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REPRESENTING NONNEGATIVE HOMOLOGY CLASSES OF $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ BY MINIMAL GENUS SMOOTH EMBEDDINGS

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Abstract. For any nonnegative class ξ in $H_2(\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2, \mathbf{Z})$, the minimal genus of smoothly embedded surfaces which represent ξ is given for $n \leq 9$, and in some cases with $n \geq 10$, the minimal genus is also given. For the finiteness of orbits under diffeomorphisms with minimal genus g, we prove that it is true for $n \leq 8$ with $g \geq 1$ and for $n \leq 9$ with $g \geq 2$.

1. Introduction

Before Donaldson's theory appeared, the main contributions for the problem of minimal genus embeddings in 4-manifolds were given by Kervaire and Milnor [KM1], Hsiang and Sczarba [HS] and Rohlin [Ro]. By using Donaldson's results, many progresses have been made for embedding 2-spheres in 4-manifolds, and most of the literature in this direction was included in the excellent survey paper of T. Lawson [L]. In particular, by using the celebrated result of Donaldson [D1], the problem for representing nonnegative homology classes of $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with $n \leq 9$ by smoothly embedded spheres was solved nearly completely. That means that for homology classes with zero square, it was solved completely for $n \leq 8$ by the author [Li], and for homology classes with positive squares, the result is complete for $n \leq 9$ by Kikuchi [K2] (Gao [G1] and Kikuchi [K1] had solved the case for $n \leq 3$). Gan [G2] gave another proof for the case of zero square with $n \leq 8$, but the problem for $n = 9$ still stands, although by Kervaire and Milnor [KM1], we know that a characteristic class in $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ with zero square cannot be represented by smoothly embedded spheres.

Seiberg-Witten invariants [Wi] shed new light on the above problem. The pioneer works of Kronheimer and Mrowka [KM2], and Morgan, Szabo and Taubes [MST] on Thom conjecture and its generalization, and the works of T.J.Li and A.K.Liu ([LL1], [LL2]) inspired by [KM2] and Taubes' works [T1]–[T4], advanced the problem from sphere embeddings to minimal genus embeddings. The minimal genus problem for S^2-bundles over surfaces has been recently solved completely by the author and T.J. Li [LiL2], and the same problem for positive classes in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with $n \leq 6$ is solved completely in [LiL1] (including partial results for $n = 7, 8$). In this paper, we give a lower bound for the minimal genus of any nonnegative class of $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with arbitrary n, and prove that if $n \leq 9$, the above lower bound is equal to the minimal genus.

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After this paper had been completed, the author was told by T.J. Li that Ruberman [Ru] got the same lower bounds of minimal genera for positive classes and part of zero classes by using a direct extension of the method of Kronheimer and Mrowka [KM2]. While the present paper shows the lower bounds work for all nonnegative classes by using a different method.

Notice that for $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$, $n = 9$ is a critical value in the following sense: if $n \leq 9$, all automorphisms preserving the intersection form are induced by diffeomorphisms, due to Wall [Wa]; while for $n > 9$, it is not the case, due to Friedman and Morgan [FM]. It was proved in [LiL1] by using Wall's result that for $n \leq 9$, any nonnegative homology class can become a reduced class by a diffeomorphism. One of the observations for this paper is that it is still true for $n > 9$, and we can consider only the reduced classes when dealing with the minimal genus problem.

If a homology class is realized by a nonsingular algebraic curve, then this curve offers a minimal genus embedding for the class. But in general, we do not know if a class is realized by such an algebraic curve (for an explanation of this, see also [Ue]). So, to prove that the lower bound that we obtain is the best, in some cases, we have to construct a smooth minimal genus embedding. This is the so-called positive technique by Donaldson in the last paragraph of his survey paper [D2] on Seiberg-Witten theory, and it turns out to be the most difficult parts of this paper.

The technique in [LiL1] to realize minimal genus embeddings was to use lines in $\mathbb{C}P^2$ or appeal to the result coming from algebraic geometry. The technique in this paper is to use lines and ellitipic curves together, it allows us to give complete results for nonnegative classes in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with $n \leq 9$ which includes the results of [Li] and [K2] obtained by different methods, and some results for $n > 9$. Fintushel and Stern [FS] created a method to get a lower bound for the number of intersection points of index $+1$ of an immersed sphere in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$, which may represents a negative class. Combining their result and our construction, we are able to obtain minimal genera for some negative classes.

Another observation here comes from symplectic topology. The reason for [LiL1] to be able to consider only the case of $n \leq 8$, i.e. for those $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with Fano surface structures, is that only for those n's, could we get complete knowledge on the symplectic cones. Here we find that the single symplectic form coming from the algebraic structure on $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ for any n, shares the function of the whole symplectic cone in [LiL1].

In [LiL1], the authors calculated the orbits of the positive classes in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with $n \leq 8$ under diffeomorphisms for minimal genera 1 and 2 which were shown to be finite, and conjectured that the finiteness should be true for positive genera. In this paper, we prove that this conjecture is not only true for $n \leq 8$, but also true for $n = 9$. An interesting phenomenon for $n = 9$ is that there are infinite orbits of square zero classes with minimal genus 1.

The minimal genus embeddings obtained in this paper are relevant to the existence of irreducible algebraic curves in $\mathbb{C}P^2$ with degrees and multiples of singularities prescribed, a hard problem in algebraic geometry. Since the minimal genus here is the same as given by the genus formula for irreducible algebraic curves, the realization of minimal genus embeddings can be viewed as the fact: the differential-topological obstruction for the existence of corresponding algebraic curves vanishes.

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2. Statements of the results

Regard $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ as the underlying differentiable manifold of any algebraic surface obtained by blowing up n different points on $\mathbb{C}P^2$, and let $H, E_1, \cdots, E_n$ be the natural basis relevant to an algebraic structure above such that $H^2 = 1, E_i^2 = -1$. We say that $\xi = aH - \sum_{i=1}^n b_i E_i \in H_2(\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2, \mathbf{Z})$ is reduced, if $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$, and

$$a \geq \begin{cases} b_1, & \text{if } n = 1, \\ b_1 + b_2, & \text{if } n = 2, \\ b_1 + b_2 + b_3, & \text{if } n \geq 3. \end{cases}$$

Proposition 1. *For any $n \geq 1$ and $\xi \in H_2(\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2, \mathbf{Z})$ with $\xi^2 \geq 0$, there is a diffeomorphism f of $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ such that $f_*\xi$ is reduced.*

We do not know if the reduced $f_*\xi$ in Proposition 1 is unique for a given ξ. Thus we set

$$\mathrm{red}_\xi = \{\, \eta \,/\, \eta \,\text{is reduced and}\, \eta = f_*\xi \,\text{for some diffeomorphism}\, f \,\}$$

and let

$$f_\xi = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum_{i=1}^n b_i(b_i - 1).$$

Denote by g_ξ the minimal genus of smoothly embedded surfaces representing ξ, then we have

Proposition 2. *For any $\xi \in H_2(\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2, \mathbf{Z})$ with $\xi^2 \geq 0$ and $\xi \neq 0$,*

$$g_\xi \geq \max_{\eta \in \mathrm{red}_\xi}\{f_\eta\} = \max\{\, f_\eta \,|\eta \cdot H > 0 \textit{ and } \eta = f_*\xi \textit{ for some diffeomorphism } f \,\}.$$

Proposition 2 means that to see if the minimal genus can be obtained by f_η, we need only to consider reduced ones, the unreduced ones could not offer better information.

Theorem 1. *Let $\xi = aH - \sum_{i=1}^n b_i E_i$ be a reduced class and $n \leq 9$, then*

$$g_\xi = \begin{cases} 0 \neq f_\xi, & \textit{if } \xi = aH - aE_1 \textit{ with } a \neq 1, \\ f_\xi, & \textit{otherwise.} \end{cases}$$

Remark. If we set $f_\xi{}' = \max\{0, f_\xi\}$, then $g_\xi = f_\xi{}'$ in Theorem 1.

Theorem 2. *For $n = 8$, and $g > 0$, the set*

$$\{\, \xi \,/\, \xi \textit{ is reduced with } g_\xi = g \,\}$$

is finite and for $n = 9$, the above set is finite for $g > 1$, while the set of reduced classes ξ with $g_\xi = 1$ and $b_9 > 0$ is

$$\{\, a(3H - \sum_{i=1}^9 E_i) \,/\, a > 0 \,\}.$$

Actually, our proof for Theorem 2 yields an algorithm to calculate all reduced classes with given g, which can be done by computer. Also, the results of [Li] and [K2] for embedding spheres are reobtained easily and included in the following:

Theorem 3. *Let $\xi = aH - \sum_{i=1}^{9} b_i E_i$ be reduced, then $g_\xi = 0$ iff*

$$\xi = \begin{cases} aH - aE_1, & a \geq 0, \\ 2H, & \\ (k+1)H - kE_1, & k \geq 0, \\ (k+1)H - kE_1 - E_2, & k \geq 1. \end{cases}$$

In the following, we give some results for $n > 9$.

Proposition 3. *If a reduced class $\xi = aH - \sum_{i=1}^{10} b_i E_i$ with $\xi^2 \geq 0$ and $b_{10} > 0$ satisfying*

$$b_9 + b_{10} \leq a - b_1 - b_2 + 1 + \min\{b_1 - b_3, 1\} + \min\{b_2 - b_3, 1\},$$

then $g_\xi = f_\xi$.

An example for Proposition 3 is $\xi = 15H - \sum_{i=1}^{8} 5e_i - 3E_9 - 3E_{10}$.

Proposition 4. *Suppose that a reduced class $\xi = aH - \sum_{i=1}^{n} b_i E_i$ for $n \geq 10$ satisties $\xi^2 \geq 0$, $b_{10} > 0, f_\xi \geq 1$, and $b_i \leq 2$ if $i \geq 10$, then $g_\xi = f_\xi$.*

Proposition 5. *If a reduced class ξ with $\xi^2 \geq 0$ has $g_\xi = f_\xi$, then for any $m > 0$, $g_{m\xi} = f_{m\xi}$.*

Proposition 5 is a special case of the following general result:

Proposition 5′. *Let X be a closed oriented 4-manifold with $b^+ > 1$ and let K be any basic class, or a 4-manifold with simplectic form ω and canonical class K and $b^+ = 1$, and $0 \neq \xi \in H_2(X, \mathbb{Z})$ satisfying $\xi^2 \geq 0$ (and $\xi \cdot [\omega] > 0$ in the latter case), then $2g_\xi - 2 = K \cdot \xi + \xi^2$ implies $2g_{m\xi} - 2 = K \cdot m\xi + m^2\xi^2$ for any $m > 0$.*

Since $aH - \sum_{i=1}^{n} E_i$ for $a > 0$ is represented by some nonsingular algebraic curve, by the adjuction formula in algebraic geometry and Proposition 5, we have

Corollary. *If $a > 0$ satisfies $a^2 \geq n \geq 10$, then for any $b > 0$,*

$$\xi = abH - b\sum_{i=1}^{n} E_i$$

has $g_\xi = f_\xi$.

In the following proposition, the homology classes may have negative squares.

Proposition 6. 1) *Let $\xi = aH - \sum_{i=1}^{n} b_i E_i \neq 0$ be a reduced class with $f_\xi = 1$ and $b_i \leq 2$ for $i > 9$, then $g_\xi = 1$.*

2) *Let $\xi' = aH - \sum_{i=1}^{9} b_i E_i \neq 0$ be a reduced class with $f_{\xi'} = 1$ and $\xi = \xi' - \sum_{i=10}^{n} E_i$, then $g_\xi = 1$.*

3. Reduced classes

In this section, we prove Propositions 1 and 2.

For Proposition 1, if $n \leq 9$, it has been proved in [LiL1]. The only difference with $n > 9$ is that, for $n > 9$, an automorphism may be not realized by a diffeomorphism, so we need to use Lemma 2 in [Wa] directly.

Let $\xi = aH - \sum b_i E_i$, it is easy to see that the trivial automorphisms are realized by diffeomorphisms for any n. And thus we may assume $a \geq b_1 \geq \cdots \geq b_n \geq 0$. Notice that Wall's automorphism

$$R \triangleq R(H - E_1 - E_2 - E_3)$$

is realized by a diffeomorphism of $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$, and

$$\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2 = (\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2)\#(n-3)\overline{\mathbb{C}P}^2.$$

Therefore, by Lemma 2 in [Wa], the automorphism

$$R'(\xi) = R(aH - b_1E_1 - b_2E_2 - b_3E_3) - \sum_{i=4}^{n} b_iE_i$$

is realized by a diffeomorphism of $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$.

Let $R'(\xi) = a'H - \sum_{i=1}^{n} b'_iE_i$, then

$$a' = 2a - b_1 - b_2 - b_3.$$

Now if ξ is not reduced, we have

$$a' < a.$$

ξ being not reduced implies $b_1 > 0$, so $\xi^2 \geq 0$ implies

$$4a^2 \geq 4\sum_{i=1}^{n} b_i^2 \geq 4(b_1^2 + b_2^2 + b_3^2) > (b_1 + b_2 + b_3)^2$$

and hence $a' > 0$. This shows that after several steps, we will obtain a diffeomorphism f such that $f_*(\xi)$ is reduced.

To prove Proposition 2, we recall first the generalized adjunction inequality, i.e. Theorem E in [LL2]:

Suppose M is a symplectic 4-manifold with $b_2^+ = 1$ and ω is a symplectic form. Let C be a connected smoothly embedded surface with a nonnegative self-intersection number. If $[C] \cdot \omega > 0$, then

$$2g(C) - 2 \geq K \cdot [C] + [C]^2.$$

In the present case, $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ has a structure as an algebraic surface, so there is a symplectic form ω such that for any algebraic curve C on $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$,

$$[C] \cdot \omega > 0.$$

The differentiable structure on $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ does not depend on the n points on $\mathbb{C}P^2$ at which we blow up. More precisely, we should explain this as follows: blowing up two groups of n points on $\mathbb{C}P^2$, we get two differentiable manifolds, M and M', with natural basis $\{H, E_1, \cdots, E_n\}$ and $\{H', E'_1, \cdots, E'_n\}$, then there is a diffeomorphism $f : M \longrightarrow M'$, such that $f_*H = H', f_*E_i = E'_i$. This can be proved in the same spirit as Wall proved in Lemma 2 in [Wa]. Therefore, we may

freely use blowing-ups on any n points. Let $h, e_1, \cdots, e_n$ be the Poincaré duals of $H, E_1, \cdots, E_n$, and

$$[\omega] = \omega_0 h - \sum_{i=1}^{n} \omega_i e_i$$

where $[\omega]$ is the de Rham cohomology class of ω. Then ω being symplectic implies $[\omega]^2 > 0$ and hence

$$\omega_0^2 - \sum_{i=1}^{n} \omega_i^2 > 0.$$

Since H is represented by algebraic curves, we have

$$\omega_0 = [\omega] \cdot H > 0.$$

Now let $\xi = aH - \sum_{i=1}^{n} b_i E_i$ with $\xi^2 \geq 0$ and $a > 0$. Then by Lemma 1.1 in [FM], we have

$$[\omega] \cdot \xi > 0.$$

Since the canonical class with respect to ω is

$$-3h + \sum_{i=1}^{n} e_i,$$

the generalized adjunction inequality yields the following:

Lemma 1. *For $\xi = aH - \sum_{i=1}^{n} b_i E_i$ with $\xi^2 \geq 0$ and $a > 0$,*

$$g_\xi \geq f_\xi = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum_{i=1}^{n} b_i(b_i - 1).$$

If $\xi \cdot H < 0$, let $\xi' = -aH - \sum_{i=1}^{n} b_i E_i$, then by Lemma 1,

$$g_\xi = g_{\xi'} \geq f_{\xi'}.$$

Recall the proof for Proposition 1, we see that if ξ with $a > 0$ is not reduced, ξ can arrive at a reduced form in the following way:

Step 1. The use of trivial automorphisms yields

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq 0.$$

Step 2. Fix b_i for $i \geq 4$, and change $aH - \sum_{i=1}^{3} b_i E_i$ to a reduced class in $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2$.

Step 3. Repeat Step 1.

Step 4. Repeat Step 2.

Since

$$f_\xi = \frac{1}{2}(-3a + \sum_{i=1}^{n} b_i + 2 + a^2 - \sum_{i=1}^{n} b_i^2)$$

and Step 1 changes f_ξ by only replacing b_i by $|b_i|$, we have $f_\xi \leq f_{\xi'}$, where ξ' is obtained by ξ via Step 1.

For Step 2, let $\eta' = a'H - \sum_{i=1}^{3} b_i' E_i$ be the reduced class obtained by $\eta = aH - \sum_{i=1}^{3} b_i E_i$. We have already proved in [LiL1] that $g_{\eta'} = f_{\eta'}$, while

$$g_\eta = g_{\eta'} \text{ and } g_\eta \geq f_\eta$$

so $f_{\eta'} \geq f_\eta$, and hence $f_\xi \leq f_{\xi'}$ for ξ' being obtained by ξ via Step 2.

Therefore, for any η with $\eta \cdot H > 0$ and $\eta = f_*\xi$ for some diffeomorphism f, there is an $\eta' \in \mathrm{red}_\xi$ such that $f_\eta \leq f_{\eta'}$. And for any $\eta' \in \mathrm{red}_\xi$, $g_\xi = g_{\eta'} \geq f_{\eta'}$ by Lemma 1, Proposition 2 is proved.

4. Minimal genera for $n \leq 9$

In this section, we give a proof for Theorem 1. First we notice that for $n \leq 9$, ξ being reduced implies $\xi^2 \geq 0$, and we may work only with $n = 9$. We have to deal with two cases.

Case 1 ($b_3 = 0$)**.** In this case, the problem reduces to the one for $n \leq 2$, and if $\xi^2 > 0$, it has been proved in [LiL1], Theorem 3, that $g_\xi = f_\xi$ (Remark 4.5 there says that it can be proved by a topological construction, without appealing to the result from algebraic geometry).

Now $\xi^2 = 0$ and

$$a^2 \geq (b_1 + b_2)^2 \geq b_1^2 + b_2^2$$

imply $b_2 = 0, a = b_1$. Hence $\xi = aH - aE_1$. Take a different lines in $\mathbb{C}P^2$ meeting at a single point A, then blow up at A, we get an embedding of a disjoint spheres in $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. By suitably using $a-1$ tubes to connect these embedded spheres, we get an embedded sphere representing ξ, here "suitably" means that we should keep the natural orientations of the spheres after tubing. Therefore, $g_\xi = 0$, while

$$f_\xi = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}a(a-1) = 0$$

iff $a = 1$, hence the theorem holds for Case 1.

Case 2 ($b_3 > 0$)**.** Let P be a nonsingular cubic curve in $\mathbb{C}P^2$, and $P_1, P_2, \cdots, P_9$ be nine different points on P. Since P has normal Euler characteristic number 9, the normal bundle has trivialization on $P \backslash \bigcup_{i=1}^9 \{P_i\}$ and $D_i, i = 1, 2, \cdots, 9$, where D_i is a closed disk with P_i as its center such that $D_i \cap D_j = \emptyset$ if $i \neq j$, and the bundle map

$$\partial D_i \times \mathbb{C} \longrightarrow (P \backslash \bigcup_{i=1}^9 \{P_i\}) \times \mathbb{C}$$

given by the identity on the normal bundle induces a map

$$\partial D_i \longrightarrow \partial D_i \times \{1\} \longrightarrow \partial D_i \times (\mathbb{C}\backslash\{0\}) \longrightarrow \mathbb{C}\backslash\{0\},$$

which has degree 1. Identify P with $S^1 \times S^1$, and let $f : S^1 \times S^1 \longrightarrow S^1 \times S^1 \times \mathbb{C}$ be given by

$$f(e^{\sqrt{-1}\theta}, e^{\sqrt{-1}\phi}) = (e^{\sqrt{-1}b_3\theta}, e^{\sqrt{-1}\phi}, e^{\sqrt{-1}\theta}),$$

we get an embedding of $S^1 \times S^1$ in the trivial bundle $P \times \mathbb{C}$ which is a b_3 sheet covering map after composition with the projection $P \times \mathbb{C} \longrightarrow P$. Thus, via an identification of the normal bundle of P with a suitably small tube neighborhood, we get an immersion of $S^1 \times S^1$ in the neighborhood such that the self-intersection points of the immersion are exactly $P_1, \cdots, P_9$ with multiple b_3 for each P_i. Furthermore, we may assume that in a neighborhood of any P_i, the immersion is given by b_3 pieces of complex lines through P_i.

Take $b_1 - b_3$ different lines $L_{1i}, i = 1, \cdots, b_1 - b_3$, through P_1 such that each L_{1i} meets $2b_3$ points transversely with the immersed $S^1 \times S^1$ other than any P_i.

Take $b_2 - b_3$ different lines $L_{2i}, i = 1, \cdots, b_2 - b_3$, through P_2 with the same property as L_{1i}.

Then we take $a-b_1-b_2-b_3$ different lines $L_{3i}, i=1,\cdots,a-b_1-b_2-b_3$, such that no three have common point, the intersection $L_{3i}\cap L_{3j}$ for $i\neq j$ is not on the immersed torus, and each L_{3i} has $3b_3$ transversal intersections with it (so, no P_i is among the intersections).

Now for $i>3$, if $b_i<b_3$, we may change the immersion around P_i so that b_3-b_i pieces of complex lines are purturbed to some positions so that the intersection point P_i of multiple b_3 splits into a point with multiple b_i and some transversal intersection points with intersection number $+1$. If $b_i>0$, we still denote by P_i the point with multiple b_i above, and if $b_i=0$, we take a point outside the image of the immersion as P_i instead of the original P_i.

Summing up what we have done, we see that we have defined an immersion in $\mathbb{C}P^2$ of the disjoint union of $(b_1-b_3)+(b_2-b_3)+(a-b_1-b_2-b_3)=a-3b_3$ spheres and a torus such that any P_i is the self-intersection point of multiple b_i (actually if $b_i=1$, it is an ordinary point, and if $b_i=0$, P_i is not on the image of the immersion), and other self-intersections are transversal with intersection number $+1$.

Now we count the number of intersections other than $P_i, i=1,2,\cdots,9$.

1) The number contributed by the intersections of the L_{1i}'s with the immersed torus is $2b_3(b_1-b_3)$.

2) The number contributed by the intersections of the L_{2i}'s with the immersed torus is $2b_3(b_2-b_3)$.

3) The number contributed by the intersections of the L_{3i}'s with the immersed torus is $3b_3(a-b_1-b_2-b_3)$.

4) The number contributed by the intersections of the $a-3b_3$ lines is $\frac{1}{2}(a-3b_3)(a-3b_3-1)-\frac{1}{2}(b_1-b_3)(b_1-b_3-1)-\frac{1}{2}(b_2-b_3)(b_2-b_3-1)$.

5) The number contributed by the perturbation around any P_i with $4\leq i\leq 9$ is $\frac{1}{2}b_3(b_3-1)-\frac{1}{2}b_i(b_i-1)$.

Therefore, the total number of the intersections other than P_i's is

$$\begin{aligned}
&2b_3(b_1-b_3)+2b_3(b_2-b_3)+3b_3(a-b_1-b_2-b_3)\\
&\quad+\frac{1}{2}(a-3b_3)(a-3b_3-1)-\frac{1}{2}(b_1-b_3)(b_1-b_3-1)\\
&\quad-\frac{1}{2}(b_2-b_3)(b_2-b_3-1)+\sum_{i=1}^{9}(\frac{1}{2}b_3(b_3-1)-\frac{1}{2}b_i(b_i-1))\\
&\quad=\frac{1}{2}a(a-1)-3b_3-\frac{1}{2}\sum_{i=1}^{9}b_i(b_i-1).
\end{aligned}$$

Notice that by our construction of the last immersion, any immersed sphere (i.e. a complex line) has intersections with the immersed torus other than any P_i. So by doing surgery on one such intersection for any sphere, we get an immersion in $\mathbb{C}P^2$ of

$$(S^1\times S^1)\#\underbrace{S^2\#\cdots\#S^2}_{a-3b_3}\cong S^1\times S^1.$$

This new immersion of $S^1 \times S^1$ has the number of intersections other than P_i's (for P_i's, the situation has not been changed) is

$$\begin{aligned}&\frac{1}{2}a(a-1) - 3b_3 - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i-1) - (a-3b_3)\\ &\quad = \frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i-1).\end{aligned}$$

At last, doing surgeries on the above intersections and blowing up at P_i's, we get an embedding of a surface in $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ (if $n < 9, b_i = 0$ for $i > n$, hence the embedding is actually in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$). The torus has genus 1, and a surgery on the torus increases genus by 1, hence the embedded surface has genus

$$\begin{aligned}&1 + \frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i-1)\\ &\quad = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i-1) = f_\xi.\end{aligned}$$

On the other hand, by Proposition 2, we have

$$g_\xi \geq f_\xi.$$

So, we have proved that $g_\xi = f_\xi$, if $b_3 > 1$.

Combining Case 1 and Case 2, we have proved Theorem 1.

5. Finiteness of the orbits and embeddings of sphere

We are going to prove Theorems 2 and 3.

The method used here **to prove Theorem 2** is a development of that used in [LiL1] for the proof of Theorem 4 there. Let $g \geq 0$ be an integer, we consider the integer solutions of the following system:

$$(*)\qquad \begin{cases} -3a + \sum_{i=1}^{9} b_i + a^2 - \sum_{i=1}^{9} b_i^2 = 2g-2, & (1)\\ a \geq b_1 + b_2 + b_3, & (2)\\ b_1 \geq b_2 \geq \cdots \geq b_9 \geq 0, & (3)\\ a > 0. & (4)\end{cases}$$

Introducing a new variable $k = a - b_1$, and replacing b_1 by $a-k$, then $(*)$ becomes

$$(*)'\qquad \begin{cases} 2(k-1)a - k^2 - k - \sum_{i=2}^{9} b_i^2 + \sum_{i=2}^{9} b_i = 2g-2, & (1)'\\ k \geq b_2 + b_3, & (2)'\\ a-k \geq b_2 \geq b_3 \geq \cdots \geq b_9 \geq 0, & (3)'\\ a > 0. & (4)'\end{cases}$$

If $k = 0$, then it is easy to see that $(*)'$ has no solution for $g > 0$. $(3)'$ means

$$-b_i^2 + b_i \geq -b_3^2 + b_3, \text{ for } i > 3$$

and $a \geq b_2 + k$. And for $k \geq 1$, the left-hand side of $(1)'$ does not decrease as a function of a, so we have

$$\text{the l.h.s. of } (1)' \geq 2(k-1)(b_2+k) - k^2 - k - b_2^2 + b_2 + 7(-b_3^2 + b_3).$$

For fixed b_3 and k, the r.h.s. of the above inequality is a function of b_2 denoted by $h(b_2)$ whose domain is $[b_3, k - b_3]$ by $(2)'$ and $(3)'$. It is obvious that h can assume its minimum only at b_3 or $k - b_3$, and

$$h(k - b_3) - h(b_3) = (k-1)(k - 2b_2).$$

$(2)'$ and $(3)'$ imply $k \geq 2b_3$, so h assumes its minimum at b_3, and we have

$$\text{the l.h.s. of } (1)' \geq 2(k-1)(b_3+k) - k^2 - k + 8(-b_3^2 + b_3) \triangleq l_k(b_3).$$

By $(2)'$ and $(3)', 0 \leq b_3 \leq [\frac{k}{2}]$. Now

$$l_k(0) = k^2 - 3k, \qquad l_k(\frac{k-1}{2}) = 3k - 5$$

and it is easy to see that for any $g \geq 1$, if $k \geq 4g$,

$$l_k(0) > 2g - 2,\ l_k(\frac{k-1}{2}) > 2g - 2.$$

Since $-l_k(b_3)$ is a convex function of b_3 , we see that if $k \geq 4g$ and $0 \leq b_3 \leq \frac{k-1}{2}$,

$$l_k(b_3) > 2g - 2.$$

Therefore, for $k \geq 4g$, only when $b_3 = \frac{k}{2}$, $(1)'$–$(3)'$ may have solutions.

Case 1 $(k = 1)$**.** The l.h.s. of $(1)' \leq -k^2 - k = -2$, and $g \geq 1$ implies $2g - 2 \geq 0$, so $(1)'$ has no solutions.

Case 2 $(1 < k < 4g)$**.** By $(2)'$ and $(3)'$, there are only finite possibilities for $b_2, b_3, \cdots, b_9$, and a is determined by them via $(1)'$, so $(*)'$ has only finite possible solutions.

Case 3 $(k \geq 4g)$**.** Then, only when $b_3 = \frac{k}{2}$ with k even, there may be solutions for $(*)'$. $b_3 = \frac{k}{2} \leq b_2$ and $b_2 + b_3 \leq k$ imply $b_2 = \frac{k}{2}$. Thus $(1)'$ becomes

$$(1)'' \qquad 2(k-1)a = 2g - 2 + \frac{3}{2}k^2 + \sum_{i=4}^{9}(b_i^2 - b_i).$$

Now $0 \leq b_i \leq b_3$ for $i \geq 4$ implies

$$\sum_{i=4}^{9}(b_i^2 - b_i) \leq 6((\frac{k}{2})^2 - \frac{k}{2}) = \frac{3}{2}k^2 - 3k,$$

and we have by $(1)''$ that $a \leq \frac{1}{2(k-1)}(2g - 2 + 3k^2 - 3k) = \frac{g-1}{k-1} + \frac{3}{2}k$. Since $k \geq 4g > g$, we have $a \leq \frac{3}{2}k$. On the other hand, $a - k \geq b_2 = \frac{k}{2}$, so

$$a = \frac{3}{2}k.$$

Thus $(1)''$ becomes

$$\sum_{i=4}^{9}(b_i^2 - b_i) = \frac{3}{2}k^2 - 3k + 2 - 2g$$

or equivalently,

$$\sum_{i=4}^{9}((\frac{k}{2})^2-\frac{k}{2}-b_i^2+b_i)=2g-2.$$

Every term in the above summation is nonnegative, so

$$(\frac{k}{2})^2-\frac{k}{2}-b_9^2+b_9\leq 2g-2$$

i.e.

$$(\frac{k}{2}-b_9)(\frac{k}{2}+b_9-1)\leq 2g-2.$$

On the other hand, $k\geq 4g$ implies

$$\frac{k}{2}+b_9-1\geq 2g-1>2g-2.$$

Therefore, $(1)''$ together with $(2)'$ and $(3)'$ has solutions for $k\geq 4g$ only when

$$b_9=\frac{k}{2},$$

and this means $b_1=b_2=\cdots=b_9=\frac{k}{2}$, and $g=1$.

Summing up Cases 1–3, we see that if $g\geq 2$ or $g=1$ but $b_9=0$, there are only finite solutions.

At last we look at cases $g=1$ and $b_9>0$. We have already seen that if $k\geq 4g=4$, the solutions of $(*)$ are $\frac{1}{2}(3k,k,\cdots,k)$ for k even.

If $k=1$, there is no solution. If $k=2$, then $b_3\leq\frac{k}{2}$ and $b_9>0$ implies $b_2=b_3=\cdots=b_9=1$, and $(1)'$ gives solution $a=3,b_1=a-k=1$.

For $k=3,b_3\leq\frac{k}{2}$ and $b_9>0$ imply $b_3=b_4=\cdots=b_9=1$, and $b_2+b_3\leq k$ implies $b_2=1$ or 2. If $b_2=1$, then the solution of $(1)'$ is $a=3$ which does not satisfy $(3)':a-k\geq b_2=1$. If $b_2=2$, $(1)'$ has no integer solution.

So far we have proved that if $g=1$ and $b_9>0$, the solutions of $*$ are exactly $b_1=b_2=\cdots=b_9$ and $a=3b_9$. In virtue of Theorem 1, Theorem 2 is proved.

To prove Theorem 3, we solve $(*)'$ for $g=0$ first.

If $k=0$, the solution of $(*)'$ is $a=1,b_2=\cdots=b_9=0$.

Since $l_k(\frac{k}{2})=0$, and $l_k(0)=k^2-3k\geq 0$ except for $k=1$ or 2, we see from the convexity of $-l_k(b_3)$ as a function of $b_3\in[0,\frac{k}{2}]$ that if $k\neq 1$ or 2,

$$\text{the l.h.s. of } (1)'\geq 0>\text{the r.h.s. of } (1)'.$$

Hence $(*)'$ has possible solutions for $k\geq 1$ only when $k=1$ or 2.

If $k=1,b_3\leq\frac{k}{2}$ implies $b_3=0$. Thus $(1)'$ becomes

$$-b_2^2+b_2=0$$

i.e. $b_2=0$ or 1. Therefore, $a=b_1+1$.

If $k=2,l_k(\frac{k}{2})=0$ and $b_3\leq\frac{k}{2}$ imply $b_3=0$. By $(1)'$ we have

$$2a-b_2^2+b_2=4.$$

Now $b_2+b_3\leq k$ implies $b_2=0,1$ or 2. If $b_2=2$, we have $a=3$, contradicting $a-k\geq b_2$. If $b_2=0$ or 1, we have $a=2$. $b_2=1$ contradicts $a-k\geq b_2$, so the only solution of $(*)'$ for $k=2$ is $a=2,b_2=\cdots=b_9=0$.

Having found all solutions of $(*)'$, we need to consider the system $(**)$ obtained from $(*)'$ by changing $(1)'$ to

$$(**) \qquad 2(k-1)a - k^2 - k - \sum_{i=2}^{9}(b_i^2 + b_i) < -2.$$

Case 1 $(k \geq 1)$**.** We have again

$$\text{the l.h.s. of } (**) \geq l_k(b_3).$$

Since $l_k(\frac{1}{2}) = 0, l_k(0) = k^2 - 3k \geq 0$ for $k \neq 1$ or 2, and $l_1(0) = l_2(0) = -2$, the convexity of $-l_k(b_3)$ implies

$$l_k(b_3) \geq -2, \quad \text{for } 0 \leq b_3 \leq \frac{k}{2}, \text{and } \ k \geq 1,$$

so $(**)$ has no solution.

Case 2 $(k = 0)$**.** The solution of $(**)$ is $a \geq 2, b_2 = \cdots = b_9 = 0$. Thus Theorem 3 is proved by using Theorem 1.

6. The case of $n > 9$

In this section, we prove Propositions 3 and 4. The proofs are essentially extensions of that for Theorem 1.

Proof of Proposition 3. First, we have an immersion of the torus in a suitably small tube neighborhood of a nonsingular cubic curve P in $\mathbb{C}P^2$ such that its self-intersection points are $P_1, P_2, \cdots, P_9$ of multiple $b_3 > 0$.

If $b_9 + b_{10} \leq b_3 + 1$, we may perturb the immersion around P_9 (Notice: it is given by b_3 lines through P_9) to get an immersion whose image in a small neighborhood of P_9 consists of: (1).b_9 lines passing through P_9, (2).b_{10} lines passing through a point P_{10} which is also on a line through P_9, and (3).the other $b_3 + 1 - b_9 - b_{10}$ lines do not pass through either P_9 and P_{10}, and can have only intersection points of multiple 2.

If $b_9 + b_{10} > b_3 + 1$, we may assume the immersion has P_9 and P_{10} as self-intersection points with multiples b_9 and $b_3 - b_9 + 1$ respectively, given by the b_3 lines as in the preceding paragraph. By the assumption,

$$b_9 + b_{10} \leq a - b_1 - b_2 + 1 + \min\{b_1 - b_3, 1\} + \min\{b_2 - b_3, 1\}$$

we have

$$\begin{aligned} a - b_1 - b_2 - b_3 + \min\{b_1 - b_3, 1\} + \min\{b_2 - b_3, 1\} \\ \geq b_{10} - (b_3 - b_9 + 1). \end{aligned}$$

Thus, we may have three disjoint sets of lines: L_1, L_2 and L_3 such that

1) all lines in these sets pass through P_{10},
2) the cardinal $|L_1| \leq \min\{b_1 - b_3, 1\}, |L_2| \leq \min\{b_2 - b_3, 1\}, |L_3| \leq a - b_1 - b_2 - b_3$, and
$$|L_1| + |L_2| + |L_3| = b_{10} - (b_3 - b_9 + 1),$$
3) the line in L_i passes through P_i for $i = 1, 2$.

Then we add lines of suitable numbers in the sets L_1, L_2 and L_3 to have lines

$$\begin{array}{ll} L_{1i}, & i = 1, \cdots, b_1 - b_3, \\ L_{2i}, & i = 1, \cdots, b_2 - b_3, \\ L_{3i}, & i = 1, \cdots, a - b_1 - b_2 - b_3, \end{array}$$

and we do the same thing around each P_i with $3 < i < 9$ as in the proof for Theorem 1. At last, we get an immersion of the disjoint union of a torus and $a - 3b_3$ spheres in $\mathbb{C}P^2$ such that all self-intersections have multiple 2 except for $P_i, i = 1, \cdots, 10$, where the immersion has self-intersection of multiple b_i.

Then blowing up at $P_i, i = 1, \cdots, 10$, and doing surgery at any other self-intersection point, we get an embedding of a surface in $\mathbb{C}P^2\#10\overline{\mathbb{C}P}^2$.

Notice that any of the $a - 3b_3$ lines still meets the immersed torus at least at a point other than $P_i, i = 1, \cdots, 10$. So the surface is connected, and a similar argument shows that the embedding is required. Proposition 3 is proved.

To prove Proposition 4, we follow the proof of Theorem 1 until we have an immersion of a torus in $\mathbb{C}P^2$ with the number of self-intersections other than $P_1, \cdots$, and P_9 being

$$\frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i - 1).$$

Let m_j be the cardinal of the set

$$\{i \ / \ 9 < i \leq n, b_i = j\ \}, \qquad j = 0, 1 \text{ or } 2,$$

then

$$f_\xi = 1 + \frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i - 1) - m_2.$$

Now $f_\xi \geq 1$ implies

$$m_2 \leq \frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i - 1).$$

Hence, we may take m_2 self-intersections which may be assumed to be given locally by two lines. Then we take m_1 points on the image of the immersion and assume that the immersion is given by a line near each of these points.

Blowing up at $P_1, \cdots, P_9$ and the above $m_1 + m_2$ points, and at m_0 points outside the image of the immersion, and doing surgery on the remaining self-intersections with number

$$\frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^{9} b_i(b_i - 1) - m_2,$$

we thus get an imbedding in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ of a surface with genus f_ξ. Obviously, the imbedding represents ξ, and $g_\xi \geq f_\xi$ by Proposition 2. So $g_\xi = f_\xi$. Proposition 4 is proved.

7. The proof of Propositions 5′ and 6

To prove Proposition 5′, we assume an embedding in X representing ξ is given, its source surface S has genus g_ξ. The normal Euler number of the embedding is ξ^2, so we may take a section S_1 of the normal bundle intersecting with the zero section transversally such that the intersection index is $+1$ for every intersection point. Therefore, the number of the intersection points is ξ^2. Let S_i be the section iS_1 for $i = 1, \cdots, m$, and perturb $S_2, \cdots, S_m$ suitably around the ξ^2 points, we may assume that regarding $S_1, \cdots, S_m$ as an immersion of the disjoint union of m copies S, the number of self-intersection points is $\frac{m(m-1)}{2}\xi^2$ and each has intersection index $+1$. If $\xi^2 > 0$, doing surgeries on the $\frac{m(m-1)}{2}\xi^2$ points, we get an embedding with the source surface having genus

$$mg_\xi + \frac{m(m-1)}{2}\xi^2 - (m-1),$$

where $m-1$ surgeries are used for making the surface connected. If $\xi^2 = 0$, we take m sheets covering surface S' of S and embed S' in the normal bundle such that each fiber intersects S' at m points (a precise construction was given in [LiL2]). By identifying the normal bundle with a tube neighborhood, we get an imbedding representing $m\xi$, and in both cases, the genus of the source surface is

$$m(g_\xi - 1) + 1 + \frac{m(m-1)}{2}\xi^2 = 1 + \frac{1}{2}(K \cdot m\xi + m^2\xi^2)$$

by a straightforward calculation. In virtue of the adjunction inequalities due to Kronheimer and Mrowka for $b^+ > 1$, and T.J. Li and A.K. Liu for $b^+ = 1$, the proof for Proposition 5′ is complete.

To prove Proposition 6, we recall Theorem 1.2 in [FS], in a slightly different form:

Let $\xi = aH - \sum_{i=1}^n b_i E_i$ be represented by an immersed sphere with all intersection points of indexes ± 1, and p is the number of positive intersections, then $p \geq f_\xi$ if $a \geq 2$.

Notice that $\xi \neq 0$, being reduced with $f_\xi = 1$, implies $a \geq 2$. Thus for Proposition 6.2), $f_\xi = f_{\xi'} = 1$ implies ξ cannot be represented by imbedded spheres, and by Theorem 1, ξ' is represented by an embedded torus in $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$. Take $n-9$ points on the embedded torus and blow them up, we then get an embedding of torus in $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ which represents ξ.

For Proposition 6.1), we notice first that $f_\xi = 1$ means ξ not being represented by embedded spheres by virtue of the above theorem [FS]. A consideration as in the proof of Proposition 4 shows that

$$m_2 = \frac{1}{2}a(a-1) - a - \frac{1}{2}\sum_{i=1}^n b_i(b_i - 1).$$

Then the same construction as for proving Proposition 4 yields an embedding of torus representing ξ. The proof for Proposition 6 is complete.

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