

SL_n -CHARACTER VARIETIES AS SPACES OF GRAPHS

ADAM S. SIKORA

ABSTRACT. An SL_n -character of a group G is the trace of an SL_n -representation of G . We show that all algebraic relations between SL_n -characters of G can be visualized as relations between graphs (resembling Feynman diagrams) in any topological space X , with $\pi_1(X) = G$. We also show that all such relations are implied by a single local relation between graphs. In this way, we provide a topological approach to the study of SL_n -representations of groups.

The motivation for this paper was our work with J. Przytycki on invariants of links in 3-manifolds which are based on the Kauffman bracket skein relation. These invariants lead to a notion of a skein module of M which, by a theorem of Bullock, Przytycki, and the author, is a deformation of the SL_2 -character variety of $\pi_1(M)$. This paper provides a generalization of this result to all SL_n -character varieties.

1. INTRODUCTION

In this paper we introduce a new method in the study of representations of groups into affine algebraic groups. Although we consider only SL_n -representations, the results of this paper can be generalized to other affine algebraic groups; see [Si].

For any group G and any commutative ring R with 1 there is a commutative R -algebra $Rep_n^R(G)$ and the universal SL_n -representation

$$j_{G,n} : G \rightarrow SL_n(Rep_n^R(G))$$

such that any representation of G into $SL_n(A)$, where A is an R -algebra, factors through $j_{G,n}$ in a unique way. This universal property uniquely determines $Rep_n^R(G)$ and $j_{G,n}$ up to an isomorphism.

$GL_n(R)$ acts on $Rep_n^R(G)$ (see Section 2), and the subring of $Rep_n^R(G)$ composed of the elements fixed by the action, $Rep_n^R(G)^{GL_n(R)}$, is called the *universal SL_n -character ring of G* . This ring contains essential information about SL_n -representations of G . In particular, if R is an algebraically closed field of characteristic 0, then there are natural bijections between the following three sets:

- the set of all R -algebra homomorphisms $Rep_n^R(G)^{GL_n(R)} \rightarrow R$,
- the set of all semisimple $SL_n(R)$ -representations of G up to conjugation, and
- the set of $SL_n(R)$ -characters of G .

It is convenient to think about $Rep_n^R(G)^{GL_n(R)}$ as the coordinate ring of a scheme, $\mathfrak{X}_n(G) = Spec(Rep_n^R(G)^{GL_n(R)})$, called the SL_n -character variety of G .

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As explained in Section 6, the algebra $Rep_n^R(G)^{GL_n(R)}$ encodes all algebraic relations between the SL_n -characters of G . Unfortunately, it is very difficult to give a finite presentation of $Rep_n^R(G)^{GL_n(R)}$ and, hence, to describe $\mathfrak{X}_n(G)$, even for groups G with relatively simple presentations.

In this paper, we present a topological approach to the study of SL_n -character varieties. We prove that $R[\mathfrak{X}_n(G)] = Rep_n^R(G)^{GL_n(R)}$ is spanned by a special class of graphs (resembling Feynman diagrams) in X , where X is any topological space with $\pi_1(X) = G$; see Theorem 3.7. Moreover, all relations between the elements of this spanning set are induced by specific local relations between the graphs, called *skein relations*.

We postpone a detailed study of applications of our graphical calculus to the theory of SL_n -representations of groups to future papers. In this paper, we content ourselves with an example, in which we apply our method to a study of SL_3 -representations of the free group on two generators. In this algebraically non-trivial example a huge reduction of computational difficulties can be achieved by the application of our geometric method.

This work is related to several areas of mathematics and physics:

Knot theory (and skein modules). Skein relations between links were used to define the famous polynomial invariants of links, like the Conway, Jones, and Homfly polynomials, [Co, Jo, FYHLMO, P-T, Ka]. In this paper we apply skein relations to the representations of groups.

The motivation for this work was our earlier work on skein modules, [PS-2]. The main theorem of this paper generalizes the Bullock-Przytycki-Sikora theorem relating the Kauffman bracket skein module of a manifold M to the $SL_2(\mathbb{C})$ -character variety of $\pi_1(M)$; see [B-2, PS-2].

Quantum invariants of 3-manifolds. We hope that this work will help understand the connections between quantum invariants of 3-manifolds and representations of their fundamental groups. It follows from the work of Yokota [Yo] that for any 3-manifold M , the SU_n -quantum invariants of M can be defined by using our graphs considered up to relations which are q -deformations of our skein relations.

Spin networks and gauge theory. The graphs considered in this paper have an interpretation as spin networks; see [Si]. They are also very similar to graphs used by physicists in non-abelian gauge theory (QCD); see [Cv].

Number theory. After a preliminary version of this paper was made available, M. Kapranov pointed out to us that our work is related to the work of Wiles and others on “pseudo-representations.” In his work (related to Fermat’s Last Theorem), Wiles gave necessary and sufficient conditions under which a complex-valued function on G is a $GL_2(\mathbb{C})$ -character of G . His ideas were developed further and generalized to all GL_n -characters by Taylor, [Ta]. See also [Ny, Ro]. These results provide a description of the coordinate ring of GL_n -character varieties quotiented by nilpotent elements. Our results are similar in spirit, but they are concerned with SL_n -representations and they are stronger, since they describe $R[\mathfrak{X}_n(G)]$ (i.e. $Rep_n^R(G)^{GL_n(R)}$) exactly (with possible nilpotent elements).

The plan of this paper is as follows. In Section 2 we introduce some basic notions and facts concerning representations of groups. In Section 3 we define the algebra $\mathbb{A}_n(X)$ in terms of graphs in X and formulate (Theorems 3.6 and 3.7) the main results of the paper asserting that $\mathbb{A}_n(X)$ is isomorphic to $Rep_n^R(G)^{GL_n(R)}$, where

$G = \pi_1(X)$. The proof requires introducing another algebra, $\mathbb{A}_n(X, x_0)$, associated with any pointed topological space (X, x_0) . The algebra $\mathbb{A}_n(X, x_0)$ is an interesting object by itself, and for $n = 2$ it already appeared in the theory of skein modules as a relative skein algebra. Sections 4 and 5 are devoted to the proof of the results of Section 3. In the final section we consider trace identities and use our results to describe the SL_3 -character variety of the free group on two generators.

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2. BACKGROUND FROM REPRESENTATION THEORY

In this section we introduce the basic elements of the theory of SL_n -representations of groups. We follow the approach of Brumfiel and Hilden, ([B-H], Chapter 8), which although formally restricted to SL_2 -representations, has a straightforward generalization to SL_n -representations for any n . Compare also [L-M], [Pro-2].

Let G be a group and let R be a commutative ring with unity. There is a commutative R -algebra $Rep_n^R(G)$, called *the universal representation algebra*, and *the universal representation*

$$j_{G,n} : G \rightarrow SL_n(Rep_n^R(G)),$$

with the following property: For any commutative R -algebra A and any representation $\rho : G \rightarrow SL_n(A)$ there is a unique homomorphism of R -algebras $h_\rho : Rep_n^R(G) \rightarrow A$ which induces a homomorphism of groups

$$SL_n(h_\rho) : SL_n(Rep_n^R(G)) \rightarrow SL_n(A)$$

such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{j_{G,n}} & SL_n(Rep_n^R(G)) \\ & \searrow \rho & \downarrow SL_n(h_\rho) \\ & & SL_n(A) \end{array}$$

This universal property uniquely determines $Rep_n^R(G)$ up to an isomorphism of R -algebras.

The universal representation algebra of G may also be constructed explicitly in the following way. Let $\langle g_i, i \in I | r_j, j \in J \rangle$ be a presentation of G such that all relations r_j are monomials in non-negative powers of generators, g_i . Such a presentation exists for every group G . Since we work with groups which are not necessarily finitely presented, I and J may be infinite. Let $P_n(I)$ be the ring of polynomials over R in variables x_{jk}^i , where $i \in I$ and $j, k \in \{1, 2, \dots, n\}$. Let A_i , for $i \in I$, be the matrix $(x_{jk}^i) \in M_n(P_n(I))$. For any word $r_j = g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_k}^{n_k}$ consider the corresponding matrix $M_j = A_{i_1}^{n_1} A_{i_2}^{n_2} \dots A_{i_k}^{n_k} \in M_n(P_n(I))$. Let \mathcal{I} be the two-sided ideal in $P_n(I)$ generated by polynomials $Det(A_i) - 1$, for $i \in I$, and by all entries of matrices $M_j - Id$, for $j \in J$, where Id is the identity matrix. We denote the quotient $P_n(I)/\mathcal{I}$ by $Rep_n^R(G)$ and the quotient map $P_n(I) \rightarrow Rep_n^R(G)$ by η . Let $\bar{x}_{jk}^i = \eta(x_{jk}^i)$ and $\bar{A}_i = (\bar{x}_{jk}^i) \in M_n(Rep_n^R(G))$.

Note that we divided $P_n(I)$ by all relations necessary for the existence of a representation

$$j_{G,n} : G \rightarrow SL_n(Rep_n^R(G))$$

such that $j_{G,n}(g_i) = \bar{A}_i$. The algebra $Rep_n^R(G)$ and the representation $j_{G,n} : G \rightarrow SL_n(Rep_n^R(G))$ are exactly the universal representation algebra and the universal SL_n -representation of G .

Let $A \in GL_n(R)$. By the definition of the universal representation ring, $Rep_n^R(G)$, there is a unique homomorphism $f_A : Rep_n^R(G) \rightarrow Rep_n^R(G)$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{j_{G,n}} & SL_n(Rep_n^R(G)) \\ & \searrow^{A^{-1}j_{G,n}A} & \downarrow^{SL_n(f_A)} \\ & & SL_n(Rep_n^R(G)) \end{array}$$

One easily observes that f_A is an automorphism of $Rep_n^R(G)$ and that the assignment $A \rightarrow f_A$ defines a left action of $GL_n(R)$ on $Rep_n^R(G)$. We denote this action by $A*$, i.e. $f_A(r) = A * r$, for any $r \in Rep_n^R(G)$ and $A \in GL_n(R)$. We call the ring $Rep_n^R(G)^{GL_n(R)}$ consisting of elements of $Rep_n^R(G)$ fixed by the action of $GL_n(R)$ the *universal character ring of G* . This term indicates a connection between the ring $Rep_n^R(G)^{GL_n(R)}$ and SL_n -characters of G , i.e. traces of SL_n -representations of G . In the simplest case, when R is an algebraically closed field of characteristic 0, and G is finitely generated, this connection can be described as follows. Every representation $\rho : G \rightarrow SL_n(R)$ induces a homomorphism $h_\rho : Rep_n^R(G) \rightarrow R$, whose restriction to $Rep_n^R(G)^{GL_n(R)}$ we denote by h'_ρ . The next proposition follows from geometric invariant theory and from the results of [L-M].

Proposition 2.1. *Under the above assumptions, the following sets are in a natural correspondence given by bijections $\rho \rightarrow h'_\rho$, and $\rho \rightarrow \chi = tr \circ \rho$:*

- the set of all semisimple SL_n -representations of G ;
- the set of all R -homomorphisms $Rep_n^R(G)^{GL_n(R)} \rightarrow R$;
- the set of all $SL_n(R)$ -characters of G .

By the proposition, we can identify the above sets and denote them by $X_n(G)$. By the second definition, $X_n(G)$ is the affine algebraic set composed of the closed points of the SL_n -character variety, $\mathfrak{X}_n(G)$, defined in the introduction. In other words,

$$R[X_n(G)] = R[\mathfrak{X}_n(G)]/\sqrt{0} = Rep_n^R(G)^{GL_n(R)}/\sqrt{0}.$$

As shown in [L-M, KM], $Rep_n^R(G)^{GL_n(R)}$ may contain nilpotent elements and, therefore, $\mathfrak{X}_n(G)$ contains more subtle information about SL_n -representations of G than $X_n(G)$. The ring $Rep_n^R(G)^{GL_n(R)}$ will be given a topological description in Section 3.

The $GL_n(R)$ action on $Rep_n^R(G)$ induces an action of $GL_n(R)$ on the ring of $n \times n$ matrices over $Rep_n^R(G)$. If $M = (m_{ij}) \in M_n(Rep_n^R(G))$ and $A \in GL_n(R)$, then

$$(2.1) \quad A * M = A \begin{pmatrix} A * m_{11} & A * m_{12} & \dots & A * m_{1n} \\ \vdots & \vdots & \dots & \vdots \\ A * m_{n1} & A * m_{n2} & \dots & A * m_{nn} \end{pmatrix} A^{-1}.$$

There is an equivalent definition of the action of $GL_n(R)$ on $Rep_n^R(G)$ and on $M_n(Rep_n^R(G))$. In order to introduce it we will first define $GL_n(R)$ -actions on $P_n(I)$ and $M_n(P_n(I))$. We can consider $P_n(I)$ as a ring of polynomial functions defined

on the product of I copies of $M_n(R)$, $M_n(R)^I \rightarrow R$, by identifying $x_{jk}^{i_0} \in P_n(I)$ with a map assigning to $(M_i)_{i \in I} \in M_n(R)^I$ the (j, k) -entry of M_{i_0} . Therefore,

$$P_n(I) = \text{Poly}(M_n(R)^I, R).$$

With this identification any entry in a matrix M in $M_n(P_n(I))$ is a polynomial function on $M_n(R)^I$. Therefore we can think of elements of $M_n(P_n(I))$ as coordinate-wise polynomial functions $M_n(R)^I \rightarrow M_n(R)$,

$$M_n(P_n(I)) = \text{Poly}(M_n(R)^I, M_n(R)).$$

If X, Y are sets with a left G -action, then the set of all functions $\text{Fun}(X, Y)$ has a natural left G -action defined for any $f : X \rightarrow Y$ and $g \in G$ by $g * f(x) = gf(g^{-1}x)$, for $x \in X$. GL_n acts on $M_n(R)$ and on $M_n(R)^I$ by conjugation, and it acts trivially on R . These actions induce $GL_n(R)$ -actions on $\text{Fun}(M_n(R)^I, R)$ and on $\text{Fun}(M_n(R)^I, M_n(R))$, which restrict to

$$P_n(I) = \text{Poly}(M_n(R)^I, R)$$

and

$$M_n(P_n(I)) = \text{Poly}(M_n(R)^I, M_n(R)).$$

The following statement is a consequence of the above definitions.

Lemma 2.2. (1) *The natural embedding of $P_n(I)$ into $M_n(P_n(I))$ as scalar matrices is $GL_n(R)$ -equivariant.*

(2) *$A_{i_0} = (x_{jk}^{i_0}) \in M_n(P_n(I))$ is invariant under the action of $GL_n(R)$, for any $i_0 \in I$.*

Now, we are going to show that $\eta : P_n(I) \rightarrow \text{Rep}_n^R(G)$ is $GL_n(R)$ -equivariant, and hence the action of $GL_n(R)$ on $P_n(I)$ induces a $GL_n(R)$ -action on $\text{Rep}_n^R(G)$ which coincides with the $GL_n(R)$ -action on $\text{Rep}_n^R(G)$ defined previously.

Proposition 2.3. *The following diagram commutes:*

$$(2.2) \quad \begin{array}{ccc} M_n(P_n(I)) & \xrightarrow{M_n(\eta)} & M_n(\text{Rep}_n^R(G)) \\ \downarrow \text{Tr} & & \downarrow \text{Tr} \\ P_n(I) & \xrightarrow{\eta} & \text{Rep}_n^R(G) \end{array}$$

and all maps appearing in it intertwine with the $GL_n(R)$ -action.

Proof. Since the commutativity of the above diagram is obvious, we will prove only that the trace functions and homomorphisms η , $M_n(\eta)$ are $GL_n(R)$ -equivariant.

- The trace map $\text{Tr} : M_n(R) \rightarrow R$ is $GL_n(R)$ -equivariant. Therefore the induced map

$$\text{Tr} : M_n(P_n(I)) = \text{Poly}(M_n(R)^I, M_n(R)) \rightarrow \text{Poly}(M_n(R)^I, R) = P_n(I)$$

is also $GL_n(R)$ -equivariant.

- If $M = (m_{ij}) \in M_n(\text{Rep}_n^R(G))$ and $A \in GL_n(R)$, then $A * M$ is given by matrix (2.1), whose trace is $\text{Tr}(A * M) = \sum_{i=1}^n A * m_{ii} = A * \text{Tr}(M)$. Therefore

$$\text{Tr} : M_n(\text{Rep}_n^R(G)) \rightarrow \text{Rep}_n^R(G)$$

is $GL_n(R)$ -equivariant.

- Recall that $P_n(I)$ is generated by elements x_{jk}^i . Therefore in order to prove that η is $GL_n(R)$ -equivariant it is enough to show that $\eta(A * x_{jk}^i) = A * \bar{x}_{jk}^i$, for any $i \in I, j, k \in \{1, 2, \dots, n\}$, where $\bar{x}_{jk}^i = \eta(x_{jk}^i) \in \text{Rep}_n^R G$.

For any $i_0 \in I, j_{G,n}(g_{i_0}) = \bar{A}_{i_0} \in SL_n(\text{Rep}_n^R(G))$. By the definition of the $GL_n(R)$ -action on $\text{Rep}_n^R(G)$, $A^{-1}j_{G,n}(g_{i_0})A$ is the matrix obtained from $j_{G,n}(g_{i_0})$ by acting on all its entries by A . Therefore

$$(2.3) \quad A * \bar{x}_{jk}^{i_0} = (j, k)\text{-entry of } A^{-1}\bar{A}_{i_0}A.$$

Having described $A * \bar{x}_{jk}^{i_0}$, we need to give an explicit description of $A * x_{jk}^{i_0} \in P_n(I)$. Recall that we identified $x_{jk}^{i_0}$ with the map $M_n(R)^I \rightarrow R$ assigning to $\{M_i\}_{i \in I}$ the (j, k) -entry of M_{i_0} . The definition of the $GL_n(R)$ -action on maps between $GL_n(R)$ -sets implies that

$$(A * x_{jk}^{i_0})(\{M_i\}_{i \in I}) = A * \left(x_{jk}^{i_0}(A^{-1} * \{M_i\}_{i \in I}) \right).$$

Since $GL_n(R)$ acts by simultaneous conjugation on $M_n(R)^I$ and it acts trivially on R , the right side of the above equation is equal to the (j, k) -entry of $A^{-1}M_{i_0}A$. But the entries of M_{i_0} are given by the values of functions $x_{jk}^{i_0}$ evaluated on $\{M_i\}_{i \in I}$. Therefore

$$(2.4) \quad A * x_{jk}^{i_0} = (j, k)\text{-entry of } A^{-1}(x_{jk}^{i_0})A.$$

Finally, (2.3) and (2.4) imply that

$$\begin{aligned} \eta(A * x_{jk}^{i_0}) &= \eta(\text{the } (j, k)\text{-entry of } A^{-1}A_{i_0}A) \\ &= \text{the } (j, k)\text{-entry of } A^{-1}\bar{A}_{i_0}A = A * \bar{x}_{jk}^{i_0}. \end{aligned}$$

- We prove that $M_n(\eta)$ is equivariant. Let $M = (m_{jk}) \in M_n(P_n(I))$. Notice that the definition of $GL_n(R)$ -action on $M_n(P_n(I))$ implies that $A * M = A(A * m_{jk})A^{-1}$. Therefore

$$M_n(\eta)(A * M) = M_n(\eta)(A(A * m_{jk})A^{-1}) = A(\eta(A * m_{jk}))A^{-1}.$$

Since η is $GL_n(R)$ -equivariant, the matrix on the right side of the above equation is $A(A * \eta(m_{jk}))A^{-1} = A * (\eta(m_{jk}))$. Therefore $M_n(\eta)$ is also $GL_n(R)$ -equivariant. \square

The above proposition implies that there exists a function

$$\text{Tr} : M_n(\text{Rep}_n^R(G))^{GL_n(R)} \rightarrow \text{Rep}_n^R(G)^{GL_n(R)}.$$

This function will be given a simple topological interpretation in the next section.

Proposition 2.4. *The image of the universal representation*

$$j_{G,n} : G \rightarrow M_n(\text{Rep}_n^R(G))$$

is invariant under the action of $GL_n(R)$.

Proof. Since the elements g_i generate G , it is sufficient to show that $j_{G,n}(g_i) \in M_n(\text{Rep}_n^R(G))^{GL_n(R)}$. By Lemma 2.2(2), $A_i \in M_n(P_n(I))^{GL_n(R)}$. The map $M_n(\eta)$ is equivariant. Therefore it takes the invariant A_i to the invariant $M_n(\eta)(A_i) = j_{G,n}(g_i)$. \square

3. SKEIN ALGEBRAS

In this section we assign to each path-connected topological space X a commutative R -algebra $\mathbb{A}_n(X)$ and to each pointed path-connected topological space (X, x_0) an R -algebra $\mathbb{A}_n(X, x_0)$. These algebras encode the most important information about the SL_n -representations of $\pi_1(X, x_0)$. We will show that if R is a field of characteristic 0 (but not necessarily algebraically closed), then $\mathbb{A}_n(X)$ is isomorphic to the universal character ring $Rep_n^R(G)^{GL_n(R)}$, where $G = \pi_1(X, x_0)$, and $\mathbb{A}_n(X, x_0)$ is isomorphic to $M_n(Rep_n^R(G))^{GL_n(R)}$.

We start with a definition of a graph which is the most suitable for our purposes. A graph $D = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ consists of a vertex-set \mathcal{V} , a set of oriented edges \mathcal{E} , and a set of oriented loops \mathcal{L} . Each edge $E \in \mathcal{E}$ has a beginning $b(E) \in \mathcal{V}$ and an end $e(E) \in \mathcal{V}$. Loops have neither beginnings nor ends. If $b(E) = v$ or $e(E) = v$, then E is *incident* with v . For any $v \in \mathcal{V}$, all edges incident to v are ordered by consecutive integers $1, 2, \dots$. Therefore the beginning and the end of each edge is assigned a number.

The sets $\mathcal{V}, \mathcal{E}, \mathcal{L}$ are finite. We topologize each graph as a CW-complex. The topology of a graph coincides with the topology of its edges $E \simeq [0, 1], E \in \mathcal{E}$, and its loops $L \simeq S^1, L \in \mathcal{L}$. There is a natural notion of isomorphism of graphs.

Let \mathcal{G} be a set of representatives of all isomorphism classes of graphs defined above. We say that a vertex v is an n -valent source of a graph D if n distinct edges of D begin at v and no edge ends at v . Similarly, we say that v is an n -valent sink of D if n distinct edges end at v and no edge begins at v . Let \mathcal{G}_n denote the set of all graphs in \mathcal{G} , all of whose vertices are either n -valent sources or n -valent sinks. We assume that the empty graph \emptyset is also an element of \mathcal{G}_n . We denote the single loop in \mathcal{G}_n , i.e. the connected graph without any vertices, by S^1 . Let \mathcal{G}'_n denote the set of all graphs $D \in \mathcal{G}$ such that D has one 1-valent source and one 1-valent sink, and all other vertices of D are n -valent sources or n -valent sinks. We denote the single edge in \mathcal{G}'_n , i.e. the connected graph without any n -valent vertices, by $[0, 1]$.

Let X be a path-connected topological space. We will call any continuous map $f : D \rightarrow X$, where $D \in \mathcal{G}_n$, a graph in X . We identify two maps $f_1, f_2 : D \rightarrow X$ if they are homotopic. Let us denote the set of all graphs in X by $\mathcal{G}_n(X)$. Similarly, we define $\mathcal{G}_n(X, x_0)$ to be the set of all maps $f : D \rightarrow X \times [0, 1]$, where $D \in \mathcal{G}'_n$ and f maps the 1-valent sink of D to $(x_0, 1)$ and the 1-valent source of D to $(x_0, 0)$. We identify maps which are homotopic relative to $(x_0, 0)$ and $(x_0, 1)$. We will call elements of $\mathcal{G}_n(X, x_0)$ *relative graphs in $X \times [0, 1]$* .

We introduce a few classes of graphs in $\mathcal{G}_n(X)$ and $\mathcal{G}_n(X, x_0)$ which will often be used later on in the paper. Let $L_\gamma : S^1 \rightarrow X$ be a graph in X which represents the conjugacy class of $\gamma \in \pi_1(X, x_0)$. We denote by E_γ a relative graph $E_\gamma : [0, 1] \rightarrow X \times [0, 1]$, $E_\gamma(0) = (x_0, 0)$, $E_\gamma(1) = (x_0, 1)$, whose projection into X ,

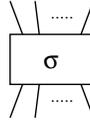
$$[0, 1] \xrightarrow{E_\gamma} X \times [0, 1] \rightarrow X,$$

represents $\gamma \in \pi_1(X, x_0)$. Let $EL_\gamma : [0, 1] \cup S^1 \rightarrow X \times [0, 1]$ be a relative graph such that $EL_\gamma(t) = (x_0, t)$ for $t \in [0, 1]$, and $EL_\gamma|_{S^1} : S^1 \rightarrow X \times [0, 1] \rightarrow X$ represents the conjugacy class of $\gamma \in \pi_1(X, x_0)$.

For any two graphs $f_1 : D_1 \rightarrow X$ and $f_2 : D_2 \rightarrow X$, $f_1, f_2 \in \mathcal{G}_n(X)$, we define a product of them to be $f_1 \cup f_2 : D_1 \cup D_2 \rightarrow X$, $f_1 \cup f_2 \in \mathcal{G}_n(X)$, where $D_1 \cup D_2$ denotes the disjoint union of D_1 and D_2 . Therefore the free R -module $R\mathcal{G}_n(X)$ on

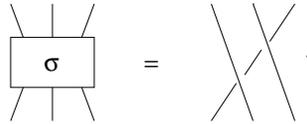
$\mathcal{G}_n(X)$ can be considered as a commutative R -algebra. The empty graph $\emptyset : \emptyset \rightarrow X$ is an identity in $R\mathcal{G}_n(X)$.

In the next definition we will represent fragments of diagrams by coupons, as depicted below:

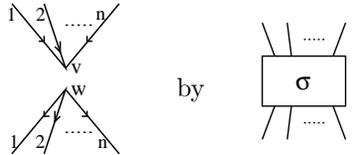


This coupon means a braid corresponding to a permutation $\sigma \in S_n$.¹

Example 3.1. If $\sigma = (1, 2, 3) \in S_3$, then



Suppose that $f : D \rightarrow X$ is a graph in X and f maps a source w and a sink v of D to the same point $x_1 \in X$. Let D_σ be a graph obtained from D by replacing



In the diagram we display v and w as separate points to accentuate the fact that they are distinct in the domain of the mapping f . There is an obvious way to modify $f : D \rightarrow X$ to a map $f_\sigma : D_\sigma \rightarrow X$. We call $(f, \{f_\sigma\}_{\sigma \in S_n})$ a family of *skein related graphs at x_1* .

Definition 3.2. Let X be a path-connected topological space, and let I be the ideal in $R\mathcal{G}_n(X)$ generated by two kinds of expressions:

- (1) $f - \sum_{\sigma \in S_n} \epsilon(\sigma) f_\sigma$, where $\epsilon(\sigma)$ denotes the sign of σ and $(f, \{f_\sigma\}_{\sigma \in S_n})$ is a family of skein related graphs at some point $x_1 \in X$.
- (2) $L_e - n$, where e is the identity element in $\pi_1(X, x_0)$ (i.e. L_e is a homotopically trivial loop).

Then the R -algebra $\mathbb{A}_n(X) = R\mathcal{G}_n(X)/I$ is called the n -th skein algebra of X .

Similarly we define $\mathbb{A}_n(X, x_0)$. Let $f_1 : D_1 \rightarrow X \times [0, 1], f_2 : D_2 \rightarrow X \times [0, 1]$ be elements of $\mathcal{G}_n(X, x_0)$. We define the product of them to be a map $f_1 \cdot f_2 : D_1 \cup D_2 \rightarrow X \times [0, 1]$, such that

$$(f_1 \cdot f_2)(d) = \begin{cases} (x, \frac{1}{2}t) & \text{if } d \in D_1 \text{ and } f_1(d) = (x, t), \\ (x, \frac{1}{2}t + \frac{1}{2}) & \text{if } d \in D_2 \text{ and } f_2(d) = (x, t). \end{cases}$$

This product extends to an associative (but generally non-commutative) product in $R\mathcal{G}_n(X, x_0)$. The identity in $R\mathcal{G}_n(X, x_0)$ is a map $f : E \rightarrow X \times [0, 1]$, where E is a single edge and f maps E onto $\{x_0\} \times [0, 1]$.

¹ Since we consider graphs up to homotopy equivalence, it does not matter which braid corresponding to σ we take.

If $f : D \rightarrow X \times [0, 1]$ is an element of $\mathcal{G}_n(X, x_0)$ such that an n -valent source w and an n -valent sink v of D are mapped to a point $x_1 \in X \times [0, 1]$, then one can define D_σ and $f_\sigma : D_\sigma \rightarrow X \times [0, 1]$, $f_\sigma \in \mathcal{G}_n(X, x_0)$, in exactly the same way as it was done for graphs in $\mathcal{G}_n(X)$ in the paragraph preceding Definition 3.2. We say, as before, that $(f, \{f_\sigma\}_{\sigma \in S_n})$ are graphs skein related at x_1 .

Definition 3.3. Let X be a path-connected topological space with a specified point $x_0 \in X$, and let I' be the ideal in $R\mathcal{G}_n(X, x_0)$ generated by expressions

- (1) $f - \sum_{\sigma \in S_n} \epsilon(\sigma) f_\sigma$, where $(f, \{f_\sigma\}_{\sigma \in S_n})$ is a family of skein related graphs at some point $x_1 \in X \times [0, 1]$.
- (2) $EL_e - n$, where e is the identity in $\pi_1(X, x_0)$.

Then the R -algebra $\mathbb{A}_n(X, x_0) = R\mathcal{G}_n(X, x_0)/I'$ is called the n -th relative skein algebra of (X, x_0) .

Note that different choices of $x_0 \in X$ give isomorphic algebras $\mathbb{A}_n(X, x_0)$.

Let $f \in \mathcal{G}_n(X)$, $f : D \rightarrow X$. Let $D' = D \cup E$, where E is an edge disjoint from D . Then D' has one 1-valent sink e_0 and one 1-valent source e_1 , $\{e_0, e_1\} = \partial E$, and $D' \in \mathcal{G}'_n$. We extend f to $f' : D' \rightarrow X \times [0, 1]$ in such a way that $f'(d) = (f(d), \frac{1}{2})$ for $d \in D$, and $f'(t) = (x_0, t)$ for $t \in [0, 1] \simeq E$, where $[0, 1] \simeq E$ is an orientation-preserving parameterization of E . This operation defines an embedding $\iota : \mathcal{G}_n(X) \rightarrow \mathcal{G}_n(X, x_0)$, $\iota(f) = f'$. Notice that ι induces a homomorphism $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$. Therefore we can consider $\mathbb{A}_n(X, x_0)$ as an $\mathbb{A}_n(X)$ -algebra.

Let $f : D \rightarrow X \times [0, 1]$ be a map, $f \in \mathcal{G}_n(X, x_0)$. Let $\overline{D} \in \mathcal{G}_n$ be a graph obtained by identification of the 1-valent sink with the 1-valent source in D . Let us compose $f : D \rightarrow X \times [0, 1]$ with a projection $X \times [0, 1] \rightarrow X$. This composition gives a map $\overline{f} : \overline{D} \rightarrow X$, $\overline{f} \in \mathcal{G}_n(X)$. Therefore, we have a function $\overline{\cdot} : \mathcal{G}_n(X, x_0) \rightarrow \mathcal{G}_n(X)$. This function can be extended to an R -linear homomorphism $\mathbb{T} : \mathbb{A}_n(X, x_0) \rightarrow \mathbb{A}_n(X)$. Notice that for any graph $D \in \mathcal{G}_n(X)$, $\mathbb{T}(\iota_*(D))$ is equal to a union of $f : D \rightarrow X$ with a contractible loop in X . Hence by Definition 3.2(2) $\mathbb{T}(\iota_*(D)) = n \cdot D$. Since graphs in X span $\mathbb{A}_n(X)$, the composition of $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$ with $\mathbb{T} : \mathbb{A}_n(X, x_0) \rightarrow \mathbb{A}_n(X)$ is equal to n times the identity on $\mathbb{A}_n(X)$. This implies the following fact.

Fact 3.4. *If $\frac{1}{n} \in R$, then $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$ is a monomorphism of rings.*

The next proposition summarizes basic properties of $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$.

Proposition 3.5. (1) *The assignment $X \rightarrow \mathbb{A}_n(X)$ (respectively, $(X, x_0) \rightarrow \mathbb{A}_n(X, x_0)$) defines a functor from the category of path-connected topological spaces (respectively, category of path-connected pointed spaces) to the category of commutative R -algebras (respectively, the category of R -algebras).*

- (2) *If $f : X \rightarrow Y, f(x_0) = y_0$, induces a surjection $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$, then the corresponding homomorphisms $\mathbb{A}_n(f) : \mathbb{A}_n(X, x_0) \rightarrow \mathbb{A}_n(Y, y_0)$ and $\mathbb{A}_n(f) : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(Y)$ are epimorphisms of R -algebras.*
- (3) *The algebra $\mathbb{A}_n(X)$ is generated by loops in X , i.e. by graphs L_γ , for $\gamma \in \pi_1(X, x_0)$.*
- (4) *The algebra $\mathbb{A}_n(X, x_0)$ is generated by graphs $E_{g_i^{\pm 1}}$ and EL_γ , where $\{g_i\}_{i \in I}$ is a set of generators of $\pi_1(X, x_0)$ and $\gamma \in \pi_1(X, x_0)$.*

Proof. Since statements (1) and (2) of Proposition 3.5 are obvious, we give a proof of (3) and (4) only.

From the definition of a graph $D \in \mathcal{G}_n$ or $D \in \mathcal{G}'_n$ we see that it has an equal number of n -valent sinks and sources. Relation (1) of Definition 3.2 and of Definition 3.3 implies that each pair of vertices of $f : D \rightarrow X$, $f \in \mathcal{G}_n(X)$ (respectively, of $f : D \rightarrow X \times [0, 1]$, $f \in \mathcal{G}_n(X, x_0)$) composed of a sink and a source can be resolved and f can be replaced by a linear combination of graphs with a smaller number of sinks and sources. Therefore, after a finite number of steps each graph in $\mathcal{G}_n(X)$ (respectively, $\mathcal{G}_n(X, x_0)$) can be expressed as a linear combination of graphs without n -valent vertices.

(3) If $f : D \rightarrow X$, $f \in \mathcal{G}_n(X)$, and D has no vertices, then D is a union of loops, $D = S^1 \cup S^1 \cup \dots \cup S^1$, and therefore $f = L_{\gamma_1} \cdot L_{\gamma_2} \cdot \dots \cdot L_{\gamma_k} \in \mathbb{A}_n(X)$, for some $\gamma_1, \gamma_2, \dots, \gamma_k \in \pi_1(X, x_0)$.

(4) If $f : D \rightarrow X \times [0, 1]$, $f \in \mathcal{G}_n(X, x_0)$, and D has no n -valent vertices, then $D = [0, 1] \cup S^1 \cup S^1 \cup \dots \cup S^1$. Suppose that $[0, 1] \xrightarrow{f} X \times [0, 1] \rightarrow X$ represents $\gamma_0 \in \pi_1(X, x_0)$, and f restricted to the j -th circle represents the conjugacy class of a $\gamma_j \in \pi_1(X, x_0)$, $j = 1, 2, \dots, k$. Then $f = E_{\gamma_0} \cdot EL_{\gamma_1} \cdot EL_{\gamma_2} \cdot \dots \cdot EL_{\gamma_k} \in \mathbb{A}_n(X, x_0)$. Therefore $\mathbb{A}_n(X, x_0)$ is generated by the elements EL_{γ} and $E_{\gamma'}$, $\gamma, \gamma' \in \pi_1(X, x_0)$. But each $E_{\gamma'}$ is a product of elements $E_{g_i^{\pm 1}}$, where $\{g_i\}$ is a set of generators of $\pi_1(X, x_0)$. □

We will see later that $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$ depend only on $\pi_1(X, x_0)$. Moreover, if $\pi_1(X, x_0)$ is a finitely generated group, then the algebras $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$ are also finitely generated.

Now we are ready to formulate the most important results of this paper.

Theorem 3.6. *Let X be any (path-connected) topological space, and let $G = \pi_1(X, x_0), x_0 \in X$. There are R -algebra homomorphisms*

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(\text{Rep}_n^R(G))^{GL_n(R)}, \quad \theta : \mathbb{A}_n(X) \rightarrow \text{Rep}_n^R(G)^{GL_n(R)},$$

uniquely determined by the following conditions:

- (1) $\Theta(E_{\gamma}) = j_{G,n}(\gamma)$ and $\Theta(EL_{\gamma'}) = \text{Tr}(j_{G,n}(\gamma'))$, for any $\gamma, \gamma' \in \pi_1(X, x_0)$.
- (2) $\theta(L_{\gamma}) = \text{Tr}(j_{G,n}(\gamma))$, for any $\gamma \in \pi_1(X, x_0)$.

Moreover, the following diagram commutes:

$$(3.1) \quad \begin{array}{ccc} \mathbb{A}_n(X, x_0) & \xrightarrow{\Theta} & M_n(\text{Rep}_n^R(G))^{GL_n(R)} \\ \downarrow \mathbb{T} & & \downarrow \text{Tr} \\ \mathbb{A}_n(X) & \xrightarrow{\theta} & \text{Rep}_n^R(G)^{GL_n(R)} \end{array}$$

Theorem 3.7. *If R is a field of characteristic 0, then*

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(\text{Rep}_n^R(G))^{GL_n(R)}, \quad \theta : \mathbb{A}_n(X) \rightarrow \text{Rep}_n^R(G)^{GL_n(R)}$$

are isomorphisms of R -algebras.

Let R be a field of characteristic 0. It can be shown that if X is a 3-manifold, then $\mathbb{A}_2(X)$ is isomorphic to the Kauffman bracket skein module of X , $\mathcal{S}_{2,\infty}(X, R, \pm 1)$. Moreover, if X is a surface, then $\mathbb{A}_2(X, x_0)$ is isomorphic to the relative Kauffman bracket skein module of X , $\mathcal{S}_{2,\infty}^{rel}(X, R, \pm 1)$. See [PS-2], [H-P], for appropriate definitions and the notational conventions. The main results of [B-1], [B-2], [PS-1]

and [PS-2] relate the Kauffman bracket skein modules of 3-manifolds with the SL_2 -representation theory of their fundamental groups. Theorem 3.7 generalizes these results to groups SL_n , for any n .

Moreover, it can be shown that in the case when X is any path-connected topological space, $\mathbb{A}_2(X, x_0)$ and $\mathbb{A}_2(X)$ can be given the following simple algebraic description: Let $G = \pi_1(X)$ and let I be the ideal in the group ring RG generated by elements $h(g + g^{-1}) - (g + g^{-1})h$, where $g, h \in G$. There is an involution τ on $H(G) = RG/I$ sending g to g^{-1} . One can show that $\mathbb{A}_2(X, x_0)$ is isomorphic to $H(G)$ and $\mathbb{A}_2(X)$ is isomorphic to $H^+(G)$, where $H^+(G)$ is the subring of $H(G)$ invariant under τ . The algebras $H(G), H^+(G)$ are introduced and thoroughly investigated in [B-H]. One of the main results of [B-H] is that $H(G) = M_n(\text{Rep}_n^R(G))^{GL_n(R)}$ and $H^+(G) = \text{Rep}_n^R(G)^{GL_n(R)}$, for $n = 2$. (Compare also [Sa-1], [Sa-2].) Theorem 3.7 can be considered as a generalization of this result to all values of n .

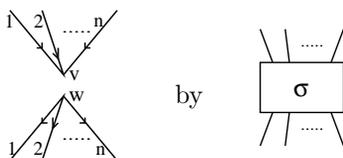
4. PROOF OF THEOREM 3.6

Before we prove Theorem 3.6 we give new definitions of $\mathbb{A}_n(X)$ and $\mathbb{A}_n(X, x_0)$ which only use $G = \pi_1(X, x_0)$.

Let X be a path-connected topological space and $x_0 \in X$. For any graph in $\mathcal{G}_n(X)$, i.e. a map $f : D \rightarrow X$ for some $D \in \mathcal{G}_n$, there is a map $f' : D \rightarrow X$ homotopic to f , which maps all vertices of D to x_0 . Therefore the homotopy class of f can be described by the graph D with each edge E labeled by an element of $\pi_1(X, x_0)$ corresponding to the map $f'_E : E \rightarrow X$ and each loop L labeled by the conjugacy class in $\pi_1(X, x_0)$ corresponding to the map $f'_{|L} : L \simeq S^1 \xrightarrow{f} X$. This description does not need to be unique.

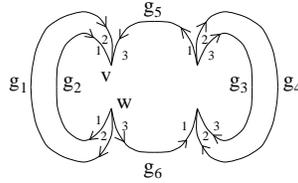
We denote the set of graphs in \mathcal{G}_n all of whose edges are labeled by elements of G and all loops are labeled by conjugacy classes in G by $\mathcal{G}_n(G)$. There is a natural multiplication operation on $\mathcal{G}_n(G)$. The product of $D_1, D_2 \in \mathcal{G}_n(G)$ is the disjoint union of D_1 and D_2 . Therefore $R\mathcal{G}_n(G)$ is a commutative R -algebra with \emptyset as the identity.

Let D be a graph in $\mathcal{G}_n(G)$. We have noticed already that D corresponds to a map $f : D \rightarrow X$ which maps all vertices of D to x_0 and restricted to edges and loops of D agrees with their labeling. Such f is unique up to a homotopy which fixes the vertices of D . Let w be a source and v be a sink in D . Since f maps v and w to the same point in X , there exists a map $f_\sigma : D_\sigma \rightarrow X$ defined for any $\sigma \in S_n$ as in the paragraph preceding Definition 3.2. Notice that f_σ maps all vertices of D_σ to $x_0 \in X$. Therefore, we can label all edges of D_σ by appropriate elements of G and all loops of D_σ by appropriate conjugacy classes in G , and hence consider D_σ as an element of $\mathcal{G}_n(G)$. Hence, we have showed that one can replace any source w and any sink v in an arbitrary graph $D \in \mathcal{G}_n(G)$

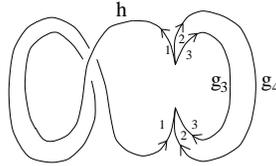


and obtain a well-defined graph $D_\sigma \in \mathcal{G}_n(G)$.

As an example consider the graph D presented below:



Replacing the vertices v, w by a coupon decorated by $\sigma = (123) \in S_3$ gives a diagram D_σ :



where $h = g_6 g_1 g_2 g_5$.

Now we are ready to define $\mathbb{A}_n(X)$ in terms of graphs in $\mathcal{G}_n(G)$. Namely, this algebra is isomorphic to $R\mathcal{G}_n(\pi_1(X, x_0))/I$, where $I \triangleleft R\mathcal{G}_n(\pi_1(X, x_0))$ is an ideal generated by relations analogous to relations (1) and (2) of Definition 3.2 and by relations following from the fact that the assignment $\mathcal{G}_n(G) \rightarrow \mathcal{G}_n(X)$ described above is onto but not 1-1. The problem comes from the fact that one can take a graph $D \in \mathcal{G}_n$ whose edges and loops are labeled in two different ways such that the corresponding maps $f, f' : D \rightarrow X$ sending the vertices of D to x_0 are homotopic but not by a homotopy relative to the vertices of D . In order to resolve this problem we need to allow an operation which moves vertices of D around paths in X beginning and ending at x_0 . Notice however that it suffices to move one vertex at a time. The following fact summarizes our observations.

Fact 4.1. *Let X be a path-connected topological space with a specified point $x_0 \in X$, and let $G = \pi_1(X, x_0)$. Let I be the ideal in $R\mathcal{G}_n(G)$ generated by expressions of the following form:*

$$(4.1) \quad \begin{array}{c} i \quad 2 \quad \dots \quad n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ 1 \quad 2 \quad \dots \quad n \end{array} - \sum_{\sigma \in S_n} \epsilon(\sigma) \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \sigma \\ \diagup \quad \diagdown \\ \dots \end{array},$$

$$(4.2) \quad \begin{array}{c} \circlearrowleft \\ e \end{array} - n,$$

$$(4.3) \quad \begin{array}{c} hg_1 \quad hg_2 \quad \dots \quad hg_n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array} - \begin{array}{c} g_1 \quad g_2 \quad \dots \quad g_n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array},$$

$$(4.4) \quad \begin{array}{c} g_1 h \quad g_2 h \quad \dots \quad g_n h \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array} - \begin{array}{c} g_1 \quad g_2 \quad \dots \quad g_n \\ \swarrow \quad \downarrow \quad \searrow \\ v \\ \swarrow \quad \downarrow \quad \searrow \\ w \\ g_1 \quad g_2 \quad \dots \quad g_n \end{array},$$

for any $g_1, g_2, \dots, g_n, h \in G$.

Then there is an isomorphism between the R -algebras $\mathbb{A}_n(X)$ and $RG_n(G)/I$ assigning to each graph $f : D \rightarrow X, f \in \mathcal{G}_n(X)$, with all vertices at x_0 the graph D with every edge E of D decorated by the element of $\pi_1(X, x_0)$ corresponding to $f|_E : E \rightarrow X$, and every loop L of D decorated by the conjugacy class in $\pi_1(X, x_0)$ represented by $f|_L : L \rightarrow X$.

Now we will state a similar fact for $\mathbb{A}_n(X, x_0)$. Let $\mathcal{G}'_n(G)$ be a set of graphs in \mathcal{G}'_n all of whose edges are labeled by elements of G and all of whose loops are labeled by conjugacy classes in G .

There is a multiplication operation defined on $\mathcal{G}'_n(G)$ in the following way. Let $D_1, D_2 \in \mathcal{G}'_n(G)$, let v_i be the 1-valent source of $D_i, i \in \{1, 2\}$, and let w_i be the 1-valent sink of D_i . Let g_i be the label of the edge of D_i joining v_i with w_i . The graph $D_1 \cdot D_2$ is obtained from the disjoint union of D_1 and D_2 by identifying v_1 with w_2 . The edge of $D_1 \cdot D_2$ joining v_2 with w_1 is labeled by $g_1 \cdot g_2$. All other edges and loops of $D_1 \cdot D_2$ inherit labels from D_1 and D_2 . A single edge labeled by $e \in G$ is the identity in $\mathcal{G}'_n(G)$.

This multiplication extends to an associative (but not commutative) multiplication in $RG'_n(G)$.

Fact 4.2. *Let X be a path-connected topological space with a specified point $x_0 \in X$, and let $G = \pi_1(X, x_0)$. Let I' be the ideal in $RG'_n(G)$ generated by the expressions (4.1),(4.3), (4.4) and*

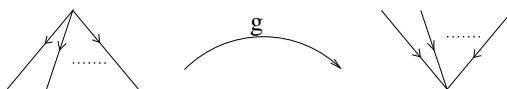
$$\begin{array}{c} \left. \begin{array}{c} \circlearrowleft \\ e \end{array} \right|_e^{-n} \end{array} .$$

Then $\mathbb{A}_n(X, x_0) \simeq RG'_n(G)/I'$.

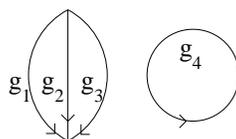
Facts 4.1 and 4.2 show that the algebras $\mathbb{A}_n(X, x_0)$ and $\mathbb{A}_n(X)$ depend only on $\pi_1(X, x_0)$. In fact 4.1 and 4.2 give us models for $\mathbb{A}_n(X, x_0)$ and $\mathbb{A}_n(X)$ built from $\mathcal{G}_n(G)$ and $\mathcal{G}'_n(G)$. In the rest of this section we will use these models.

Let us fix a commutative ring R and a positive integer n and a topological space X with $x_0 \in X, \pi_1(X, x_0) = G$. Let $\mathcal{R} = Rep_n^R(G)$ and let $V = \mathcal{R}^n$ be a free n -dimensional module over \mathcal{R} with the standard basis, $\{e_1, e_2, \dots, e_n\}, e_i = (0, 0, \dots, 1, \dots, 0)$. The dual space V^* has the dual basis $e^1, e^2, \dots, e^n, e^i(e_j) = \delta_{i,j}$. We will always use the standard bases and therefore identify $V^* \otimes V \simeq End_{\mathcal{R}}(V) \simeq M_n(\mathcal{R})$.

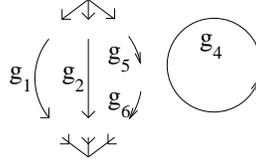
Let D be an element of $\mathcal{G}_n(G)$ or $\mathcal{G}'_n(G)$. We can decompose D into arcs, sources and sinks:



Example 4.3.



can be decomposed to



where $g_1, g_2, g_3, g_4, g_5, g_6 \in G$, $g_3 = g_6g_5$.

Notice that the decomposition of a graph is not unique, since we can cut each edge or loop into many pieces.

Let us assign to each n -valent source the tensor

$$\sum_{\sigma \in S_n} \epsilon(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \dots \otimes e_{\sigma(n)} \in V^{\otimes n},$$

and to each n -valent sink the tensor

$$\sum_{\sigma \in S_n} \epsilon(\sigma) e^{\sigma(1)} \otimes e^{\sigma(2)} \otimes \dots \otimes e^{\sigma(n)} \in (V^*)^{\otimes n}.$$

To each edge labeled by g we assign a tensor in $V^* \otimes V \simeq \text{End}_{\mathcal{R}}(V)$ corresponding to $j_{G,n}(g) \in SL_n(\mathcal{R}) \subset M_n(\mathcal{R})$.

Let D_0 denote a graph D decomposed into pieces. We assign to D_0 the tensor product of tensors corresponding to them. We denote this tensor by $T(D_0)$. Notice that $T(D_0) \in V^{\otimes N} \otimes (V^*)^{\otimes N}$, where $N =$ the number of 1-valent sources in $D_0 =$ the number of 1-valent sinks in D_0 .

Now we glue all components of D_0 together to get the graph D back. Whenever we glue an end of one piece to a beginning of another piece in D_0 , we make the corresponding contraction on $T(D_0)$. More specifically, suppose that the two free ends glued together correspond to the two underlined components:

$$T(D_0) \in V \otimes \dots \otimes \underline{V} \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes \underline{V^*} \otimes \dots \otimes V^*.$$

By applying to this tensor space the contraction map $\underline{V} \otimes \underline{V^*} \rightarrow \mathcal{R}$ (which is the evaluation map $(v, f) \rightarrow f(v)$), we send $T(D_0) \in V^{\otimes N} \otimes (V^*)^{\otimes N}$ to an element of $V^{\otimes N-1} \otimes (V^*)^{\otimes N-1}$. By repeating this process until we get the graph D back, we obtain an element of \mathcal{R} , if $D \in \mathcal{G}_n(G)$, or an element of $M_n(\mathcal{R})$, if $D \in \mathcal{G}'_n(G)$. Notice that the above construction does not depend on the particular decomposition of D into pieces. Therefore, we have defined functions

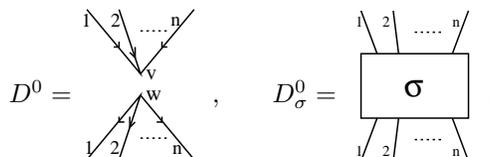
$$(4.5) \quad \Theta : \mathcal{G}'_n(G) \rightarrow M_n(\mathcal{R}), \quad \theta : \mathcal{G}_n(G) \rightarrow \mathcal{R}.$$

Lemma 4.4. *Let D be a graph in $\mathcal{G}_n(G)$ or in $\mathcal{G}'_n(G)$. Let w be an n -valent source of D and v an n -valent sink of D . Let D_σ , for $\sigma \in S_n$, be defined as at the beginning of Section 4. If $D \in \mathcal{G}_n(G)$, then $\theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma) \theta(D_\sigma)$. If $D \in \mathcal{G}'_n(G)$, then $\Theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma) \Theta(D_\sigma)$.*

Proof. We will prove Lemma 4.4 only for $D \in \mathcal{G}_n(G)$. For $D \in \mathcal{G}'_n(G)$ the proof is identical.

Decompose D and D_σ into sources, sinks, and edges. We denote the fragment of the decomposition of D composed of the source w and the sink v by D^0 . We

may assume that the decomposition of D_σ is identical to that of D , except that it contains a coupon D_σ^0 instead of D^0 :



We order the 1-valent sources and sinks of D_σ^0 consistently with the ordering of the 1-valent vertices of D^0 .

Let $T(D^0)$ (respectively, $T(D_\sigma^0)$) be the tensor associated to D^0 (respectively, D_σ^0). We assume that the i -th coordinate of $T(D^0) \in V^{\otimes n} \otimes V^{*\otimes n}$ corresponds to the i -th source of D^0 , if $1 \leq i \leq n$, or to the $(i - n)$ -th sink of D^0 , if $n < i \leq 2n$.

Recall that $\theta(D), \theta(D_\sigma) \in \mathcal{R}$ are results of contractions of tensors associated with elements of decompositions of D and D_σ . Since the decompositions of D and D_σ chosen by us differ only by elements D^0, D_σ^0 , the proof of Lemma 4.4 can be reduced to a local computation on tensors. Namely, it is enough to prove that

$$(4.6) \quad T(D^0) = \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0).$$

Notice that each edge of D_σ^0 is labeled by the identity map in $End_R(V)$. This map is represented by $\sum_{i=1}^n e_i \otimes e^i$ in $V \otimes V^* \simeq End_R(V)$. Therefore, if $\sigma = id \in S_n$, then

$$T(D_\sigma^0) = \sum_{i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_n}.$$

Similarly, for any $\sigma \in S_n$, we have

$$T(D_\sigma^0) = \sum_{i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{i_{\sigma(1)}} \otimes e^{i_{\sigma(2)}} \otimes \dots \otimes e^{i_{\sigma(n)}}.$$

Therefore,

$$(4.7) \quad \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0) = \sum_{\substack{\sigma \in S_n \\ i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}}} \epsilon(\sigma) e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{i_{\sigma(1)}} \otimes e^{i_{\sigma(2)}} \otimes \dots \otimes e^{i_{\sigma(n)}}.$$

Note that we can assume that the numbers i_1, i_2, \dots, i_n appearing on the right side of (4.7) are all different. Indeed, if $i_j = i_k, j \neq k$, then there is an equal number of even and odd permutations contributing the term

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_n} \otimes e^{j_1} \otimes e^{j_2} \otimes \dots \otimes e^{j_n}$$

to the sum on the right side of (4.7), for any j_1, j_2, \dots, j_n .

Therefore, we can assume that the numbers (i_1, i_2, \dots, i_n) appearing in each term of the sum on the right side of (4.7) form a permutation τ of $(1, 2, \dots, n)$. Hence we

have

$$\begin{aligned} & \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0) \\ &= \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) e_{\tau(1)} \otimes e_{\tau(2)} \otimes \dots \otimes e_{\tau(n)} \otimes e^{\tau(\sigma(1))} \otimes e^{\tau(\sigma(2))} \otimes \dots \otimes e^{\tau(\sigma(n))}. \end{aligned}$$

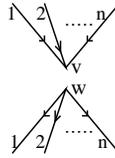
Substitute τ' for $\tau \circ \sigma$. Then $\epsilon(\sigma) = \epsilon(\tau)\epsilon(\tau')$, and we get

$$\begin{aligned} & \sum_{\sigma \in S_n} \epsilon(\sigma) T(D_\sigma^0) \\ &= \sum_{\tau, \tau' \in S_n} \epsilon(\tau)\epsilon(\tau') e_{\tau(1)} \otimes e_{\tau(2)} \otimes \dots \otimes e_{\tau(n)} \otimes e^{\tau'(1)} \otimes e^{\tau'(2)} \otimes \dots \otimes e^{\tau'(n)}. \end{aligned}$$

Notice that the right side of the above equation is equal to

$$\left(\sum_{\tau \in S_n} \epsilon(\tau) e_{\tau(1)} \otimes e_{\tau(2)} \otimes \dots \otimes e_{\tau(n)} \right) \otimes \left(\sum_{\tau' \in S_n} \epsilon(\tau') e^{\tau'(1)} \otimes e^{\tau'(2)} \otimes \dots \otimes e^{\tau'(n)} \right).$$

But the expression above is exactly the tensor assigned to:



Therefore we have proved (4.6) and completed the proof of Lemma 4.4. □

The study of SL_n -actions on linear spaces was one of the main objectives of classical invariant theory. In particular, Weyl ([We]) determined all invariants of the action of $SL(V)$ on $V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*$ and gave a full description of relations between them. The set of “typical” invariants consists of brackets $[v_1, \dots, v_n] = \text{Det}(v_1, \dots, v_n)$, $[\phi_1, \dots, \phi_n]^* = \text{Det}(\phi_1, \dots, \phi_n)$, where $v_1, \dots, v_n \in V$, $\phi_1, \dots, \phi_n \in V^*$, and contractions $\phi_j(v_i)$. The identity

$$(4.8) \quad [v_1, \dots, v_n][\phi_1, \dots, \phi_n]^* = \text{Det}(\phi_j(v_i))_{i,j=1}^n$$

is one of the fundamental identities relating the typical invariants. The bracket $[\cdot, \cdot, \dots, \cdot]$ is a skew symmetric linear functional on $V \otimes V \otimes \dots \otimes V$ and hence an element of $\wedge^n V^*$. Similarly, $[\cdot, \cdot, \dots, \cdot]^* \in \wedge^n V$. Note that the sources and sinks of graphs considered by us are labeled exactly by the tensors $[\cdot, \cdot, \dots, \cdot]^*$ and $[\cdot, \cdot, \dots, \cdot]$. (However, V is in our case a free module over $\mathcal{R} = \text{Rep}_n^R(G)$.)

It follows from the proof of Lemma 4.4 that

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \left[\begin{array}{c} \diagdown \quad \dots \quad \diagup \\ \boxed{\sigma} \\ \diagup \quad \dots \quad \diagdown \end{array} \right]$$

represents the tensor in $\text{Hom}(V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*, \mathcal{R}) = V^* \otimes \dots \otimes V^* \otimes V \otimes \dots \otimes V$ assigning to $(v_1, v_2, \dots, v_n, \phi_1, \phi_2, \dots, \phi_n)$ the value $\text{Det}(\phi_j(v_i))_{i,j=1}^n$. Therefore, the

identity

$$\theta(D) = \sum_{\sigma \in S_n} \epsilon(\sigma)\theta(D_\sigma)$$

is essentially equivalent to (4.8).

Lemma 4.5. *Let L_g, E_g, EL_g be graphs defined as in Section 3 but considered as elements of $\mathcal{G}_n(G)$ and $\mathcal{G}'_n(G)$, i.e.*

- (1) $L_g \in \mathcal{G}_n(G)$ is a single loop labeled by the conjugacy class of $g \in G$,
- (2) $E_g \in \mathcal{G}'_n(G)$ is a single edge labeled by $g \in G$, and
- (3) $EL_g \in \mathcal{G}'_n(G)$ is a graph composed of an edge labeled by the identity in G and of a loop labeled by the conjugacy class of $g \in G$.

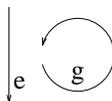
Under the above assumptions the functions Θ and θ satisfy conditions (1) and (2) of Theorem 3.6.

Proof. (1) L_g can be decomposed into a single arc



which has associated the tensor $j_{G,n}(g) \in SL_n(\mathcal{R}) \subset V^* \otimes V$. The contraction of this tensor gives $\theta(L_g) = Tr(j_{G,n}(g))$.

- (2) E_g is a single arc. Therefore $\Theta(E_g) = T(E_g) = j_{G,n}(g)$.
- (3) EL_g can be decomposed into:



The tensor associated with this decomposition is $id \otimes j_{G,n}(g) \in End(V) \otimes End(V)$. After making a contraction corresponding to the identification of the ends of the arc, we get

$$\Theta(EL_g) = id \cdot Tr(j_{G,n}(g)) \in End(V).$$

□

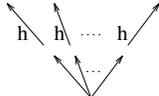
Lemma 4.6. *Let $D, D' \in \mathcal{G}_n(G)$ be two graphs which are identical as unlabeled graphs and which have the same labeling of edges and loops except the labeling of edges incident to a vertex v . Moreover, suppose that*

- (1) if v is a source, then the edges in D incident to v are labeled by g_1, g_2, \dots, g_n and the edges in D' incident to v are labeled by g_1h, g_2h, \dots, g_nh for some $g_1, g_2, \dots, g_n, h \in G$;
- (2) if v is a sink, then the edges in D incident to v are labeled by g_1, g_2, \dots, g_n and the edges in D' incident to v are labeled by hg_1, hg_2, \dots, hg_n for some $g_1, g_2, \dots, g_n, h \in G$.

Under the above assumptions $\theta(D) = \theta(D')$. An analogous fact is true for graphs in $\mathcal{G}'_n(G)$.

Proof. We prove part (1) only. The proof of part (2) is analogous.

Let v be a source. Notice that D and D' have identical decompositions into sinks, sources, and arcs except that $D_0 = \begin{matrix} & \swarrow & \dots & \searrow \\ & & & \end{matrix}$ is an element of the decomposition of

D and the diagram $D'_0 =$  is a fragment of a decomposition of D' .

Therefore we need to show that the tensors assigned to the above diagrams are identical. Notice that the tensor associated to D_0 , $T(D_0)$, is an element of the one-dimensional \mathcal{R} -linear space of skew-symmetric tensors $\bigwedge^n V \subset V^n$. Let $A : V \rightarrow V$ be an endomorphism given in the standard coordinates of V by $j_{G,n}(h) \in SL_n(\mathcal{R})$. A induces an endomorphism $\wedge A : \bigwedge^n V \rightarrow \bigwedge^n V$ with the property that $\wedge A(T(D_0)) = Det(A)T(D_0) \in \bigwedge^n V$. Notice that $\wedge A(T(D_0))$ is exactly the tensor associated to D'_0 . Since $Det(A) = 1$, the tensors associated to D_0 and D'_0 are equal. \square

Proof of Theorem 3.6. Let us extend the functions θ and Θ to all R -linear combinations of graphs in $\mathcal{G}_n(G)$ and $\mathcal{G}'_n(G)$ respectively. Facts 4.1 and 4.2 and Lemmas 4.4, 4.5, and 4.6 imply that these functions descend to R -linear homomorphisms

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(\mathcal{R}), \quad \theta : \mathbb{A}_n(X) \rightarrow \mathcal{R}.$$

By Lemma 4.5, Θ and θ satisfy conditions (1) and (2) of Theorem 3.6.

We have showed in Proposition 3.5(4) that $\mathbb{A}_n(X, x_0)$ is generated by elements E_γ and $EL_{\gamma'}$, for $\gamma, \gamma' \in G = \pi_1(X, x_0)$. By Proposition 2.4 and the paragraph preceding it, $\Theta(E_\gamma) = j_{G,n}(\gamma)$ and $\Theta(EL_{\gamma'}) = Tr(j_{G,n}(\gamma'))$ belong to

$$M_n(Rep_n^R(G))^{GL_n(R)}.$$

Therefore the image of Θ lies in $M_n(Rep_n^R(G))^{GL_n(R)}$. We show analogously that the image of θ lies in $Rep_n^R(G)^{GL_n(R)}$. Therefore the proof will be completed if we show that the diagram of Theorem 3.6 commutes.

Let $D \in \mathcal{G}'_n(G)$, $G = \pi_1(X, x_0)$, represent an element of $\mathbb{A}_n(X, x_0)$. Then $\Theta(D) \in M_n(Rep_n^R(G))^{GL_n(R)}$ is the result of a contraction of tensors associated with elements of some decomposition of D . Notice that $\mathbb{T}(D)$ is an element of $\mathbb{A}_n(X)$ represented by the diagram D with its 1-valent vertices identified.² Hence $\theta(\mathbb{T}(D))$ is a contraction of $\Theta(D)$, i.e. $\theta(\mathbb{T}(D)) = Tr(\Theta(D))$. Since the elements $D \in \mathcal{G}'_n(G)$ span $\mathbb{A}_n(X, x_0)$, the diagram of Theorem 3.6 commutes. \square

5. PROOF OF THEOREM 3.7

Now we assume that R is a field of characteristic 0.

We start by stating the first and second fundamental theorems of invariant theory, following the approach of Procesi, [Pro-1] (compare also [Ra]).

Let I be an infinite set. Let $P_n(I)$ and A_i be as before,

$$P_n(I) = R[x_{jk}^i, j, k \in \{1, 2, \dots, n\}, i \in I], \quad A_i = (x_{jk}^i) \in M_n(P_n(I)).$$

We are going to present Procesi's description of the ring $M_n(P_n(I))^{GL_n(R)}$.

Let T be a commutative R -algebra freely generated by the symbols

$$Tr(X_{i_1} X_{i_2} \dots X_{i_k}),$$

where $i_1, i_2, \dots, i_k \in I$. We adopt the convention that $Tr(M) = Tr(N)$ if and only if the monomial N is obtained from M by a cyclic permutation. Let $T\{X_i\}_{i \in I}$ be a non-commutative T -algebra freely generated by variables X_i , $i \in I$. We

²Recall that \mathbb{T} was defined in the paragraph preceding Fact 3.4.

have a natural T -linear homomorphism $Tr : T\{X_i\}_{i \in I} \rightarrow T$ which assigns to $X_{i_1}X_{i_2}\dots X_{i_k} \in T\{X_i\}_{i \in I}$ an element $Tr(X_{i_1}X_{i_2}\dots X_{i_k}) \in T$.

There is a homomorphism $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))$ uniquely determined by the conditions:

- $\pi(X_i) = A_i$,
- $\pi(Tr(X_{i_1}X_{i_2}\dots X_{i_k})) = Tr(A_{i_1}A_{i_2}\dots A_{i_k}) \in P_n(I) \subset M_n(P_n(I))$.³

Proposition 2.3 and Lemma 2.2 imply that the image of π is fixed by the $GL_n(R)$ -action on $M_n(P_n(I))$, i.e. $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$.

Notice that the following diagram commutes:

$$\begin{array}{ccc} T\{X_i\}_{i \in I} & \xrightarrow{\pi} & M_n(P_n(I))^{GL_n(R)} \\ \downarrow Tr & & \downarrow Tr \\ T & \xrightarrow{\pi|_T} & P_n(I)^{GL_n(R)} \end{array}$$

The following version of *The First Fundamental Theorem* of invariant theory of $n \times n$ matrices is due to Procesi, [Pro-1].

Theorem 5.1. $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$ is an epimorphism.

Before we state the second fundamental theorem of invariant theory of $n \times n$ matrices, we need some preparations.

Suppose that $\{1, 2, \dots, m\} \subset I$ and specify $i_0 \in \{1, 2, \dots, m\}$. We can present any $\sigma \in S_m$ as a product of cycles in such a way that i_0 is the first element of the first cycle, $\sigma = (i_0, i_1, \dots, i_s)(j_0, j_1, \dots, j_t)\dots(k_0, k_1, \dots, k_v)$. We define $\Phi_{\sigma, i_0}(X_1, X_2, \dots, X_m)$ to be equal to

$$X_{i_0}X_{i_1}\dots X_{i_s}Tr(X_{j_0}X_{j_1}\dots X_{j_t})\dots Tr(X_{k_0}X_{k_1}\dots X_{k_v}) \in T\{X_i\}_{i \in I}.$$

We also define another expression which does not depend on i_0 :

$$\Phi_\sigma(X_1, X_2, \dots, X_m) = Tr(X_{i_0}X_{i_1}\dots X_{i_s})Tr(X_{j_0}X_{j_1}\dots X_{j_t})\dots Tr(X_{k_0}X_{k_1}\dots X_{k_v}) \in T.$$

Let

$$F(X_1, X_2, \dots, X_m) = \sum_{\sigma \in S_m} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, \dots, X_m) \in T.$$

$F(X_1, X_2, \dots, X_{n+1})$ is called the fundamental trace identity of $n \times n$ matrices.

Procesi argues that there exists a unique element $G(X_1, X_2, \dots, X_n) \in T\{X_i\}_{i \in I}$, involving only the variables X_1, \dots, X_n and the traces of monomials in these variables, such that

$$F(X_1, X_2, \dots, X_{n+1}) = Tr(G(X_1, X_2, \dots, X_n)X_{n+1}) \in T\{X_i\}_{i \in I}.$$

Procesi gives an explicit formula for $G(X_1, X_2, \dots, X_n)$, but we want to give a different formula, which will be more suitable for our purposes.

Lemma 5.2.

$$\begin{aligned} G(X_1, X_2, \dots, X_n) &= \sum_{\sigma \in S_n} \epsilon(\sigma)\Phi_\sigma(X_1, X_2, \dots, X_n) - \sum_{\substack{i \in \{1, 2, \dots, n\} \\ \sigma \in S_n}} \epsilon(\sigma)\Phi_{\sigma, i}(X_1, X_2, \dots, X_n). \end{aligned}$$

³We identify $P_n(I)$ with the scalar matrices in $M_n(P_n(I))$.

Proof. It follows from the remarks preceding Lemma 5.2 that it is enough to show that if we multiply the right side of the equation of Lemma 5.2 by X_{n+1} , then the trace of it will be equal to $F(X_1, \dots, X_{n+1})$, i.e. we have to prove that

$$(5.1) \quad \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(X_1, X_2, \dots, X_n) Tr(X_{n+1}) - \sum_{\substack{i \in \{1, 2, \dots, n\} \\ \sigma \in S_n}} \epsilon(\sigma) Tr(\Phi_{\sigma, i}(X_1, X_2, \dots, X_n) X_{n+1}) = F(X_1, X_2, \dots, X_{n+1}).$$

Notice that $\epsilon(\sigma) \Phi_\sigma(X_1, X_2, \dots, X_n) Tr(X_{n+1}) = \epsilon(\sigma') \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1})$, where $\sigma' \in S_{n+1}$, $\sigma'(i) = \sigma(i)$, for $i \in \{1, 2, \dots, n\}$, and $\sigma'(n+1) = n+1$.

Similarly we can simplify $Tr(\Phi_{\sigma, i}(X_1, X_2, \dots, X_n) X_{n+1})$. Suppose that

$$\sigma = (i_0, i_1, \dots, i_s)(j_0, j_1, \dots, j_t) \dots (k_0, k_1, \dots, k_v) \in S_n,$$

where $i_0 = i$. Then

$$Tr(\Phi_{\sigma, i}(X_1, X_2, \dots, X_n) X_{n+1}) = \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1}),$$

for $\sigma' = (i_0, i_1, \dots, i_s, n+1)(j_0, j_1, \dots, j_t) \dots (k_0, k_1, \dots, k_v) \in S_{n+1}$. Notice that $\epsilon(\sigma') = -\epsilon(\sigma)$. Therefore the left side of the equation (5.1) is equal to

$$\sum_{\substack{\sigma' \in S_{n+1}, \\ \sigma'(n+1)=n+1}} \epsilon(\sigma') \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1}) + \sum_{\substack{i \in \{1, 2, \dots, n\}, \sigma' \in S_{n+1} \\ \text{such that } \sigma'(n+1)=i}} \epsilon(\sigma') \Phi_{\sigma'}(X_1, X_2, \dots, X_n, X_{n+1}).$$

The above expression is obviously equal to $F(X_1, X_2, \dots, X_{n+1})$. □

Now we are ready to state *The Second Fundamental Theorem* of invariant theory of $n \times n$ matrices, [Pro-1].

Theorem 5.3. *The kernel of π is generated by elements $G(M_1, M_2, \dots, M_n)$ and $F(N_1, N_2, \dots, N_{n+1})$, where $M_1, M_2, \dots, M_n, N_1, N_2, \dots, N_{n+1}$ are all possible monomials in the variables $X_i, i \in I$.*

Let X be a path-connected topological space. We choose a presentation $\langle g_i, i \in I | r_j, j \in J \rangle$ of $G = \pi_1(X, x_0)$ such that

- I is an infinite set,
- the inverse of every generator is also a generator, and
- the defining relations r_j are products of non-negative powers of generators.

Note that such a presentation always exists (even if G is finitely generated).

Let $\psi : T\{X_i\}_{i \in I} \rightarrow \mathbb{A}_n(X, x_0)$ be an R -homomorphism such that $\psi(X_i) = E_{g_i}$ and $\psi(Tr(X_{i_1} X_{i_2} \dots X_{i_k})) = EL_{g_{i_1} g_{i_2} \dots g_{i_k}}$. Recall that, by Proposition 3.4, $\mathbb{A}_n(X)$ can be considered as a subalgebra of $\mathbb{A}_n(X, x_0)$ in such a way that $L_\gamma \in \mathbb{A}_n(X)$ is identified with $EL_\gamma \in \mathbb{A}_n(X, x_0)$. Hence $\psi(Tr(X_{i_1} X_{i_2} \dots X_{i_k})) \in \mathbb{A}_n(X)$ and ψ restricts to $\psi : T \rightarrow \mathbb{A}_n(X)$. Moreover, the following diagram commutes:

$$(5.2) \quad \begin{array}{ccc} T\{X_i\}_{i \in I} & \xrightarrow{\psi} & \mathbb{A}_n(X, x_0) \\ \downarrow Tr & & \downarrow Tr \\ T & \xrightarrow{\psi} & \mathbb{A}_n(X) \end{array}$$

We are going to show that the kernel of $\psi : T\{X_i\}_{i \in I} \rightarrow \mathbb{A}_n(X, x_0)$ contains the kernel of $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$ and therefore ψ descends to a homomorphism $M_n(P_n(I))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0)$.

We will need the following fact, due to Formanek (Proposition 45 [For]).

Proposition 5.4. *For any matrix $A \in M_n(R)$*

$$Det(A) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) Tr(A^{c_1}) Tr(A^{c_2}) \dots Tr(A^{c_k}),$$

where c_1, c_2, \dots, c_k denote the lengths of all cycles in σ .

For completeness we sketch a proof of Proposition 5.4. A multilinearization of the determinant, $Det : M_n(R) \rightarrow R$, gives a function on n -tuples of $n \times n$ matrices,

$$\mathcal{M}(X_1, \dots, X_n) = \sum_{\sigma \in S_n} Det(X_\sigma),$$

where X_σ is a matrix whose i -th row is the i -th row of $X_{\sigma(i)}$. Note that $\mathcal{M}(A, \dots, A) = n! Det(A)$, and therefore the identity of Proposition 5.4 is a special case of the following identity:

$$\mathcal{M}(X_1, \dots, X_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(X_1, \dots, X_n),$$

where Φ_σ was defined in the second paragraph after Theorem 5.1. Formanek gives the following proof of the above identity. Assume that $1, 2, \dots, n \in I$. Since $A_1, \dots, A_n \in M_n(P_n(I))$ represent generic matrices, in order to prove the above identity it is enough to show it for $X_1 = A_1, \dots, X_n = A_n$. Since $\mathcal{M}(A_1, \dots, A_n)$ is an invariant polynomial function on n -tuples of matrices, the First Fundamental Theorem implies that $\mathcal{M}(A_1, \dots, A_n)$ can be expressed in terms of traces of monomials in A_1, \dots, A_n . Since $\mathcal{M}(A_1, \dots, A_n)$ is linear with respect to A_1, \dots, A_n , it is a linear combination of terms $Tr(A_{i_1} \dots A_{i_s}) \dots Tr(A_{j_1} \dots A_{j_t})$, where $i_1, \dots, i_s, \dots, j_1, \dots, j_t$ form a permutation of $1, 2, \dots, n$. Therefore

$$\mathcal{M}(A_1, \dots, A_n) = \sum_{\sigma \in S_n} \alpha_\sigma \Phi_\sigma(A_1, \dots, A_n),$$

and hence

$$(5.3) \quad \mathcal{M}(X_1, \dots, X_n) = \sum_{\sigma \in S_n} \alpha_\sigma \Phi_\sigma(X_1, \dots, X_n),$$

for any $n \times n$ matrices X_1, \dots, X_n . We need to prove that $\alpha_\sigma = \epsilon(\sigma)$. If we restrict the above equation to matrices $A_1, \dots, A_n \in M_{n-1}(P_{n-1}(I))$ embedded into $M_n(P_{n-1}(I))$ in the standard, non-unit-preserving way, we will get the following polynomial identity on $(n - 1) \times (n - 1)$ matrices:

$$\sum_{\sigma \in S_n} \alpha_\sigma \Phi_\sigma(A_1, \dots, A_n) = 0.$$

It is not difficult to see that the Second Fundamental Theorem implies that $F(A_1, \dots, A_n)$ is the only (up to scalar) n -linear trace identity of degree n on matrices $A_1, \dots, A_n \in M_{n-1}(P_{n-1}(I))$. Therefore $\alpha_\sigma = \epsilon(\sigma)c$, for some fixed c . Substituting the matrix (x_{ij}) with a single non-zero entry $x_{ii} = 1$ for X_i in (5.3), we get $c = 1$. Thus the proof of Proposition 5.4 is finished.

The specialization $A = Id \in M_n(R)$ in Proposition 5.4 yields the following corollary.

Corollary 5.5. *For any positive integer n ,*

$$\sum_{\sigma \in S_n} \epsilon(\sigma) n^{c(\sigma)} = n!,$$

where $c(\sigma)$ is the number of cycles in the cycle decomposition of σ .

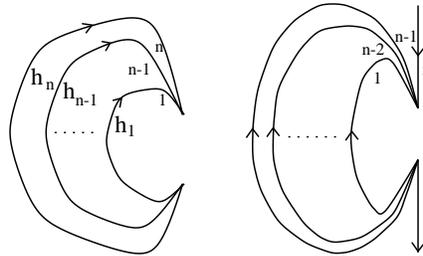
From the definition of $T\{X_i\}_{i \in I}$ it immediately follows that for any family of matrices $\{M_i\}_{i \in I} \subset M_n(R)$ there is a well-defined substitution

$$X_i \rightarrow M_i, \quad \text{Tr}(X_{i_1} X_{i_2} \dots X_{i_k}) \rightarrow \text{Tr}(M_{i_1} M_{i_2} \dots M_{i_k}) \in R \subset M_n(R),$$

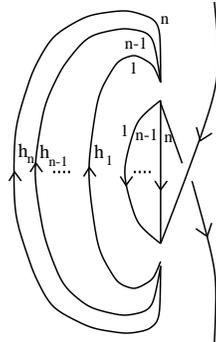
which can be extended to the whole ring $T\{X_i\}_{i \in I}$. Therefore, if $H(X_{i_1}, \dots, X_{i_k})$ is an element of $T\{X_i\}_{i \in I}$ involving variables X_{i_1}, \dots, X_{i_k} , then $H(M_{i_1}, \dots, M_{i_k})$ is a well-defined matrix in $M_n(R)$.

Lemma 5.6. *If N_1, N_2, \dots, N_n are any monomials in the variables X_i , $i \in I$, then $\psi(G(N_1, N_2, \dots, N_n)) = 0$.*

Proof. By the definition of ψ (given in the second paragraph after Theorem 5.3), $\psi(N_i) = E_{h_i}$, for some $h_1, h_2, \dots, h_n \in G$. Consider the following graph D in $\mathcal{G}'_n(G)$:



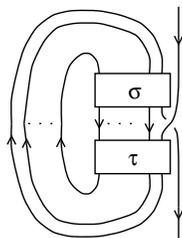
in which we omitted labels of edges labeled by the identity in G . Notice that D can also be presented in the following way:



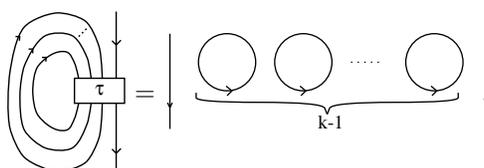
Since the vertices of D can be resolved in two possible ways (corresponding to the two diagrams above), we obtain the following equation:

$$(5.4) \quad \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) \epsilon(\tau) \left(\begin{array}{c} \text{Diagram } \sigma \\ \sigma \end{array} \right) \left(\begin{array}{c} \text{Diagram } \tau \\ \tau \end{array} \right) = \sum_{\sigma, \tau \in S_n} \epsilon(\sigma) \epsilon(\tau) D_{\sigma, \tau},$$

where $D_{\sigma,\tau}$ is a graph of the form:



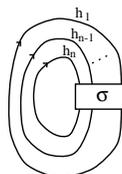
If $\tau \in S_n$ decomposes into $k = c(\tau)$ cycles, then



Therefore, by Corollary 5.5,

$$\sum_{\tau \in S_n} \epsilon(\tau) = \left(\text{graph with } \tau \text{ box and loop} \right) = \sum_{\tau \in S_n} \epsilon(\tau) n^{c(\tau)-1} = (n-1)!$$

Notice moreover that



is equal to $\psi(\Phi_\sigma(N_1, N_2, \dots, N_n))$. Therefore the left side of (5.4) is equal to the value of ψ on

$$(n-1)! \sum_{\sigma \in S_n} \epsilon(\sigma) \Phi_\sigma(N_1, N_2, \dots, N_n).$$

Now we are going to consider the right side of (5.4). Notice that the single arc in $D_{\sigma,\tau}$ is labeled by an element $h_{i_s} \dots h_{i_1} h_{i_0} \in G$, where $i_0 = \tau(n)$, $i_1 = \tau\sigma(i_0)$, \dots , $i_s = \tau\sigma(i_{s-1})$, and $\sigma(i_s) = n$. Since $\tau(\sigma(i_s)) = \tau(n) = i_0$, $(i_s, i_{s-1}, \dots, i_1, i_0)$ is a cycle of the permutation $(\tau\sigma)^{-1} \in S_n$.

Note that every loop in $D_{\sigma,\tau}$ is labeled by the conjugacy class of $h_{j_t} h_{j_{t-1}} \dots h_{j_1} h_{j_0}$, where $(j_t, j_{t-1}, \dots, j_1, j_0)$ is a cycle of $(\tau\sigma)^{-1} \in S_n$ disjoint from $(i_s, i_{s-1}, \dots, i_1, i_0)$. Therefore $D_{\sigma,\tau}$ is the value of ψ on

$$N_{i_s} \dots N_{i_1} N_{i_0} Tr(N_{j_t} \dots N_{j_1} N_{j_0}) \dots Tr(N_{k_v} \dots N_{k_1} N_{k_0}),$$

where

$$(i_s, \dots, i_1, i_0)(j_t, \dots, j_1, j_0) \dots (k_v, \dots, k_1, k_0)$$

is the cycle decomposition of $(\tau\sigma)^{-1}$. The above expression is equal to

$$\Phi_{(\tau\sigma)^{-1}, i_s}(N_1, N_2, \dots, N_n) = \Phi_{(\tau\sigma)^{-1}, \sigma^{-1}(n)}(N_1, N_2, \dots, N_n).$$

Therefore, the right side of (5.4) is the value of ψ on

$$\sum_{\sigma, \tau \in S_n} \epsilon(\sigma)\epsilon(\tau)\Phi_{(\tau\sigma)^{-1}, \sigma^{-1}(n)}(N_1, N_2, \dots, N_n) \in T\{X_i\}_{i \in I}.$$

Let us replace $(\tau\sigma)^{-1}$ by κ in the expression above. Then we get

$$\sum_{\sigma, \kappa \in S_n} \epsilon(\kappa)\Phi_{\kappa, \sigma^{-1}(n)}(N_1, N_2, \dots, N_n) = (n-1)! \sum_{\kappa \in S_n, i \in \{1, 2, \dots, n\}} \epsilon(\kappa)\Phi_{\kappa, i}(N_1, N_2, \dots, N_n).$$

After comparing the above algebraic descriptions of the two sides of (5.4) we see that for any monomials N_1, N_2, \dots, N_n the following element of $T\{X_i\}_{i \in I}$ belongs to $Ker \psi$:

$$\sum_{\sigma \in S_n} \epsilon(\sigma)\Phi_{\sigma}(N_1, N_2, \dots, N_n) - \sum_{\kappa \in S_n, i \in \{1, 2, \dots, n\}} \epsilon(\kappa)\Phi_{\kappa, i}(N_1, N_2, \dots, N_n).$$

By Lemma 5.2 the above expression is equal to $G(N_1, N_2, \dots, N_n)$. Therefore $\psi(G(N_1, N_2, \dots, N_n)) = 0$. □

Lemma 5.7. *Let N_1, N_2, \dots, N_{n+1} be any monomials in the variables $X_i, i \in I$. Then $\psi(F(N_1, N_2, \dots, N_{n+1})) = 0$.*

Proof. By definition, $F(N_1, N_2, \dots, N_{n+1}) = Tr(G(N_1, N_2, \dots, N_n)N_{n+1})$. By (5.2), ψ commutes with the trace function. Therefore

$$\begin{aligned} \psi(F(N_1, N_2, \dots, N_{n+1})) &= \psi(Tr(G(N_1, N_2, \dots, N_n)N_{n+1})) \\ &= Tr(\psi(G(N_1, N_2, \dots, N_n))\psi(N_{n+1})) = 0. \end{aligned}$$

□

Lemmas 5.6 and 5.7 and the Second Fundamental Theorem imply that the kernel of $\psi : T\{X_i\}_{i \in I} \rightarrow \mathbb{A}_n(X, x_0)$ contains the kernel of $\pi : T\{X_i\}_{i \in I} \rightarrow M_n(P_n(I))^{GL_n(R)}$. Therefore we have the following corollary.

Corollary 5.8. *There exists an R -algebra homomorphism $\psi' : M_n(P_n(I))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0)$ such that $\psi'(A_i) = E_{g_i}$ and $\psi'(Tr(A_{i_1}A_{i_2}\dots A_{i_k})) = EL_{g_{i_1}g_{i_2}\dots g_{i_k}}$, for any $i_1, i_2, \dots, i_k \in I$.*

The epimorphism $\eta : P_n(I) \rightarrow Rep_n^R(G)$ introduced in Section 2 induces an epimorphism $M_n(\eta) : M_n(P_n(I)) \rightarrow M_n(Rep_n^R(G))$ and, therefore, by restriction, a homomorphism $M_n(\eta)^{GL_n(R)} : M_n(P_n(I))^{GL_n(R)} \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$. Our goal is to show that ψ' descends to

$$\psi'' : M_n(Rep_n^R(G))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0)$$

such that the following diagram commutes:

$$(5.5) \quad \begin{array}{ccc} M_n(P_n(I))^{GL_n(R)} & & \\ \downarrow M_n(\eta)^{GL_n(R)} & \searrow \psi' & \\ M_n(Rep_n^R(G))^{GL_n(R)} & \xrightarrow{\psi''} & \mathbb{A}_n(X, x_0) \end{array}$$

In order to prove this fact we need to show that $Ker M_n(\eta)^{GL_n(R)} \subset Ker \psi'$. We will use the following lemma.

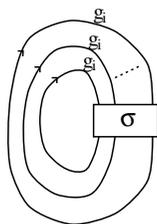
Lemma 5.9. (1) $Det(A_i) \in P_n(I)^{GL_n(R)} \subset M_n(P_n(I))^{GL_n(R)}$.
 (2) $\psi'(Det(A_i)) = 1$, for any $i \in I$.

Proof. (1) By Proposition 5.4, $Det(A_i)$ can be expressed as a linear combination of traces of powers of A_i . By Lemma 2.2(2), $A_i^k \in M_n(P_n(I))^{GL_n(R)}$, and hence, by Proposition 2.3, $Tr(A_i^k) \in P_n(I)^{GL_n(R)}$. Finally, by Lemma 2.2(1) there is a natural embedding $P_n(I)^{GL_n(R)} \subset M_n(P_n(I))^{GL_n(R)}$.

(2) If c_1, c_2, \dots, c_k are the lengths of all cycles of $\sigma \in S_n$, then ψ' maps

$$Tr(A_i^{c_1})Tr(A_i^{c_2})\dots Tr(A_i^{c_k})$$

to a graph



Therefore, by Proposition 5.4 and Fact 4.1,

$$\psi'(Det(A_i)) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \left[\text{Diagram with } \sigma \text{ box and } c_i \text{ loops} \right] = \frac{1}{n!} \left[\text{Diagram with } g_i \text{ loops} \right].$$

Analogously,

$$1 = \psi'(Det(\mathbf{1})) = \frac{1}{n!} \left[\text{Diagram with } e \text{ loops} \right],$$

where e is the identity in G . But, by (4.3) (or, equivalently, (4.4)),

$$\left[\text{Diagram with } g_i \text{ loops} \right] = \left[\text{Diagram with } e \text{ loops} \right].$$

Therefore $\psi'(Det(A_i)) = 1$. □

The next proposition is due to Procesi. Since the proof of this proposition is hidden in the proof of Theorem 2.6 in [Pro-2], we will recall it here for completeness of this paper.⁴

Proposition 5.10. *Let $\mathcal{J} \triangleleft M_n(P_n(I))^{GL_n(R)}$ be a two-sided ideal and let \mathcal{J}' be the ideal in $P_n(I)$ generated by the entries of elements of $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \triangleleft M_n(P_n(I))$. Then:*

- (1) $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) = M_n(\mathcal{J}') \triangleleft M_n(P_n(I))$.
- (2) *There is a unique $GL_n(R)$ -action on $M_n(P_n(I)/\mathcal{J}')$ such that the natural projection $i : M_n(P_n(I)) \rightarrow M_n(P_n(I)/\mathcal{J}')$ is $GL_n(R)$ -equivariant.*
- (3) *i induces a homomorphism*

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \rightarrow M_n(P_n(I)/\mathcal{J}')^{GL_n(R)}$$

which is an isomorphism of R -algebras.

Proof. (1) This follows from the basic algebraic fact that every ideal \mathcal{I} in $M_n(R)$, for any ring R with 1, is of the form $M_n(\mathcal{I}')$, where \mathcal{I}' is the ideal in R generated by the entries of a generating set of the ideal \mathcal{I} .

(2) Let $B \in GL_n(R)$, and let $B*$ denote the action of B on $M_n(P_n(I))$. B leaves $M_n(\mathcal{J}')$ invariant. Indeed, any element $C \in M_n(\mathcal{J}')$ is of the form $\sum_i M_i C_i N_i$, where $M_i, N_i \in M_n(P_n(I))$, $C_i \in \mathcal{J}$, and therefore

$$B * C = \sum_i (B * M_i)(B * C_i)(B * N_i) \in M_n(\mathcal{J}').$$

This implies that the action of $GL_n(R)$ on $M_n(P_n(I)/\mathcal{J}')$ is well defined. All other statements of (2) are obvious consequences of this fact.

(3) For any rational action of $GL_n(R)$ on any R -vector space N there exists a linear projection $\nabla : N \rightarrow N^{GL_n(R)}$, called the Reynolds operator, with the following properties:

- (i) $\nabla(x) = x$ for $x \in N^{GL_n(R)}$, and, therefore, ∇ is an epimorphism.
- (ii) ∇ is natural with respect to $GL_n(R)$ -equivariant maps $N \rightarrow N'$.
- (iii) If N is an algebra, then $\nabla(xy) = x\nabla(y)$ and $\nabla(yx) = \nabla(y)x$ for $x \in N^{GL_n(R)}$ and $y \in N$.

For more information about this operator, see [MFK] or a more elementary text [Fog].

The homomorphism i restricted to $M_n(P_n(I))^{GL_n(R)}$ induces a homomorphism

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \rightarrow M_n(P_n(I)/\mathcal{J}')^{GL_n(R)}.$$

By Property (i) of ∇ , j is an epimorphism. It remains to prove that j is injective.

Choose $i_0 \in I$. For any monomial m in $P_n(I) = R[x_{jk}^i, i \in I, j, k = 1, 2, \dots, n]$ we define the degree of m to be the number of appearances of the variables $x_{jk}^{i_0}$, $j, k \in \{1, 2, \dots, n\}$, in m . This induces a grading on $P_n(I)$. We can extend this grading on $M_n(P_n(I))$ as follows. For any matrix $A = (a_{jk}) \in M_n(P_n(I))$ with a single non-zero entry a_{st} , $deg(A) = deg(a_{st})$. Note that the degree of the matrix $A_{i_0} = (x_{jk}^{i_0}) \in M_n(P_n(I))$ considered in Section 2 is 1.

⁴Compare also Proposition 9.5 in [B-H].

Let $B \in GL_n(R)$. By the definition of the $GL_n(R)$ -action on $M_n(P_n(I))$ and by Lemma 2.2(2),

$$B \begin{pmatrix} B * x_{11}^{i_0} & B * x_{12}^{i_0} & \dots & B * x_{1n}^{i_0} \\ \vdots & \vdots & \dots & \vdots \\ B * x_{n1}^{i_0} & B * x_{n2}^{i_0} & \dots & B * x_{nn}^{i_0} \end{pmatrix} B^{-1} = \begin{pmatrix} x_{11}^{i_0} & x_{12}^{i_0} & \dots & x_{1n}^{i_0} \\ \vdots & \vdots & \dots & \vdots \\ x_{n1}^{i_0} & x_{n2}^{i_0} & \dots & x_{nn}^{i_0} \end{pmatrix}.$$

Therefore $B * x_{jk}^{i_0}$ is a linear combination of the variables $x_{j'k'}^{i_0}$, $j', k' = 1, 2, \dots, n$, and hence the action of $GL_n(R)$ preserves the grading of $P_n(I)$. For any $M \in M_n(P_n(I))$, $B * M$ is a matrix obtained by applying the action of B to all entries of M and then conjugating the resulting matrix by B . Therefore the action of $GL_n(R)$ also preserves the grading of $M_n(P_n(I))$. The naturality of the Reynolds operators $\nabla : P_n(I) \rightarrow P_n(I)^{GL_n(R)}$ and $\nabla : M_n(P_n(I)) \rightarrow M_n(P_n(I))^{GL_n(R)}$ implies that they also preserve the gradings. This fact will be an important element of the proof of Proposition 5.10(3).

We need to show that

$$\mathcal{J} = M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \cap M_n(P_n(I))^{GL_n(R)}.$$

However, it is sufficient to show that

$$\mathcal{J} \supset M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \cap M_n(P_n(I))^{GL_n(R)},$$

since the opposite inclusion is obvious.

Let $c = \sum_i a_i c_i b_i \in M_n(P_n(I))^{GL_n(R)}$, where $a_i, b_i \in M_n(P_n(I)), c_i \in \mathcal{J}$. We will show that $c \in \mathcal{J}$. Since c involves only finitely many variables x_{jk}^i and I is infinite, we can choose $i_0 \in I$ such that $x_{jk}^{i_0}$, $j, k = 1, 2, \dots, n$, do not appear in a_i, b_i, c_i . Thus $\text{deg } a_i = \text{deg } b_i = \text{deg } c_i = 0$.

Consider $\text{Tr}(cA_{i_0})$. By our assumptions about c and by Lemma 2.2(2), $cA_{i_0} \in M_n(P_n(I))^{GL_n(R)}$. Proposition 2.3 states that $\text{Tr} : M_n(P_n(I)) \rightarrow P_n(I)$ is $GL_n(R)$ -equivariant, and therefore $\text{Tr}(cA_{i_0}) \in M_n(P_n(I))^{GL_n(R)}$. Thus

$$\begin{aligned} \text{Tr}(cA_{i_0}) &= \text{Tr}(\nabla(cA_{i_0})) = \text{Tr}\left(\sum_i \nabla(a_i c_i b_i A_{i_0})\right) \\ &= \text{Tr}\left(\sum_i \nabla(b_i A_{i_0} a_i c_i)\right) = \text{Tr}\left(\sum_i \nabla(b_i A_{i_0} a_i) c_i\right). \end{aligned}$$

Note that $b_i A_{i_0} a_i$ has degree 1 and, since ∇ preserves the grading, $\nabla(b_i A_{i_0} a_i)$ is also of degree 1. By The First Fundamental Theorem of Invariant Theory (Theorem 5.1), $M_n(P_n(I))^{GL_n(R)}$ is generated by the elements A_i and $\text{Tr}(M)$, where M varies over the set of monomials composed of non-negative powers of matrices $A_i, i \in I$. By our definition of degree,

$$\text{deg}(A_i) = \begin{cases} 1 & \text{if } i = i_0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\text{deg}(\text{Tr}(M)) = \text{number of appearances of } A_{i_0} \text{ in } M.$$

Therefore, $\nabla(b_i A_{i_0} a_i)$ can be presented as

$$\sum_j p_{ij} A_{i_0} q_{ij} + \sum_k Tr(s_{ik} A_{i_0}) t_{ik},$$

for some elements $p_{ij}, q_{ij}, s_{ik}, t_{ik} \in M_n(P_n(I))^{GL_n(R)}$ of degree 0. Thus

$$\begin{aligned} Tr(cA_{i_0}) &= Tr\left(\sum_i \sum_j p_{ij} A_{i_0} q_{ij} c_i + \sum_i \sum_k Tr(s_{ik} A_{i_0}) t_{ik} c_i\right) \\ &= Tr\left(\sum_i \sum_j p_{ij} A_{i_0} q_{ij} c_i\right) + \sum_i \sum_k Tr(s_{ik} A_{i_0}) Tr(t_{ik} c_i) \\ &= Tr\left(\left(\sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik}\right) A_{i_0}\right). \end{aligned}$$

Therefore

$$Tr\left(\left[c - \left(\sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik}\right)\right] A_{i_0}\right) = 0$$

in $M_n(P_n(I))$. The expression in brackets above has degree 0. Note that if $deg d = 0$, $d \in M_n(P_n(I))$, then $Tr(dA_{i_0}) = 0$ if and only if $d = 0$. Therefore

$$c = \sum_i \sum_j q_{ij} c_i p_{ij} + \sum_i \sum_k Tr(t_{ik} c_i) s_{ik},$$

and hence $c \in \mathcal{J}$. This completes the proof of Proposition 5.10. □

Let $\mathcal{J} \triangleleft M_n(P_n(I))^{GL_n(R)}$ be the ideal generated by elements $Det(A_i) - 1$, $i \in I$, and elements $A_{i_1} A_{i_2} \dots A_{i_k} - 1$ corresponding to all defining relations $r_j = g_{i_1} g_{i_2} \dots g_{i_k}$ of G . By Lemma 5.9(1) and Lemma 2.2(2), $Det(A_i) - 1$ and $A_{i_1} A_{i_2} \dots A_{i_k} - 1$ are indeed elements of $M_n(P_n(I))^{GL_n(R)}$ and therefore \mathcal{J} is well defined. By Proposition 5.10(1) the ideal $M_n(P_n(I))\mathcal{J}M_n(P_n(I)) \triangleleft M_n(P_n(I))$ is equal to $M_n(\mathcal{J}')$, where $\mathcal{J}' \triangleleft P_n(I)$ is the ideal generated by coefficients of matrices belonging to \mathcal{J} . Notice that \mathcal{J}' is exactly the kernel of the epimorphism $\eta : P_n(I) \rightarrow Rep_n^R(G)$ introduced in Section 2. Therefore by Proposition 5.10 the homomorphism

$$M_n(\eta)^{GL_n(R)} : M_n(P_n(I))^{GL_n(R)} \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$$

considered in diagram (5.5) descends to an isomorphism

$$j : M_n(P_n(I))^{GL_n(R)} / \mathcal{J} \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}.$$

Proposition 5.11. $M_n(Rep_n^R(G))^{GL_n(R)}$ is generated by the elements $j_{G,n}(g_i)$ and $Tr(j_{G,n}(g_{i_1} g_{i_2} \dots g_{i_k}))$, where $i, i_1, i_2, \dots, i_k \in I$.

Proof. From the paragraph preceding Proposition 5.11 it follows that $M_n(\eta)^{GL_n(R)}$ is an epimorphism. By Theorem 5.1, $M_n(P_n(I))^{GL_n(R)}$ is generated by the elements A_i and $Tr(A_{i_1} A_{i_2} \dots A_{i_k})$, where $i, i_1, i_2, \dots, i_k \in I$. The homomorphism $M_n(\eta)^{GL_n(R)}$ carries these elements to $j_{G,n}(g_i)$ and $Tr(j_{G,n}(g_{i_1} g_{i_2} \dots g_{i_k}))$, respectively. □

This proposition and Theorem 3.6 imply that Θ is an epimorphism. We will show that it is also a monomorphism.

We have shown in Lemma 5.9 that $Det(A_i) - 1 \in Ker \psi'$.⁵ Moreover, by the definition of ψ' , $A_{i_1}A_{i_2}\dots A_{i_k} - 1 \in Ker \psi'$, for any i_1, i_2, \dots, i_k such that $g_{i_1}g_{i_2}\dots g_{i_k} = e$ in G . Therefore $\mathcal{J} \subset Ker \psi'$, and we can factor ψ' to

$$\psi'' : M_n(Rep_n^R(G))^{GL_n(R)} \rightarrow \mathbb{A}_n(X, x_0),$$

such that diagram (5.5) commutes and, by Corollary 5.8,

- $\psi''(j_{G,n}(g_i)) = E_{g_i}$,
- $\psi''(Tr(j_{G,n}(g_{i_1}g_{i_2}\dots g_{i_k}))) = EL_{g_{i_1}g_{i_2}\dots g_{i_k}}$, for any $i_1, i_2, \dots, i_k \in I$.

Recall that our assumptions about the presentation of G (stated in the paragraph following Theorem 5.3) say that the inverse of any generator of G is also a generator and that every element of G is a product of non-negative powers of generators. Thus, by Proposition 3.5(4), $\mathbb{A}_n(X, x_0)$ is generated by the elements E_{g_i} and $EL_{g_{i_1}g_{i_2}\dots g_{i_k}}$, for $i, i_1, i_2, \dots, i_k \in I$. Since $\psi'' \circ \Theta$ is the identity on the generators of $\mathbb{A}_n(X, x_0)$, it also is the identity on $\mathbb{A}_n(X, x_0)$. Therefore Θ is a monomorphism.

In order to complete the proof of Theorem 3.7, we need to show that θ is also an isomorphism.

Fact 3.4 implies that we have an embedding $\iota_* : \mathbb{A}_n(X) \rightarrow \mathbb{A}_n(X, x_0)$, $\iota_*(L_g) = EL_g$, for $g \in G$. Therefore we can consider $\mathbb{A}_n(X)$ as a subring of $\mathbb{A}_n(X, x_0)$. Moreover, by Theorem 3.6, θ is just the restriction of

$$\Theta : \mathbb{A}_n(X, x_0) \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$$

to $\mathbb{A}_n(X)$. Therefore θ is a monomorphism.

In order to show that θ is an epimorphism we use once again an argument from invariant theory. By the naturality of the Reynolds operators $\nabla : M_n(Rep_n^R(G)) \rightarrow M_n(Rep_n^R(G))^{GL_n(R)}$ and $\nabla' : Rep_n^R(G) \rightarrow Rep_n^R(G)^{GL_n(R)}$, the following diagram commutes:

$$\begin{array}{ccc} M_n(Rep_n^R(G)) & \xrightarrow{Tr} & Rep_n^R(G) \\ \downarrow \nabla & & \downarrow \nabla' \\ M_n(Rep_n^R(G))^{GL_n(R)} & \xrightarrow{Tr} & Rep_n^R(G)^{GL_n(R)} \end{array}$$

Since $Tr : M_n(Rep_n^R(G)) \rightarrow Rep_n^R(G)$ and all Reynolds operators are epimorphic, $Tr : M_n(Rep_n^R(G))^{GL_n(R)} \rightarrow Rep_n^R(G)^{GL_n(R)}$ is also epimorphic. But now commutativity of (3.1) implies that θ is an epimorphism as well.

Therefore we have shown that θ is an isomorphism. This completes the proof of Theorem 3.7.

6. SL_n-CHARACTER VARIETIES

In this section we present one of the possible applications of Theorem 3.7 to a study of SL_n-character varieties.

Let X be a path-connected topological space whose fundamental group $G = \pi_1(X)$ is finitely generated. Let K be an algebraically closed field of characteristic 0. Recall that we noticed in Section 2 that the set of all SL_n(K)-characters of G , denoted by $X_n(G)$, is an algebraic set whose coordinate ring is $Rep_n^R(G)^{GL_n(K)}/\sqrt{0}$.

Let $\chi_g = Tr(j_{G,n}(g)) \in Rep_n^R(G)^{GL_n(K)}/\sqrt{0}$, for any $g \in G$. It is not difficult to check that χ_g , considered as an element of $K[X_n(G)]$, is a function which

⁵Recall that the map ψ' was defined in Corollary 5.8.

assigns to a character χ the value $\chi(g)$. By Proposition 3.5(3) and Theorem 3.7, $Rep_n^R(G)^{GL_n(K)}$ is generated by the elements $Tr(j_{G,n}(g))$. Therefore the functions χ_g generate $K[X_n(G)]$.

By an SL_n -trace identity for G we mean a polynomial function in variables $\chi_g, g \in G$, which is identically equal to 0 on $X_n(G)$. For example,

$$\chi_g \chi_h = \chi_{gh} + \chi_{gh^{-1}}$$

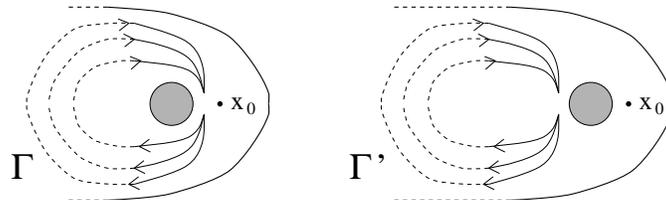
is the famous Fricke-Klein SL_2 -trace identity, valid for any group G and any $g, h \in G$. From the above discussion it follows that the coordinate ring of $X_n(G)$ can be considered as the quotient of the ring of polynomials in formal variables $\chi_g, g \in G$, by the ideal of all SL_n -trace identities for G . Therefore Theorem 3.7 implies the following corollary.

Corollary 6.1. *There is an isomorphism $\Lambda : \mathbb{A}_n(X)/\sqrt{0} \rightarrow K[X_n(G)]$ such that $\Lambda(L_g) = \chi_g$. Under this isomorphism the identities on graphs in X induced by skein relations correspond to SL_n -trace identities for G .*

The above corollary is very useful in the study of trace identities, since it makes it possible to interpret them geometrically. Consider for example the following SL_3 -trace identity, which holds for any $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in G$ and any $\chi \in X_3(G)$, where G is an arbitrary group:

$$\begin{aligned} & \chi(\gamma_1)\chi(\gamma_2)\chi(\gamma_3) - \chi(\gamma_1)\chi(\gamma_2\gamma_3) - \chi(\gamma_2)\chi(\gamma_1\gamma_3) - \chi(\gamma_3)\chi(\gamma_1\gamma_2) \\ & + \chi(\gamma_1\gamma_2\gamma_3) + \chi(\gamma_1\gamma_3\gamma_2) - \chi(\gamma_1\gamma_0)\chi(\gamma_2\gamma_0)\chi(\gamma_3\gamma_0) \\ (6.1) \quad & + \chi(\gamma_1\gamma_0)\chi(\gamma_2\gamma_0\gamma_3\gamma_0) + \chi(\gamma_2\gamma_0)\chi(\gamma_1\gamma_0\gamma_3\gamma_0) + \chi(\gamma_3\gamma_0)\chi(\gamma_1\gamma_0\gamma_2\gamma_0) \\ & - \chi(\gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0) - \chi(\gamma_1\gamma_0\gamma_3\gamma_0\gamma_2\gamma_0) = 0. \end{aligned}$$

Our theory provides the following interpretation of this identity. Let $x_0 \in X$ and $G = \pi_1(X, x_0)$. Let γ_0 be a path in X representing a non-trivial element of $\pi_1(X, x_0)$. We assume that γ_0 goes along a ‘‘hole’’ in X presented in the picture below. Consider the following two, obviously equivalent, graphs Γ and Γ' in X :



The graph Γ' is obtained from Γ by pulling its vertices along the ‘‘hole’’ in X . The obvious resolution of vertices in Γ and Γ' gives an equation involving closed loops in X . This equation corresponds to the trace identity (6.1).

There is a large body of literature about SL_2 -character varieties and their applications. However, very little is known about SL_n -character varieties for $n > 2$. The reason for this is that the SL_n -trace identities, like (6.1), are intractable by classical (algebraic) methods. Since our theory often gives a simple geometric interpretation to complicated trace identities, it can be applied to a more detailed study of character varieties. This idea was already used in [PS-2] and [PS-3] to study SL_n -character varieties for $n = 2$. A generalization of these results for $n > 2$, which is based on our skein method, will appear in future papers. In this work we test our method on the simplest non-trivial example – we study SL_3 -character

variety of the free group on two generators, $F_2 = \langle g_1, g_2 \rangle$. The basic problem is to determine the minimal dimension of the affine space in which $X_3(F_2)$ is embedded, or equivalently, the minimal number of generators of $K[X_3(F_2)]$. A result of Procesi (Theorem 3.4(a) [Pro-1]) implies that $K[X_3(F_2)]$ is generated by the elements $\chi_{g_{i_1} g_{i_2} \dots g_{i_j}}$, where $j \leq 7$ and $i_1, i_2, \dots, i_j \in \{1, 2\}$. A direct calculation shows that, after identifying words in g_1, g_2 which are related by cyclic permutations, we get a set of 57 generators of $K[X_3(F_2)]$. It is difficult to obtain any further reduction of this set in any simple algebraic manner. However, our geometric method allows us to reduce this problem to the study of 3-valent graphs in the twice-punctured disc. By playing with pictures of such graphs one can reduce the number of generators of $K[X_3(F_2)]$ to nine! These are

$$\chi_{g_1}, \chi_{g_2}, \chi_{g_1^2}, \chi_{g_2^2}, \chi_{g_1 g_2}, \chi_{g_1^2 g_2}, \chi_{g_1 g_2^2}, \chi_{g_1^2 g_2^2}, \chi_{g_1^2 g_2^2 g_1 g_2}.$$

Moreover, it is possible to show that this is the minimal number of generators, and $X_3(F_2) \subset K^9$ is a solution set of one irreducible polynomial.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742
E-mail address: asikora@math.umd.edu