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EFFECTIVE ESTIMATES ON THE VERY AMPLENESS OF THE CANONICAL LINE BUNDLE OF LOCALLY HERMITIAN SYMMETRIC SPACES

SAI-KEE YEUNG

ABSTRACT. We study the problem about the very ampleness of the canonical line bundle of compact locally Hermitian symmetric manifolds of non-compact type. In particular, we show that any sufficiently large unramified covering of such manifolds has very ample canonical line bundle, and give estimates on the size of the covering manifold, which is itself a locally Hermitian symmetric manifold, in terms of geometric data such as injectivity radius or degree of coverings.

Let L be an ample line bundle on an algebraic manifold M. From Kodaira's Embedding Theorem, we know that mL is very ample if m is sufficiently large so that the sections of mL give rise to an embedding of M into some projective space. A natural question is on the estimates of such m so that mL is very ample. In particular, one may ask the same question for the canonical line bundle of a manifold with negative first Chern class, or of general type. Of particular interest among such manifolds is the class of locally Hermitian symmetric spaces. The main purpose of this article is to show that for compact locally Hermitian symmetric manifolds of non-compact type, K is very ample if the injectivity radius of the manifold is greater than some effective constant which can be estimated in terms of some geometric data.

Effective estimates for $mK = K + (m-1)K$ with $m \geq 2$ in terms of injectivity radius have been obtained in [HT] and [Y2]. The difficulty in the case of $m = 1$ can be seen from the fact that in constructing holomorphic sections from the Bochner-Kodaira technique, one always need $K+L$ for some extra positivity from L. In [Y2], it is shown that for a tower of coverings of compact Hermitian locally symmetric manifolds M_i, K_{M_i} becomes very ample when i approaches to ∞. To compensate for the extra positivity required for the Bochner-Kodaira technique, we prove our result by showing that the limit of the Bergman kernel on M_i approaches the corresponding one on the universal covering. However, the limiting process makes the whole approach highly non-effective and requires an infinite sequence of normal coverings of the original manifold. The main purpose of this article is to show the very ampleness of K_M for any unramified covering M of M_o with injectivity radius or degree of the coverings greater than some effectively estimable constant. For this purpose, an alternate approach using heat kernel estimates combined with Atiyah's

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Covering Index Theorem is taken to bypass the limiting process. The main difficulty lies in the uniform control of the trace of all the global holomorphic sections and its derivatives at each point of the manifold. This delicate point explains the length of the arguments in Section 3. The main results are stated as Theorem 2 in Section 3, and Theorems 3 and 4 in Section 4.

The arguments of this paper work for other algebraic manifolds whose universal covering admits a lot of holomorphic functions so that the Bergman kernel is well behaved. Apart from locally Hermitian symmetric manifolds, the arguments in particular can be applied to those Kähler manifolds whose Riemannian sectional curvature is bounded between two negative constants.

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1. Preliminaries

Let $V^i = \wedge^i T_M^*$ be the space of exterior differential forms on M. Let Δ_i be the Laplacian acting on V^i. We have $\Delta_i = dd^* + dd^*$, where $d : \wedge^i T^* \to \wedge^{i+1}$ is given by

$$d(\sum_{j_1,\ldots,j_i} f_{j_1\cdots j_i} dx^{j_i} \wedge \cdots \wedge dx^{j_i}) = \sum_{j,j_1,\ldots,j_i} \frac{\partial f_{j_1\ldots j_i}}{\partial x_j} dx^j \wedge dx^{j_i} \wedge \cdots \wedge dx^{j_i}$$

and d^* is the formal adjoint of d.

As the principal symbol of Δ_i is a scalar multiple in the sense that $\sigma_{\Delta_i}(v,v)s = -g(v,v)s$ for any local exterior i-form s and $v \in T^*M$, the usual construction of heat kernel can be applied to the heat equation $(\frac{\partial}{\partial t} + \Delta_i)k_i(t,x,y) = 0$. The heat kernel $k_i(t,x,y)$ is defined to be $Hom(V_x^i, V_y^i)$-valued functions on $(0,\infty) \times \tilde{M} \times \tilde{M}$ satisfying the above heat equation and other properties similar to those in ([P], page 241) and ([Do1], page 489). As the metric h on V induces an isomorphism of the dual V^{i*} of V to V, we may identify $Hom(V_x^i, V_y^i)$ with $V_x^i \otimes V_y^i$. For $x = y$, we can then define the trace of a homomorphism in $Hom(V_x^i, V_x^i)$ by taking the inner product with respect to the metric h, which is the same as the trace of the homomorphism with respect to an orthonormal basis at x. The trace of the heat kernel $Trk_i(t,x,x)$ is then defined and is the same as its magnitude with respect to the metric involved.

Lemma 1. *Let $\tilde{M}$ be a complete manifold which is the universal covering of a compact manifold M. Consider the heat kernel $\tilde{k}^i$ of the laplacian Δ_i obtained with the lift of the metric from M. Let $T > 0$ be an arbitrary fixed number. Then the heat kernel on $\tilde{M}$ satisfies the following estimates for $T \geq t \geq 1$:*

$$\|\tilde{k}_i(t,x,y)\| \leq c_1 exp[-\frac{d^2(x,y)}{4t}],$$

where c_1 is a constant depending only on T.

Proof. This follows from the standard construction of [P], [BGM] and especially [Do1]. ∎

Lemma 2. *Let $\tilde{k}_i(t,x,y)$ be the heat kernel of the Laplacian Δ_i on i-forms of a Riemannian manifold $\tilde{M}$ which covers a compact manifold. Let $\tilde{\mathcal{H}}_i$ be the space of harmonic i-forms with L^2-norm 1 on M. Let ϕ_j be an orthonormal basis for $\tilde{\mathcal{H}}_i$. Let $\tilde{H}_i(x,y) = \sum_{\phi_j \in \tilde{\mathcal{H}}_i} \phi_j(x) \otimes \phi_j(y)$. Then, given any $\delta > 0$, there is a computable* *constant r_2 depending only on $\tilde{M}$ such that for all $t > r_2$ and all points $x \in \tilde{M}$, $|Tr\tilde{k}_i(t,x,x) - Tr\tilde{H}_i(x,x)| < \delta$.*

Proof. First we note that as $\tilde{M}$ covers a compact manifold M, and the heat kernel is invariant under isometry, $k_i(t,x,x)$ is bounded for all $t \geq 1$. From spectral decomposition we get

$$k_i(t,x,y) = \int_{\sigma(\Delta_i)} e^{-\lambda t} dE^i_{x,y}(\lambda),$$

where $\sigma(\Delta_i)$ is the spectrum of Δ_i on $\tilde{M}$, and the spectral measure $dE^i_{x,y}(\lambda)$ is the spectral projection valued measure from y to x. Hence

$$Trk_i(t,x,x) = \int_{\sigma(\Delta_i)} e^{-\lambda t} dE^i_{x,x}(\lambda) = Tr\tilde{H}_i(x,x) + \int_{\sigma(\Delta_i)\cap(0,\infty)} e^{-\lambda t} dE_{x,x}(\lambda).$$

For $\epsilon > 0$, define $F(\epsilon,t) = \int_{\sigma(\Delta_i)\cap(0,\epsilon)} e^{-\lambda t} dE_{x,x}(\lambda)$. Note that $F(\epsilon,1)$ is a monotonic increasing function in ϵ and $\lim_{\epsilon\to 0} F(\epsilon,1) = 0$ as the domain of integration approaches to an empty set. In fact it is estimated by [Lü] that $F(1,\epsilon) \leq c\frac{1}{-\log(\epsilon)}$ for some controllable constant $c > 0$.

Now given any $\delta > 0$, we can find an $\epsilon > 0$ such that

$$F(\epsilon,1) = \int_{\sigma(\Delta_i)\cap(0,\epsilon)} e^{-\lambda} dE_{x,x}(\lambda) < \frac{\delta}{2}.$$

$\lambda \geq \epsilon$ implies that $e^{-\lambda t} \leq e^{-\frac{\epsilon}{2}t} e^{-\frac{\lambda}{2}t}$, and hence

$$\int_{\sigma(\Delta_i)\cap[\epsilon,\infty)} e^{-\lambda t} dE_{x,x}(\lambda) \leq e^{-\frac{\epsilon}{2}t} Trk_i(\frac{t}{2},x,x) \leq e^{-\frac{\epsilon}{2}t} C.$$

If t is chosen to be so large that $e^{-\frac{\epsilon}{2}t} < \frac{\delta}{2}$, we conclude that

$$\begin{aligned} &|Tr\tilde{k}_i(t,x,x) - Tr\tilde{H}_i(x,x)| \\ &\quad = \int_{\sigma(\Delta_i)\cap(0,\epsilon)} e^{-\lambda} dE_{x,x}(\lambda) + \int_{\sigma(\Delta_i)\cap[\epsilon,\infty)} e^{-\lambda t} dE_{x,x}(\lambda) < \delta. \end{aligned}$$

□

Lemma 3. *Let $\tilde{M}$ be the universal cover of a compact manifold M whose sectional curvature is bounded from below by $-b$ and the eigenvalues of the curvature operator are bounded from above by b_1. Let $T > 1$ be a fixed number. There exists a constant $c = c(T)$ such that if the injectivity radius $\tau = \tau(M)$ of M satisfies $\tau > c$, then for all $0 \leq t \leq T$ and $x \in \tilde{M}$*

$$|Trk_i(t,x,x) - Tr\tilde{k}_i(t,x,x)| \leq exp(-\frac{\tau^2}{12t})$$

for all points $x \in M$. Here $c(T)$ depends only on the curvature bounds.

Proof. The idea is similar to the arguments in [Do1]. It essentially follows from the fact that

$$k_i(t,x,x) = \sum_{\gamma\in\Gamma} \tilde{k}(t,x,\gamma x) = \tilde{k}(t,x,x) + \sum_{\gamma\in\Gamma-\{1\}} \tilde{k}(t,x,\gamma x)$$

in the sense of uniform convergence, and estimates of the second term above from Lemma 1. □

2. Effective estimates on the Bergman kernel

Proposition 1. *Assume that the L^2 cohomology group of the universal covering $\tilde{M}$ of M satisfies $\tilde{h}_i(\tilde{M}) = 0$ for $0 \le i \le n-1$, where n is the complex dimension of M. Then, given any $\epsilon > 0$, there exists $r = r(\epsilon)$, depending on $\tilde{M}$, such that the estimates*

$$|TrH_i(x,x)| \le \epsilon$$

hold for all $x \in M$ if the injectivity radius satisfies $\tau > r$.

Proof. Let $T = r_1$ be the constant given by Lemma 2, so that for $t \ge T$

$$|Tr\tilde{k}_i(t,x,x) - Tr\tilde{H}_i(x,x)| \le \delta.$$

Let $c(T)$ be the constant in Lemma 3. Choose $r_2(T) = \max(c(T), \sqrt{-12T(\log\delta)}+1)$ so that $exp(-\frac{(i\tau)^2}{12T}) < \delta$ and hence $|k_i(T,x,x) - \tilde{k}_i(T,x,x)| < \delta$ for $\tau > r_2(T)$. Let $r_3(T) = \max(T, c(T), r_2(T))$. Then for $\tau > r_3(T)$, we conclude that

$$\begin{aligned}
0 &\le TrH_i(x,x) \le Trk_i(r_3,x,x)\\
&\le |Trk_i(r_3,x,x) - Tr\tilde{k}_i(r_3,x,x)| + |Tr\tilde{k}_i(r_3,x,x) - Tr\tilde{H}_i(x,x)| + Tr\tilde{H}_i(x,x)\\
&\le 2\delta
\end{aligned}$$

This concludes the proof of the proposition (we choose $\epsilon = 2\delta$). □

Theorem 1. *Assume that the L^2 cohomology group of the universal covering $\tilde{M}$ of M satisfies $\tilde{h}_i(\tilde{M}) = 0$ for $0 \le i \le n-1$. Suppose M is a holomorphic covering of M_o. Assume that $vol(M_o) = V_o$ is given. Given any $\alpha > 0$, there is a constant r, depending only on $\tilde{M}$ and V_o, such that*

$$|TrH_n(x,x)| \ge (1-\alpha)|Tr\tilde{H}_n(x,x)|$$

for all $x \in M$ if the injectivity radius satisfies $\tau > r$.

Proof. Consider the Euler characteristics $\chi(M) = \sum_{i=0}^n h^i(M)$ on M and the L^2-analogue on the universal covering $\tilde{M}$, $\chi^{(2)}(\tilde{M}) = \sum_{i=0}^n \tilde{h}^i(\tilde{M})$. Atiyah's Covering Index Theorem [A] implies that $\chi(M) = \chi^{(2)}(\tilde{M})$, and hence

$$\tilde{h}^n(\tilde{M}) - h^n(M) = \sum_{i=0}^{n-1} h^i(M).$$

Recall that $h^i(M) = \int_M TrH_i(x,x)$ and $\tilde{h}^i(\tilde{M}) = \int_M Tr\tilde{H}_i(x,x)$, where the second integral is taken over a fundamental domain of M on $\tilde{M}$. Since the Galois transformation group on M corresponding to the covering map of M to M_o is a biholomorphism and the Schwarz kernels are invariant under biholomorphism, this can also be rewritten as

$$\int_{M_o} (Tr\tilde{H}^n(\tilde{M}) - TrH^n(M)) = \sum_{i=0}^{n-1} \int_{M_o} TrH^i(M).$$

On the other hand, it follows from the previous proposition that

$$|\sum_{i=0}^{n-1} \int_{M_o} TrH^i(M)| \le [\frac{n}{2}]\epsilon V_o$$

if $\tau(M) \geq r(\epsilon)$. We also note that $\tilde{H}^n(\tilde{M})(x,x)$ is invariant under biholomorphism on $\tilde{M}$, and hence is a contant function.

Let $x \in M$ be any point. Without loss of generality, we may assume that the injectivity radius of M, $\tau(M) \geq 1$. Let $\sigma < 1$. Let $B_\sigma(x)$ be a ball of radius σ around x. Consider a complex geodesic coordinate in a normal neighbourhood of x. In terms of the Euclidean coordinates, consider the Euclidean metric g^E, which agrees with our metric g up to second order at x, so that $g^E(x) = g(x) + \mathcal{O}(|x|^2)$. There exist two Euclidean balls of optimal radius $r_1^E(\sigma)$ and $r_2^E(\sigma)$ respectively so that $B^E_{r_1(\sigma)}(x) \subset B_\sigma(x) \subset B^E_{r_2(\sigma)}(x)$, and as $\sigma < 1$, we can find absolute constants a_1, a_2 so that

$$a_1 vol^E(B^E_{r_1(\sigma)}(x)) \leq vol^g(B_\sigma(x)) \leq a_2 vol^E(B^E_{r_1(\sigma)}(x)),$$

the constants depending only on the curvature bound.

Recall that

$$H^n(M)(x,x) = \sum_{i=1}^N \|s_i\|^2 = \sum_{i=1}^N |f_i(x)|^2 h,$$

where $s_i = f_i dz^1 \wedge \cdots \wedge dz^n$ is an orthonormal bases of holomorphic n-forms, and h is the metric of the canonical line bundle. We have $0 < e_1 < h < e_2$ for some constants e_1, e_2 on $B_\sigma(x)$ depending only on the curvature bounds of M. From our earlier lemma, we conclude easily that

$$H^n(M)(x,x) \leq \tilde{H}^n(\tilde{M})(x,x) = c$$

for all x. Hence we conclude that $\sum_{i=1}^N |f_i(x)|^2 \leq \frac{c}{e_1}$. From Cauchy's integral formula we get

$$f'(x) = \int_{\partial\sigma(x)} \frac{f(s)}{s-x} ds,$$

and hence

$$\begin{aligned} |\sum_{i=1}^N |f_i'(x) f_i(x)|^2 &= |\int_{\partial\sigma(x)} \sum_i \frac{f_i(s) f_i(x)}{s-x} ds| \\ &\leq \int_{\partial\sigma(x)} \sum_i \frac{|f_i(s)|^2 + |f_i(x)|^2}{2\sigma} ds \\ &\leq \frac{2\pi c}{e_1}. \end{aligned}$$

It follows that the Euclidean derivative of $\sum_{i=1}^N |f_i(y)|^2$ is bounded by $\frac{4\pi c}{e_1}$ for all $y \in B_\sigma(x)$, and hence, by direction integration from x to y along a real straight line segment,

$$\sum_{i=1}^N |f_i(y)|^2 \leq \sum_{i=1}^N |f_i(x)|^2 + \frac{4\pi c}{e_1}|y-x|.$$

Suppose now that $\|H^n(M)(x,x)\| \leq \eta$. Then

$$\begin{aligned}
& \int_{B_\sigma(x)} |H^n(M)(y,y)|_g dv_g \\
= & \int_{B_\sigma(x)} |H^n(M)(y,y)|_{g^E} dv_{g^E} \\
= & \int_{B_\sigma(x)} \sum_{i=1}^N |f_i(y)|^2 dv_{g^E} \\
\leq & \int_{B_\sigma(x)} \sum_{i=1}^N [|f_i(x)|^2 + \frac{4\pi c e_2}{(n+1)e_1} r_2] dv_{g^E} \\
= & [\eta + \frac{4\pi c e_2}{(n+1)a_1 e_1} r_2] vol(B_\sigma)
\end{aligned}$$

We conclude that

$$\begin{aligned}
\int_{M_o} |H^n(M)(y,y)|_g dv_g &= \int_{M_o - B_{\sigma(x)}} H^n(M)(y,y) + \int_{B_{\sigma(x)}} H^n(M)(y,y) \\
&\leq (V_o - vol(B_{\sigma(x)}))c + [\eta + \frac{4\pi c e_2}{(n+1)a_1 e_1} r_2] vol(B_\sigma).
\end{aligned}$$

On the other hand, by our previous constructions, we have

$$\int_{M_o} |H^n(M)(y,y)|_g \geq V_o c - \epsilon.$$

Hence

$$\eta \geq c - [\frac{4\pi c e_2}{(n+1)a_1 e_1} r_2] - \frac{\epsilon}{vol(B_\sigma)}.$$

Recall that $r_2 = r_2(\sigma)$ is the the radius of the smallest Euclidean ball at x containing $B_\sigma(x)$. Choose σ and hence r_2 so small that $c - [\frac{4\pi c e_2}{(n+1)a_1 e_1} r_2] \geq (1 - \frac{\alpha}{2})c$. Then choose ϵ so small that $\frac{\epsilon}{vol(B_\sigma)} < \frac{\alpha}{2}c$. Now from the previous lemma, we can find ϵ satisfying the latter condition if the injectivity radius τ of M is sufficiently large. This concludes the proof of the theorem. □

3. Effective very ampleness

First we prove the following lemma.

Lemma 4. *In the same notation as in Theorem 1, assume that the Bergman kernels $H^n(x,y)$ of M and $\tilde{H}^n(x,y)$ of $\tilde{M}$ satisfy $\|\tilde{H}^n(x,x) - H^n(x,x)\| < \epsilon$. Then*

$$\|\tilde{H}^n(x,y) - H^n(x,y)\| < 4\epsilon$$

for all $(x,y) \in M$ with the distance $d(x,y) < \frac{\tau}{2}$ if the injectivity radius τ is greater than an effective constant.

Proof. First we note that, pulling back to the universal covering $\tilde{M}$, which is a bounded domain, the addition or subtraction of the heat and Bergman kernels involved in the following discussions makes sense, since only line bundle is involved and it can be trivialized by the standard canonical section on C^n.

Similarly to our arguments of Proposition 1, we consider

$$\begin{aligned} & |H_n(x,y) - \tilde{H}_n(x,y)| \\ \leq \quad & |H_n(x,y) - k_n(t,x,y)| + |k_n(t,x,y) - \tilde{k}_n(t,x,y)| + |\tilde{k}_n(t,x,y) - \tilde{H}_n(x,y)| \end{aligned}$$

Let us begin with the second term $|k_n(t,x,y) - \tilde{k}_n(t,x,y)|$, where

$$k_n(t,x,y) = \tilde{k}_n(t,x,y) + \sum_{\gamma\in\Gamma-\{1\}} \tilde{k}_n(t,x,\gamma y).$$

From the assumption that $d(x,y) < \frac{\tau}{2}$, we know that $d(x,\gamma y) > \frac{\tau}{2}$ for all $\gamma \in \Gamma - \{1\}$. Hence, as in Lemma 3,

$$|Trk_i(t,x,y) - Tr\tilde{k}_i(t,x,x)| \leq exp(-\frac{\tau^2}{48t}).$$

For the third term, given $\delta > 0$, the argument of Lemma 2 gives

$$|\tilde{k}_n(t,x,y) - \tilde{H}_n(x,y)| < \delta$$

for $t > r_2$, the notation being the same as in Lemma 2.

Hence, as in the proof of Proposition 1, for

$$\tau > r_3(T) = \max(T, c(T), \sqrt{-48T\log\delta} + 1)$$

and T determined by r_1 in Lemma 2, we conclude that

$$|k_n(t,x,y) - \tilde{k}_n(t,x,y)| + |\tilde{k}_n(t,x,y) - \tilde{H}_n(x,y)| \leq 2\delta.$$

We choose $\delta = \epsilon$.

For the first term, in terms of an orthonormal set of eigenvectors,

$$\begin{aligned} & |k_n(t,x,y) - H_n(x,y)| \\ = \quad & |\sum_{\lambda_i>0} e^{-\lambda_i t}\phi_i(x)\phi_i(y)| \\ \leq \quad & |\sum_{\lambda_i>0} e^{-\lambda_i t}\phi_i(x)\phi_i(x)|^{\frac{1}{2}} |\sum_{\lambda_i>0} e^{-\lambda_i t}\phi_i(y)\phi_i(y)|^{\frac{1}{2}} \\ = \quad & |k_n(t,x,x) - H_n(x,x)|^{\frac{1}{2}} |k_n(t,y,y) - H_n(y,y)|^{\frac{1}{2}}. \end{aligned}$$

But again

$$\begin{aligned} & |k_n(t,x,x) - H_n(x,x)| \\ \leq \quad & |k_n(t,x,x) - \tilde{k}_n(t,x,x)| + |\tilde{k}_n(t,x,x) - \tilde{H}_n(x,x)| + |\tilde{H}_n(x,x) - H_n(x,x)| \\ \leq \quad & 3\epsilon, \end{aligned}$$

where the last term is estimated by Theorem 1. Hence $|k_n(t,x,x) - H_n(x,x)|^{\frac{1}{2}} < 3\epsilon$ and so $|k_n(t,x,y) - H_n(x,y)| \leq 3\epsilon$.

Together with the estimates for the other two terms, we conclude that

$$|H_n(x,y) - \tilde{H}_n(x,y)| \leq 4\epsilon.$$

□

Proposition 2. *In the same notation as before, let M cover M_o, whose diameter is bounded from above by d_o. Then there is an effective constant r such that if the radius $r(M)$ of M is larger than r and $x \in M$ is arbitrary, there exists a canonical section $s(x)$ of K_M such that $\nabla s \neq 0$.*

Proof. By homogeneity of $\tilde{M}$, we may assume that x is the origin and identify M_o with a fundamental domain on $\tilde{M}$ containing 0. Pulling back to $\tilde{M}$, a section s on M is a holomorphic section of $\tilde{M}$, locally but not globally L^2. Let W be a vector of unit length on $\tilde{M}$, which is realized as a bounded domain on C^n. We use $\frac{\partial}{\partial w}$ or ∂_W to denote differentiation on $\tilde{M} \subset C^n$, and $C_{W,x}$ to denote the complex line through x in the direction of W.

Let $h(x,y) = \sum_i f_i(x)\overline{f_j(y)}$ and similarly define $\tilde{h}(x,y)$ for the coefficients of the Bergman kernels on M and $\tilde{M}$. We define

$$D_W h(x,y) = \sum_{i=1}^{N} \frac{\partial f}{\partial w}(x)\frac{\partial f}{\partial w}(y), \qquad D_W \tilde{h}(x,x) = \sum_{i=1}^{\infty} \frac{\partial \tilde{f}}{\partial w}(x)\frac{\partial \tilde{f}}{\partial w}(y).$$

As in the proof of the previous lemma, it suffices for us to do the estimates on a fundamental domain of M_o on $\tilde{M}$ and consider the lower bound for $Dh(x,x)$. We are going to show that Dh differs at most by the order of ϵ from $D\tilde{h}(x,x)$.

From the holomorphicity and antiholomorphicity of $h(x,y)$ and $\tilde{h}(x,y)$ in x and y respectively, we have the integral representation

$$\begin{aligned}
&|D_W h(x,y)| \\
&= |\sum_i \int_{s\in\partial B_r\cap C_{W,x}} \int_{t\in\partial B_r\cap C_{W,y}} [\frac{f_i(s)-f_i(x)}{s-x}][\overline{\frac{f_i(t)-f_i(y)}{t-y}}]| \\
&= |\sum_i \int_{s\in\partial B_r\cap C_{W,x}} \int_{t\in\partial B_r\cap C_{W,y}} [\frac{f_i(s)\overline{f_i(t)} - f_i(x)\overline{f_i(t)} - f_i(s)\overline{f_i(y)} + f_i(x)\overline{f_i(y)}}{(r-x)\overline{(s-y)}}] \\
&= \frac{1}{r^2}\int_{s\in\partial B_r\cap C_{W,x}} \int_{t\in\partial B_r\cap C_{W,y}} |h(s,t)-h(x,t)-h(t,y)+h(x,y)| \\
&\geq \frac{1}{r^2}\int_{s\in\partial B_r\cap C_{W,x}} \int_{t\in\partial B_r\cap C_{W,y}} [|\tilde{h}(s,t)-\tilde{h}(x,t)-\tilde{h}(t,y)+\tilde{h}(x,y)| - 4c\epsilon] \\
&= |D_W\tilde{h}(x,y)| - 4c\epsilon.
\end{aligned}$$

Here Lemma 4 was used. In particular,

$$D_W h(x,x) \geq D_W \tilde{h}(x,x) - 4c\epsilon.$$

Suppose $W = \sum_\alpha w^\alpha \frac{\partial}{\partial w^\alpha}$ in terms of local coordinates $w^\alpha, 1 \leq \alpha \leq n$. We define a section

$$s(x) = \frac{1}{\int_{\tilde{M}} |\bar{w}^\alpha z^\alpha dV|^2}[\sum_\alpha \bar{w}^\alpha z^\alpha dV]$$

on $\tilde{M}$. Then $\frac{\partial}{\partial w} f = \frac{1}{Vol^E(\tilde{M})}$ is bounded from below, as $\tilde{M}$ is a bounded domain in C^n. This concludes the proof of the lemma. □

Remark. For a complete manifold $\tilde{M}$ with injectivity radius bounded from below, we can always get a lower bound on $D_W\tilde{h}$ by considering appropriate L^2 sections using L^2-estimates as in [Y2].

Now we state the main theorem.

Theorem 2. *Let $\tilde{M}$ be an Hermitian symmetric space and M a cocompact quotient. Assume that there exists a holomorphic action of a finite group on M whose* *quotient is M_o. Then the canonical line bundle K_M of M is very ample if the injectivity radius of M is bounded from below by an effective constant r. The effective constant r depends on the following data.*

(i) The rate of the convergence of the heat kernel $k_i(t,x,y)$ to the Bergman kernel $\mathcal{H}(x,x)$ as $t \to \infty$. It would be sufficient to estimate ϵ such that

$$\int_{\sigma(\Delta_i)\cap(0,\epsilon)} e^{-\lambda} dE_{x,x}(\lambda) < 1$$

as estimated in Lemma 2.

(ii) The upper bound of the diameter of M_o.

Proof. a. Base point freeness. This follows from Theorem 1, since the existence of a positive lower bound of the Bergman kernel on M implies the non-vanishing of a certain canonical section at every point on M.

b. Immersion. Let ∇ be the Hermitian connection on the canonical line bundle. We need to show that for each point $x \in M$, and vector W, there exist sections s_i and s_j so that

$$\nabla_W(\frac{s_i}{s_j})(x) = \frac{s_j \nabla_W s_i - s_j \nabla_W s_i}{s_j^2}(x) \neq 0.$$

Since the Bergman metric is defined independently of choice of a basis, we may assume that s_1 is chosen among all canonical sections of M to take the supremum norm at x, i.e.

$$|s_1|_g = \sup_{\|f\|_g=1, f\in\Gamma(K)} |f|_g.$$

We then complete the section to an orthonormal basis by adding in sections $s_2, \ldots, s_N$. Let $A_W(x) = \sum_{j\geq 2} |\nabla_W s_j(x)|_g^2$. We are going to prove that $A_W(x)$ is bounded away from 0.

Assume on the contrary that $|\nabla_W s_j|_g = 0$ for all $i \geq 2$. It follows from the definition that $s_j(x) = 0$ for $j \geq 2$. In this way

$$\begin{aligned} |H^n(x,x)|_g &= |s_1(x)|_g^2, \\ \partial_W |H^n(x,x)|_g &= (\nabla_W s_1(x), s_1(x))_g, \end{aligned}$$

where g denotes the Hermitian metric on K induced by the original metric g and the second entry is conjugate holomorphic. Applying the same argument to L^2 canonical sections on $\tilde{M}$, we get

$$\begin{aligned} |\tilde{H}^n(x,x)|_g &= |\tilde{s}_1(x)|_g^2, \\ \partial_W |\tilde{H}^n(x,x)|_g &= (\nabla_W \tilde{f}_1(x), \tilde{s}_1(x))_g. \end{aligned}$$

We note that

$$0 \leq |\tilde{s}_1(x)|_g^2 - |s_1(x)|_g^2 = |\tilde{H}^n(x,x)|_g - |H^n(x,x)|_g \leq c\epsilon$$

with some effective absolute constant c. Hence $|s_1|_g \geq |\tilde{s}_1|_g - c\epsilon$.

As $|\tilde{H}(x,x)|_g$ is invariant under a biholomorphism, we get

$$(\nabla_W \tilde{s}_1(x), \tilde{s}_1(x))_g = \partial_W |\tilde{H}(x,x)|_g = 0$$

and hence $\nabla_W \tilde{s}_1(x) = 0$. The arguments of the previous proposition and Cauchy's integral formula imply that

$$c\epsilon \geq |\partial_W |\tilde{H}^n(x,x)|_g - \partial_W |H^n(x,x)|_g| \\ = |\sum_i (\nabla_W \tilde{s}_i(x), \tilde{s}_i(x))_g - \sum_j (\nabla_W s_j(x), s_j(x))_g|.$$

for some absolute constant c. Hence we conclude that $|(\nabla_W s_1(x), s_1(x))_g| \leq c\epsilon$, which can be rewritten as $|\nabla_W s_1(x)|_g |s_1(x)|_g \leq c\epsilon$. As a result,

$$|\nabla_W s_1|_g^2 \leq (\frac{c\epsilon}{|H(x,x)|_g - c\epsilon})^2.$$

However, from the previous proposition,

$$|D_W H(x,x)|_g \geq |D_W \tilde{H}(x,x)|_g - c\epsilon.$$

Defining $A_W(x) = |D_W H(x,x)|_g - |\nabla_W s_1|_g^2$,we conclude that

$$A_W(x) \geq |D_W \tilde{H}(x,x)|_g - c\epsilon - (\frac{c\epsilon}{|H_W(x,x)|_g - c\epsilon})^2.$$

As $|D_W \tilde{H}(x,x)|_g$ is bounded uniformly from below on M_o, it follows that $A_W(x) = \sum_{j\geq 2} |\nabla_W s_j(x)|_g^2$ is non-zero for ϵ sufficiently small. This implies that $|\nabla_W s_j(x)|_g \neq 0$ for some $j \geq 2$, which implies that the sections of K gives an immersion of M. As before, ϵ depends only on the radius of M and the diameter of M_o.

c. Separation of points. For this we need some lemmas

Lemma 5. *There exists a constant $r_1 > 0$ such that for all $x \in M$, $\Gamma(M,K)$ separates points on $B_{r_1}(x)$.*

Proof. Consider the function $A_W(x) = \sum_{j\geq 2} |\nabla_W s_j(x)|_g^2$ in the study of immersion of the canonical map, where s_1 takes the supremum value at x among all canonical sections of M whose L^2-norm is 1. Let $s_1^\perp$ be the space of norm 1 canonical sections in the orthogonal complement of s_1 with respect to g. We have $A_W(x) = \sum_{f\in s_1^\perp} |\nabla_W f|_g^2$. The estimate on the lower bound of A_W in the previous section implies that there is a function $s \in s_1^\perp$ so that

$$|\nabla_W s|_g \geq C_1 = |D_W \tilde{H}(x,x)|_g - c\epsilon - (\frac{c\epsilon}{|H(x,x)|_g - c\epsilon})^2.$$

Hence

$$|\nabla_W (\frac{s}{s_1})(x)| = |\frac{\nabla_W s}{s_1}(x)| = \frac{|\nabla_W s|_g(x)}{\sqrt{H}(x,x)} \geq \frac{C_1}{\sqrt{H(x,x)}}.$$

From here on when doing estimates we used the same notation x to denote a point on M and a point of its lift to the universal covering. Consider a small Euclidean ball $B^E_{r_1}(x)$ around x with r_1 sufficiently small, to be determined later, assuming for the time being only that $r_1 < 2\tau = \frac{1-|x|}{2}$. Pulling back to $\tilde{M}$, we can write $s = f dV, s_1 = f_1 dV$ so that f, f_1 are holomorphic functions on $B_{r_1}(x)$; the metric g is related to g_E by $c_1 g_E \leq g \leq c_2 g_E$. Since $\|s_1^2(x)\| = trB(x,x) = a^2$ at the point x, from our choice of s_1 we conclude that $c_1 a \leq |f_1(x)| \leq c_2 a$. For any holomorphic function f with L^2-norm 1, the mean value inequality implies that $|f(w)| < C$ for all $w \in B^E_{2\tau}(x)$ for some absolute constant C, since the injectivity radius of M is bounded from below by an absolute constant. From this we conclude that $|f'(w)| \leq \frac{C}{\tau}$ for $w \in B^E_\tau(x)$, from Cauchy type estimates. It follows from direct integration that $|f(y)-f(x)| \leq |y-x|\frac{C}{\tau} \leq r_1\frac{C}{\tau}$. Using Cauchy's estimates again, we conclude that

$$|\frac{\partial f}{\partial w}(y) - \frac{\partial f}{\partial w}(x)| \leq r_1\frac{C}{\tau^2},$$

for all $y \in B_1^E(x)$. Similarly, $|\frac{\partial f_1}{\partial w}(y) - \frac{\partial f_1}{\partial w}(x)| \leq r_1\frac{C}{\tau^2}$. Hence for $y \in B_{r_1}^E(x)$

$$\begin{aligned} & |\nabla_W(\frac{s}{s_1})(y) - \nabla_W(\frac{s}{s_1})(x)| \\ = & |\partial_W(\frac{f}{f_1})(y) - \partial_W(\frac{f}{f_1})(x)| \\ = & |[\frac{\partial_W f}{f_1^2}(y) - \frac{\partial_W f}{f_1^2}(x)] - [\frac{f\partial_W f_1}{f_1^2}(y) - \frac{f\partial_W f_1}{f_1^2}(x)]| \end{aligned}$$

As $c_1 a \leq |f_1(x)| \leq c_2 a$ and thus $|f_1(y)| \geq c_1 a - r_1\frac{C}{\tau}$, we conclude that

$$|\nabla_W(\frac{s}{s_1})(y) - \nabla_W(\frac{s}{s_1})(x)| \leq r_1 C_2$$

with C_2 an absolute constant depending only on τ and a, assuming that $c_1 a - r_1\frac{C}{\tau} > 0$. Integrating the expression along any geodesic rays from the origin on $B_{r_1}(x)$, we conclude that

$$\begin{aligned} |\frac{s}{s_1}(y)| &= |\int_0^1 \nabla_W(\frac{s}{s_1})(x+t(y-x))dt|y-x| + \frac{s}{s_1}(x)| \\ &= |\int_0^1 \nabla_W(\frac{s}{s_1})(x)dt|y-x| \\ &\quad + \int_0^1 [\nabla_W(\frac{s}{s_1})(x+t(y-x)) - \nabla_W(\frac{s}{s_1})(x)]dt|y-x| + 0| \\ &\geq |\int_0^1 \nabla_W(\frac{s}{s_1})(x)dt|y-x|| \\ &\quad - \int_0^1 |\nabla_W(\frac{s}{s_1})(x+t(y-x)) - \nabla_W(\frac{s}{s_1})(x)|dt|y-x| \\ &\geq \frac{C_1}{\sqrt{H}(x,x)}|y-x| - r_1 C_2|y-x|. \end{aligned}$$

Choosing

$$r_1 = \min(\frac{C_1}{2C_2\sqrt{H}(x,x)}, \frac{c_1 a\tau}{2C}),$$

we conclude that, $\frac{s}{s_1}(y) \neq 0 = \frac{s}{s_1}(x)$ on $B_{r_1}(x)$. As x is arbitrary on M, this implies the separation of point in each small neighbourhood of uniform size determined by r_1.

Now we consider the separation of points whose distance apart is at least r_1.

Lemma 6. *Assume that for any $c > 0$, there exists a number $\kappa > 0$ such that for every pair of points $x, y \in \tilde{M}$ of distance $d(x,y) \geq r_1$, there is always a holomorphic section $s \in \Gamma^{(2)}(\tilde{M}, K)$ satisfying $\|s\|_{L^2} = 1$, $s(x) = 0, \|s(y)\| \geq \kappa$. Then the canonical map of M takes different values at x and y if $d(x,y) > r_1$, provided that the injectivity radius of M is at least r for some effective r.*

Proof. From the lemma on the estimate of the difference between H^n and $\tilde{H}^n$, we know that for each $x \in M$, there exists a canonical section s_x such that $|s_x|_g(x) \geq c_1$ for some absolute constant $c_1 = \frac{1}{2}|\tilde{H}^n(x,x)|_g$ which is independent of x. From the mean-value inequality, as argued in the earlier proposition, we also have $|s_x|_g(x) \leq c_2$ for some absolute contant c_2. For any pair of points $x, y \in M$, we can find a linear combination $s_{xy} = rs_x + (1-r)s_y$ so that $c_3 \leq |s_{xy}|_g(x) \leq c_4$. For simplicity of notation, we denote $s = s_{xy}$.

Let $h(z,w) = \frac{H^n(x,y)}{s(x)\overline{s(y)}}$ and $\tilde{h}(z,w) = \frac{\tilde{H}^n(z,w)}{s(z)\overline{s(w)}}$. Let $s_i, i = 1, \dots, n$, be an orthonormal basis for M and $\tilde{s}_i, i \geq 1$, an orthonormal basis for $\tilde{M}$. Let $f_i = \frac{s_i}{s}$ and $\tilde{f}_i = \frac{\tilde{s}_i}{s}$. From the assumption on the universal covering, we know that $\sum_i |\tilde{f}_i(x) - \tilde{f}_i(y)|^2 \geq c$ for some fixed contant c determined by c_3 and c_4. By the earlier lemma on the estimate of the difference betweeen the Bergman kernel of M and $\tilde{M}$, we know that

$$\begin{aligned}
& \sum_i |f_i(x) - f_i(y)|^2 \\
= \quad & h(x,x) - h(y,x) - h(x,y) + h(y,y) \\
\geq \quad & \tilde{h}(x,x) - \tilde{h}(y,x) - \tilde{h}(x,y) + \tilde{h}(y,y) - 4C\epsilon \\
= \quad & \sum_i |\tilde{f}_i(x) - \tilde{f}_i(y)|^2 - 4C\epsilon \\
\geq \quad & c - 4C\epsilon
\end{aligned}$$

Hence if the radius of M is sufficiently large so that ϵ is small, as before, we conclude that $\sum_i |f_i(x) - f_i(y)|^2 \neq 0$ and hence the canonical section of M separates x and y. This concludes the proof of Lemma 6. □

The condition of Lemma 6 on the existence of a holomorphic section $s \in \Gamma^{(2)}(\tilde{M}, K)$ satisfying $\|s\|_{L^2} = 1$, $s(x) = 0, \|s(y)\| \geq \kappa$ is satisfied for Hermitian symmetric manifolds. In fact, it suffices to consider bounded linear holomorphic functions.

This concludes the proof of Lemma 5 and hence the proof of Theorem 2. □

4. Examples

Let M be a locally Hermitian symmetric space. It is well-known that the fundamental group Γ of M is residually finite in the sense that there is a tower of normal subgroups $\Gamma_1 = \Gamma$, $\Gamma_i > \Gamma_{i+1}$ and $\bigcap_{i=1}^{\infty} \Gamma_i = \infty$. Then $M_l = \tilde{M}/\Gamma_l$ is a $[\Gamma, \Gamma_l]$-sheeted covering of M. In this section, we would like to discuss the effective estimate of $[\Gamma, \Gamma_l]$ to guarantee the very ampleness of K_{M_l}. To apply results in earlier sections, we are going to consider M to be M_l with l to be estimated, and M_o to be the lowest manifold in the tower.

First we consider the arithmetic lattices Γ of $SU(n,1)$ defined by Hermitian quadratic forms. This gives rise to arithmetic complex ball quotients

$$\Gamma \backslash B_C^n = SU(n,1)/S(U(1) \times U(n)).$$

Let k be a totally real algebraic number field with places $\sigma_0 = 1, \sigma_1, ..., \sigma_m$. Let Q be an Hermitian quadratic form defined on C^n over k of signature $(1,n)$ such that the conjugates of Q by the $\sigma_i, i > 1$, are all negative definite. Let $\mathcal{Z}$ be the ring of integers of k and $G(Q)$ the group of elements in $GL(n+1)$ preserving Q. An arithmetic lattice Γ of $G = SU(n,1)$ arising from Q is defined to be a lattice commensurable with $G(Q, \mathcal{Z}) = GL(n+1, \mathcal{Z}) \cap G(Q)$, the group of units of Q. We can diagonalize Q by an element $g \in GL(n+1,k)$ such that $G(Q,\mathcal{Z})$ is commensurable with $G(Q^g,\mathcal{Z})$. Hence for our purpose, it suffices to consider Γ to be $G(Q,\mathcal{Z})$ for some diagonal Q of the form

$$l_o|x_0|^2 - l_1|x_1|^2 - \cdots - l_n|x_n|^2,$$

where $l_i \in k$ and $l_i > 0$. The sublattices are given by $\Gamma_j = \Gamma(q_j) = \{\gamma \in \Gamma : \gamma \equiv I \pmod{q_j}\}$ for a sequence of nested ideals q_j with norm approaching infinity. We would call such a sequence a tower of congruence subgroups. We also recall that if $q = t_1^{a_1}\cdots t_s^{a_s}$ is the decomposition of an ideal q into prime ideals and t_i lies above a rational prime p_i with residue class degree r_i, the norm of q is given by $|q| = \prod p_i^{a_i r_i}$.

Let $\tau(M_l)$ to be the injectivity radius of M_j. Lemma 2.2.1 of [Y1] allows us to estimate $\tau(M_l)$ in terms of $[\Gamma,\Gamma_l]$.

Lemma 7 ([Y1], Lemma 2.2.1)**.**

$$\tau(M_l)) \geq c + \frac{2}{(n+1)^2-1}\log([\Gamma,\Gamma_l]),$$

for an effective constant c independent of l. c depends only on the size of the $l_i, i = 0,\ldots,n$, in the defining equation of Γ and the diameter of M_o.

We note that $[\Gamma,\Gamma_l]$ is actually the ratio $\frac{vol(M_l)}{vol(M_o)}$ and is also the ratio of the degree of M_l to the degree of M_o. Recall that the degree of an algebraic manifold is the number $K_M^n = \int_M c_1(K)^n$. The ratio $[\Gamma,\Gamma_l]$ gives a lower bound of the norm of q_l, which in turn controls the lower bound of the injectivity radius of M_l from the explicit formula of the distance between two points and elementary arithmetic considerations.

This lemma and the theorem of the last section allow us to conclude the following statements.

Theorem 3. *For a tower of lattices from Hermitian quadratic forms of complex balls obtained as above, $\Gamma(M_j,K_j)$ is very ample on M_j as long as the degree of M_j is greater than a certain contant d which is an effective constant depending only on the injectivity radius and diameter of M_o.*

Here the degree of M_j is the integral of the top power of the Chern class of the canonical line bundle of M_j, and is related to the degree of M_o by a multiple corresponding to the order of the covering. Note that in applying the theorems of the previous sections, the volume V_o, of M_o can be estimated in terms of the injectivity radius and diameter of M_o.

Remark. For complex ball quotients, there are examples of non-arithmetic lattices in dimension 2 and 3. However, for higher rank Hermitian symmetric spaces $\tilde{M}$, according to the Margulis Arithmeticity Theorem, every cocompact lattice Γ of $\tilde{M}$ is arithmetic, as constructed in [B]. Hence we can consider a tower of congruence subgroups defined similarly to the above case of complex hyperbolic balls by considering a nested sequence of ideals q_i in the defining number field of Γ, and regarding $\Gamma_j = \Gamma(q_j) = \{\gamma \in \Gamma : \gamma \equiv I \pmod{q_j}\}$. In this way, $\tilde{M}/\Gamma_j$ gives rise to a tower of normal coverings of M_o. As we also have an explicit formula for the distance function between two points for each (global) Hermitian symmetric space, the argument of the above lemma, or Lemma 2.2.1 of [Y1], allows us to conclude that

$$\tau(M_l)) \geq c + f(n)\log([\Gamma, \Gamma_l])$$

for some constant $f(n)$ depending on the type of the symmetric space $\tilde{M}$, and c depending on M_o as before. Now we conclude that $\Gamma(M_j, K_j)$ is very ample on M_j as long as the degree of M_j is greater than a certain contant d which is an effective constant depending only on the injectivity radius and diameter of M_o, and the type of the Hermitian symmetric space $\tilde{M}$.

Theorem 4. *Suppose M is a compact Hermitian locally symmetric space of non-compact type of rank at least 2. There exists an unramified covering M' of M of controllable order of covering such that the canonical line bundle of M' is very ample. The order of the covering can be estimated in terms of the defining number field of the lattice of M on its universal covering $\tilde{M}$.*

This follows from the previous discussions. Note that the injectivity radius and diameter of M are determined by the lattice.

Finally, we add a few more remarks.

Remarks. **1.** The arguments of this article can easily be generalized to statements about estimating the injectivity radius so that the canonical section would generate a certain fixed k-jet of the manifold.

2. All the arguments can be applied to Kähler manifolds whose Riemannian sectional curvature is bounded by two negative constants. This follows from the fact that there are a lot of L^2-holomorphic functions on the universal covering, as explained in [Y2]. However, apart from locally Hermitian symmetric spaces, the only such example that we know of in higher dimension cases are the ones constructed by Mostow and Siu in [MS].

3. One consequence of the argument is that, given a hyperelliptic curve M, there is an unramified covering M' of M which is not hyperelliptic. Moreover, the order of the covering can be estimated in terms of some geometric data as stated in the theorems above.

References

[A] Atiyah, M. F., Elliptic operators, discrete groups, and von Neumann algebras, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), Astérisque No. 32–33, Soc. Math France, Paris, 1976, pp. 43–72. MR **54:**8741

[B] Borel, A., Compact Clifford-Klein forms of symmetric spaces. Topology 2 (1963), 111–122. MR **26:**3823

[BGM] Berger, M., Gauduchon, P., Mazet, E., Le Spectre d'une Varieté Riemannienne, Springer Lecture Notes in Mathematics No. 194, Springer-Verlag, N.Y., 1971. MR **43:**8025

[Do1] Donnelly, H., Asymptotic expansions for the compact quotients of properly discontinuous group actions, Illinois J. Math. 23 (1979), 485–496. MR **80h:**58049

[Do2] Donnelly, H., Elliptic operators and covers of Riemannian manifolds, Math. Zeit. 223(1996), 303-308. MR **98a:**58163

[HT] Hwang, J.-M. and To, W.-K., On Seshadri constants of canonical bundles of compact complex hyperbolic spaces, Compositio Math. 118 (1999), 203–215. MR **2000i:**32034

[Lü] Lück, W., Approximating L^2-invariants by their finite-dimensional analogues. Geom. Funct. Anal. 4 (1994), 455–481. MR **95g:**58234

[MP] Minakshisundaram, S., Pleijel, A., Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. Canadian J. Math. 1, (1949). 242–256. MR **11:**108b

[MS] Mostow, G. D., Siu, Y.-T., A compact Kähler surface of negative curvature not covered by the ball. Ann. of Math. (2) 112 (1980), no. 2, 321–360. MR **82f:**53075

[P] Patodi, V. K., Curvature and the eigenforms of the Laplace operator, J. Diff. Geom. 5(1971) 233-249. MR **45:**1201

[Y1] Yeung, S. K., Betti numbers on a tower of coverings, Duke Math. J. 73(1994), 201-226. MR **95e:**58182

[Y2] Yeung, S. K., Very ampleness of line bundles and canonical embedding of coverings of manifolds, to appear in Compositio. Math.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1395
E-mail address: yeung@math.purdue.edu

Department of Mathematics, The University of Hong Kong, Hong Kong