W_2^1(\Omega)} = \|u\|_{W_2^1(\Omega)}$. Also, $C(\overline{\Omega})$ and $L_2(\partial\Omega)$ are lattices, and $\||u|\|_{2,\partial\Omega} = \|u\|_{2,\partial\Omega}$. Hence, $|u| \in V_1$ for all $u \in V_1$, where V_1 is the space defined in (b). Since the map defined by $u \mapsto |u|$ is continuous on V_1 it follows that $\tilde{V}$ is a lattice with the claimed properties.

(d) Suppose that Ω is a domain with a boundary whose $(N-1)$-dimensional Hausdorff measure is locally infinite; that is, for all $x \in \partial\Omega$ and $\varepsilon > 0$ we have that

$$\sigma\big(B_\varepsilon(x) \cap \partial\Omega\big) = \infty\,. \tag{3.7}$$

Then, $V = \tilde{V} = \mathring{W}_2^1(\Omega)$ up to an equivalent norm. In this case our boundary value problem coincides with the Dirichlet problem. If (3.7) only holds on part of the boundary, Dirichlet boundary conditions are satisfied on that part. Examples of such domains are domains with fractional boundaries such as, for instance, the interior of the well known "snowflake curve" (e.g. [11]). To prove our claim we first show that $V_0 \subset \mathring{W}_2^1(\Omega)$. To do so let $u \in V \cap C(\overline{\Omega})$ be arbitrary. Since u is continuous on $\overline{\Omega}$, it follows from (3.7) that $u|_{\partial\Omega} = 0$. By splitting u in positive and negative parts (which by (c) also belong to $V \cap C(\overline{\Omega})$) we can assume without loss of generality that $u \geq 0$. By the uniform continuity of u on $\overline{\Omega}$ it is clear that $u_\varepsilon := \max\{u-\varepsilon, 0\} \in \mathring{W}_2^1(\Omega)$ for all $\varepsilon > 0$. It is then easily seen that u_ε tends to u in V as ε goes to zero. Finally, note that $\|\nabla u\|_2$ is an equivalent norm on $\mathring{W}_2^1(\Omega)$, and thus u_ε also converges in $\mathring{W}_2^1(\Omega)$. Hence, it follows that $V = \mathring{W}_2^1(\Omega)$.

(e) The above remark together with Proposition 3.3 show that wherever the boundary is bad, either in the sense that its measure is locally infinite or that it allows j to have a kernel, the weak solutions of (2.1) satisfy Dirichlet boundary conditions in some sense. In the case of a nontrivial kernel we are not sure whether we really have Dirichlet boundary conditions in the weak sense because we do not know whether we can approximate the solutions by elements of V_0 being zero in a neighbourhood of that part of the boundary.

(f) Suppose that $K \subset \partial\Omega$ is a compact set of capacity zero and that $\partial\Omega \setminus K$ is locally Lipschitz in the sense that it is locally the graph of a Lipschitz function. In that case every $u \in \tilde{V}$ has locally a trace $\gamma_0 u$, and for all $x \in \partial\Omega \setminus K$ there exists a ball $B_\varepsilon(x)$ in $\mathbb{R}^N$ and a constant $c(\varepsilon, x)$ such that

$$\Big(\int_{\partial\Omega \cap B_\varepsilon(x)} |u(y)|^2\, d\sigma\Big)^{\frac{1}{2}} \leq c(\varepsilon, x)\|u\|_{W_2^1(\Omega)} \tag{3.8}$$

for all $u \in V_0$ (see e.g. [22]). This implies that $\gamma_0 u$ is in $L_{2,\mathrm{loc}}(\partial\Omega \setminus K)$. Then, it turns out that

$$\tilde{V} = V = \{u \in W_2^1(\Omega) \colon \gamma_0 u \in L_2(\partial\Omega \setminus K)\}\,. \tag{3.9}$$

This characterisation shows that, in general, the space $\tilde{V}$ is much smaller than $W_2^1(\Omega)$ even if the measure of $\partial\Omega$ is finite and the domain is smooth except for one point. As an example, consider a domain with an outward pointing cusp. Using (3.1) it is clear that $\tilde{V} \hookrightarrow W_2^1(\Omega)$. If we had equality, the open mapping theorem would imply that the two spaces coincide up to an equivalent norm. This is not possible since $\tilde{V} \hookrightarrow L_{\frac{2N}{N-1}}$ but $W_2^1(\Omega) \not\subset L_p$ for all $p > 2$ if the domain has an exponential cusp (see [1, Theorem 5.32]).

To prove (3.9) it is sufficient to prove that

$$V_1 := \{u \in W_2^1(\Omega) \cap C^\infty(\overline{\Omega}) \colon u = 0 \text{ in a neighbourhood of } K\}$$

is dense in $W_2^1(\Omega)$ as well as in V_0. It is well known that it is dense in $W_2^1(\Omega)$. Since K has zero capacity, there exists a sequence $\varphi_k \in \mathcal{D}(\mathbb{R}^N)$ with $0 \leq \varphi_k \leq 1$, $\varphi_k = 1$ in a neighbourhood of K such that $\|\varphi_k\|_{W_2^1(\Omega)}$ converges to zero as k tends to infinity. Let $u \in V_0$ be arbitrary and set $u_k := (1-\varphi_k)u$. Then obviously $u_k \in V_1$ and $\nabla u_k \to \nabla u$ in L_2 as k tends to infinity. Since K has zero capacity, we have that $\sigma(K)=0$ (e.g. [31, Theorem 2.6.16]) and thus φ_k converges to zero σ-almost everywhere on $\partial\Omega$. Since $|\varphi_k u|\le |u|$ on $\partial\Omega$, it follows from the Dominated Convergence Theorem that u_k converges to u in $L_2(\partial\Omega)$. This shows that V_1 is dense in V_0 and hence in V. This proves our claim.

(g) A sufficient condition making sure that j is injective is that $W_2^1(\Omega)$ allows, at least locally, a trace operator. More precisely, assume that for σ-almost all $x\in\partial\Omega$ (3.8) holds. Then, $u_n|_{\partial\Omega}$ converges to zero in $L_2(\partial\Omega\cap B_\varepsilon(x))$ if u_n converges to zero in $W_2^1(\Omega)$. From this we conclude that $u=0$ σ-almost everywhere on $\partial\Omega$, whence $u=0\in\tilde V$ and our claim follows.

We conclude this section by proving Lemma 3.4

Proof of Lemma 3.4. In a first step we assume that $g\in C^\infty(\overline{\Omega})$. If $v\in\ker j$ take $v_n\in V_0$ such that $v_n\to v$ in $\tilde V$. Clearly $gv_n\in V_0$ and by a simple calculation

$$\|gv_n\|_{W_2^1(\Omega)}\le(\|g\|_\infty+\|\nabla g\|_\infty)\|v_n\|_{W_2^1(\Omega)}\,.$$

Thus, since $\|v_n\|_{W_2^1(\Omega)}\to 0$ by (3.4), we conclude that $\|gv_n\|_{W_2^1(\Omega)}\to 0$. Moreover,

$$\|g(v_n-v)\|_{2,\partial\Omega}\le\|g\|_\infty\|v_n-v\|_{2,\partial\Omega}\to 0$$

since n goes to infinity. By (3.4) this implies that $gv\in\ker j$.

In a second step we assume that $g\in C(\overline{\Omega})$. Then, by Tietze's Theorem there exists a continuous extension of g to $\mathbb{R}^N$ which we denote again by g. This continuous function can then be approximated by smooth functions g_k uniformly on $\overline{\Omega}$. By the first step $g_k v\in\ker j$ for all $k\in\mathbb{N}$. Moreover, since

$$\|g_k v-g_\ell v\|_{\tilde V}^2=\int_{\partial\Omega}|g_k-g_\ell|^2v^2\,d\sigma \tag{3.10}$$

for all $k,\ell\in\mathbb{N}$, it turns out that $g_k v$ is a Cauchy sequence in $\tilde V$ converging to gv. Since $\ker j$ is closed $gv\in\ker j$.

We next try to approximate $g\in L_\infty(\partial\Omega)$ by continuous functions in a suitable way. To do so first note that the restriction of the $(N-1)$-dimensional Hausdorff measure σ to $\partial\Omega$ is an inner regular Borel measure. Fix $v\in\ker j$ and set for $k\in\dot{\mathbb{N}}:=\mathbb{N}\setminus\{0\}$,

$$A_k:=\{x\in\partial\Omega\colon |v(x)|\ge k^{-1}\}\qquad\text{and}\qquad A_0:=\{x\in\partial\Omega\colon |v(x)|\ne 0\}.$$

Then, the union of the A_k is A_0. Since $v\in L_2(\partial\Omega)$, it follows that $|A_k|:=\sigma(A_k)<\infty$ for all $k\in\dot{\mathbb{N}}$. By Lusin's Theorem (see [12, Theorem 2.3]) we find for every $k\in\dot{\mathbb{N}}$ a compact set C_k such that

$$|A_k\setminus C_k|<k^{-1} \tag{3.11}$$

and $g|_{C_k}$ is continuous. By Tietze's Theorem there exists a sequence of functions $g_k\in C(\partial\Omega)$ such that

$$\|g_k\|_\infty\le\|g\|_\infty \tag{3.12}$$

and $g_k=g$ on C_k. Without loss of generality, we can choose C_k in such a way that $C_{k+1}\supset C_k$ for all $k\in\dot{\mathbb{N}}$. We prove now that $g_k\to g$ σ-almost everywhere on A_0. Suppose this is not the case. Then, there exists a measurable set $B\subset A_0$ with $|B|>0$ such that $g_k\not\to g$ on B. Since the union of A_k is A_0, there exists $k_0\in\mathbb{N}$ and $\delta>0$ such that $|B\cap A_k|\ge\delta$ for all $k\ge k_0$. On the other hand, $|B\cap A_k|=|B\cap A_k\setminus C_k|+|B\cap C_k|>\delta$ for $k\ge k_0$, whence $|B\cap C_k|>\delta-k^{-1}>0$ by (3.11) for k large enough. Since $g=g_k$ on C_k, this is a contradiction. This shows that $g_k v \to gv$ almost everywhere on $\partial\Omega$. By step two, $g_k v \in \ker j$ for all $k \in \dot{\mathbb{N}}$. Moreover, by (3.12) the estimate $|g_k - g_\ell|^2 v^2 \leq 2\|g\|_\infty v^2$ holds. Hence, it follows from (3.10) and the Dominated Convergence Theorem that $g_k v$ is a Cauchy sequence in $\tilde{V}$ with limit gv. Since $\ker j$ is closed, this concludes the proof of the lemma. $\square$

4. Global estimates for weak solutions

In this section we prove global estimates for weak (sub-, super-) solutions of equation (2.1). As usual $u \in V$ is called a (weak) subsolution of (2.1) if

$$a(u, v) \leq \langle f, v\rangle \tag{4.1}$$

for all nonnegative $v \in V$, where V is as defined in Proposition 3.3. If the reverse inequality holds, u is called a (weak) supersolution. Further, we denote by u^+ and u^- the positive and negative parts of u, respectively; that is, $u^\pm := \max\{\pm u, 0\}$. Equivalently, we can replace V by $\tilde{V}$ in our definition, which is more convenient for practical purposes.

Theorem 4.1. *Suppose* $(\mathcal{A}, \mathcal{B})$ *satisfy* (2.2)–(2.5). *Then, if* $p \in [2, N)$*, there exists a constant* $c > 0$ *depending only on* N, p *and an upper bound for* $|\Omega|$ *such that for any weak solution of* (2.1)

$$\|u\|_{\frac{Np}{N-p}} + \|u\|_{\frac{(N-1)p}{N-p},\partial\Omega} \leq c\gamma\big(\|f\|_p + \gamma^\mu \delta^{1+\mu}\|u\|_2\big)\,, \tag{4.2}$$

where γ, δ *are as in* (2.9) *and* $\mu := N(p-2)/2p$. *If* $p > N$*, there exists a constant* $c > 0$ *depending on the same quantities as above such that*

$$\|u\|_\infty \leq c\gamma\big(\|f\|_p + \gamma^\mu \delta^{1+\mu}\|u\|_2\big)\,. \tag{4.3}$$

Moreover,

$$\|u\|_{\infty,\partial\Omega} \leq \|u\|_\infty. \tag{4.4}$$

Finally, if u *is a sub- or supersolution, the above assertions hold for* u *replaced by* u^+ *or* u^-*, respectively.*

Assertion (4.4) is not completely obvious since we do not know how much the trace of a function $u \in V$ on $\partial\Omega$ can "disconnect" from the function in Ω as mentioned in Remark 3.5(a).

Proof of Theorem 4.1. Note that it is sufficient to give a proof for subsolutions. For supersolutions the assertion follows since $-u$ is a subsolution and for solutions by combining the two inequalities. For every $m, t \geq 1$ we define the function

$$G_{t,m}(\xi) := \begin{cases} 0 & \text{if } \xi \leq 0, \\ \xi^t & \text{if } \xi \in (0, m), \\ m^{t-1}\xi & \text{if } \xi \geq m. \end{cases}$$

Clearly $G_{t,m}$ is piecewise smooth and has a bounded derivative. Hence, $G_{t,m} \circ u \in W_2^1(\Omega)$ for all $u \in W_2^1(\Omega)$ by [13, Theorem 7.8]. It is also not hard to show that the substitution operator induced by $G_{t,m}$ is continuous on $W_2^1(\Omega)$ and on $L_2(\partial\Omega)$ (see also [17]). Hence, it is continuous on V_0 and therefore on V. To simplify notation we set $v := G_{q-1,m}(u)$ and $w := G_{q/2,m}(u)$ for some fixed $m \geq 1$ and $q \geq 2$. Then, taking into account [13, Theorem 7.8] we get that

$$\partial_j w \partial_i w = \frac{q^2}{4(q-1)} \partial_j u \partial_i v, \quad w \partial_i w = \frac{q}{2} v \partial_i u = \frac{q}{2(q-1)} u \partial_i v \quad \text{and} \quad w^2 = uv$$

if $u(x) \leq m$, and

$$\partial_j w \partial_i w = \partial_j u \partial_i v, \quad w \partial_i w = v \partial_i u = u \partial_i v \quad \text{and} \quad w^2 = uv$$

if $u(x) \geq m$. Using this as well as (2.4) we get that

$$\alpha_0 \|\nabla w\|_2^2 \leq \frac{q^2}{4(q-1)} a_0(u,v) + \frac{q}{2}\big(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty\big)\|w\|_2 \|\nabla w\|_2 + \frac{q}{2}\|c_0^-\|_\infty \|w\|_2^2 \,,$$

where $a_0(\cdot\,,\cdot)$ is defined as (2.7) but without the boundary integral. Using the elementary inequality $2\xi\eta \leq \varepsilon^{-1}\xi^2 + \varepsilon\eta^2$ for all $\xi, \eta \geq 0$ and $\varepsilon > 0$, and choosing $\varepsilon > 0$ appropriately we obtain

$$\alpha_0 \|\nabla w\|_2 \leq \frac{q^2}{2(q-1)} a_0(u,v) + \frac{q^2}{2}\delta \|w\|^2 \leq q\big(a_0(u,v) + \delta(q-1)\|w\|_2^2\big).$$

Obviously, we have that

$$\beta_0 \|w\|_{2,\partial\Omega} \leq q \int_{\partial\Omega} b_0 w^2 \, d\sigma = q \int_{\partial\Omega} b_0 uv \, d\sigma.$$

Combining the two estimates we get that

$$\|w\|_V^2 \leq \gamma q\Big(a(u,v) + \delta(q-1)\|w\|_2^2\Big). \tag{4.5}$$

Using that u is a subsolution of (2.1), Hölder's inequality as well as (3.1), we arrive at

$$\|w\|_{\frac{2d}{d-2}}^2 \leq \bar{c}\gamma^{-1}\|w\|_V^2 \leq \bar{c}q^2\Big(\|f\|_p \|v\|_{p'} + \delta\|w\|_2^2\Big)\,, \tag{4.6}$$

where we have set $d := 2N$ and $\bar{c} := \gamma c(N, |\Omega|)^2$. Here, $c(N, |\Omega|)$ is the constant from (3.1). Furthermore, if $p \geq 2$, then $r := 4p(3p-2)^{-1} \leq 2$, and by a well known interpolation inequality (e.g. [13, p. 146]) we obtain

$$\|w\|_2^2 \leq \|w\|_{\frac{2d}{d-2}}^{2(1-\theta)} \|w\|_r^{2\theta}$$

with θ satisfying $2^{-1} = \theta r^{-1} + (1-\theta)(d-2)(2d)^{-1}$. Applying Young's inequality (e.g. [13, p. 147]) and (3.1), this implies that for all $\varepsilon > 0$

$$\|w\|_2^2 \leq \varepsilon c(N, |\Omega|)^2 \|w\|_V^2 + \varepsilon^{-\mu}\|w\|_r^2$$

with $\mu := d(p-2)(4p)^{-1}$. Furthermore, by Hölder's inequality and the fact that $w^{2/q} \leq u^+$ we have

$$\|w\|_r^2 = \|w^{\frac{2}{q} + \frac{2}{q}(q-1)}\|_{\frac{r}{2}} \leq \|w^{\frac{2}{q}}\|_2 \|w^{\frac{2}{q}(q-1)}\|_{p'} \leq \|u^+\|_2 \|w^{\frac{2}{q}}\|_{p'(q-1)}^{q-1}.$$

Combining all this and setting $\varepsilon := (2\bar{c}\delta q^2)^{-1}$ we conclude from (4.6) that

$$\|w^{\frac{2}{q}}\|_{\frac{dq}{d-2}}^q \leq \bar{c}\gamma^{-1}\|w\|_V^2 \leq \bar{c}q^{2(1+\mu)}\Big(\|f\|_p + \bar{c}^\mu \delta^{1+\mu}\|u^+\|_2\Big)\|w^{\frac{2}{q}}\|_{p'(q-1)}^{q-1}\,. \tag{4.7}$$

To estimate $\|v\|_{p'}$ we also used that $v \leq w^{2(q-1)/q}$. Inequality (4.7) is the basis for all the estimates. If $p \in [2, d/2)$, we set $q := p(d-2)(d-2p)^{-1}$ and divide inequality (4.7) by $\|w^{2/q}\|_{\frac{dp}{d-2p}}^{q-1}$. Note that the sequence of functions $w^{2/q} = [G_{q/2,m}(u)]^{2/q}$ is increasing as m increases. It converges to u^+ as m goes to infinity. Hence the monotone convergence theorem implies that

$$\|u^+\|_{\frac{dp}{d-2p}} \le C\bar{c}\big(\|f\|_p + \bar{c}^\mu \delta^{1+\mu}\|u^+\|_2\big)\,, \tag{4.8}$$

where C depends on d and p only. To show that the L_∞-estimate (4.3) holds for $p > N = d/2$, we set $q_0 := 2$ and $q_{n+1} := 1 + \eta q_n$ for all $n \in \mathbb{N}$, where

$$\eta := \frac{d(p-1)}{p(d-2)}\,. \tag{4.9}$$

Moreover, set $\bar{u} := u^+ M^{-1}$ with $M := \bar{c}\big(\|f\|_p + \bar{c}^\mu \delta^{1+\mu}\|u\|_2\big)$ if $M \neq 0$. By letting m to infinity in (4.7) we obtain

$$\|\bar{u}\|^{q_{n+1}}_{\frac{dq_{n+1}}{d-2}} \le q_{n+1}^{2(1+\mu)}\|\bar{u}\|^{\eta q_n}_{\frac{dq_n}{d-2}}\,,$$

from which we get by induction that

$$\|\bar{u}\|_{\frac{dq_{n+1}}{d-2}} \le \prod_{k=1}^{n+1} q_k^{2(1+\mu)\frac{\eta^{n-k+1}}{q_{n+1}}}\|\bar{u}\|^{2\frac{\eta^{n+1}}{q_{n+1}}}_{\frac{2d}{d-2}}\,. \tag{4.10}$$

Observe now that $\eta \le q_{n+1}/q_n \le 2\eta$, and therefore $\eta^n \le q_n \le (2\eta)^n$. Furthermore, by induction, $q_{n+1} = \eta^{n+1} + \sum_{k=0}^{n+1}\eta^k$. Hence, it follows from the above that

$$\|\bar{u}\|_{\frac{dq_{n+1}}{d-2}} \le (2\eta)^{2(1+\mu)\sum_{k=1}^{n+1}k\eta^{-k}}\|\bar{u}\|^{2(1+\sum_{k=0}^{n+1}\eta^{-k})^{-1}}_{\frac{2d}{d-2}}\,.$$

Since $\eta > 1$ for the range of p under consideration, we can let n to infinity to get

$$\|\bar{u}\|_\infty \le C\|\bar{u}\|^\theta_{\frac{2d}{d-2}}\,, \tag{4.11}$$

where

$$\theta := 1 - \frac{1}{2\eta - 1} \in (0,1) \qquad \text{and} \qquad C := (2\eta)^{2(1+\mu)\sum_{k=1}^\infty k\eta^{-k}} < \infty.$$

depend on d and p only. Since $|\Omega| < \infty$, this implies that

$$\|\bar{u}\|_\infty \le \big(C|\Omega|^{\theta(\frac{1}{2}-\frac{1}{d})}\big)^{1-\theta} \tag{4.12}$$

and thus (4.3) follows. It remains to prove the estimate on the boundary. By definition of $\|\cdot\|_V$ and setting $q := p(N-1)(N-p)^{-1}$ it follows from (4.7) that

$$\|\bar{u}\|^q_{\frac{p(N-1)}{N-p},\partial\Omega} \le \|\bar{u}\|^{q-1}_{\frac{Np}{N-p}}\,.$$

Combining this with (4.8) we conclude that (4.2) holds. To prove (4.4) note that as a consequence of (4.7) we have that

$$\|\bar{u}\|_{q,\partial\Omega} \le \big(q^{\frac{1}{q}}\big)^{2(1+\mu)}\|\bar{u}\|_\infty^{1-\frac{1}{q}}$$

for all $q \ge 2$ and our assertion follows by letting q to infinity as the constants in front of $\|\bar{u}\|_\infty$ tend to one. This proves Theorem 4.1. $\square$

Corollary 4.2. *Under the assumptions of the above theorem there exists a constant $C > 0$ only depending on N, p and upper bounds for $\delta, |\Omega|$ and γ such that any weak solution of problem* (2.10) *with $\lambda \ge \delta$ satisfies the a priori estimate*

$$\|u\|_{\frac{Np}{N-p}} + \|u\|_{\frac{(N-1)p}{N-p},\partial\Omega} \le C\|f\|_p \tag{4.13}$$

if $p \in [2, N)$*, and*

$$\|u\|_\infty \leq C\|f\|_p \tag{4.14}$$

if $p > N$*.*

Proof. Note first that any weak solution of (2.10) satisfies the a priori estimates (4.2) and (4.3) with the same constants c and γ. Also, by replacing $\mathcal{A}$ by $\mathcal{A} + \lambda$ for $\lambda \geq 0$ the constant δ can be chosen to be the same. Therefore, it remains to estimate $\|u\|_2$. To achieve this, note that by (4.5) with $q = 2$, and (3.1) for any weak solution of (2.10)

$$\|u\|^2_{\frac{2d}{d-2}} \leq 2\bar{c}\big(a(u,u) + \delta\|u\|_2^2\big) = 2\bar{c}\big(\langle f, u\rangle - (\lambda - \delta)\|u\|_2^2\big) \leq 2\bar{c}\|f\|_2\|u\|_2,$$

where for the last inequality we used the Cauchy Schwarz inequality and the assumption that $\lambda \geq \delta$. Hence, by using the embeddings for L_p-spaces we get that

$$\|u\|_2 \leq 2\bar{c}|\Omega|^{\frac{1}{2N}}\|f\|_2 \leq 2\bar{c}|\Omega|^{\frac{N+1}{2N} - \frac{1}{p}}\|f\|_p$$

whenever $p \geq 2$. This proves our claim. $\square$

Under an additional structure condition we get an L_∞-estimate for solutions of (2.1) of the form (4.14). However, we are not able to deduce the corresponding estimate (4.13) for $p \in [2, N)$ without losing control over the constant C. The same problems occur in the case of the Dirichlet problem treated in [13, Theorem 8.16].

Theorem 4.3. *Let* $(\mathcal{A}, \mathcal{B})$ *satisfy* (2.2)–(2.5) *with* $c_0 \geq 0$*. Suppose that, in addition, the structure condition*

$$\int_\Omega \sum_{i=1}^N a_i \partial_i \varphi + c_0 \varphi \, dx \geq 0 \tag{4.15}$$

is satisfied for all nonnegative $\varphi \in W_1^1(\Omega)$*. Then, if* $p > N$*, there exists a constant* $C > 0$ *depending only on* N, p *and upper bounds for* $|\Omega|$*,* γ *and* δ *such that for any weak solution* u *of* (2.1) *the estimate* (4.14) *holds. If* u *is a sub- or a supersolution, the inequality remains true if we replace* u *by* u^+ *and* u^-*, respectively.*

Proof. As in Theorem 4.1 it is sufficient to prove our claim for subsolutions. The proof of (4.14) is based on a modification of a test function argument due to Trudinger [29] (see [13, proof of Theorem 8.16]). For Robin boundary conditions we need to take extra care of the boundary integral. To estimate it we need the elementary inequality

$$\Big(\log\frac{r}{r-s}\Big)^2 \leq \frac{s^2}{r(r-s)} \tag{4.16}$$

which holds for $r > 0$ and $s \in [0, r)$. To prove it we set $t := s/r$. Then, (4.16) is equivalent to $(\log(1-t))^2 \leq t^2(1-t)^{-1}$ for $t \in [0, 1)$. For $t = 0$ the latter inequality is obviously true. For $t \in (0, 1)$ it is equivalent to $g(t) := t^{-2}(1-t)\big(\log(1-t)\big)^2 \leq 1$. Using de l'Hôpital's rule we find that $\lim_{t\to 0} g(t) = 1$. Since g is nonnegative, it is therefore sufficient to show that

$$g'(t) = -t^{-2}\big(2 - t^{-1}(t-2)\log(1-t)\big)\log(1-t) < 0$$

for all $t \in (0,1)$, which is the case if and only if $t^{-1}(t-2)\log(1-t) > 2$. Using the Taylor expansion of $\log(1-t)$ we obtain

$$t^{-1}(t-2)\log(1-t) = 2 + \sum_{k=2}^{\infty}\Big(\frac{2}{k+1} - \frac{1}{k}\Big)t^k > 2$$

for all $t \in (0,1)$ completing the proof of (4.16). Set

$$v := \frac{u^+}{m+k-u^+} \qquad \text{and} \qquad w := \log\frac{m+k}{m+k-u^+}\,,$$

where $m := \|u^+\|_\infty$ and $k := \|f\|_p$. We already know from Theorem 4.1 that $u \in L_\infty$ so the above definitions make sense. Setting $r := m+k$ and $s = u^+$ we conclude from (4.16) that

$$\int_{\partial\Omega} w^2\,d\sigma \le \frac{1}{m+k}\int_{\partial\Omega}\frac{(u^+)^2}{m+k-u^+}\,d\sigma = \frac{1}{m+k}\int_{\partial\Omega} uv\,d\sigma\,.$$

Then, by an elementary calculation and (4.15) we get that

$$\begin{aligned}
\|w\|_V^2 &\le \frac{\gamma}{m+k}\Big(\int_\Omega a_{ij}\partial_j u\partial_i v\,dx + \int_{\partial\Omega} b_0 uv\,d\sigma\Big)\\
&\le \frac{\gamma}{m+k}\Big(a(u,v) + \int_\Omega\sum_{i=1}^N (a_i - b_i)u^+\partial_i w\,dx - \int_\Omega\sum_{i=1}^N a_i\partial_i(uv) + c_0 uv\,dx\Big)\\
&\le \gamma\int_\Omega |f|k^{-1}\,dx + \gamma\int_\Omega\sum_{i=1}^N |a_i - b_i||\nabla w|\,dx\\
&\le \gamma c + \gamma^2\|\mathbf{a}-\mathbf{b}\|_2^2 + \frac{1}{2}\|w\|_V^2,
\end{aligned}$$

where c just depends on $|\Omega|$. Here we used the product rule for functions in V, which holds since V is a subspace of $W_2^1(\Omega)$. This together with (3.1) yields $\|w\|_2 \le C$, where C just depends on N and upper bounds for δ, γ and $|\Omega|$. Now set

$$v := \frac{\varphi}{m+k-u^+}$$

for all nonnegative $\varphi \in V_0$ with $uv \ge 0$. Further, note that

$$\log\frac{r}{r-s} \le \frac{s}{r-s}$$

for all $r > 0$ and $s \in [0,r)$. Indeed, this follows from the fact that $\log(1+t) \le t$ for $t \ge 0$ by setting $t := s(r-s)^{-1}$. Setting $r := m+k$ and $s = u^+$ the above inequality shows that $w\varphi \le uv$. Using this to estimate the boundary integral it follows by an elementary calculation that

$$\begin{aligned}
a_1(w,\varphi) &:= \int_\Omega\Big(\sum_{i,j=1}^N a_{ij}\partial_j w\partial_i\varphi + \sum_{i=1}^N (b_i - a_i)\varphi\partial_i w\Big)\,dx + \int_{\partial\Omega} b_0 w\varphi\\
&\le a(u,v) - \int_\Omega\varphi\sum_{i,j=1}^N a_{ij}\partial_j w\partial_i w\,dx - \int_\Omega\sum_{i=1}^N a_i\partial_i(uv) + c_0 uv\,dx\\
&\le \langle f, v\rangle \le \langle |f|k^{-1}, \varphi\rangle
\end{aligned}$$

for all nonnegative $\varphi \in V$ satisfying $u\varphi \ge 0$. Here we also used (2.4) and (4.15). Since $u\varphi \ge 0$ if and only if $w\varphi \ge 0$, this shows that $a_1(w,\varphi) \le \langle |f|k^{-1}, \varphi\rangle$ for all nonnegative $\varphi \in V_0$ such that $w\varphi \ge 0$. Next note that in the proof of Theorem 4.1 we just needed that (4.1) holds for all $0 \le v \in V$ such that $uv \ge 0$. Hence, as

$a_1(\cdot\,,\cdot)$ corresponds to an operator of the form (2.2), we may apply Theorem 4.1 to estimate $\|w\|_\infty$ in terms of $\|k^{-1}f\|_p = 1$ and $\|w\|_2$. Using the bound on $\|w\|_2$, proved above, we finally obtain (4.14) by taking exponentials. $\square$

We conclude this section with a few remarks. Note that these remarks also apply to the parabolic situation which we treat in Section 6.

Remark 4.4. (a) For smooth domains it is known that the solution u of (2.1) is in $W_p^1(\Omega)$ and its trace $\gamma_0 u$ in $W_p^{1-1/p}(\partial\Omega)$ if $f \in L_p$. Hence, by embedding theorems for Sobolev spaces $u \in L_{\frac{Np}{N-p}}(\Omega)$ and $\gamma_0 u \in L_{\frac{p(N-1)}{N-p}}(\partial\Omega)$. Inequality (4.2) tells us that $(u, \gamma_0 u) \in L_{\frac{Np}{N-p}}(\Omega) \times L_{\frac{p(N-1)}{N-p}}(\partial\Omega)$ for any bounded domain Ω even if the solution is not in the above Sobolev space. Note that even not in the case of a polygonal domain and smooth f we can expect the solution of the Robin problem to lie in $W_p^1(\Omega)$ for all $p > 2$. For a counterexample, see [14, Theorem 4.4.3.7], where the $S_{j,m}$ turn out to be not in $W_p^1(\Omega)$ for p large enough.

(b) The above results remain true if we have Dirichlet boundary conditions on a closed subset Γ_0 of $\partial\Omega$. In this case we replace the space V_0 introduced in Section 3 by

$$V_{0,\Gamma_0} := \{u \in V_0 \colon \operatorname{supp} u \in \overline{\Omega} \setminus \Gamma_0\}.$$

Then, (3.1) still holds for all $u \in V_{0,\Gamma_0}$ and hence, for all u in the completion V_{Γ_0} of V_{0,Γ_0} with respect to the norm $\|\cdot\|_V$. Since the proofs of the above theorems are based on (3.1), everything remains true.

(c) If we restrict the class of domains we can get better estimates than those in Theorem 4.1. For instance, if we take domains in the class $\mathcal{K}_{\alpha,1}$, $\alpha < 1$, introduced by Maz'ja (see [20, Section 3.6.1]), which are determined by the same constant $\mathcal{E}$, it turns out that $V \hookrightarrow L_{\frac{2d}{d-2}}$ for $d := 2(1-\alpha)^{-1}$ (replace u by u^2 in [20, Theorem 3.6.3]). In the above proofs we just have to replace $d = 2N$ by the new value of d and in the assertion N by $d/2$. The classes of domains mentioned above allow us to deal with nonsmooth domains having cusps of restricted sharpness. The sharpness of the cusp determines the class $\mathcal{K}_{\alpha,1}$ ([20, Section 3.6]). Above we only considered the class $\mathcal{K}_{1-\frac{1}{N},1}$ where all bounded domains belong to with the same constant $\mathcal{E}$ which is given by the isoperimetric constant (see [20, Example 3.6.2/1]).

(d) By using (3.1) we obviously can weaken the assumptions on the coefficients a_i, b_i and c_0. It is sufficient to assume that $a_i, b_i \in L_{\frac{r}{2}}$ and $c_0 \in L_r$ for some $r > N$ (r may vary from one coefficient to another). We just have to modify the definition of δ in an appropriate way.

5. The L_p-theory

By means of the results in the previous section and some facts on forms, we establish an L_p-theory for Robin boundary value problems. In particular, we prove Theorems 2.1, 2.2 and 2.3 stated in Section 2.

Suppose that V is as in Proposition 3.3, and that $a(\cdot\,,\cdot)$ is the form defined by (2.7). We already proved in Section 2 that $a(\cdot\,,\cdot)$ extends to a bounded bilinear form on V, and that it satisfies (3.2) for some $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$. Further, we put $H := L_2(\Omega)$. By means of the Riesz isomorphism we identify H with its dual space. Then, $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$. In this identification we use the duality pairing $\langle \cdot\,,\cdot \rangle$

on V induced by the L_2-inner product given by (2.8). The form $a(\cdot\,,\cdot)$ induces an operator $A \in \mathcal{L}(V, V')$ satisfying

$$a(u,v) = \langle Au, v\rangle \tag{5.1}$$

for all $u, v \in V$. Indeed, as V is reflexive, the linear operator from V' to V' defined by $v \mapsto \langle v, \cdot\rangle$ is a topological isomorphism. Since $a(u,\cdot) \in V'$, it follows that there exists $w \in V'$ such that $a(u,v) = \langle w, v\rangle$ for all $v \in V$. We then define $Au := w$. It is easy to see that $A\colon V \to V'$ is a linear operator. By definition of the dual norm on V' and the boundedness of the form $a(\cdot\,,\cdot)$ we have

$$\|Au\|_{V'} = \sup_{\substack{\|v\|_V=1\\ v\in V}} |\langle Au, v\rangle| = \sup_{\substack{\|v\|_V=1\\ v\in V}} |a(u,v)| \leq c\|u\|_v$$

for all $u \in V$ which shows that $A \in \mathcal{L}(V, V')$. In this framework we can write (2.1) in the abstract form

$$Au = f \tag{5.2}$$

for all $f \in V'$. By a solution of (5.2) we mean a function $u \in V$ satisfying (5.2). We saw in Section 3 that $f \in L_p(\Omega)$ can be identified with some $f \in V'$ if $p \geq 2N(N+1)^{-1}$. For such f it is easily verified that u is a weak solution of (2.1) if and only if it is a solution of (5.2). For $p \geq 2N(N+1)^{-1}$ we define the L_p-realization A_p of A by

$$A_p u = Au \qquad \text{for all } u \in D(A_p) := \{u \in V \cap L_p \colon Au \in L_p\}. \tag{5.3}$$

This makes sense since for the range of p under consideration $L_p \hookrightarrow V'$. It is easy to see that A_p is a closed operator on L_p. Since $\mathcal{D}(\Omega) \subset V$, clearly $V \cap L_p$ is dense in $L_p(\Omega)$ for all $p \in [1,\infty)$. Using this, it turns out that $D(A_p)$ is dense in V and thus A_p is densely defined if $p < \infty$. For a proof we refer to Lemma A.2 in the appendix. Next we define the adjoint form $a^\sharp(\cdot\,,\cdot)$ by

$$a^\sharp(u,v) := a(v,u) \tag{5.4}$$

for all $u, v \in V$. This form corresponds to the formal adjoint boundary value problem $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ to $(\mathcal{A}, \mathcal{B})$ given by

$$\mathcal{A}^\sharp u := -\sum_{i=1}^N \partial_i\Big(\sum_{j=1}^N a_{ji}(x)\partial_j u + b_i(x)u\Big) + \sum_{i=1}^N a_i(x)\partial_i u + c_0(x)u$$

and

$$\mathcal{B}^\sharp u := \sum_{i=1}^N\Big(\sum_{j=1}^N a_{ji}(x)\partial_j u + b_i(x)u\Big)\nu_i + b_0(x)u.$$

Note that $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ has the same structure as $(\mathcal{A}, \mathcal{B})$. Therefore, the operator $A^\sharp$ induced by $a^\sharp(\cdot\,,\cdot)$ on V' and its L_p-realization $A_p^\sharp$ have the same properties as A and A_p, respectively. By the above facts it makes sense to define

$$A_p := (A^\sharp_{p'})' \tag{5.5}$$

for $p \in (1,2]$. It turns out that A_q is the maximal restriction of A_p to L_q, and that A_q is the closure of A_p in L_q, whenever $1 < p < q < \infty$. Note that the operators A_p were defined already for $p \in [2N(N+1)^{-1}, 2]$. Hence for this range of p the two definitions are consistent. Moreover, A_p is closable in L_1, and its closure A_1 is independent of $p > 1$. Similar assertions are true for $A_p^\sharp$. Finally, (5.5) holds for all $p \in [1, \infty)$. The proofs are contained in Appendix C.

Definition 5.1. Let $p \in [1, \infty]$. We say that u is a solution of

$$A_p u = f \tag{5.6}$$

if $f \in L_p(\Omega)$ and $u \in D(A_p)$. A solution of the above equation we called a generalised solution of (2.1).

Clearly, if $p \geq 2N(N+1)^{-1}$, a generalised solution of (2.1) is a weak solution. The following theorem contains the assertions of Theorem 2.1 in an abstract form.

Theorem 5.2. *For all $p \in [1, \infty]$ we have that $D(A_p) \subset L_{m(p)}(\Omega)$ and for all $\lambda \in \varrho(-A_p)$*

$$(\lambda + A_p)^{-1} \in \mathcal{L}(L_p, L_{m(p)}), \tag{5.7}$$

where $m(p)$ is as in Theorem 2.1. In particular, any generalised solution of (2.1) *with $f \in L_p$ is in $L_{m(p)}$.*

Proof. If $p \geq 2$, a generalised solution is a weak solution of (2.1). Hence, Theorem 4.1 applies, and it follows that $u \in L_{m(p)}(\Omega)$. In particular, this implies that $D(A_p) \subset L_{m(p)}$ and (5.7) follows from the closed graph theorem. Since similar statements are true for the adjoint problem, we get from (5.5) that $D(A_p) \subset L_{m(p)}$ also for $p \in [1, 2]$. This proves (5.7). □

Theorem 5.3. *For all $p \in (1, \infty)$ the operators A_p have compact resolvent.*

Proof. Suppose that $\lambda \in \varrho(-A_p)$ and that A_p has compact resolvent. Using (5.7) and a compactness property of the Riesz Thorin interpolation (see [15]) it follows that A_q has compact resolvent on L_q for all $q \in [p, m(p))$. For $p = 2$ compactness of the resolvent follows from the fact that V is compactly embedded into $L_2(\Omega)$ (see [20, Corollary 4.11.1/3]) and thus by a bootstrapping argument this follows for all $p \in [2, \infty)$. A similar argument shows the assertion for $p \in (1, 2]$. □

Corollary 5.4. *For the solutions of* (5.6) *the Fredholm alternative holds.*

Proof. If $\lambda \in \varrho(A_p)$, then u is a solution of (5.6) if and only if it is a solution of

$$u - \lambda(\lambda + A_p)^{-1} u = (\lambda + A_p)^{-1} f. \tag{5.8}$$

Further, since A_p has compact resolvent, $\mathrm{id} - \lambda(\lambda + A_p)^{-1}$ is a compact perturbation of the identity and hence is Fredholm of index zero. Therefore, the Fredholm alternative holds for solutions of (5.8) and thus the same is true for (5.6). □

Corollary 5.5. *The spectrum of A_p is independent of $p \in [1, \infty]$ and consists of eigenvalues of finite algebraic multiplicity. Moreover, all eigenfunctions lie in $L_\infty(\Omega)$.*

Proof. The p-independence of the spectrum follows from Theorem 5.2 and Theorem C.3(c) in Appendix C. That the spectrum consists of eigenvalues of finite algebraic multiplicity, and that the eigenfunctions are in L_∞ follows from Theorem 5.3 and 5.2. □

We next establish a more general abstract version of Theorem 2.3.

Theorem 5.6. *Suppose that δ and $m(p)$ are as in Theorem 2.1. Then, for all $p \in [1,\infty]$ we have $[\delta,\infty) \subset \varrho(-A_p)$, and there exists a constant $C > 0$ depending only on the quantities listed in Theorem 2.3 such that*

$$\|(\lambda + A_p)^{-1}\|_{p,m(p)} \leq C \tag{5.9}$$

for all $\lambda \geq \delta$. Suppose, in addition, that the structure condition (4.15) *is satisfied. Then, $[0,\infty) \subset \varrho(-A_p)$ and* (5.9) *holds for all $\lambda \geq 0$ and $p > N$ with a constant C depending on the same quantities.*

Proof. It follows from (4.5) by setting $q = 2$ that (3.2) is true for $\lambda_0 \geq \delta$ if we choose α suitably. By the Lax Milgram Theorem it follows that for all $\lambda \geq \delta$ the equation $Au + \lambda u = f$ has a unique weak solution for every $f \in L_p(\Omega) \hookrightarrow V'$, $p \geq 2$ and thus $[\delta,\infty) \subset \varrho(-A)$. Moreover, by Corollary 4.2 it follows that (5.9) holds. Since the adjoint problem has the same properties, and the norm of the dual operator is the same as for the original one, the inequality (5.9) is true for $p \in [1,2]$ as well. Suppose now that the structure condition (4.15) is satisfied. Then, it follows from Theorem 4.3 that for $\lambda \geq 0$ the solution of $Au + \lambda u = f$, if it exists, is unique. By the Fredholm alternative proved in Corollary 5.4 there exists a unique solution of (5.6) for all $f \in L_p(\Omega)$ with $p \geq N$. The a priori estimate follows from Theorem 4.3. In particular, we conclude that $[0,\infty) \subset \varrho(-A_p)$ for $p > N$. ☐

Remark 5.7. If the structure condition (4.15) is satisfied, one could get an estimate of the form (5.9) with $\lambda = 0$ for all $p \in [1,\infty]$ by using the fact that $0 \in \varrho(-A_p)$. However, by this argument we do not get control over the constant C. A similar problem arises in the case of Dirichlet boundary conditions (see [13, p. 191]).

The following corollary is a generalised version of Corollary 2.4 concerning the spectral bound $\mathrm{s}(-A_p) := \inf\{\operatorname{Re}\lambda \colon \lambda \in \sigma(-A_p)\}$ of $-A_p$. Since by Corollary 5.5 the spectrum is independent of $p \in [1,\infty]$, it does not matter for which p we prove the result.

Corollary 5.8. *Suppose that either the coefficients a_i or b_i satisfy the structure condition* (4.15)*. Then there exists $\lambda_* > 0$ depending only on N and upper bounds for γ, δ and $|\Omega|$ such that $\mathrm{s}(-A_p) \geq \lambda_*$ for all $p \in [1,\infty]$.*

Proof. Suppose that a_i satisfies the structure condition. Then by Theorem 5.6 $\|A_p^{-1}\|_{\mathcal{L}(L_p)}$ is bounded by a constant, depending only on the quantities listed in the corollary if $p > N$. Further, observe that the spectral radius of A_p^{-1} is given by $1/\mathrm{s}(-A_p)$ whence, $1/\mathrm{s}(-A_p) \leq \|A_p^{-1}\|_{\mathcal{L}(L_p)}$. Taking the infimum over all p we get the existence of $\lambda_* > 0$ with the required properties. Since the spectrum is independent of p, the claim holds for all $p \in [1,\infty]$. If b_i satisfies the structure condition, we consider the formal adjoint operator which brings us back to the previously considered case. Since the spectrum of the original and the adjoint operators are the same, the claim follows. ☐

Proposition 5.9. *The resolvent $(\lambda + A_p)^{-1}$ is positive and irreducible for all $\lambda \geq \delta$ and $p \in [1,\infty]$.*

Proof. Positivity of weak solutions of $(\lambda + A)u = f$ for $f \geq 0$ in $L_p(\Omega)$, $p \geq 2N(N+1)^{-1}$, follows by a simple test function argument (take the negative part of the solution $u := (\lambda + A)^{-1} f$ as a test function and conclude that it is zero). By a density argument it turns out that $(\lambda + A_p)^{-1}$ is positive for all $p \in [0,\infty]$.

Suppose now that $g \geq 0$ but $f \neq 0$. To show irreducibility it is sufficient to prove that there exists $k \in \mathbb{N}$ such that $U := (\lambda + A_p)^{-k} f$ is strictly positive almost everywhere in Ω, which means that u is a quasi interior point of the positive cone (see [26, Section III.8]). Using the smoothing properties of the resolvent established in Theorem 5.2 we find $n \in \mathbb{N}$ such that $(\lambda + A_p)^{-n}(L_1) \subset L_\infty$. Clearly, $v := (\lambda + A_p)^{-n} f \geq 0$ and $v \neq 0$. As $v \in L_\infty \cap W_2^1(\Omega)$ is a strict supersolution of $(\lambda + \mathcal{A})w = 0$ in Ω in the usual sense the Harnack inequality (cf. [13, Theorem 8.18]) implies that $(\lambda + A_p)^{-1} v > 0$ almost everywhere. Setting $k = n + 1$ our claim follows. $\square$

We next discuss some consequences of the above results on the spectrum of A_p.

Corollary 5.10. *The spectral bound of $-A_p$ is an algebraically simple eigenvalue. It is the only eigenvalue having a positive eigenfunction.*

Proof. Since $(\lambda + A)^{-1}$ is compact, positive and irreducible for λ large enough by the above proposition and Theorem 5.3, the first assertion on the spectral bound is a consequence of the abstract theory in [26, Section V.5]. $\square$

The following proposition shows that (5.7) is optimal.

Theorem 5.11. *There exists a bounded domain Ω with smooth boundary except for one point, and* (5.7) *fails if we replace $m(p)$ by any larger exponent.*

Proof. First note that there exists a domain Ω such that the embedding of V into $L_{\frac{2N}{N-1}}$ is not compact. Such a domain is depicted in Figure 1 (see [20, pp. 259–260]). This domain can easily be modified in such a way that its boundary is of class C^∞ except for one point. Suppose that (5.7) is true if we replace $m(p)$ by $q > m(p)$. Since A_p has compact resolvent, it follows from a compactness property of the Riesz Thorin interpolation (see [15]) that (5.7) is compact. If $p > 2N(N+1)^{-1}$, we choose $1 < r < 2N(N+1)^{-1}$. Then, by (5.7) we have $(\lambda + A_r)^{-1} \in \mathcal{L}(L_r, L_{m(r)})$ and hence by using the compactness property of the Riesz Thorin interpolation, again we conclude that $(\lambda + A)^{-1} \in \mathcal{L}(L_{\frac{2N}{N+1}}, L_{\frac{2N}{N-1}})$ is compact. Setting $E := L_{\frac{2N}{N-1}}$ it follows from Proposition A.4 that V is compactly embedded into $L_{\frac{2N}{N-1}}$. Recall that Ω was chosen in such a way that this embedding is not compact. Since this is a contradiction, the proof of the theorem is complete. $\square$

Remark 5.12. Using domain perturbation methods and the above theorem it is shown in [7, Theorem 5.1] that the estimate (5.9) cannot even be improved for

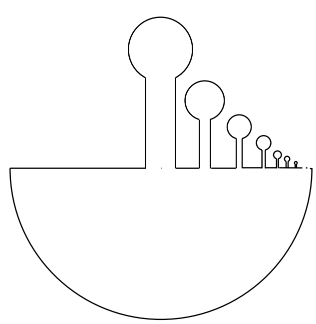

FIGURE 1.

smooth domains without making the upper bound C in (5.9) dependent on the geometry rather than the measure of the domain Ω.

6. Parabolic problems

In this section we shall briefly discuss some consequences on parabolic problems of the form (2.13). We start off with the homogeneous abstract Cauchy problem

$$\begin{aligned} \dot{u} + A_p u &= 0 \qquad \text{in } (0,\infty), \\ u(0) &= u_0\,, \end{aligned} \tag{6.1}$$

where $\dot{u} := du/dt$. By well known abstract results $-A_2$ generates a strongly continuous analytic semigroup $T_2(t) := e^{-tA_2}$ on $L_2(\Omega)$ (see e.g. [9, Proposition XVII.6/3]) and hence, (6.1) has a unique solution for all $u_0 \in L_2$ which is given by $u(t) = T_2(t)u_0$. We will show that $T_2(\cdot)$ acts on all L_p-spaces. To do so note that by a well known smoothing property of analytic semigroups $T_2(t)(L_2) \subset D(A_2^k)$ for all $k \in \mathbb{N}$ and $t > 0$. It follows from Theorem 5.2 and a bootstrapping argument that $D(A_2^k) \subset L_\infty$ for k large enough and therefore,

$$T_2(t) \in \mathcal{L}(L_2, L_\infty) \tag{6.2}$$

for all $t > 0$. In particular, this shows that T_2 acts on L_p for all $p \in [2,\infty]$ and thus $T_p(t) := T_2(t)|_{L_p}$ is a semigroup on L_p for that range of p. Obviously, we can do the same with the semigroup $T_2^\sharp$ generated by the formal adjoint $-A_2^\sharp$. Moreover, since $A_2' = A_2^\sharp$, we have that $T_2' = T_2^\sharp$ (see e.g. [24, Corollary 1.10.6]). Hence, it makes sense to set

$$T_p(t) := \bigl(T_{p'}^\sharp(t)\bigr)' \tag{6.3}$$

for all $p \in (1,2]$. Due to the above remarks we have for all $u, v \in L_2$ and $t > 0$,

$$|\langle T_2(t)u, v\rangle| = |\langle u, T_2^\sharp(t)v\rangle| \le \|u\|_1\|v\|_2\|T_2^\sharp(t)\|_{2,\infty}\,,$$

whence, $\|T_2(t)u\|_2 \le \|T_2^\sharp(t)\|_{2,\infty}\|u\|_1$. For this reason, $T_2(t)$ has a unique continuous extension to L_1 which we denote by $T_1(t)$. A simple density argument shows that $T_1(\cdot)$ is a semigroup on L_1. Often the definition of "analytic semigroup" includes strong continuity at zero. In this paper, by an analytic semigroup on the Banach space E we just mean a semigroup for which $T(\cdot)\colon (0,\infty) \to E$ is an analytic function. Hence, by the smoothing properties of $T_p(\cdot)$ we see that for each $\varepsilon > 0$ the map $t \mapsto T_p(t) = T_p(\varepsilon/2)T_2(t-\varepsilon)T_p(\varepsilon/2)$ is analytic from (ε,∞) to $L_p(\Omega)$ for $p \in [1,\infty]$, and thus $T_p(\cdot)$ is an analytic semigroup on L_p for all $p \in [1,\infty]$. In the following problem we collect some more properties of these semigroups.

Theorem 6.1. *For $1 \le p \le \infty$ the semigroup $T_p(\cdot)$ is compact positive irreducible and analytic on L_p. For $1 < p < \infty$ it is strongly continuous at zero, and its generator is $-A_p$. Moreover, for $2 \le p \le q \le \infty$, $p < \infty$ and γ, δ as in (2.9), we have that*

$$\|T_p(t)\|_{p,q} \le c(1+\delta t)^{N(\frac{1}{p}-\frac{1}{q})} t^{-N(\frac{1}{p}-\frac{1}{q})} e^{(p-1)\delta t}, \tag{6.4}$$

where c just depends on N and upper bounds for γ and $|\Omega|$.

Proof. We already know that T_2 is a strongly continuous analytic semigroup. Irreducibility of $T_2(t)$ for $t > 0$ follows since the resolvent of its generator $-A_2$ has the same property by Proposition 5.9 (see [21, Section C-III.3]). For $p \neq 2$ the compactness and irreducibility follow from the above by writing $T_p(t) =$

$T_2(t/3)T_2(t/3)T_p(t/3)$ and taking into account the smoothing properties of T_2. We next show that T_p is exponentially bounded on L_p. For all $q \geq 2$ set $u_q := \operatorname{sign} u|u|^{q-1}$. Applying (4.5) to u and $-u$, using the definition of A and letting m to infinity we get that

$$\|u\|^q_{\frac{qN}{N-1}} \leq q\bar{c}\langle Au + (q-1)\delta u, u_q\rangle \tag{6.5}$$

holds whenever $u \in V$ and the right-hand side is finite. Set $u(t) = e^{-(p-1)\delta t}T_p(t)u_0$ for $u_0 \in L_p$ and observe that $u \in C^\infty((0,\infty), L_p)$ for all p by (6.2). Differentiating $\|u(t)\|_p^p$ and using (6.5) we get the differential inequality $\frac{d}{dt}\|u(t)\|_p^p \leq -\|u(t)\|_p^p$ from which we conclude that for all $p \in [2,\infty)$ and $t \geq 0$

$$\|T_p(t)\|_{p,p} \leq e^{(p-1)\delta t}. \tag{6.6}$$

Strong continuity of $T_p(\cdot)$ at $t = 0$ for $p \in [2,\infty)$ follows by (6.6) by interpolation since $T_2(\cdot)$ is strongly continuous.

We know that $-A_2$ is the generator of T_2. To show that $-A_p$ generates T_p for $p \in (2,\infty)$ we prove that $D(A_p)$ is a core for the generator of T_p; that is, $D(A_p)$ is invariant under T_p, it is dense in L_p and it is contained in the domain of the definition of the generator. Since A_p is closed, it follows that $-A_p$ is the generator of T_p (e.g. [21, Proposition I.1.9]). Since $D(A_p) \subset D(A_2)$, we get for all $u \in D(A_p)$,

$$\frac{d}{dt}T_p(t)u = \frac{d}{dt}T_2(t)u = -A_2T_2(t)u = -T_2(t)A_pu.$$

By the smoothing property (6.2) it follows therefore that $D(A_p)$ is invariant under T_p. By letting t to zero and taking into account the strong continuity of T_p it also follows that $u \in D(A_p)$ is in the domain of the definition of the generator of T_p. Finally, since $-A_p$ is densely defined, it is the generator of T_p. For $p \in (1,2)$ this follows from (5.5) by duality using the adjoint semigroup (see [24, Theorem 1.10.6]).

It remains to prove (6.4). We first consider the case $q = \infty$. For $p = 2$ the result is a consequence of (6.5) and [30, Theorem II.3.5]. For $p \in (2,\infty)$ we have to modify the proof given there. The proof is based on (6.6) and an iteration process. This iteration process is started with $p = 2$. However, we could as well start with any given $p \in (2,\infty)$ and use the same arguments. More precisely, in their proof we have to put $p_\nu := pk^\nu$ for all $\nu \in \mathbb{N}$, where $k := N(N-1)^{-1}$ (note that $n = 2N$ in our case). It is left to the reader to check the details. This proves (6.4) for $q = \infty$. For finite q we use (6.6), the estimate we just proved, and the Riesz Thorin Interpolation Theorem. □

Corollary 6.2. *Suppose that $1 \leq p \leq q \leq \infty$ with $p \neq q$, $q = 1$ or $p = \infty$. Then for every $t_0 > 0$, there exists a constant $C(p, t_0)$ depending only on N, p, t_0 and upper bounds for γ, δ and $|\Omega|$ such that*

$$\|T_p(t)\|_{p,q} \leq C(p,t_0)t^{-N(\frac{1}{p}-\frac{1}{q})}. \tag{6.7}$$

Proof. For $p, q \in [2,\infty]$ this follows immediately from (6.4). For $p, q \in (1,2]$ it follows by duality. Finally, if $p < 2 < q$, the assertion follows by writing $\|T_p(t)\|_{p,q} \leq \|T_p(t/2)\|_{p,2}\|T_2(t/2)\|_{2,q}$ and using the previous estimates. □

Remark 6.3. The estimate (6.4) is optimal. By representing the resolvent of A by means of the Laplace transform a weaker singularity in t would imply better smoothing properties of the resolvent. This is not possible for general bounded domains as Theorem 5.11 shows.

Using the above results it is easy to deal with the abstract parabolic equation

$$\begin{aligned} \dot{u} + A_p u &= f \qquad \text{in } (0,\infty), \\ u(0) &= u_0 \,. \end{aligned} \tag{6.8}$$

As usual we call

$$u(t) := T_p(t)u_0 + \int_0^t T_p(t-s)f(s)\,ds \tag{6.9}$$

a mild solution of (6.8) (e.g. [24]). If u is a mild solution of (6.8), we say that u is a generalised solution of (2.13). We have the following abstract version of Theorem 2.6.

Theorem 6.4. *Let $f \in L_\infty((0,T), L_q)$ and $u_0 \in L_p$, where $1 < p \leq q < \infty$ satisfy* (2.14)*. Then,* (6.8) *has a unique mild solution u in $C([0,t], L_q)$.*

Proof. Using (6.7) and (2.14) it is clear that (6.9) exists. This proves the assertion. □

Remark 6.5. It is easy to see that a solution u of (6.8) is a weak solution of the parabolic problem $\partial_t u + \mathcal{A}u = f$ in $\Omega \times (0,\infty)$ in the usual sense. Hence, all the "interior regularity" results for parabolic equations apply to our situation as well.

Remark 6.6. By (6.4) with $p = 2$ and duality we get that for some constant $c > 0$

$$\|T_1(t)\|_{1,\infty} \leq c(1+\delta t)^N t^{-N} e^{\delta t}, \tag{6.10}$$

Therefore, T_1 has has a representation of the form

$$T_1(t)u(x) = \int_\Omega k_t(x,y)u(y)\,dy$$

where $k_t(\cdot\,,\cdot) \in L_\infty(\Omega) \otimes L_1(\Omega)$ is called the heat kernel (e.g. [25, Appendix C.1]). It is also well known that the estimate (6.10) leads to the estimate

$$|k_t(x,y)| \leq c(1+\delta t)^N t^{-N} e^{\delta t}\,.$$

Moreover, $k_t(x,y) \to \delta(x-y)$ weakly as $t \to 0$ (e.g. [25, Appendix C.1]). By the positivity of the semigroup we have that $k(x,y,t) > 0$ for all $x, y \in \Omega$ and $t > 0$.

Remark 6.7. For operators with $c_0 \geq 0$ and $a_i = b_i = 0$ for all $i = 1, \dots, N$ the results are simpler. In that case $\delta = 0$ and we have that for all $1 \leq p \leq q$,

$$\|T_p(t)\|_{p,q} \leq Ct^{-N(\frac{1}{p}-\frac{1}{q})}$$

with a constant only depending on N and upper bounds for γ and $|\Omega|$. For $2 \leq p \leq q \leq \infty$ this is even true if we only assume that $a_i = 0$. Indeed, if we analyse the calculations in the proof of Theorem 4.1 leading to (6.5), we see that if $a_i = 0$ for all $i = 1, \dots, N$, then (6.5) holds without the factor $(p-1)$ in front of δ. Hence, for $p \geq 2$ we have $\|T_p(t)\|_{p,p} \leq e^{\delta t}$ rather than (6.6). Hence, for $2 \leq p \leq q$ we have (6.4) with the factor $e^{\delta t}$ rather than $e^{(p-1)\delta t}$.

The following remark deals with the selfadjoint problem; that is, $a_{ij} = a_{ji}$ and $a_i = b_i = 0$ for all $i, j = 1, \dots, N$.

Remark 6.8. (a) For selfadjoint operators with $c_0 \leq 0$ we can prove that the heat kernel has an upper Gaussian bound of the form

$$0 \leq k_t(x,t) \leq Ct^{-N} e^{-\frac{|x-y|^2}{Ct}}, \tag{6.11}$$

where C depends on the same quantities as c in (6.10). The difference to the usual one is that N is replaced by $2N$. The proof is a modification of the one for the Dirichlet or Neumann problem such as given in [10] for the Dirichlet or Neumann problem. More precisely, The proof is a simple modification of the proof for the Dirichlet problem. First of all, note that by (6.10) (with $\delta = 0$) and the abstract Lemma 2.2.3 in [10] a logarithmic Sobolev inequality with $\beta(\varepsilon) = c - (N/2)\log\varepsilon$ and a constant c depending only on the quantities listed in the theorem holds for all $0 \leq u \in V \cap L_\infty \cap L_1$. Note also that $V_0 \cap V$ is a form core for $a(\cdot\,,\cdot)$. Now we can repeat the calculations in [10, Section 3.3], with minor modifications including the boundary term of the form $a(\cdot\,,\cdot)$.

(b) By using the Laplace transform we get from (6.11) the estimate

$$0 \leq g(x,y) \leq \frac{C^N\Gamma(N-1)}{|x-y|^{2(N-1)}}$$

for the kernel $g(\cdot\,,\cdot)$ of A^{-1}, where C is the same as in (6.11) and $\Gamma(\cdot)$ the Gamma function. The right-hand side of the above inequality is not integrable on a ball, but still the best possible bound since all the other bounds were optimal. This is no contradiction because this just means that this kind of singularity may only occur near a bad point of the boundary, and that a better interior estimate holds.

Appendix A. Maximal restrictions and duality

Assume that V, H are Hilbert spaces with $V \overset{d}{\hookrightarrow} H$. Identifying H with its dual we have that $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$. We further suppose that $\langle\cdot\,,\cdot\rangle$ is the duality pairing on V induced by the inner product $(\cdot|\cdot)$ of H; that is, $\langle u, v\rangle = (u|v)$ holds for all $u \in H \subset V'$ and $v \in V$. The duality pairing $\langle\cdot\,,\cdot\rangle$ is well defined since V is dense in H. Finally, we assume that $a(\cdot\,,\cdot)\colon V \times V \to \mathbb{R}$ is a continuous sesquilinear form and that there exist $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$ such that (3.2) holds. As shown at the beginning of Section 5 the form $a(\cdot\,,\cdot)$ induces an operator $A \in \mathcal{L}(V, V')$ satisfying (5.1) for all $u, v \in V$. Due to (3.2) and the Lax-Milgram Theorem $\lambda + A$ has a bounded inverse for all $\lambda \geq \lambda_0$. Since $V \overset{d}{\hookrightarrow} V'$, we may consider A as a densely defined operator on V' with a domain of definition V. It follows from the boundedness of $a(\cdot\,,\cdot)$ and (3.2) that the graph norm $\|u\|_{D(A)} = \|Au\|_{V'} + \|u\|_{V'}$ is equivalent to the norm on V. Since $A \in \mathcal{L}(V, V')$, it follows that A is a closed operator on V' with domain $D(A) = V$.

Definition A.1. If F is a Banach space with $F \hookrightarrow V'$, the F-realization A_F of A defined by

$$A_F u := Au \qquad \text{for all } u \in D(A_F) := \{v \in V \cap F\colon Av \in F\}$$

is a closed operator on F.

The operator A_F is also called the maximal restriction of A to F, or the part of A in F.

Lemma A.2. *Let $F \overset{d}{\hookrightarrow} V'$ and $(\lambda + A)^{-1}(F) \subset F$ for some $\lambda \in \varrho(-A)$. Then $D(A_F)$ is dense in V.*

Proof. First note that under the present assumptions $\varrho(A) \subset \varrho(A_F)$ so, in particular, $\lambda_0 \in \varrho(-A_F)$. It is sufficient to show that the orthogonal complement of $D(A_F)$ with respect to the bilinear form $a(\cdot\,,\cdot) + \lambda_0(\cdot \mid \cdot)_H$ is zero. To see this, note that $0 = a(u,v) + \lambda_0(u|v)_H = \langle(\lambda_0 + A_F)u, v\rangle$ for all $u \in D(A_F)$ implies that $v = 0$ since the range of $\lambda_0 + A_F$ is F which is dense in V'. □

The following proposition, which turns out to be useful later, is a consequence of a result due to Arendt [2].

Proposition A.3. *Let us suppose that $F \hookrightarrow V'$, that $(\lambda + A)^{-1}(F) \subset F$ and $(\lambda + A)^{-k}(V') \subset F$ for some $\lambda \in \varrho(-A)$ and $k \in \mathbb{N}$. Then, $\varrho(A) = \varrho(A_F)$ and $(\lambda + A_F)^{-1} = (\lambda + A)^{-1}|_F$ for all $\lambda \in \varrho(-A)$.*

Proposition A.4. *Suppose that E is a Banach space with $V \overset{d}{\hookrightarrow} E$. Then, the embedding $V \hookrightarrow E$ is compact if and only if $(\lambda_0 + A_{E'})^{-1} \in \mathcal{L}(E', E)$ is compact.*

Proof. That the condition is necessary is clear. To prove the reverse we can assume that $\lambda_0 = 0$ in (3.2) by replacing A by $\lambda_0 + A$. Doing so, we get that for all $u \in E' \hookrightarrow V'$,

$$\alpha\|A^{-1}u\|_V^2 \leq a(A^{-1}u, A^{-1}u) = \langle u, A^{-1}u\rangle \leq \|u\|_{E'}\|A^{-1}u\|_E\,. \tag{A.1}$$

Suppose that $B \subset E'$ is bounded and that w_n is a sequence in $A^{-1}(B)$; that is, $w_n = A^{-1}u_n$ for some $u_n \in B$. Since A^{-1} is compact as an operator from E' to E by hypotheses, it follows that w_n has a convergent subsequence in E. Applying (A.1) to $u = u_k - u_\ell$ we get that

$$\|w_k - w_\ell\|_V^2 \leq \alpha^{-1}\|u_k - u_\ell\|_{E'}\|w_k - w_\ell\|_E$$

for all $k, \ell \in \mathbb{N}$. Since B is bounded in E', this shows that this subsequence also converges in V. Consequently, $A^{-1}(B)$ is relatively compact in V, whence $A^{-1} \in \mathcal{L}(E', V)$ is compact. Therefore, the embedding $E' \hookrightarrow V'$ given by AA^{-1} is compact which is equivalent to $V \hookrightarrow E$ being compact. This concludes the proof of the proposition. □

Next we define extensions of A to Banach spaces which are not necessarily subspaces of V'. To do this let us consider the dual form $a^\sharp(\cdot\,,\cdot)\colon V \times V \to \mathbb{R}$ given by (5.4) for all $u, v \in V$. Denote by $A^\sharp$ the closed operator induced by $a^\sharp(\cdot\,,\cdot)$ on V' and let F be a Banach space such that $F' \overset{d}{\hookrightarrow} V'$ and $(\lambda + A^\sharp)^{-1}(F') \subset F'$. Assume, in addition, that $D(A^\sharp_{F'})$ is dense in F'. Then, we can define

$$A_F := (A^\sharp_{F'})', \tag{A.2}$$

where the domain of the definition of A_F is given by

$$D(A_F) = \left\{u \in F\colon v \mapsto \langle A^\sharp_{F'}v, u\rangle \text{ is continuous on } D(A^\sharp_{F'}) \subset F'\right\}.$$

Then the following proposition holds.

Proposition A.5. *Suppose F and A_F are defined as above. Moreover, let E be a Banach space such that $E \hookrightarrow F$, $E \overset{d}{\hookrightarrow} V'$ and $(\lambda + A)^{-1}(E) \subset E$ for some $\lambda \in \varrho(-A)$. Then, $(A_F)_E = A_E$, that is, the E-realization of A_F coincides with A_E. In particular, A_F is an extension of A_E.*

Proof. Let us assume without loss of generality that $0 \in \varrho(A) \cap \varrho(A^\sharp)$ by replacing A and $A^\sharp$ by $\lambda_0 + A$ and $\lambda_0 + A^\sharp$, respectively. By our assumptions it is clear that $0 \in \varrho(A_E) \cap \varrho(A^\sharp_{F'})$. Let $u \in D(A_E)$. Then, for all $v \in D(A^\sharp_{F'})$

$$|\langle A^\sharp_{F'}v, u\rangle| = |a^\sharp(v,u)| = |a(u,v)| = |\langle Au, v\rangle| \leq \|Au\|_F\|v\|_{F'} \leq c\|A_Eu\|_E\|v\|_{F'}\,,$$

where we used that $E \hookrightarrow F$. Hence, $u \in D(A_F)$. The above calculation also shows that $\langle A_F u, v\rangle = \langle A_E u, v\rangle$ for all $v \in D(A^\sharp_{F'})$, and therefore by density of $D(A^\sharp_{F'})$ in F' we conclude that $A_F u = A_E u$ for all $u \in D(A_E)$; that is, A_F is an extension of A_E. Next assume that $u \in D((A_F)_E)$; that is, $u \in D(A_F) \cap E$ such that $A_F u \in E$. Since $0 \in \varrho(A_E)$ and $A_F u \in E \hookrightarrow V'$, there exists $w \in D(A_E) \subset V$ such that $A_F u = A_E w \in E$. Hence, by the definition of A_F,

$$\langle A^\sharp_{F'} v, u\rangle = \langle A_F u, v\rangle = \langle A_E w, v\rangle = a(w,v) = a^\sharp(v,w) = \langle A^\sharp_{F'} v, w\rangle$$

for all $v \in D(A^\sharp_{F'})$. Since the range of $A^\sharp_{F'}$ is F' (recall that $0 \in \varrho(A^\sharp_{F'})$), it follows that $u = w \in V \cap E$ and $A_F u = A_E u \in E$ which completes the proof of the proposition. □

Remark A.6. (a) Taking $E = F = V'$ in the above proposition we get that $(A^\sharp)' = A$ which is of course well known. In particular, we have that $\varrho(A) = \varrho(A^\sharp)$.

(b) Let F be as in the above proposition, and suppose that $(\lambda + A^\sharp)^{-k}(V') \subset F'$ for some $\lambda \in \varrho(-A^\sharp)$ and $k \in \mathbb{N}$. Then, by definition (A.2) and Proposition A.3 we get that $\varrho(A_F) = \varrho(A^\sharp_{F'}) = \varrho(A^\sharp) = \varrho(A)$.

Appendix B. A priori estimates

Suppose that (X, m) is a finite measure space and let $L_p = L_p(X, m)$. We consider the special case that $H = L_2$, where the inner product is, as usual, given by $\int_X uv\,dm$. Further, assume that if $u \in V$, then also $|u| \in V$. Finally, let $a(\cdot\,,\cdot)$ be a form and let $A \in \mathcal{L}(V, V')$ be the induced operator as discussed in Appendix A. We suppose that there exist constants $d > 2$, $\bar c, \delta \geq 0$ and nondecreasing functions $c_1, c_2 \colon [2,\infty) \to [1,\infty)$ of polynomial growth such that

$$\|u\|^2_{\frac{qd}{d-2}} \leq \bar c c_1(q)\langle Au + c_2(q)\delta u, u_q\rangle \tag{B.1}$$

for all $u \in D(A)$ and $q \geq 2$ for which the right-hand side makes sense. Here, $u_q := \operatorname{sign} u |u|^{q-1}$. We show that this inequality leads to a priori estimates for solutions of the equation $Au = f$.

Theorem B.1. *Suppose u is the solution of the equation $Au = f$ with $f \in L_p$ and that* (B.1) *holds. Set $m(p) := dp(d-2p)^{-1}$ if $p \in (1, d/2)$ and $m(p) = \infty$ if $p > d/2$. Then, if $p \in [2, d/2) \cup (d/2, \infty]$,*

$$\|u\|_{m(p)} \leq C\bar c\big(\|f\|_p + \bar c^\mu \delta^{1+\mu}\|u\|_2\big), \tag{B.2}$$

where $C > 0$ only depends on d and p and an upper bound for $m(X)$, and $\mu = d(p-2)/4p$.

Proof. The proof of the above theorem is almost identical to part of the proof of Theorem 4.1. In this case we set $v := \operatorname{sign} u|u|^{q-1}$ and $w := |u|^{\frac{q}{2}}$. Then, if f is a solution of (5.2) we get the estimates (4.6) and (4.7) for $\|w\|^2_{\frac{2d}{d-2}}$ with q^2 replaced by $c_3(q) := \max\{c_1(q), c_2(q)\}$. If $p > d/2$ we proceed exactly as in the proof of Theorem 4.1. In case $p \in [2, d/2)$ we have to be more careful as we do not know a priori that $u \in L_{p'(q-1)}$ and hence we cannot just set $q := \bar q := p(d-2)(d-2p)^{-1}$ and divide inequality (4.7) by $\|u\|^{\bar q - 1}_{\frac{dp}{d-2p}}$ to get the result. The main problem therefore is to show that u has a finite $L_{\frac{dp}{d-2p}}$-norm. To do so we define q_n and η as in (4.9).

As for that range of p under consideration $q_n < \bar{q}$ and $\eta < 1$ we have that

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} \eta^n + \sum_{k=0}^{n} \eta^k = (1-\eta)^{-1} = \bar{q}.$$

Using these facts we deduce from (4.10) that

$$\|u\|_{\frac{dq_{n+1}}{d-2}} \le c_3(\bar{q})^{\frac{1+\mu}{2}\sum_{k=1}^{n+1}\eta^k} \|u\|_{\frac{2d}{d-2}}^{2(1+\sum_{k=0}^{n+1}\eta^{-k})^{-1}}$$

for all $n \in \mathbb{N}$. Letting n to infinity we obtain

$$\|u\|_{\frac{dp}{d-2p}} = \|u\|_{\frac{d\bar{q}}{d-2}} \le c_3(\bar{q})^{\frac{1+\mu}{2}\bar{q}} \|u\|_{\frac{2d}{d-2}}^{\theta} \tag{B.3}$$

with $\theta := (2\eta - 2)(2\theta - 1)^{-1}$. This completes the proof in the case $p \in [2, d/2)$. $\square$

Remark B.2. With a slight modification the above theorem remains true if we omit the assumption that $m(X) < \infty$. In that case we have to assume that $f \in L_2 \cap L_p$, and we get that for $p \in [2, d/2) \cup (d/2, \infty]$,

$$\|u\|_{m(p)} \le C\bar{c}^{1+\mu}\big(\|f\|_p + \delta^{1+\mu}\|u\|_2\big) + C\|u\|_2\,, \tag{B.4}$$

where $C > 0$ just depends on d and p. To see this, note first that the only place we used that the volume is finite is to pass from (4.11) and (B.3) to (B.2). So we only have to prove (B.4). To do so note that (4.11) and (B.3) imply that $\|u\|_{m(p)} \le M^{1-\theta}\|u\|_{\frac{2d}{d-2}}^{\theta}$ with M defined as in the proof of Theorem 4.1. By Young's inequality (e.g. [13, p. 145]) we get that $\|u\|_{m(p)} \le M + \|u\|_{\frac{2d}{d-2}}$. Finally, by a well known interpolation inequality (e.g. [13, p. 146]) we have that

$$\|u\|_{\frac{2d}{d-2}} \le \frac{1}{2}\|u\|_{m(p)} + 2^{\frac{1}{2}(\frac{d-2}{2d} - \frac{1}{m(p)})^{-1}}\|u\|_2$$

which completes the proof of (B.4). Note that (B.2) cannot be true on unbounded domains. As a counterexample consider the equation $-\Delta u = f$ in $\mathbb{R}^N$. In that case $\delta = 0$ and (B.2) would imply that $(-\Delta)^{-1}$ is bounded from L_p to $L_{m(p)}$, which is not true.

Appendix C. Smoothing properties of the resolvent

Throughout, in what follows, we make the same assumptions as in Appendix B. In addition, we suppose that

$$V \cap L_p \quad \text{is dense in} \quad L_p \tag{C.1}$$

for all $p \in [2, \infty)$. We further suppose that the operators A and $A^\sharp$ induced by the forms $a(\cdot\,,\cdot)$ and $a^\sharp(\cdot\,,\cdot)$ satisfy (B.1) with the same constants. Observe that by (B.1) with $q = 2$, (5.1) and the continuity of $a(\cdot\,,\cdot)$ on $V \times V$ we get that $\|u\|_{\frac{2d}{d-2}} \le c_1\|u\|_V$ and therefore $V \overset{d}{\hookrightarrow} L_{\frac{2d}{d-2}}$. Hence, by duality and the embeddings for L_p-spaces we conclude that

$$L_p \overset{d}{\hookrightarrow} V' \tag{C.2}$$

for all $p \in [2d(d+2)^{-1}, \infty]$.

Lemma C.1. *For all $p \in [2,\infty]$ we have that*

$$(\lambda_0 + A)^{-1} \in \mathcal{L}(L_p, L_{m(p)}) , \tag{C.3}$$

where $m(p)$ is as in Theorem B.1, and λ_0 from (3.2)*. Moreover, there exists $k \in \mathbb{N}$ only depending on d such that*

$$(\lambda_0 + A)^{-k}(V') \subset L_\infty .$$

The assertion is the same if we replace A by $A^\sharp$.

Proof. Clearly $\lambda_0 \in \varrho(-A) = \varrho(-A^\sharp)$ and $(\lambda_0 + A)^{-1}(V') \subset V \hookrightarrow L_2$. Hence, there exists a constant $c > 0$ such that $\|u\|_2 \le c\|f\|_p$ for $p \ge 2$, where $u := (\lambda_0 + A)^{-1} f$. Using (B.2) with A replaced by $A + \lambda_0$ the first assertion follows. For the second assertion we again use that $(\lambda_0 + A)^{-1}(V') \subset V \hookrightarrow L_2$ and iterate the first result a finite number of times depending on d only. □

For $p \ge 2$ we denote by A_p and $A_p^\sharp$ the L_p-realization of A and $A^\sharp$, respectively. Recall that $(L_p)' = L_{p'}$ with equal norms for all $p \in [1,\infty)$, where as usual $p^{-1} + (p')^{-1} = 1$. Thus, if $p \in (1,2]$, $(L_p)' \overset{d}{\hookrightarrow} V'$, and by the above lemma $(\lambda + A^\sharp)^{-1}(L_{p'}) \subset L_{p'}$. Hence, by assumption (C.1) and Lemma A.2 $D(A_{p'}^\sharp)$ is dense in $L_{p'}$. Therefore, for all $p \in (1,2)$ we can define A_p by (5.5) for $p \in (1,2]$. As a consequence of Proposition A.5 A_p is an extension of A_q for all $q > p$. On the other hand, A_p is the closure of A_q for such p, q. To see this it is sufficient to show that $D(A_\infty)$ is dense in $D(A_p)$ for all $p > 1$. Indeed, since L_∞ is dense in L_p, there exists a sequence v_n in L_∞ approaching $(\lambda_0 + A)u \in L_p$. Set $u_n := (\lambda_0 + A_\infty)^{-1} v_n$. Since $(\lambda_0 + A_\infty)^{-1}|_{L_p} = (\lambda_0 + A_p)^{-1}$, this shows the assertion.

Note that $D(A_\infty^\sharp)$ is not dense in L_∞ in general, and that $(L_\infty)' \ne L_1$. For this reason we cannot define A_1 to be the dual of $A_\infty^\sharp$. Instead, we define A_1 to be the closure of A_p for some arbitrary $p > 1$. The next lemma shows that this is possible.

Lemma C.2. *For $p > 1$ the operator A_p is closable in L_1 and the closure is independent of p. If we denote its closure by A_1 we have that $(A_1)' = A_\infty^\sharp$.*

Proof. Let $p \in (1,\infty)$ be arbitrary and suppose that $u_n \in L_1 \cap D(A_p)$ is a sequence such that $A_p u_n \to g$ in L_1 and $u_n \to 0$ in L_1 as n tends to infinity. For $w \in D(A_\infty^\sharp) \subset D(A_{p'}^\sharp)$ we have by (5.5) that $\langle A_p u_n, w\rangle = \langle u_n, A_{p'}^\sharp w\rangle$. Since $A_{p'}^\sharp w \in L_\infty$, it follows that the last term tends to zero, whereas the first tends to $\langle g, w\rangle$ as n tends to infinity. We know from (C.2) and Lemma A.2 that $D(A_\infty^\sharp)$ is dense in V and therefore dense in L_q for $q \le 2d(d-2)^{-1}$. Hence, we can approximate every function $v \in L_\infty$ pointwise by a sequence $w_n \in D(A_\infty^\sharp)$ with $\|w_n\|_\infty$ uniformly bounded and the Dominated Convergence Theorem yields $0 = \langle g, w_n\rangle \to \langle g, w\rangle = 0$, which implies that $g = 0$. This shows that A_p is closable. The closure is independent of p because A_p is the closure of A_∞ for all $p > 1$.

It remains to show that $(A_1)' = A_\infty^\sharp$. Let $u \in D(A_\infty^\sharp)$, that is, $u \in V \cap L_\infty$ and $A^\sharp u \in L_\infty$, and suppose that $v \in D(A_p) \subset D(A_1)$ for some $p \in (1,\infty)$. Using (5.5) we obtain

$$|\langle u, A_1 v\rangle| = |\langle u, A_p v\rangle| = |\langle A_{p'}^\sharp u, v\rangle| \le \|A_\infty^\sharp u\|_\infty \|v\|_1 .$$

Since $D(A_p)$ is dense in L_1, this implies that $u \in D((A_1)')$ and $(A_1)'u = A_\infty^\sharp u$. Suppose now that $u \in D((A_1)')$. Therefore, for all $v \in D(A_p)$ we have that

$$|\langle u, A_p v\rangle| = |\langle u, A_1 v\rangle| \le c\|v\|_1 \le c_1 \|v\|_p$$

for some constants $c, c_1 > 0$ independent of $v \in D(A_p)$. Hence, $u \in D((A_p)') = D(A^\sharp_{p'})$. By (5.5) we conclude that $(A_1)'u = (A_p)'u = A^\sharp_{p'}u = A^\sharp u$. Since $u \in L_\infty$ and $(A_1)'u \in L_\infty$, this implies that $u \in D(A^\sharp_\infty)$. This concludes the proof of the lemma. ☐

Theorem C.3. *Let A_p and $A^\sharp_p$ be defined as above for all $p \in [1,\infty]$. Then, the following assertions hold:*

(a) *For $p \in [1,\infty]$ the operator A_p is closed. It is densely defined if $p < \infty$. Moreover, A_p is the closure of A_q, and A_q the L_q-realization of A_p whenever $1 \leq p \leq q \leq \infty$.*

(b) *$A'_p = A^\sharp_{p'}$ for all $p \in [1,\infty)$.*

(c) *The spectrum of A_p is independent of $p \in [1,\infty]$. Moreover, if $V \hookrightarrow L_2$ compactly, then A_p has compact resolvent for all $p \in (1,\infty)$, and if A_p has compact resolvent for some $p \in (1,\infty)$, this is true for all p, and $V \hookrightarrow L_2$ compactly.*

(d) *For $p \in [1,\infty]$ and $\lambda \in \varrho(-A)$ (C.3) holds, where $m(p)$ is as in Theorem B.1 if $p \in (1, d/2) \cup (d/2, \infty]$ and $m(1) \in [1, d(d-2)^{-1})$ arbitrary if $p = 1$. Furthermore,*

$$\|(\lambda + A_p)^{-1}\|_{p,q} = \|(\lambda + A^\sharp_{q'})^{-1}\|_{q',p'} \tag{C.4}$$

whenever $p, q \in [1,\infty]$ and one of the norms is finite.

Proof. (a) is clear from Lemma C.2 and the preceeding discussion, and (b) follows from definition (5.5) and Lemma C.2. Independence of the spectrum of A_p from $p \in (1,\infty]$ is a consequence of the definition of A_p and Remark A.6(b). That $\varrho(A_1)$ is the same follows by duality from (b). Compactness of the resolvent in the case $p = 2$ is clear as by assumption $V \hookrightarrow L_2$ compactly. Compactness of the resolvent of A_p follows as in the proof of Theorem 5.3. If the resolvent is compact for some p, it follows by interpolation that A_2 has compact resolvent and therefore by Proposition A.4 that $V \hookrightarrow L_2$ compactly and we are back to the situation considered before. This proves (c). To prove (d) note that $[(\lambda + A_p)^{-1}]' = (\lambda + A^\sharp_{p'})^{-1}$ for all $p \in [1,\infty)$. Since an operator and its dual have the same norm, this implies (C.4). The remaining assertion follows from this together with (C.3). ☐

Note added in proof. In Proposition 3.3 it is not clear whether the set S is measurable in general. As we do not make use of the existence of S this does not affect the other results in the paper.

References

1. Robert A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975. MR **56:**9247
2. W. Arendt, *Gaussian estimates and interpolation of the spectrum in L_p*, Differential and Integral Equations **7** (1994), 1153–1168. MR **95e:**47066
3. J. Thomas Beale, *Scattering frequencies of resonators*, Comm. Pure Appl. Math. **26** (1973), 549–563. MR **50:**5217
4. M-H. Bossel, *Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimérique de Raleigh-Faber-Krahn*, Z. Angew. Math. Phys. **39** (1988), 733–742. MR **90a:**73103
5. E. N. Dancer, *The effect of domain shape on the number of positive solutions of certain nonlinear equations*, J. Differential Equations **74** (1988), 120–156. MR **89h:**35256
6. E. N. Dancer and D. Daners, *A priori bounds for solutions and spectra of Robin boundary value problems*, Tech. Report No. 94-33, University of Sydney, 1994.
7. ———, *Domain perturbation for elliptic equations subject to Robin boundary conditions*, J. Differential Equations **138** (1997), 86–132. MR **98e:**35017

8. Daniel Daners, *Heat kernel estimates for operators with boundary conditions*, Math. Nachr., to appear.
9. R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 5, Springer, Berlin, 1992. MR **92k:**00006
10. E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989. MR **90e:**35123
11. K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, Chichester, 1990. MR **92j:**28008
12. Herbert Federer, *Geometric Measure Theory*, Springer, Berlin, 1969. MR **41:**1976
13. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, Berlin, 1983. MR **86c:**35035
14. P. Grisvard, *Elliptic Equations in Nonsmooth Domains*, Pitman, London, 1985. MR **86m:**35044
15. M. A. Krasnosel'skiĭ, *On a theorem of M. Riesz*, Soviet Math. Dokl. **1** (1960), 229–231. MR **22:**9852
16. Peter D. Lax and Ralph S. Phillips, *On the scattering frequencies of the Laplace operator for exterior domains*, Comm. Pure Appl. Math. **25** (1972), 85–101. MR **45:**5531
17. M. Marcus and V. J. Mizel, *Every superposition operator mapping one Sobolev Space into another is continuous*, J. Funct. Anal. **33** (1979), 217–229. MR **80h:**47039
18. V. G. Maz'ja, *Classes of regions and embedding theorems for function spaces*, Soviet Math. Dokl. **1** (1960), 882–885. MR **23A:**3448
19. ———, *Zur Theorie Sobolewscher Räume*, Teubner-Texte zur Mathematik, Teubner, Leipzig, 1981.
20. ———, *Sobolev Spaces*, Springer, Berlin, 1985. MR **87g:**46056
21. R. Nagel et al., *One-parameter semigroups of positive operators*, Lecture Notes in Mathematics, vol. 1184, Springer, Berlin, 1986. MR **88i:**47022
22. Jindřich Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967. MR **37:**3168
23. L. E. Payne and H. F. Weinberger, *Lower bounds for vibration frequencies of elastically supported membranes and plates*, SIAM J. **5** (1957), 171–182. MR **19:**1110c
24. A. Pazy, *Semigroups of lieear operators and applications to partial differential equations*, Springer, New York, 1983. MR **85g:**47061
25. Derek W. Robinson, *Elliptic Operators on Lie Groups*, Oxford University Press, Oxford, 1991. MR **92m:**58133
26. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin, 1974. MR **54:**11023
27. René P. Sperb, *Untere und obere Schranken für den tiefsten Eigenwert der elastisch gestützten Membran*, Z. Angew. Math. Phys. **23** (1972), 231–244. MR **47:**1355
28. ———, *Maximum Principles and Their Applications*, Academic Press, New York, 1981. MR **84a:**35033
29. N. S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa **27** (1973), 265–308. MR **51:**6113
30. N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, 1992. MR **95f:**43008
31. William P. Ziemer, *Weakly Differentiable Functions*, Springer, New York, 1989. MR **91e:**46046

School of Mathematics and Statistics, University of Sydney, New South Wales 2006, Australia

E-mail address: D.Daners@maths.usyd.edu.au