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THE STRUCTURE OF THE BRAUER GROUP AND CROSSED PRODUCTS OF $C_0(X)$-LINEAR GROUP ACTIONS ON $C_0(X,\mathcal{K})$

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Abstract. For a second countable locally compact group G and a second countable locally compact space X let $\mathrm{Br}_G(X)$ denote the equivariant Brauer group (for the trivial G-space X) consisting of all Morita equivalence classes of spectrum fixing actions of G on continuous-trace C^*-algebras A with spectrum $\widehat{A}=X$. Extending recent results of several authors, we give a complete description of $\mathrm{Br}_G(X)$ in terms of group cohomology of G and Čech cohomology of X. Moreover, if G has a splitting group H in the sense of Calvin Moore, we give a complete description of the $C_0(X)$-bundle structure of the crossed product $A\rtimes_\alpha G$ in terms of the topological data associated to the given action $\alpha: G\to \operatorname{Aut} A$ and the bundle structure of the group C^*-algebra $C^*(H)$ of H.

1. Introduction and statement of the main results

Assume that G is a second countable locally compact group acting continuously on the second countable locally compact space X. The equivariant Brauer group $\mathrm{Br}_G(X)$ of Crocker, Kumjian, Raeburn and Williams (e.g., see [CKRW]) is the set of all $C_0(X)$-Morita equivalence classes of dynamical systems (A,G,α), in which A is a continuous-trace C^*-algebra with spectrum $\widehat{A}=X$ and $\alpha: G\to\operatorname{Aut} A$ is an (strongly continuous) action of G on A which induces the given action on X. The group operation is given by $[(A,\alpha)][(B,\beta)]=[(A\otimes_{C_0(X)}B,\alpha\otimes_X\beta)]$. There has been quite some effort in recent years to describe the group $\mathrm{Br}_G(X)$ in terms of a combination of Moore's group cohomology and Čech cohomology (see [CKRW, P, PRW, EW1, RW2]).

If G acts trivially on X, then there is a natural splitting $\mathrm{Br}_G(X)=\check{H}^3(X,\mathbb{Z})\oplus\mathcal{E}_G(X)$, where $\mathcal{E}_G(X)$ denotes the subgroup of $\mathrm{Br}_G(X)$ consisting of all exterior equivalence classes of $C_0(X)$-linear actions of G on $C_0(X,\mathcal{K})$, where $\mathcal{K}=\mathcal{K}(\mathcal{H})$ is the algebra of compact operators on the separable infinite dimensional Hilbert space $\mathcal{H}$ (see [EW1, Proposition 5.1]). If $G=\mathbb{R}^n\times\mathbb{Z}^m\times\mathbb{T}^l\times F$, F finite abelian, it was shown by Packer in [P, PRW] that $\mathcal{E}_G(X)$ has a (non-canonical) splitting as the direct sum $\check{H}^1(X,H^1(G,\mathbb{T}))\oplus C(X,H^2(G,\mathbb{T}))$ (if N is a topological group, we denote by $\check{H}^n(X,N)$ the nth cohomology of the sheaf of germs of continuous N-valued functions on X). This result was extended in [EW1] to all groups G with compactly generated abelianization $G_{\mathrm{ab}}=G/\overline{[G,G]}$ such that G is *smooth* in the sense of Moore ([M2]), i.e., there exists a central extension $1\to Z\to H\to G\to 1$ of G such that the transgression map $\mathrm{tg}:\widehat{Z}=H^1(Z,\mathbb{T})\to H^2(G,\mathbb{T})$ in the

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corresponding Lyndon-Hochschild-Serre spectral sequence is an isomorphism. The group H in this extension is called a *representation group* of G and its group algebra $C^*(H)$ has the structure of a (in general semi-continuous) C^*-bundle over $H^2(G,\mathbb{T})$ with fibres isomorphic to the twisted group algebras $C^*(G,\omega)$, $[\omega]\in H^2(G,\mathbb{T})$. Now if α is a $C_0(X)$-linear action of G on $A=C_0(X,\mathcal{K})$ (or on any continuous-trace algebra A with spectrum X), then the crossed product $A\rtimes_\alpha G$ is a bundle over X with fibres stably isomorphic to twisted group algebras of G, and the main result of [EW2] gives a description of this bundle in terms of the universal bundle $C^*(H)$ and the topological data associated to α.

Although many groups are smooth (see [M2] and the discussion in [EW1, §4]) it is easy to construct examples of connected Lie groups which are not (see [M2, p. 85] and [EW1, §7]). However, it was shown by Moore in [M2] that every almost connected group satisfies the weaker condition of having a *splitting extension* $1\to Z\to H\to G\to 1$ in which the transgression map $\mathrm{tg}:\widehat{Z}\to H^2(G,\mathbb{T})$ is surjective. It is the main task of this paper to obtain analogues of the results of [EW1, EW2] for groups which have a splitting extension H.

In order to do this we start with a more general construction in §2: Suppose that the action of G is *locally inner*. This means that there exists an open cover $\boldsymbol{U}=\{U_i\}$ of X and a family of Borel maps $\pi_i: G\to C(U_i,\mathcal{U}(\mathcal{H}))$ such that π_i implements α on $C_0(U_i,\mathcal{K})$. We then say that $\pi:=\{\pi_i\}$ is an element of $C^1(G,\check{C}^0(\boldsymbol{U},\mathcal{U}(\mathcal{H})))$ which *locally implements* α. Note that by a result of Rosenberg (see the proof of [R, Corollary 2.2]), *any $C_0(X)$-linear action of G on $C_0(X,\mathcal{K})$ is locally inner if $G_{ab}=G/\overline{[G,G]}$ is compactly generated.*

For $n>0, m\ge 0$ we write

$$C^n(G,\check{C}^m(\boldsymbol{U},\mathbb{T})):=\prod_{i_0,\dots,i_m} C^n(G,C(U_{i_0,\dots,i_m},\mathbb{T})),$$

where $C^n(G,C(V,\mathbb{T}))$ denotes the set of all Borel maps of G^n into the trivial G-module $C(V,\mathbb{T})$ equipped with the compact open topology (of course, the notation $C^1(G,\check{C}^0(\boldsymbol{U},\mathcal{U}(\mathcal{H})))$ used above is motivated by this definition). We obtain a double complex

$$\begin{array}{ccccccc}
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G,\check{C}^2(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & C^2(G,\check{C}^2(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G,\check{C}^1(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & C^2(G,\check{C}^1(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G,\check{C}^0(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & C^2(G,\check{C}^0(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & ,
\end{array} \tag{1.1}$$

where ∂_G denotes the usual group coboundary and $\check{\partial}$ denotes Čech coboundary. Let $\mathbb{H}^k(G,\boldsymbol{U})$ denote the kth cohomology of the associated total complex

$$\left(\bigoplus_{n+m=k} C^m(G,\check{C}^n(\boldsymbol{U},\mathbb{T})),\big((-1)^{m+1}\check{\partial}\big)\oplus\partial_G\right),$$

and define

$$\mathbb{H}^k(G,X) := \lim_{\boldsymbol{U}} \mathbb{H}^k(G,\boldsymbol{U})$$

as the inductive limit with respect of taking refinements. As we will show later (see §2), these groups control locally inner actions. More precisely, if $\alpha \in \mathcal{E}_G(X)$ is locally inner, and if $\pi \in C^1(G, \check{C}^0(\boldsymbol{U}, \mathcal{U}(\mathcal{H})))$ locally implements α, then we can formally apply the Čech coboundary $\check{\partial}$ and the group coboundary ∂_G to π. Since π locally implements α, it follows that $\check{\partial}\pi$ and $\partial_G \pi$ take values in the centers $\check{C}^1(\boldsymbol{U},\mathbb{T}) \subseteq \check{C}^1(\boldsymbol{U},\mathcal{U}(H))$ and $\check{C}^0(\boldsymbol{U},\mathbb{T}) \subseteq \check{C}^0(\boldsymbol{U},\mathcal{U}(\mathcal{H}))$, respectively, i.e.,

$$(\check{\partial}\pi, \partial_G\pi) \in C^1(G, \check{C}^1(\boldsymbol{U},\mathbb{T})) \oplus C^2(G, \check{C}^0(\boldsymbol{U},\mathbb{T})),$$

and we get

Theorem 1.1. *Assume that G_{ab} is compactly generated. Let $\alpha \in \mathcal{E}_G(X)$ and let $\pi \in C^1(G, \check{C}^0(\boldsymbol{U}, \mathcal{U}(\mathcal{H})))$ such that π locally implements α. Then $(\check{\partial}\pi, \partial_G\pi) \in \boldsymbol{Z}^2(G,\boldsymbol{U})$ and the map $[\alpha] \mapsto [(\check{\partial}\pi, \partial_G\pi)] \in \mathbb{H}^2(G,X)$ induces an isomorphism between $\mathcal{E}_G(X)$ and $\mathbb{H}^2(G,X)$. In particular, for any trivial G-space X we get a canonical isomorphism*

$$\mathrm{Br}_G(X) \cong \check{H}^3(X,\mathbb{Z}) \oplus \mathbb{H}^2(G,X).$$

If G_{ab} is not necessarily compactly generated, we obtain at least a classification of the group $\mathcal{E}_G^{li}(X)$ of classes of locally inner actions via $\mathbb{H}^2(G,X)$.

Remark 1.2. At this point we should remark that, in the literature, an inner action $\alpha : G \to \mathrm{Aut}\, A$ is sometimes understood as an action which is implemented by a strictly continuous *homomorphism* $\pi : G \to \mathcal{U}M(A)$. Of course this condition is much stronger than the one we (locally) impose here, namely that every single automorphism α_s is implemented be some unitary $\pi_s \in \mathcal{U}M(A)$ (if A is separable, then it follows from [RR, §0] that $s \mapsto \pi_s$ can then be chosen to be Borel). If α is implemented by a strictly continuous homomorphism, then (following the convention of [PhR] and others) we say that α is *unitary*. Note that an action α is unitary if and only if it is exterior equivalent to the trivial action (via π). We come back to (locally) unitary actions in Proposition 3.6 below.

The double complex above has two natural filtrations which play an important, though not necessarily explicit, role in what follows.

The first one is given by

$$\mathcal{F}^n = \bigoplus_{k \geq n, i \geq 1} C^i(G, \check{C}^k(\boldsymbol{U},\mathbb{T})).$$

The corresponding spectral sequence has the E_2 term of the form

$$\begin{array}{cccc}
* & * & * & * \\
\check{\mathcal{H}}^2(\boldsymbol{U}, H^1(G,\mathbb{T})) & \check{\mathcal{H}}^2(\boldsymbol{U}, H^2(G,\mathbb{T})) & * & * \\
\check{\mathcal{H}}^1(\boldsymbol{U}, H^1(G,\mathbb{T})) & \check{\mathcal{H}}^1(\boldsymbol{U}, H^2(G,\mathbb{T})) & * & * \\
\check{\mathcal{H}}^0(\boldsymbol{U}, H^1(G,\mathbb{T})) & \check{\mathcal{H}}^0(\boldsymbol{U}, H^2(G,\mathbb{T})) & * & *
\end{array} \tag{1.2}$$

with the boundary map

$$d_2 : \check{\mathcal{H}}^0(\boldsymbol{U}, H^2(G,\mathbb{T})) \to \check{\mathcal{H}}^2(\boldsymbol{U}, H^1(G,\mathbb{T})).$$

Here the groups $\check{\mathcal{H}}^i(\boldsymbol{U}, H^j(G,\mathbb{T}))$ have to be understood as the cohomology groups of the *mapping cone* of

$$B^j(G, \check{\mathcal{C}}^*(\boldsymbol{U},\mathbb{T})) \to Z^j(G, \check{\mathcal{C}}^*(\boldsymbol{U},\mathbb{T})),$$

where $B^j \to Z^j$ is the inclusion of the group of coboundaries to the group of cocycles.

Remark 1.3. Note that, at least in low dimensions, it is possible to write down a natural evaluation map

$$\check{\mathcal{H}}^i(\boldsymbol{U}, H^j(G,\mathbb{T})) \to \check{H}^i(\boldsymbol{U}, H^j(G,\mathbb{T})).$$

Unfortunately, this map is not an isomorphism in general, which is basically due to the fact that the topology on $H^*(G,\mathbb{T})$ is not well behaved (e.g., it is not hard to use [EW1, Example 7.3] to get an example in which the natural map $\check{\mathcal{H}}^0(X, H^2(G,\mathbb{T})) \to \check{H}^0(X, H^2(G,\mathbb{T}))$ is not injective). However, for $j = 1$ the evaluation map gives an identification of $\check{\mathcal{H}}^i(\boldsymbol{U}, H^1(G,\mathbb{T}))$ with $\check{H}^i(\boldsymbol{U}, \widehat{G}_{ab})$. Here and later $\widehat{G}_{ab} = H^1(G,\mathbb{T})$ denotes the Pontrjagin dual of the abelianization $G_{ab} = G/\overline{[G,G]}$ of G.

In particular, the E_2-term of our spectral sequence gives for the $\mathbb{H}^2$-term (after taking the limit over coverings) an exact sequence of the form

$$0 \to \check{H}^1(X, \widehat{G}_{ab}) \to \mathcal{E}^{li}_G(X) \to \check{\mathcal{H}}^0(X, H^2(G,\mathbb{T})) \xrightarrow{d_2} \check{H}^2(X, \widehat{G}_{ab}). \tag{1.3}$$

If G has a splitting extension $1 \to Z \to H \to G \to 1$, the cohomology of the mapping cone of

$$\check{\mathcal{C}}^*(\boldsymbol{U}, \widehat{H}_{ab}) \to \check{\mathcal{C}}^*(\boldsymbol{U}, \widehat{Z})$$

maps naturally to $\check{\mathcal{H}}^0(X, H^2(G,\mathbb{T}))$. In fact, as we prove in section 4, for the computation of $\mathbb{H}^2(G,X)$ we can replace the above double complex by the (inductive limit over coverings of the) double complex

$$\begin{array}{ccccccc}
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & \check{C}^2(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}}) & \xrightarrow{\mathrm{res}_*} & \check{C}^2(\boldsymbol{U}, \widehat{Z}) & \longrightarrow & 0 \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & \check{C}^1(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}}) & \xrightarrow{\mathrm{res}_*} & \check{C}^1(\boldsymbol{U}, \widehat{Z}) & \longrightarrow & 0 \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & \check{C}^0(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}}) & \xrightarrow{\mathrm{res}_*} & \check{C}^0(\boldsymbol{U}, \widehat{Z}) & \longrightarrow & 0,
\end{array} \tag{1.4}$$

where $\mathrm{res} : \widehat{H}_{\mathrm{ab}} \to \widehat{Z}; \chi \mapsto \chi|_Z$ is the restriction map.

Since the second cohomology of this complex classifies the isomorphism classes of the *equivariant pairs* $(\mathcal{P}, F)$ in which $p : \mathcal{P} \to X$ is a principal $\widehat{H}_{\mathrm{ab}}$-bundle over X and $F : \mathcal{P} \to \widehat{Z}$ is a continuous $\widehat{H}_{\mathrm{ab}}$-equivariant map, this gives

Theorem 1.4. *Let $1 \to Z \to H \to G \to 1$ be a splitting extension for G such that $H_{ab} = H/\overline{[H,H]}$ is compactly generated. Then there is a natural one-to-one correspondence between $\mathcal{E}_G(X)$ and the set of isomorphism classes of all equivariant pairs $(\mathcal{P}, F)$.*

Remark 1.5. If H is actually a representation group for G, then the above result reduces to the splitting $\mathcal{E}_G(X) = \check{H}^1(X, H^1(G,\mathbb{T})) \oplus C(X, H^2(G,\mathbb{T}))$ of [EW1] (see the discussion above).

We then proceed by giving a description of the crossed product $C_0(X,\mathcal{K}) \rtimes_\alpha G$ in terms of an equivariant pair $(\mathcal{P}, F)$ corresponding to α and the (universal) bundle $C^*(H)$ over $\widehat{Z}$. We can use the continuous map $F : \mathcal{P} \to \widehat{Z}$ to construct the pull-back $C_0(\mathcal{P}) \otimes_F C^*(H) := C_0(\mathcal{P}) \otimes_{C_0(\widehat{Z})} C^*(H)$, where the $C_0(\widehat{Z})$-action on $C_0(\mathcal{P})$ is induced by F in the obvious way. Now $\widehat{H}_{\mathrm{ab}}$ acts on $C_0(\mathcal{P})$ by the translation action τ and on $C^*(H)$ via the usual dual action, say γ. The $\widehat{H}_{\mathrm{ab}}$-equivariance of $F : \mathcal{P} \to \widehat{Z}$ then implies that the diagonal action on $C_0(\mathcal{P}) \otimes C^*(H)$ factors through an action $\tau \otimes_{\widehat{Z}} \gamma$ of $\widehat{H}_{\mathrm{ab}}$ on $C_0(\mathcal{P}) \otimes_F C^*(H)$.

Theorem 1.6. *Let $1 \to Z \to H \to G \to 1$ be a splitting extension for G such that $H_{ab} = H/\overline{[H,H]}$ is compactly generated. Let $[\alpha] \in \mathcal{E}_G(X)$ and let $(\mathcal{P}, F)$ be an equivariant pair corresponding to α via Theorem 1.4. Then $C_0(X,\mathcal{K}) \rtimes_\alpha G$ is stably isomorphic to $\big(C_0(\mathcal{P}) \otimes_F C^*(H)\big) \rtimes_{\tau \otimes_{\widehat{Z}} \gamma} \widehat{H}_{ab}$.*

As a consequence we get a description of the spectrum $\big(C_0(X,\mathcal{K}) \rtimes_\alpha G\big)^\frown$ in terms of $(\mathcal{P}, F)$, using the spectrum $\widehat{H}$ of $C^*(H)$ as the universal object. Note that since Z is central in H, any irreducible representation U of H restricts to a multiple of some character $\chi \in \widehat{Z}$. Thus, we get a well-defined restriction map $\mathrm{res} : \widehat{H} \to \widehat{Z}$.

Corollary 1.7. *Let $1 \to Z \to H \to G \to 1$, α and $(\mathcal{P}, F)$ be as in Theorem 1.6. Let*

$$\mathcal{P} \times_F \widehat{H} = \{(y, U) \in \mathcal{P} \times \widehat{H} : F(y)^{-1} = \mathrm{res}\, U\},$$

and let $\widehat{H}_{ab}$ act on $\mathcal{P} \times_F \widehat{H}$ by $\chi \cdot (y, U) = (\chi \cdot y, \overline{\chi} \otimes U)$. Then $\big(C_0(X,\mathcal{K}) \rtimes_\alpha G\big)^\frown$ is homeomorphic to $(\mathcal{P} \times_F \widehat{H})/\widehat{H}_{ab}$.

Remark 1.8. If we use the second filtration (by columns) for the double complex (1.4), then the E_2-term of the corresponding spectral sequence gives an exact sequence

$$C(X, \widehat{H}_{ab}) \to C(X, \widehat{Z}_{ab}) \to \mathcal{E}_G^{li}(X) \to \check{H}^1(X, \widehat{H}_{ab}) \to 0. \tag{1.5}$$

While it plays a role in the study of the local behavior of actions, we will not use it in this paper.

All our results extend to give classifications of locally inner actions on arbitrary (stable) continuous trace algebras A with spectrum X, and even to locally inner actions on stable $\mathcal{CR}(X)$-algebras as considered in [EW1, EW2]. In fact, we show in the appendix that any locally inner action on such an algebra is exterior equivalent to an action of the form $\beta \otimes_X \mathrm{id}$, where $\beta \in \mathcal{E}_G^{li}(X)$ and $\beta \otimes_X \mathrm{id}$ denotes the diagonal action of G on $C_0(X,\mathcal{K}) \otimes_{C_0(X)} A \cong A$ (see Proposition 6.1). Since

$$\big(C_0(X,\mathcal{K}) \otimes_{C_0(X)} A\big) \rtimes_{\beta \otimes_X \mathrm{id}} G \cong \big(C_0(X,\mathcal{K}) \rtimes_\beta G\big) \otimes_{C_0(X)} A$$

by [EW2, Proposition 4.3], we also obtain a description of the crossed products in the case where G has a splitting extension H with H_{ab} compactly generated.

2. The double complex

If G is a second countable group and X is a second countable locally compact space, then $H^n(G, C(X,\mathbb{T}))$ denotes Moore's group cohomology with Borel cochains with values in the trivial G-module $C(X,\mathbb{T})$ (see [M1, M3]). Note that $C(X,\mathbb{T})$ is a polish group when equipped with the compact open topology. If $f \in Z^n(G, C(X,\mathbb{T}))$ we write $f(x) \in Z^n(G,\mathbb{T})$ for the evaluation of f at x.

Recall that an action $\alpha : G \to \mathrm{Aut}(C_0(X,\mathcal{K}))$ is *inner* if (and only if) there exists a Borel map $\pi : G \to \mathcal{U}M(C_0(X,\mathcal{K})) \cong C(X,\mathcal{U}(\mathcal{H}))$ (where $\mathcal{U}(\mathcal{H})$ is equipped with the strong operator topology) such that $\alpha_s = \mathrm{Ad}\,\pi_s$ for all $s \in G$. If π is such a map, then $\pi(s)\pi(t)$ and $\pi(st)$ both implement α_{st} on $C_0(X,\mathcal{K})$. Thus $(s,t) \mapsto \partial_G\pi(s,t) = \pi(s)\pi(t)\pi(st)^*$ is a Borel map with values in the center $\mathcal{Z}\mathcal{U}M(C_0(X,\mathcal{K})) \cong C(X,\mathbb{T})$ of $\mathcal{U}M(C_0(X,\mathcal{K}))$. The usual computations show that $\partial_G\pi \in Z^2(G, C(X,\mathbb{T}))$. It follows from [RR, Corollary 0.12] that the exterior equivalence classes of inner actions are classified via the corresponding classes $[\partial_G\pi] \in H^2(G, C(X,\mathbb{T}))$. Since we need this construction several times, we briefly recall how to reconstruct an inner action from a given cocycle $f \in Z^2(G, C(X,\mathbb{T}))$: If $\omega \in Z^2(G,\mathbb{T})$, then we denote by L_ω the left regular ω-representation on $L^2(G)$ given by the formula

$$\big(L_\omega(s)\xi\big)(t) = \omega(s, s^{-1}t)\xi(s^{-1}t). \tag{2.1}$$

It follows from [HORR, Proposition 3.1] that

$$L_f : G \to C(X, \mathcal{U}(L^2(G)));\ L_f(s)(x) := L_{f(x)}(s), \tag{2.2}$$

is a Borel map such that $\partial_G L_f = f$. Thus $\alpha^f = \mathrm{Ad}\,L_f$ is an inner action corresponding to f.

An action $\alpha : G \to \mathrm{Aut}\,C_0(X,\mathcal{K})$ is called *locally inner* if each $x \in X$ has an open neighborhood U such that α restricts to an inner action on the ideal $C_0(U,\mathcal{K})$ of $C_0(X,\mathcal{K})$. Such actions are automatically $C_0(X)$-linear and we denote by $\mathcal{E}_G^{li}(X)$ the subgroup of $\mathcal{E}_G(X)$ consisting of all equivalence classes of locally inner actions of G on $C_0(X,\mathcal{K})$. The following well-known fact follows from the arguments given in the proof of [R, Corollary 2.2].

Proposition 2.1. *Suppose that the abelianization G_{ab} of G is compactly generated, or that each $x \in X$ has a neighborhood U with $\check{H}^2(U,\mathbb{Z}) = 0$ (e.g., if X is locally euclidean). Then every $C_0(X)$-linear action on $C_0(X,\mathcal{K})$ is automatically locally inner, i.e., $\mathcal{E}_G^{li}(X) = \mathcal{E}_G(X)$.*

For each open cover $\boldsymbol{U}$ of X consider the double complex

$$\begin{array}{ccccccc}
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G,\check{C}^2(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & C^2(G,\check{C}^2(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G,\check{C}^1(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & C^2(G,\check{C}^1(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G,\check{C}^0(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} & C^2(G,\check{C}^0(\boldsymbol{U},\mathbb{T})) & \xrightarrow{\partial_G} &
\end{array}$$

as in the introduction. The group coboundary ∂_G in this complex is defined via the usual group coboundary $\partial_G : C^m(G, C(U_{i_0,\dots,i_n}, \mathbb{T})) \to C^{m+1}(G, C(U_{i_0,\dots,i_n}, \mathbb{T}))$ and the Čech coboundary $\check{\partial}$ is defined by $(\check{\partial} f)(s) = \check{\partial}(f(s))$, where on the right-hand side we applied the usual Čech coboundary $\check{\partial} : \check{C}^n(\boldsymbol{U}, \mathbb{T}) \to \check{C}^{n+1}(\boldsymbol{U}, \mathbb{T})$ to the evaluation $f(s) \in \check{C}^n(\boldsymbol{U}, \mathbb{T})$. The two-cocycles of the corresponding total complex

$$\left(\bigoplus_{n+m=k} C^m(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})), \big((-1)^{m+1}\check{\partial}\big) \oplus \partial_G\right).$$

consist of pairs $(g, f) \in C^1(G, \check{C}^1(\boldsymbol{U}, \mathbb{T})) \oplus C^2(G, \check{C}^0(\boldsymbol{U}, \mathbb{T}))$ satisfying

$$f_i \in Z^2(G, C(U_i, \mathbb{T})), \quad g(s) \in \check{Z}^1(\boldsymbol{U}, \mathbb{T}), \quad \text{and} \quad \partial_G g = \check{\partial} f. \tag{2.3}$$

If $\alpha : G \to \operatorname{Aut} C_0(X, \mathcal{K})$ is locally inner, we find an open cover $\boldsymbol{U} = \{U_i\}$ of X and a family of Borel maps $\pi_i : G \to C(U_i, \mathcal{U}(\mathcal{H}))$ such that $\alpha^i = \operatorname{Ad} \pi_i$, where α^i denotes the restriction of α to $C_0(U_i, \mathcal{K})$. Thus, we may view $\pi = \{\pi_i\}$ as an element of $C^1(G, \check{C}^0(\boldsymbol{U}, \mathcal{U}(\mathcal{H})))$ and we say that π *locally implements* α.

Proposition 2.2. *If $\pi \in C^1(G, \check{C}^0(\boldsymbol{U}, \mathcal{U}(\mathcal{H})))$ locally implements an action $\alpha : G \to \operatorname{Aut} C_0(X, \mathcal{K})$, then $(\check{\partial}\pi, \partial_G \pi) \in \boldsymbol{Z}^2(G, \boldsymbol{U})$, and*

$$\Phi : \mathcal{E}_G^{li}(X) \to \mathbb{H}^2(G, X); [\alpha] \mapsto [(\check{\partial}\pi, \partial_G \pi)],$$

is a well-defined injective homomorphism of abelian groups, where the group structure on $\mathbb{H}^2(G, X)$ is given by pointwise multiplication of cocycles.

Proof. The above discussion on inner actions implies that $\partial_G \pi_i \in Z^2(G, C(U_i, \mathbb{T}))$ for all $i \in I$. Moreover, if we restrict π_i and π_j to the overlap $U_{ij} = U_i \cap U_j$, then both implement the restriction α^{ij} of α to $C_0(U_{ij}, \mathcal{K})$. Thus, the difference $\pi_i(s)\pi_j^*(s)$ (defined on U_{ij}) takes values in the center $C(U_{ij}, \mathbb{T})$ of $C_0(U_{ij}, \mathcal{K})$. This shows that $\check{\partial}\pi \in C^1(G, \check{C}^1(\boldsymbol{U}, \mathbb{T}))$, and since $\check{\partial}^2 = 0$, it follows that $\check{\partial}\pi(s) \in \check{Z}^1(\boldsymbol{U}, \mathbb{T})$ for all $s \in G$. A straightforward computation shows that $\partial_G(\check{\partial}\pi) = \check{\partial}(\partial_G \pi)$, and therefore $(\check{\partial}\pi, \partial_G \pi) \in \boldsymbol{Z}^2(\boldsymbol{U}, G)$.

To see that Φ is well defined assume that β is exterior equivalent to α. Then there exists a strictly continuous map $v : G \to C(X, \mathcal{U}(\mathcal{H}))$ satisfying

$$\alpha_s = \operatorname{Ad} v_s \circ \beta_s \quad \text{and} \quad v_{st} = v_s \beta_s(v_t)$$

for all $s, t \in G$. Thus, if $\{\pi_i\}$ and $\{\rho_i\}$ implement α and β on a common cover $\boldsymbol{U}$ of X, then (after multiplying $\{\pi_i\}$ with some element in $C^1(G, \check{C}^0(\boldsymbol{U}, \mathbb{T}))$ if necessary) we may assume that $\pi_i(s) = v(s)\rho_i(s)$ for all $s \in G$. This implies that $\check{\partial}\pi = \check{\partial}\rho$. Moreover, using the relation $v(st) = v(s)\beta_s(v(t)) = v(s)\rho_i(s)v(t)\rho_i(s)^*$ on each U_i, we compute

$$\begin{aligned}
\partial_G(\pi_i)(s,t) &= v(s)\rho_i(s)v(t)\rho_i(t)\rho_i(st)^* v(st)^* \\
&= v(st)\rho_i(s)\rho_i(t)\rho_i(st)^* v(st)^* \\
&= v(st)\partial_G \rho_i(s,t) v(st)^* = \partial_G(\rho_i)(s,t).
\end{aligned}$$

Thus, we also have $\partial_G \pi = \partial_G \rho$.

In order to check that Φ is a group homomorphism we just observe that if α and β are locally implemented by $\pi = \{\pi_i\}$ and $\rho = \{\rho_i\}$, then the product action $\alpha \otimes_X \beta$ of G on $C_0(X, \mathcal{K}) \otimes_{C_0(X)} C_0(X, \mathcal{K}) \cong C_0(X, \mathcal{K} \otimes \mathcal{K})$ is locally implemented by $\pi \otimes \rho := \{\pi_i \otimes \rho_i\}$ (taking pointwise tensor products). It then follows that $\check{\partial}(\pi \otimes \rho) = \check{\partial}\pi \cdot \check{\partial}\rho$ and $\partial_G(\pi \otimes \rho) = \partial_G\pi \cdot \partial_G\rho$.

We now show that Φ is injective: For this let α be given by a family of local maps $\pi_i : G \to C(U_i, \mathcal{U}(\mathcal{H}))$ such that $(\check{\partial}\pi, \partial_G\pi)$ gives the trivial class in $\mathbb{H}^2(G, X)$. Then, after taking a refinement if necessary and multiplying π with a suitable element $f \in C^1(G, \check{C}^0(\boldsymbol{U}, \mathbb{T}))$, we may assume that $(\check{\partial}\pi, \partial_G\pi) = (1, 1)$. But $\partial_G\pi = 1$ implies that each $\pi_i : G \to C(U_i, \mathcal{U}(\mathcal{H}))$ is a group homomorphism and $\check{\partial}\pi = 1$ implies that these homomorphisms coincide on the overlaps $U_{ij} = U_i \cap U_j$. Thus α is implemented by a global homomorphism $\tilde{\pi} : G \to C(X, \mathcal{U}(\mathcal{H}))$, which means that α is exterior equivalent to the trivial action. □

Recall that if $\boldsymbol{U}$ is an open cover of X and $V_{ij} : U_{ij} \to \mathcal{U}(\mathcal{H})$ is a set of strongly continuous transition functions satisfying $(\check{\partial}V)_{ijk} = V_{ij}V_{jk}V_{ik}^* = 1$ on the triple overlaps U_{ijk}, then there exists a locally trivial bundle $q : \mathcal{A} \to X$ of compact operators over X with local trivializations $h_i : U_i \times \mathcal{K} \to q^{-1}(U_i)$ such that

$$h_i^{-1} \circ h_j(y, T) = (y, \operatorname{Ad} V_{ij}(y)(T))$$

for all $(y, T) \in U_{ij} \times \mathcal{K}$. Note that $\mathcal{A}$ is the continuous field of elementary C^*-algebras associated to the continuous field of Hilbert spaces with fibre $\mathcal{H}$ and transition functions V_{ij} as described in [D, Chapter 10]. Since $\mathcal{U}(\mathcal{H})$ is contractible by Kuiper's theorem, it follows that this bundle of Hilbert spaces is trivial (e.g., see [RW2, Corollary 4.79]). Thus $\mathcal{A}$ is isomorphic to the trivial bundle $X \times \mathcal{K}$, and therefore its algebra $C_0(X, \mathcal{A})$ of C_0-sections is isomorphic to $C_0(X, \mathcal{K})$; but it is important for us to work with the local trivializations given above.

Assume that for each $i \in I$ there is a $C_0(U_i)$-linear action $\alpha^i : G \to \operatorname{Aut} C_0(U_i, \mathcal{K})$. Then we can conjugate each α^i with the isomorphisms $h_i : U_i \times \mathcal{K} \to q^{-1}(U_i)$ to get actions $\tilde{\alpha}_i$ of G on $C_0(U_i, q^{-1}(U_i))$. Some routine fiddling with the transition functions reveals that there exists a global $C_0(X)$-linear action $\alpha : G \to \operatorname{Aut} C_0(X, \mathcal{A})$ which restricts to $\tilde{\alpha}_i$ on the ideals $C_0(U_i, q^{-1}(U_i))$ if and only if

$$\alpha_s^i \circ \operatorname{Ad} V_{ij} = \operatorname{Ad} V_{ij} \circ \alpha_s^j, \quad s \in G, \tag{2.4}$$

on $C_0(U_{ij}, \mathcal{K})$. We need

Lemma 2.3. *Suppose that $g \in C^1(G, \mathbb{T})$ and $\omega \in Z^2(G, \mathbb{T})$. Let $V_g \in \mathcal{U}(L^2(G))$ be the unitary operator given by $(V_g\xi)(s) = g(s)\xi(s)$, $s \in G$, and let L_ω and $L_{\partial_G g \cdot \omega}$ denote the left regular representations corresponding to ω and $\partial_G g \cdot \omega$ (see (2.1)). Then*

$$V_g L_\omega(s) = g(s) L_{\partial_G g \cdot \omega}(s) V_g \quad \text{for all } s \in G.$$

Proof. For $\xi \in L^2(G)$ and $s, t \in G$ we just compute

$$\begin{aligned}\big(V_g L_\omega(s) V_g^* \xi\big)(t) &= g(t)\omega(s, s^{-1}t)\overline{g(s^{-1}t)}\xi(s^{-1}t)\\ &= g(s)\partial_G g(s, s^{-1}t)\omega(s, s^{-1}t)\xi(s^{-1}t)\\ &= g(s)\big(L_{\partial_G g \cdot \omega}(s)\xi\big)(t).\end{aligned}$$

□

We are now ready to finish the proof of Theorem 1.1:

Proposition 2.4. *The map $\Phi : \mathcal{E}_G^{li}(X) \to \mathbb{H}^2(G, X)$ of Proposition 2.2 is surjective.*

Proof. Consider a cocycle $(g, f) \in \boldsymbol{Z}^2(G, \boldsymbol{U})$, i.e., (g, f) satisfies the conditions of (2.3). For each $y \in U_{ij}$ let $V_{ij}(x)$ denote the unitary operator on $L^2(G)$ given by pointwise multiplication with $g_{ij}(x) \in C^1(G, \mathbb{T})$. Since $g_{ij}(x_n)$ converges pointwise to $g(x)$ if $x_n \to x$, it follows that the maps $V_{ij} : U_{ij} \to \mathcal{U}(L^2(G))$ are strongly continuous. Moreover, $\check{\partial} g = 1$ implies $\check{\partial} V = 1$, so we can use the V_{ij} as transition functions to build a bundle $q : \mathcal{A} \to X$ with local trivializations $h_i : U_i \times \mathcal{K}(L^2(G)) \to q^{-1}(U_i)$ as in the discussion above. If G is infinite, then $C_0(X, \mathcal{A})$ is isomorphic to $C_0(X, \mathcal{K})$; otherwise, we stabilize in order to get the trivial bundle.

We use the family $\{f_i\}$ of elements in $Z^2(G, C(U_i, \mathbb{T}))$ to define an action α^i of G on $C_0(U_i, \mathcal{K}(L^2(G)))$ as in (2.2), i.e.,

$$\alpha^i_s(\varphi)(x) = \operatorname{Ad} L_{f_i(x)}(s)(\varphi(x)), \quad \varphi \in C_0(U_i, \mathcal{K}(L^2(G))).$$

In order to see that these actions induce a global action on $C_0(X, \mathcal{A})$ we have to show that they satisfy (2.4) above, which in our situation is equivalent to the equation

$$\operatorname{Ad}\big(L_{f_i(x)}(s) V_{ij}(x)\big) = \operatorname{Ad}\big(V_{ij}(x) L_{f_j(x)}(s)\big) \tag{2.5}$$

for all $s \in G$, $x \in U_{ij}$. The equation $\partial_G g = \check{\partial} f$ implies that $f_i(x) = \partial_G g_{ij}(x) f_j(x)$, and it follows from Lemma 2.3 that

$$V_{ij}(x) L_{f_j(x)}(s) = g_{ij}(s, x) L_{f_i(x)}(s) V_{ij}(x) \tag{2.6}$$

for all $s \in G$, $x \in U_{ij}$. But this implies (2.5).

Finally, if we choose any $C_0(X)$-linear isomorphism between $C_0(X, \mathcal{A})$ and $C_0(X, \mathcal{K})$ we may conjugate our global action on $C_0(X, \mathcal{A})$ to a $C_0(X)$-linear action $\alpha : G \to \operatorname{Aut} C_0(X, \mathcal{K})$. This action is locally implemented by $C_0(U_i)$-linear conjugates π_i of L_{f_i}. But then $\partial_G \pi_i = \partial_G L_{f_i} = f_i$ for all $i \in I$, and on the overlaps we use (2.6) to compute $(\check{\partial}\pi)_{ij}(s) = L_{f_i}(s) V_{ij} L_{f_j}(s) V_{ij}^* = g_{ij}(s)$. Thus, $(\check{\partial}\pi, \partial_G \pi) = (g, f)$. □

Remark 2.5. In our use of group cohomology we broke the common conventions by setting $C^0(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})) = 0$ instead of using $C^0(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})) = \check{C}^n(\boldsymbol{U}, \mathbb{T})$. Since all our G-modules carry the trivial action of G, we get in both cases the zero map for the coboundary operator $\partial_G : C^0(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})) \to C^1(G, \check{C}^n(\boldsymbol{U}, \mathbb{T}))$. Thus, if we would use the usual group complex instead of the one we use here, we would get $\check{H}^2(X, \mathbb{T}) \cong \check{H}^3(X, \mathbb{Z})$ as an extra direct summand in the second cohomology of the corresponding total complex. This summand corresponds to the factor $\check{H}^3(X, \mathbb{Z})$ in the splitting of the full Brauer group $\operatorname{Br}_G(X) = H^3(X, \mathbb{Z}) \oplus \mathcal{E}_G(X)$ as discussed in the introduction. Certainly, if one would try to extend the results of this section to nontrivial G-spaces X, one would have to work with an analogue of the "full" double complex. Actually, we plan to come back to this point in another publication, and we omit further details here.

3. Splitting groups

Suppose that $1 \to Z \to H \to G \to 1$ is a short exact sequence of second countable locally compact groups such that Z is central in H (in particular Z is abelian). Then (see [M1, p. 53]) we obtain an exact sequence in cohomology

$$0 \to H^1(G, \mathbb{T}) \overset{\text{inf}}{\to} H^1(H, \mathbb{T}) \overset{\text{res}}{\to} H^1(Z, \mathbb{T}) \overset{\text{tg}}{\to} H^2(G, \mathbb{T}) \overset{\text{inf}}{\to} H^2(H, \mathbb{T}), \tag{3.1}$$

where inf and res denote the inflation and restriction maps, and where $\mathrm{tg} : \widehat{Z} = H^1(Z, \mathbb{T}) \to H^2(G, \mathbb{T})$ is the transgression map, which can most conveniently be described as follows: Choose a Borel section $c : G \to H$ for the quotient map $q : H \to G$. Then $\partial_G c(s,t) = c(s)c(t)c(st)^{-1}$ is a Borel cocycle in $Z^2(G, Z)$ which in turn determines a cocycle $\mu \in Z^2(G, C(\widehat{Z}, \mathbb{T}))$ by defining

$$\mu(s,t)(\chi) := \chi(c(s)c(t)c(st)^{-1}), \quad \chi \in \widehat{Z}. \tag{3.2}$$

Then $\mathrm{tg}(\chi)$ is the cohomology class of the evaluation $\mu(\chi) \in Z^2(G, \mathbb{T})$ (note that the cocycle $\mu \in Z^2(G, C(\widehat{Z}, \mathbb{T}))$ constructed above will play a very important role in the following section). Note that all maps in the above sequence are continuous with respect to Moore topology (in the sense of [M3]; see also the discussion below). We recall the following definitions of Moore ([M2]):

Definition 3.1. A central extension $1 \to Z \to H \to G \to 1$ of G by Z is called a *splitting extension* for G (and $\mathbb{T}$) if the transgression map $\mathrm{tg} : H^1(Z, \mathbb{T}) \to H^2(G, \mathbb{T})$ is surjective. The group H is then called a *splitting group* for G.

A *representation group* for G is a splitting group H such that $\mathrm{tg} : \widehat{Z} \to H^2(G, \mathbb{T})$ is an isomorphism. G is called *smooth* if G has a representation group.

It follows from the discussions in [M2] and [EW1, §4] that the class of smooth groups is quite large: it contains all compactly generated abelian groups, all discrete groups, all compact groups and all simply connected and connected Lie groups. On the other hand, it is not difficult to construct examples of connected Lie groups which are not smooth (e.g., see [M2, p. 85]). However, it is shown in [M2, Chapter 1] that any almost connected group G (i.e., G/G_0 is compact) has a splitting group H. Note that if H is a splitting extension of G, then it follows from [M4, Theorem 6] that the transgression map induces a topological isomorphism $\dot{\mathrm{tg}} : \widehat{Z}/\ker \mathrm{tg} \to H^2(G, \mathbb{T})$.

In what follows we denote by $\underline{C}^n(G, \mathbb{T})$ the quotient of $C^n(G, \mathbb{T})$, where we identify functions which coincide almost everywhere. If $d \in C^n(G, \mathbb{T})$ we denote by $\underline{d}$ the corresponding class in $\underline{C}^n(G, \mathbb{T})$. Note that $\underline{C}^n(G, \mathbb{T})$ is a polish group when equipped with the topology of convergence in measure (with respect to the measure class of Haar measure on G). The group coboundary $\partial_G : C^n(G, \mathbb{T}) \to C^{n+1}(G, \mathbb{T})$ determines a (continuous) coboundary map $\underline{\partial}_G : \underline{C}^n(G, \mathbb{T}) \to \underline{C}^{n+1}(G, \mathbb{T})$, and it follows from [M3, Corollary 1] that the maps $c \mapsto \underline{c}$ induce isomorphisms between the cohomology groups $H^n(G, \mathbb{T})$ and $\underline{H}^n(G, \mathbb{T})$. The *Moore topology* (see [M3]) on $H^n(G, \mathbb{T})$ is the quotient topology of $\underline{Z}^n(G, \mathbb{T})/\underline{B}^n(G, \mathbb{T})$. Thus $H^n(G, \mathbb{T})$ is Hausdorff if and only if $\underline{B}^n(G, \mathbb{T})$ is closed in $\underline{Z}^n(G, \mathbb{T})$.

Every element $f \in C^n(G, C(X, \mathbb{T}))$ can be viewed as a function from X into $C^n(G, \mathbb{T})$ via evaluation, and the resulting map $\underline{f} : X \to \underline{C}^n(G, \mathbb{T})$ is continuous. It is not clear at all whether any continuous function from X into $\underline{C}^n(G, \mathbb{T})$ comes from evaluating an element of $C^n(G, C(X, \mathbb{T}))$. The following nontrivial result (which deals with this question in case $n = 1$) will play a crucial role in this paper. It is basically a consequence of the arguments given in the proof of [R, Theorem 2.1], some more detailed arguments can be found in the proof of [RW1, Proposition 3.4].

Lemma 3.2. *Suppose that $f \in Z^2(G, C(X, \mathbb{T}))$ and $\mu : X \to \underline{C}^1(G, \mathbb{T})$ is a continuous map such that $\underline{\partial}_G\big(\mu(x)\big) = \underline{f}(x)$ for every $x \in X$. Then there exists a unique element $g \in C^1(G, C(X, \mathbb{T}))$ such that $\mu = \underline{g}$ and $\partial_G g = f$. In particular, $f \in B^2(G, C(X, \mathbb{T}))$.*

If $1 \to Z \to H \to G \to 1$ is a splitting extension for G, then the inflation map $\inf : H^2(G,\mathbb{T}) \to H^2(H,\mathbb{T})$ is the zero map. Thus $\inf(\underline{Z}^2(G,\mathbb{T}))$ can be identified with the subgroup $\mathcal{C}$ of $\underline{B}^2(H,\mathbb{T})$ consisting of all functions which are constant on cosets of Z almost everywhere. Let $\mathcal{D} := \underline{\partial}_H{}^{-1}(\mathcal{C}) \subseteq \underline{C}^1(H,\mathbb{T})$. Since $\underline{\partial}_G$ is continuous, it follows that $\mathcal{D}$ is a closed subgroup of $\underline{C}^1(H,\mathbb{T})$. For further use note that $\ker \underline{\partial}_H = \underline{Z}^1(H,\mathbb{T}) \cong \widehat{H}_{\mathrm{ab}}$, and therefore $\mathcal{C} \cong \mathcal{D}/\widehat{H}_{\mathrm{ab}}$. The inflation map $\inf : \underline{C}^1(G,\mathbb{T}) \to \mathcal{D}$ defines a canonical inclusion of $\underline{C}^1(G,\mathbb{T})$ into $\mathcal{D}$ and there is also an inclusion $\iota : \widehat{Z} \to \mathcal{D}$ given by

$$\iota(\chi)(c(s)z) = \chi(z), \quad \chi \in \widehat{Z}, s \in G, z \in Z. \tag{3.3}$$

The proof of the following lemma is part of the proof of [M4, Theorem 6].

Lemma 3.3. *The map*

$$\Psi : \underline{C}^1(G,\mathbb{T}) \times \widehat{Z} \to \mathcal{D}; \Psi(g,\chi) = \inf(g) \cdot \iota(\chi)$$

is a (topological) isomorphism of groups.

In what follows we denote by $p_{\widehat{Z}} : \mathcal{D} \to \widehat{Z}$ and $p_{\underline{C}^1} : \mathcal{D} \to \underline{C}^1(G,\mathbb{T})$ the projections of $\mathcal{D}$ onto $\widehat{Z}$ and $\underline{C}^1(G,\mathbb{T})$, respectively given by the above splitting of $\mathcal{D}$. Recall also, that for $\chi \in \widehat{Z}$ we denoted by $\mu(\chi)$ the two-cocycle on G defined by

$$\mu(\chi)(s,t) = \big(\chi \circ \partial_G c\big)(st) = \chi(c(s)c(t)c(st)^{-1}).$$

Lemma 3.4. *Let $1 \to Z \to H \to G \to 1$ and $c : G \to H$ be as above. Let $f \in Z^2(G,\mathbb{T})$ and let $g \in C^1(H,\mathbb{T})$ such that $\partial_H g = \inf f$. Further, let $\underline{g}$ be the element of $\mathcal{D}$ determined by g. Then the following is true:*

1. *The restriction $g|_Z$ is a character of Z and $p_{\widehat{Z}}(\underline{g}) = g|_Z$.*
2. *We have $p_{\underline{C}^1}(\underline{g}) = \underline{g \circ c}$.*
3. *$\partial_G(g \circ c) = f \cdot \mu(g|_Z)$ and $\partial_H(\iota(\chi)) = \inf \mu(\chi)^{-1}$ for all $\chi \in \widehat{Z}$.*

Here we view $\iota(\chi)$ as an element of $C^1(H,\mathbb{T})$ (and not of $\underline{C}^1(H,\mathbb{T})$ as above).

Proof. Since $1 = \inf f(z,h) = \partial_H g(z,h) = g(z)g(h)g(zh)^{-1}$, we get $g(hz) = g(h)g(z)$ for all $(h,z) \in H \times Z$. But this implies that $g|_Z$ is a character, and that $g(c(s)z) = g(c(s))g|_Z(z) = \inf(g \circ c) \cdot \iota(g|_Z)$. This proves (1) and (2). In order to get (3) we use the equation $c(s)c(t)c(st)^{-1} = c(st)^{-1}c(s)c(t)$ (which follows by conjugating the central element $c(s)c(t)c(st)^{-1}$ by $c(st)^{-1}$) to compute

$$\begin{aligned}
f(s,t) &= \inf f(c(s),c(t)) = \partial_H g(c(s),c(t)) \\
&= g(c(s))g(c(t))g(c(s)c(t))^{-1} \\
&= g(c(s))g(c(t))g(c(st)c(s)c(t)c(st)^{-1})^{-1} \\
&= \big(g(c(s))g(c(t))g(c(st))^{-1}\big) \\
&\quad \cdot g(c(s)c(t)c(st)^{-1})^{-1} \quad (\text{since } c(s)c(t)c(st)^{-1} \in Z) \\
&= \partial_G(g \circ c)(s,t) \cdot \mu(g|_Z)(s,t)^{-1}.
\end{aligned}$$

A similar computation shows that $\partial_H(\iota(\chi)) = \inf \mu(\chi)^{-1}$ for all $\chi \in \widehat{Z}$. □

A cocycle $f \in Z^n(G, C(X,\mathbb{T}))$ is called *locally trivial*, if every $x \in X$ has an open neighborhood U such that the restriction $f|_U \in Z^2(G, C(U,\mathbb{T}))$ of f to U is trivial, i.e., $f|_U \in B^2(G, C(U,\mathbb{T}))$.

Proposition 3.5. *Let* $1 \to Z \to H \to G \to 1$ *be a splitting extension for* G *such that* $H_{ab} = H/\overline{[H,H]}$ *is compactly generated, and let* $f \in Z^2(G, C(X,\mathbb{T}))$*. Then the inflation* $\inf f \in Z^2(H, C(X,\mathbb{T}))$ *of* f *is locally trivial.*

Proof. Let $\underline{f} : X \to \underline{Z}^2(G,\mathbb{T})$ be the continuous map given by evaluation of f. Then $x \mapsto \inf \underline{f}(x)$ is a continuous map of X into $\mathcal{C} \cong \mathcal{D}/\ker \underline{\partial}_H$ with $\ker \underline{\partial}_H = \widehat{H}_{\mathrm{ab}}$. Since H_{ab} is compactly generated, it follows that $\widehat{H}_{\mathrm{ab}}$ is a Lie group. Hence, Palais' slice theorem [Pal] implies that there exist local continuous sections from $\mathcal{C} \to \mathcal{D}$ which invert $\underline{\partial}_H$. Thus, if $x \in X$ is given, we find an open neighborhood U of x and a continuous function $\underline{\tilde{f}} : U \to \mathcal{D} \subseteq \underline{C}^1(H,\mathbb{T})$ such that $\underline{\partial}_H \underline{\tilde{f}}(x) = \inf \underline{f}(x)$ for all $x \in U$. It follows then from Lemma 3.2 that there exists an element $\tilde{f} \in C^1(H, C(U,\mathbb{T}))$ such that $\partial_H \tilde{f} = \inf f$. □

If A is a C^*-algebra with continuous trace and spectrum $\widehat{A} = X$, then an action $\alpha : G \to \operatorname{Aut} A$ is called *locally unitary* if each $x \in X$ has an open neighborhood U such that α restricts to a unitary action on A_U, where A_U denotes the ideal corresponding to U (for the definition of unitary actions see Remark 1.2 above). A $C_0(X)$-linear action α is called *pointwise unitary* if all actions $\alpha^x : G \to \operatorname{Aut} A_x$ induced by α on the fibres A_x are unitary. Note that the crossed product by a locally unitary action α can be described particularly well via the Phillips-Raeburn obstruction $\zeta(\alpha) \in \check{H}^1(X, \widehat{G}_{\mathrm{ab}})$ (see [PhR, RR, R, EW1, EW2]). Hence, there has been substantial interest in the question under what conditions pointwise unitary actions are automatically locally unitary. Using the above results we are now able to extend the known results on automatic local unitarity by the following

Proposition 3.6. *Suppose that* G *has a splitting extension* $1 \to Z \to H \to G \to 1$ *such that* H_{ab} *is compactly generated and such that* $D := \ker \mathrm{tg}$ *is countable (it is shown in* [M2, Chapter 2] *that every almost connected second countable group satisfies these assumptions). Suppose further that* A *is a separable continuous-trace* C^**-algebra with locally connected spectrum* $X = \widehat{A}$*. Then every pointwise unitary action* $\alpha : G \to \operatorname{Aut} A$ *is automatically locally unitary.*

Proof. Since G_{ab} is a quotient of H_{ab}, it follows from our assumptions that G_{ab} is compactly generated. Thus we can follow the arguments given in the proof of [R, Corollary 2.2] to see that it suffices to show that every pointwise trivial cocycle $f \in Z^2(G, C(X,\mathbb{T}))$ is locally trivial (a cocycle $f \in Z^2(G, C(X,\mathbb{T}))$ is called *pointwise trivial*, if $f(x) \in B^2(G,\mathbb{T})$ for all $x \in X$).

So let $f \in Z^2(G, C(X,\mathbb{T}))$ be pointwise trivial and let $x_0 \in X$ be given. By Proposition 3.5 we find a neighborhood U of x_0 and an element $\tilde{f} \in C^1(H, C(U,\mathbb{T}))$ such that $\partial_H \tilde{f} = \inf f|_U$. Part (1) of Lemma 3.4 then implies that $x \mapsto \tilde{f}(x)|_Z$ is a continuous function of U into $\widehat{Z}$. Since f is pointwise trivial, it follows now from part (3) of Lemma 3.4 that

$$\mathrm{tg}(\tilde{f}(x)|_Z) = [f(x)][\mu(\tilde{f}(x)|_Z)] = [\partial_G(\tilde{f}(x) \circ c)] = [1]$$

in $H^2(G,\mathbb{T})$ for all $x \in U$. Thus the map $x \mapsto \tilde{f}(x)|_Z$ takes its values in $D = \ker \mathrm{tg}$. In particular, since $\ker \mathrm{tg} = \mathrm{res}(H^1(H,\mathbb{T}))$, we find a character $\chi \in \widehat{H}_{\mathrm{ab}}$ such that $\chi|_Z = f(x_0)|_Z$, and by multiplying each $\tilde{f}(x)$ with χ^{-1} we may assume that $\tilde{f}(x_0)|_Z \equiv 1$.

Since X is locally connected, we may assume without loss of generality that U is compact and connected. Since D is countable by assumption, it follows then that the image under the map $x \mapsto \tilde{f}(x)|_Z$ is a connected, compact, and countable subset of the Hausdorff space X. But an easy application of Baire's theorem then implies that $x \mapsto \tilde{f}(x)|_Z$ is constant on U. Thus we get $\tilde{f}(x)|_Z \equiv 1$ for all $x \in U$, which implies that $\tilde{f} = \inf g$ for some $g \in C^1(G, C(U, \mathbb{T}))$. Thus we get $f|_U = \partial_G g$. $\square$

Remark 3.7. Note that the above proposition is not true in this generality if we omit the assumption on the local connectivity of the spectrum X of A. In fact, in [EW1, §7] the authors constructed a pointwise unitary action of a connected two-step nilpotent group G on $C_0(\mathbb{N} \cup \{\infty\}, \mathcal{K})$ which is not locally unitary.

4. The reduced double complex

In this section we will show that if G has a splitting extension $1 \to Z \to H \to G \to 1$ such that H_{ab} is compactly generated, then $\mathcal{E}_G(X)$ can be described in terms of continuous $\widehat{H}_{\mathrm{ab}}$-equivariant maps from principal $\widehat{H}_{\mathrm{ab}}$-bundles over X into $\widehat{Z}$.

Let $\psi : N \to M$ be a continuous homomorphism of the topological abelian group N into the topological abelian group M and let X be a paracompact topological space. Then for each open cover $\boldsymbol{U}$ of X the cone of $\check{C}^*(X, \psi)$ is the double complex

$$\begin{array}{ccccccc}
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & \check{C}^2(\boldsymbol{U}, N) & \xrightarrow{\psi_*} & \check{C}^2(\boldsymbol{U}, M) & \longrightarrow & 0 \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & \check{C}^1(\boldsymbol{U}, N) & \xrightarrow{\psi_*} & \check{C}^1(\boldsymbol{U}, M) & \longrightarrow & 0 \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & \check{C}^0(\boldsymbol{U}, N) & \xrightarrow{\psi_*} & \check{C}^0(\boldsymbol{U}, M) & \longrightarrow & 0.
\end{array} \tag{4.1}$$

We denote by $\mathbb{H}^2(\boldsymbol{U}, \mathcal{C}(\psi))$ its second cohomology and put

$$\mathbb{H}^2(X, \mathcal{C}(\psi)) = \lim_{\boldsymbol{U}} \mathbb{H}^2(\boldsymbol{U}, \mathcal{C}(\psi)).$$

Let $(\gamma, f) \in \boldsymbol{Z}^2(\boldsymbol{U}, \mathcal{C}(\psi))$ be a representative of some class in $\mathbb{H}^2(X, \mathcal{C}(\psi))$. This means that (γ, f) satisfies the conditions

$$\gamma \in \check{Z}^1(\boldsymbol{U}, N),\ f \in \check{C}^0(\boldsymbol{U}, M), \quad \text{and} \ \ \psi_*(\gamma) = \check{\partial} f.$$

By standard techniques we can construct a principal N-bundle $\mathcal{P}$ over X with local trivializations $U_i \times N$ and transition functions γ_{ij}. To be more precise, if $p : \mathcal{P} \to X$ denotes the projection, then there exist N-equivariant homeomorphisms $h_i : U_i \times N \to p^{-1}(U_i)$ such that $h_j^{-1} \circ h_i(y, n) = (y, n\gamma_{ij}(y))$ for all $(y, n) \in U_{ij} \times N$. Note that the bundle $\mathcal{P}$ is unique up to isomorphism and that the isomorphism class of $\mathcal{P}$ only depends on the class $[\gamma] \in \check{H}^1(X, N)$.

Since $\psi_*(\gamma) = \check{\partial} f$, it is straightforward to check that the equation

$$F(h_i(x, n)) = f_i(x)\psi(n), \qquad (x, n) \in U_i \times N, \tag{4.2}$$

determines a continuous N-equivariant map $F : \mathcal{P} \to M$. Thus the pair $(\mathcal{P}, F)$ is an equivariant pair in the sense of

Definition 4.1. Let $\psi : N \to M$ and X be as above. Let $\mathcal{P}$ be a principal (i.e., locally trivial) N-bundle over X and let $F : \mathcal{P} \to M$ be a continuous N-equivariant map. Then $(\mathcal{P}, M)$ is called an *equivariant pair* with respect to X and $\psi : N \to M$. Two equivariant pairs $(\mathcal{P}, F)$ and $(\mathcal{Q}, G)$ are called *isomorphic* if there exists a bundle isomorphism $\phi : \mathcal{P} \to \mathcal{Q}$ such that $F = G \circ \phi$.

The proof of the following proposition is completely routine, so we omit the details.

Proposition 4.2. *If $(\gamma, f) \in \boldsymbol{Z}^2(X, \mathcal{C}(\psi))$ and $(\mathcal{P}, F)$ denote the equivariant pair constructed from (γ, f) as in the discussion above, then the assignment $[(\gamma, f)] \to [(\mathcal{P}, F)]$ defines a one-to-one correspondence between $\mathbb{H}^2(X, \mathcal{C}(\psi))$ and the set of all isomorphism classes of equivariant pairs with respect to X and $\psi : N \to M$.*

Assume now that $1 \to Z \to H \to G \to 1$ is a splitting extension for G such that H_{ab} is compactly generated. Since G_{ab} is a quotient of H_{ab}, this automatically implies that G_{ab} is compactly generated, and hence we get $\mathcal{E}_G(X) = \mathcal{E}_G^{li}(X)$ by Proposition 2.1. Let $\mathrm{res} : \widehat{H}_{\mathrm{ab}} \to \widehat{Z}$ denote the restriction map. In what follows we want to show that there is a canonical isomorphism between $\mathbb{H}^2(X, \mathcal{C}(\mathrm{res}))$ and $\mathbb{H}^2(G, X) \cong \mathcal{E}_G(X)$, thus, giving a classification of $\mathcal{E}_G(X)$ in terms of equivariant pairs with respect to X and $\mathrm{res} : \widehat{H}_{\mathrm{ab}} \to \widehat{Z}$ via Proposition 4.2 above.

Let us fix once and for all a Borel section $c : G \to H$. Let $\mu \in Z^2(G, C(\widehat{Z}, \mathbb{T}))$ be the cocycle constructed from c as in (3.2) (i.e., $\mu(s,t)(\chi) = \chi(c(s)c(t)c(st)^{-1})$ and let $\nu \in C^1(G, C(\widehat{H}_{\mathrm{ab}}, \mathbb{T}))$ be given by

$$\nu(s)(\chi) = \chi(c(s)), \quad s \in G, \chi \in \widehat{H}_{\mathrm{ab}}. \tag{4.3}$$

If X is any second countable locally compact space, and U is an open subset of X, then ν and μ induce homomorphisms

$$\nu_U : C(U, \widehat{H}_{\mathrm{ab}}) \to C^1(G, C(U, \mathbb{T})); \quad \mu_U : C(U, \widehat{Z}) \to Z^2(G, C(U, \mathbb{T}))$$

by

$$\nu_U(f)(s)(x) := \nu(s)(f(x)); \quad \mu_U(g)(s,t)(x) := \mu(s,t)(g(x)). \tag{4.4}$$

Thus, for any open cover $\boldsymbol{U}$ of X we get obvious homomorphisms

$$\nu_* : \check{C}^n(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}}) \to C^1(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})); \quad \mu_* : \check{C}^n(\boldsymbol{U}, \widehat{Z}) \to Z^2(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})).$$

Now observe that for the computation of $\mathcal{E}_G(X)$, i.e., of the second cohomology of the double complex of Theorem 1.1 (for a given open cover $\boldsymbol{U}$), we may reduce our attention to the complex

$$\begin{array}{ccccccc}
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G, \check{C}^2(\boldsymbol{U}, \mathbb{T})) & \xrightarrow{\partial_G} & Z^2(G, \check{C}^2(\boldsymbol{U}, \mathbb{T})) & \longrightarrow & 0 \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G, \check{C}^1(\boldsymbol{U}, \mathbb{T})) & \xrightarrow{\partial_G} & Z^2(G, \check{C}^1(\boldsymbol{U}, \mathbb{T})) & \longrightarrow & 0 \\
\uparrow & & \check{\partial}\uparrow & & \check{\partial}\uparrow & & \\
0 & \longrightarrow & C^1(G, \check{C}^0(\boldsymbol{U}, \mathbb{T})) & \xrightarrow{\partial_G} & Z^2(G, \check{C}^0(\boldsymbol{U}, \mathbb{T})) & \longrightarrow & 0
\end{array}$$

It is easily checked that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^1(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})) & \xrightarrow{\partial_G} & Z^2(G, \check{C}^n(\boldsymbol{U}, \mathbb{T})) & \longrightarrow & 0 \\ & & {\scriptstyle \nu_*}\big\uparrow & & {\scriptstyle \mu_*}\big\uparrow & & \\ 0 & \longrightarrow & \check{C}^n(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}}) & \xrightarrow{\mathrm{res}_*} & \check{C}^n(\boldsymbol{U}, \widehat{Z}) & \longrightarrow & 0 \end{array}$$

commutes. Hence, since (ν_*, μ_*) also commutes with $\check{\partial}$ it follows that (ν_*, μ_*) induces a chain map between the total complexes of the corresponding double complexes. The proof of Theorem 1.6, as stated in the introduction, will follow from Proposition 4.2 together with

Theorem 4.3. *Assume that $1 \to Z \to H \to G \to 1$ is a splitting extension for the second countable locally compact group G such that H_{ab} is compactly generated. Then the chain map (ν_*, μ_*) constructed above induces a group isomorphism Ψ : $\mathbb{H}^2(X, \mathcal{C}(\mathrm{res})) \to \mathbb{H}^2(G, X) \cong \mathcal{E}_G(X)$.*

Proof. We first show that Ψ is injective. For this let (φ_{ij}, h_i) be a cocycle in $\boldsymbol{Z}^2(\boldsymbol{U}, \mathcal{C}(\mathrm{res}))$ such that $(\nu_*(\varphi), \mu_*(h)) = (\check{\partial}\eta, \partial_G\eta)$ for some $\eta \in C^1(G, C(\boldsymbol{U}, \mathbb{T}))$. We have to construct an element $\psi \in \check{C}^0(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}})$ such that $(\varphi, h) = (\check{\partial}\psi, \mathrm{res}_*\,\psi)$.

For each $i \in I$ we define $\tilde{h}_i \in C^1(H, C(U_i, \mathbb{T}))$ by $\tilde{h}_i(c(s)z) = h_i(z)$. Thus we have $\tilde{h}_i(x) = \iota(h_i(x))$, where $\iota : \widehat{Z} \to C^1(H, \mathbb{T})$ is as in Lemma 3.4. Moreover, let $\inf \eta_i$ denote the inflation of $\eta_i \in C^1(G, C(U_i, \mathbb{T}))$ to H. Since $\partial_G \eta_i = \mu(h_i)$, it follows from Lemma 3.4, (3) that

$$\begin{aligned} \partial_H(\inf \eta_i(x)) &= \inf(\partial_G \eta_i(x)) = \inf(\mu(h_i)(x)) \\ &= \partial_H(\iota(h_i(x)))^{-1} = \partial_H(\tilde{h}_i(x))^{-1}. \end{aligned}$$

Thus we get $\inf \eta_i \cdot \tilde{h}_i \in Z^1(H, C(U_i, \mathbb{T})) \cong C(U_i, \widehat{H}_{\mathrm{ab}})$. We then have

$$\mathrm{res}\, \psi_i = \mathrm{res}(\inf \eta_i \cdot \tilde{h}_i) = h_i,$$

for all $i \in I$. Moreover, since $\check{\partial}\eta = \varphi$ and $\check{\partial}h = \mathrm{res}\,\varphi$ we get

$$\begin{aligned} \big(\psi_i(x)\psi_j(x)^{-1}\big)(c(s)z) &= \eta_i(x)(s)\eta_j(x)(s)^{-1} h_i(x)(z) h_j(x)(z)^{-1} \\ &= \varphi_{ij}(x)(c(s))\varphi_{ij}(x)(z) \\ &= \varphi_{ij}(x)(c(s)z), \end{aligned}$$

for all $x \in U_{ij}$, $s \in G$ and $z \in Z$. This proves that $(\varphi, h) = (\check{\partial}\psi, \mathrm{res}_*\,\psi)$.

We now show surjectivity of Ψ. For this let (g_{ij}, f_i) be a representative of some class in $\mathbb{H}^2(G, X)$ with respect to some open cover $\boldsymbol{U} = \{U_i\}_{i \in I}$. It follows then from Proposition 3.5 that, after passing to a refinement if necessary, there exist elements $\tilde{f}_i \in C^1(H, C(U_i, \mathbb{T}))$ such that $\partial_H \tilde{f}_i = \inf f_i$.

Since $\partial_G g = \check{\partial} f$ it follows that on each overlap U_{ij}, we get

$$\partial_H(\inf g_{ij}(y)) = \inf f_i(y) \inf f_j^{-1}(y) = \partial_H(\tilde{f}_i \tilde{f}_j^{-1}(y)).$$

Thus there exist unique continuous functions $\varphi_{ij} : U_{ij} \to \widehat{H}_{\mathrm{ab}}$ satisfying

$$\tilde{f}_i(y)\tilde{f}_j(y)^{-1} = \varphi_{ij}(y) \cdot \inf g_{ij}(y) \tag{4.5}$$

for all $y \in U_{ij}$. Moreover, since $\tilde{f}_i(x)|_Z$ is a character by Lemma 3.4, (1), we get $h_i := \tilde{f}_i|_Z \in Z^1(Z, C(U_i, \mathbb{T})) \cong C(U_i, \widehat{Z})$. It follows directly from our constructions that $(\varphi, h) \in \boldsymbol{Z}^2(\boldsymbol{U}, \mathcal{C}(\mathrm{res}))$, i.e., that $\check{\partial}\varphi = 1$ and $\mathrm{res}\,\varphi = \check{\partial}h$.

We claim that $(\nu_*(\varphi)^{-1}, \mu_*(h)^{-1})$ is cohomologous to (g, f). To see this we have to produce an element $\eta \in C^1(G, \check{C}^0(\boldsymbol{U}, \mathbb{T}))$ such that $(\check{\partial}\eta, \partial_G\eta) = (g \cdot \nu_*(\varphi), f \cdot \mu_*(h))$. For this let $\tilde{f}_i \in C^1(H, C(U_i, \mathbb{T}))$ be as above. Define $\eta_i \in C^1(G, C(U_i, \mathbb{T}))$ by $\eta_i = \tilde{f}_i \circ c$. Since $\partial_H \tilde{f}_i = \inf f_i$, it follows from Lemma 3.4, (3) that

$$\partial_G \eta_i(x) = f_i(x) \cdot \mu(\tilde{f}_i(x)|_Z) = f_i(x) \cdot \mu(h_i(x)),$$

for all $x \in U_i$. This proves $\partial_G\eta = f \cdot \mu_*(h)$. On the other hand, we have for all $y \in U_{ij}$ and $s \in G$:

$$\begin{aligned}\big(\eta_i(y)\eta_j(y)\big)(s) &= \big(\tilde{f}_i(y)\tilde{f}_j(y)\big)(c(s)) \\ &\overset{(4.5)}{=} \big(\varphi_{ij}(y) \cdot \inf g_{ij}(y)\big)(c(s)) \\ &= \big(\nu(\varphi_{ij})(y) \cdot g_{ij}(y)\big)(s).\end{aligned}$$

Thus, we also have $\check{\partial}\eta = g \cdot \nu_*(\varphi)$, which completes the proof. □

If H is actually a representation group of G, then $\mathrm{res}_* : \check{C}^n(\boldsymbol{U}, \widehat{H}_{\mathrm{ab}}) \to \check{C}^n(\boldsymbol{U}, \widehat{Z})$ is the zero map. Thus we get a natural splitting $\mathbb{H}^2(X, \mathcal{C}(\mathrm{res})) = \check{H}^1(X, \widehat{G}_{\mathrm{ab}}) \oplus C(X, \widehat{Z})$. Since in this situation the transgression map $\mathrm{tg} : \widehat{Z} \to H^2(G, \mathbb{T})$ is a topological isomorphism and $H_{\mathrm{ab}} = G_{\mathrm{ab}}$, we obtain an alternative proof of [EW1, Theorem 5.4]:

Corollary 4.4. *Suppose that G is smooth and G_{ab} is compactly generated. Then $\mathcal{E}_G(X) \cong \check{H}^1(X, \widehat{G}_{ab}) \oplus C(X, H^2(G, \mathbb{T}))$.*

We should note that the splitting in the above corollary depends on the given choice of a representation group H of G, thus the splitting is not canonical!

It is certainly interesting to see how one can directly recover the class $[\alpha] \in \mathcal{E}_G(X)$ from a corresponding equivariant pair $(\mathcal{P}, F)$, and in fact we shall need such a construction later. So in what follows let $p : \mathcal{P} \to X$ be a principal $\widehat{H}_{\mathrm{ab}}$-bundle and let $F : \mathcal{P} \to \widehat{Z}$ be an $\widehat{H}_{\mathrm{ab}}$-equivariant continuous map. Let $c : G \to H$ be a Borel section and consider the action $\beta : G \to \mathrm{Aut}\,\mathcal{K}$, $\mathcal{K} = \mathcal{K}(L^2(G))$ (or any stabilization of it if G is finite), given by $\beta_\chi = \mathrm{Ad}\,V_\chi$, where

$$(V_\chi \xi)(s) = \chi(c(s))\xi(s), \quad \xi \in L^2(G).$$

Let $\tau \otimes \beta$ denote the diagonal action of $\widehat{H}_{\mathrm{ab}}$ on $C_0(\mathcal{P}, \mathcal{K})$ given by

$$(\tau \otimes \beta)_\chi(\psi(y)) = \beta_\chi(\psi(\overline{\chi} \cdot y)), \quad \psi \in C_0(\mathcal{P}, \mathcal{K}).$$

This action extends to an (probably not strongly continuous) action on $C_b(\mathcal{P}, \mathcal{K})$ and the *induced algebra* $\mathrm{Ind}_\beta^{\mathcal{P}}\,\mathcal{K}$ is the fixed point algebra of $C_0(X) \cdot C_b(\mathcal{P}, \mathcal{K}) \subseteq C_b(\mathcal{P}, K)$ with respect to $\tau \otimes \beta$ (here the action of $C_0(X)$ on $C_b(\mathcal{P}, \mathcal{K})$ is given via the canonical extension of functions of X to functions of $\mathcal{P}$).

We now use the function $F : \mathcal{P} \to \widehat{Z}$ to construct an action $\alpha : G \to \mathrm{Aut}(\mathrm{Ind}_\beta^{\mathcal{P}}\,\mathcal{K})$. For this we consider the cocycle $\mu(F) = \mu_{\mathcal{P}}(F) \in Z^2(G, C(\mathcal{P}, \mathbb{T}))$ as given in (4.4). Then as in (2.2) we obtain an action $\alpha^F : G \to \mathrm{Aut}\,C_0(\mathcal{P}, \mathcal{K})$ by defining

$$\big(\alpha_s^F(\psi)\big)(y) = \mathrm{Ad}\,L_{\mu(F)(y)}(s)\big(\psi(y)\big), \quad \psi \in C_0(\mathcal{P}, \mathcal{K}).$$

We need

Lemma 4.5. *The actions $\alpha^F : G \to \operatorname{Aut} C_0(\mathcal{P},\mathcal{K})$ and $\tau\otimes\beta : \widehat{H}_{ab} \to \operatorname{Aut} C_0(\mathcal{P},\mathcal{K})$ commute.*

Proof. For $s,t \in G$, $\chi \in \widehat{H}_{\mathrm{ab}}$ and $y \in \mathcal{P}$ we get

$$\begin{aligned} \mu(F)(\overline{\chi}\cdot y)(s,t) &= F(\overline{\chi}\cdot y)(c(s)c(t)c(st)^{-1}) = \big(\overline{\chi}|_Z \cdot F(y)\big)(c(s)c(t)c(st)^{-1}) \\ &= \partial_G(\overline{\chi}\circ c)(s,t)\mu(F)(y)(s,t). \end{aligned} \tag{4.6}$$

Using this, it follows from Lemma 2.3 that

$$L_{\mu(F)(y)}(s)V_\chi = \overline{\chi}(c(s))V_\chi L_{\mu(F)(\overline{\chi}\cdot y)}(s), \quad \chi \in \widehat{H}_{\mathrm{ab}}, s \in G. \tag{4.7}$$

Thus, for $s \in G$, $\chi \in \widehat{H}_{\mathrm{ab}}$ and $\psi \in C_0(\mathcal{P},\mathcal{K})$ we get

$$\begin{aligned} \big((\tau\otimes\beta)_\chi \alpha^F_s(\psi)\big)(s) &= \operatorname{Ad} V_\chi\big(\alpha^F_s(\psi)(\overline{\chi}\cdot y)\big) \\ &= \operatorname{Ad}(V_\chi L_{\mu(F)(\overline{\chi}\cdot y)}(s))\big(\psi(\overline{\chi}\cdot y)\big) \\ &\overset{(4.7)}{=} \operatorname{Ad}(L_{\mu(F)(y)}(s)V_\chi)\big(\psi(\overline{\chi}\cdot y)\big) \\ &= \big(\alpha^F_s(\tau\otimes\beta)_\chi(\psi)\big)(y). \end{aligned}$$

This finishes the proof. □

The above lemma implies that for each $s \in G$ the automorphism α^F_s (extended to an automorphism of $C_0(X)\cdot C_b(\mathcal{P},\mathcal{K})$) restricts to an automorphism on the fixed point algebra $\operatorname{Ind}^{\mathcal{P}}_\beta \mathcal{K}$ with respect to $\tau\otimes\beta$.

Proposition 4.6. *Let $(\mathcal{P},F)$ be an equivariant pair and for each $s \in G$ let α_s denote the restriction of α^F_s to $\operatorname{Ind}^{\mathcal{P}}_\beta \mathcal{K}$. Then $\operatorname{Ind}^{\mathcal{P}}_\beta \mathcal{K}$ is ($C_0(X)$-linearly) isomorphic to $C_0(X,\mathcal{K})$ and $\alpha : G \to \operatorname{Aut}(\operatorname{Ind}^{\mathcal{P}}_\beta \mathcal{K})$ is a representative of the class in $\mathcal{E}_G(X)$ which corresponds to the isomorphism class of $(\mathcal{P},F)$ via Theorem 1.6.*

Proof. Let $\boldsymbol{U} = \{U_i\}$ be an open cover of X such that there are local trivializations $k_i : U_i \times \widehat{H}_{\mathrm{ab}} \to p^{-1}(U_i)$ with transition functions $\varphi_{ij} : U_{ij} \to \widehat{H}_{\mathrm{ab}}$, i.e., we have

$$k_j^{-1}\circ k_i(x,1) = (x,\varphi_{ij}(x)), \quad x \in U_{ij}.$$

Further, let $f_i : U_i \to \widehat{Z}$ be defined by $f_i(x) = F(k_i(x,1))$. Then $(\varphi,f) \in \boldsymbol{Z}^2(X,\mathcal{C}(\mathrm{res}))$ is a cocycle representing $(\mathcal{P},F)$ in $\mathbb{H}^2(X,\mathcal{C}(\mathrm{res}))$ via Proposition 4.2, and Theorem 4.3 tells us that $(\nu_*\varphi,\mu_*f) \in \boldsymbol{Z}^2(G,X)$ is a representative for the class in $\mathcal{E}_G(X)$ corresponding to the isomorphism class of $(\mathcal{P},F)$.

We show that our action α coincides (up to isomorphism) with the action corresponding to $(\nu_*\varphi,\mu_*f)$ as constructed in the proof of Proposition 2.4. For this we first observe that $\operatorname{Ind}^{\mathcal{P}}_\beta \mathcal{K}$ can be identified with the algebra of C_0-sections of the bundle

$$\mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K} := (\mathcal{P}\times\mathcal{K})/\widehat{H}_{\mathrm{ab}},$$

where $\widehat{H}_{\mathrm{ab}}$ acts diagonally on $\mathcal{P}\times\mathcal{K}$ and the projection $q : \mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K} \to X$ is given by $q([y,T]) = p(y)$. In fact, if $\psi \in \operatorname{Ind}^{\mathcal{P}}_\beta \mathcal{K}$, then the corresponding section $\tilde{\psi} \in C_0(X,\mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K})$ is given by $\tilde{\psi}(p(y)) = [y,\psi(y)]$ (see [EW2, Proposition 3.7]). Now, $\mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K}$ has local trivializations

$$h_i : U_i\times\mathcal{K} \to q^{-1}(U_i);\ h_i(x,T) = [k_i(x,1),T].$$

On the overlaps we get

$$\begin{aligned}h_i^{-1}\circ h_j(x,T) &= h_i^{-1}([k_j(x,1),T]) = h_i^{-1}([k_i(x,\varphi_{ij}(x)^{-1}),T])\\ &= h_i^{-1}([\varphi_{ij}(x)^{-1}\cdot k_i(x,1),T]) = h_i^{-1}([k_i(x,1),\beta_{\varphi_{ij}(x)}(T)])\\ &= (x,\operatorname{Ad}V_{\varphi_{ij}(x)}(T)).\end{aligned}$$

But by definition we have

$$(V_{\varphi_{ij}(x)}\xi)(s) = \varphi_{ij}(x)(c(s))\xi(s) = \nu(\varphi_{ij})(x)(s)\xi(s), \quad \xi\in L^2(G),$$

and this implies that the bundle $\mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K}$ coincides (up to isomorphism of bundles over X) with the bundle $\mathcal{A}$ constructed from $(\nu_*\varphi,\mu_*f)$ in Proposition 2.4. As observed in the discussions preceding Proposition 2.4, this implies that $\operatorname{Ind}_\beta^{\mathcal{P}}\mathcal{K}\cong C_0(X,\mathcal{A})\cong C_0(X,\mathcal{K})$.

The action, say $\tilde\alpha$, of G on $\mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K}$ corresponding to our action α on

$$C_0(X,\mathcal{P}\times_{\widehat{H}_{\mathrm{ab}}}\mathcal{K})\cong\operatorname{Ind}_\beta^{\mathcal{P}}\mathcal{K}$$

is given by $\tilde\alpha_s([y,T]) = [y,\operatorname{Ad}L_{\mu(F)(y)}(s)(T)]$. We then get

$$\begin{aligned}\tilde\alpha_s(h_i(x,T)) &= [k_i(x,1),\operatorname{Ad}L_{\mu(F)(k_i(x,1))}(s)(T)]\\ &= [k_i(x,1),\operatorname{Ad}L_{\mu(f_i)(x)}(s)(T)]\\ &= h_i\big(x,\operatorname{Ad}L_{\mu(f_i)(x)}(s)(T)\big).\end{aligned}$$

But this shows that the actions $\alpha^i: G\to\operatorname{Aut}C_0(U_i,\mathcal{K})$ coming from α via the local trivializations h_i are precisely the inner actions corresponding to the cocycles $\mu(f_i)\in Z^2(G,C(U_i,\mathbb{T}))$ via (2.2). Thus α is precisely the action corresponding to $(\nu_*\varphi,\mu_*f)$ as constructed in the proof of Proposition 2.4. $\square$

5. The crossed products

In this section we want to use our description of $\mathcal{E}_G(X)$ in terms of equivariant pairs $(\mathcal{P},F)$ (in the presence of a splitting extension $1\to Z\to H\to G\to 1$) to describe the bundle structure of the crossed products $C_0(X,\mathcal{K})\rtimes_\alpha G$.

For this we first have to recall some basic facts on $C_0(X)$-algebras. If X is a locally compact space, then a $C_0(X)$-algebra is a C^*-algebra A together with a non-degenerate $*$-homomorphism $\phi: C_0(X)\to\mathcal{Z}M(A)$. We usually write $\varphi\cdot a$ for $\phi(\varphi)a$ if confusion seems unlikely. If $x\in X$, then the *fibre* of A over x is the quotient $A_x := A/I_x$, with $I_x = C_0(X\setminus\{x\})\cdot A$. Note that $C_0(X)$-algebras are the C^*-algebra analogues of topological bundles $p: Y\to X$, where p is just a continuous map, and in this respect the fibres A_x correspond to the fibres $Y_x = p^{-1}(\{x\})$ of such topological bundles. In particular, if Y is a locally compact space and $p: Y\to X$ is a continuous map, then $C_0(Y)$ becomes a $C_0(X)$-algebra via the map $p^*: C_0(X)\to C_b(Y) = M(C_0(Y)); p^*(\varphi)=\varphi\circ p$. This allows us to define the *pull-back* $C_0(Y)\otimes_p A$ of a $C_0(X)$-algebra A via a continuous map $p: Y\to X$ as

$$C_0(Y)\otimes_p A := C_0(Y)\otimes_{C_0(X)} A$$

(see [RW]).

If A is a $C_0(X)$-algebra and $\alpha: G\to\operatorname{Aut}A$ is an action of G by $C_0(X)$-linear automorphisms, then the crossed product $A\rtimes_\alpha G$ carries a natural structure of a $C_0(X)$-algebra via the composition $i_A\circ\phi: C_0(X)\to M(A\rtimes_\alpha G)$, where $i_A: A\to M(A\rtimes_\alpha G)$ denotes the canonical embedding. The fibres $(A\rtimes_\alpha G)_x$ are then canonically isomorphic to $A_x \rtimes_{\alpha^x} G$, where α^x is the action on the G-invariant quotient $A_x = A/I_x$ induced from α (e.g., see [N] for more details).

In particular, if $\alpha : G \to \operatorname{Aut} C_0(X, \mathcal{K})$ is $C_0(X)$-linear, then $C_0(X, \mathcal{K}) \rtimes_\alpha G$ can be viewed as a "bundle" over X with fibres $\mathcal{K} \rtimes_{\alpha^x} G$. Moreover, if $[\omega_x] \in H^2(G, \mathbb{T})$ denotes the Mackey obstruction of α^x (i.e., the class $[\omega_x] \in H^2(G, \mathbb{T})$ corresponding to the inner action $\alpha^x : G \to \operatorname{Aut} \mathcal{K}$), then $\mathcal{K} \rtimes_{\alpha^x} G$ is isomorphic to $\mathcal{K} \otimes C^*(G, \overline{\omega}_x)$, where $C^*(G, \overline{\omega}_x)$ is the twisted group algebra with respect to the inverse $[\overline{\omega}_x]$ of $[\omega_x]$ (see [G, Theorem 18]). Note that the isomorphism class of $C_0(X, \mathcal{K}) \rtimes_\alpha G$ (as a $C_0(X)$-algebra) only depends on the class $[\alpha] \in \mathcal{E}_G(X)$ (e.g., see [EW2, Remark 4.4]).

Suppose now that $1 \to Z \to H \to G \to 1$ is a central extension of G and let $i_H : H \to \mathcal{U}M(C^*(H))$ denote the canonical map. Then the integrated form of $i_H|_Z : Z \to \mathcal{U}M(C^*(H))$ is a non-degenerate $*$-homomorphism $\phi : C^*(Z) \to \mathcal{Z}M(C^*(H))$. Thus, if we identify $C^*(Z)$ with $C_0(\widehat{Z})$ via the Gelfand transform we see that $C^*(H)$ is a $C_0(\widehat{Z})$-algebra in a natural way.

Remark 5.1. In this paper we use the formula

$$\widehat{f}(\chi) = \int_Z f(z)\overline{\chi(z)}\, dz \tag{5.1}$$

for the Fourier transform of $f \in C_c(Z) \subseteq C^*(Z)$. Notice that this convention differs from the convention used in [EW2, Lemma 6.3], where the alternative formula $\widehat{f}(\chi) = \int_Z f(z)\chi(z)\, dz$ was used (without explicitly writing it down). This convention implies that the structure of $C^*(H)$ as a $C_0(\widehat{Z})$-algebra we use here differs from the structure obtained in [EW2, Lemma 6.3] by applying the homeomorphism $\chi \mapsto \overline{\chi}$ to the base space $\widehat{Z}$. Thus it follows from [EW2, Lemma 6.3] that our fibres $C^*(H)_\chi$ are isomorphic to $C^*(G, \operatorname{tg}(\overline{\chi}))$ (see also Remark 5.5 below).

Lemma 5.2. *Let $1 \to Z \to H \to G \to 1$ be a central extension of G, let $\mathcal{P}$ be a locally compact $\widehat{H}_{ab}$-space and let $F : \mathcal{P} \to \widehat{Z}$ be a continuous $\widehat{H}_{ab}$-equivariant map. Further, let $\tau : \widehat{H}_{ab} \to \operatorname{Aut} C_0(\mathcal{P})$ be the translation action and let $\gamma : \widehat{H}_{ab} \to \operatorname{Aut} C^*(H)$ be defined by $\gamma_\chi(\varphi)(h) = \chi(h)\varphi(h)$ for $\varphi \in C_c(H) \subseteq C^*(H)$. Then the diagonal action $\tau \otimes \gamma$ of $\widehat{H}_{ab}$ on $C_0(\mathcal{P}) \otimes C^*(H)$ factors through an action $\tau \otimes_{\widehat{Z}} \gamma : \widehat{H}_{ab} \to \operatorname{Aut}(C_0(\mathcal{P}) \otimes_F C^*(H))$.*

Proof. Let $\delta : \widehat{H}_{\mathrm{ab}} \to \operatorname{Aut} C_0(\widehat{Z})$ be the action $\delta_\chi(f)(\mu) = f(\overline{\chi}|_Z \cdot \mu)$. Then it is enough to show that the maps $F^* : C_0(\widehat{Z}) \to \mathcal{Z}M(C_0(\mathcal{P})) \cong C_b(\mathcal{P})$ and $\phi : C_0(\widehat{Z}) \to \mathcal{Z}M(C^*(H))$ are both $\widehat{H}_{\mathrm{ab}}$-equivariant, since this will imply that the closed ideal generated by

$$\{g \cdot f \otimes a - g \otimes f \cdot a : g \in C_0(\mathcal{P}), f \in C_0(\widehat{Z}), a \in C^*(H)\}$$

is $\widehat{H}_{\mathrm{ab}}$-invariant.

The equivariance of F^* follows directly from the equivariance of F. To prove the equivariance of ϕ let $g \in C_c(Z)$. Then it follows from (5.1) that

$$\delta_\chi(\widehat{g})(\mu) = \widehat{g}(\overline{\chi}|_Z \cdot \mu) = (\chi|_Z \cdot g)^\wedge(\mu).$$

Using this we compute for $g \in C_c(Z)$ and $\varphi \in C_c(H)$:

$$\begin{aligned}\gamma_\chi(\phi(\widehat{g})\varphi)(h) &= \chi(h)\int_Z g(z)\varphi(z^{-1}h)\,dz = \int_Z (\chi|_Z \cdot g)(z)(\chi\cdot\varphi)(z^{-1}h)\,dz \\ &= \big(\phi(\chi|_Z \cdot g)^\wedge \gamma_\chi(\varphi)\big)(h) = \big(\phi(\delta_\chi(g))\gamma_\chi(\varphi)\big)(h).\end{aligned}$$

Thus ϕ is $\widehat{H}_{\mathrm{ab}}$-equivariant. □

Remark 5.3. If in the above lemma $p : \mathcal{P} \to X$ is a principal $\widehat{H}_{\mathrm{ab}}$-bundle over X, then the composition of $p^* : C_0(X) \to C_b(\mathcal{P})$ composed with the canonical embedding $C_b(\mathcal{P}) \to \mathcal{Z}M(C_0(\mathcal{P}) \otimes_F C^*(H))$ turns $C_0(\mathcal{P}) \otimes_F C^*(H)$ into a $C_0(X)$-algebra. It is easy to check that the action $\tau \otimes_{\widehat{Z}} \gamma$ of the lemma is $C_0(X)$-linear. Thus the crossed product $\big(C_0(\mathcal{P}) \otimes_F C^*(H)\big) \rtimes_{\tau\otimes_{\widehat{Z}}\gamma} \widehat{H}_{\mathrm{ab}}$ carries a natural structure as a $C_0(X)$-algebra.

We now recall from the introduction what will be the main result of this section.

Theorem 5.4. *Suppose that* $1 \to Z \to H \to G \to 1$ *is a splitting extension for* G *such that* H_{ab} *is compactly generated. Let* $[\alpha] \in \mathcal{E}_G(X)$ *and let* $(\mathcal{P}, F)$ *be in the isomorphism class of equivariant pairs corresponding to* $[\alpha]$ *via Theorem 1.6. Then* $C_0(X, \mathcal{K}) \rtimes_\alpha G$ *is stably* $C_0(X)$*-isomorphic to* $(C_0(\mathcal{P}) \otimes_F C^*(H)) \rtimes_{\tau\otimes_{\widehat{Z}}\gamma} \widehat{H}_{ab}$.

Remark 5.5. Notice, that in the special situation where H is a representation group of G, the map $F : \mathcal{P} \to \widehat{Z}$ is the lift of the Mackey obstruction map $\varphi^\alpha : X \to \widehat{Z} \cong H^2(G, \mathbb{T})$ for α. Thus our result is comparable to [EW2, Theorem 6.6]. In fact, a stabilized (and hence weaker) version of that theorem can be deduced from our result above; we omit the details. Note that in [EW2, Theorem 6.6] the pull-back was taken with respect to the function $f(x) = \varphi^\alpha(x)^{-1}$. However, by our change of convention for the Fourier transform (see Remark 5.1 above) we can directly work with F here.

The proof of the theorem requires several steps. First of all it follows from Proposition 4.6 that we may work with the action $\alpha : G \to \mathrm{Aut}(\mathrm{Ind}_\beta^{\mathcal{P}} \mathcal{K})$ constructed from $(\mathcal{P}, F)$ as in that proposition. Notice that the $C_0(X)$-module action on $\mathrm{Ind}_\beta^{\mathcal{P}} \mathcal{K}$ is given via the canonical action of $C_0(X)$ on $C_0(\mathcal{P})$. We further need to work with the actions

$$\alpha^F : G \to \mathrm{Aut}\, C_0(\mathcal{P}, \mathcal{K}), \quad \tau \otimes \beta : \widehat{H}_{\mathrm{ab}} \to \mathrm{Aut}\, C_0(\mathcal{P}, \mathcal{K})$$

which we used for the construction of α (see the discussions preceding Proposition 4.6). Note that all these actions are $C_0(X)$-linear with respect to the canonical $C_0(X)$-action on $C_0(\mathcal{P}, \mathcal{K})$. Lemma 4.5 implies that α^F and $\tau \otimes \beta$ commute. Thus we get a $C_0(X)$-linear product action $\alpha^F \times (\tau \otimes \beta)$ of $G \times \widehat{H}_{\mathrm{ab}}$ on $C_0(\mathcal{P}, \mathcal{K})$. Via the usual decomposition of crossed products by product actions, we then get natural $C_0(X)$-linear isomorphisms between the crossed product $C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\alpha^F \times (\tau\otimes\beta)} (G \times \widehat{H}_{\mathrm{ab}})$ and the iterated crossed products

$$\big(C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\alpha^F} G\big) \rtimes_{(\tau\otimes\beta)^{\mathrm{dec}}} \widehat{H}_{\mathrm{ab}} \quad \text{and} \quad \big(C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}}\big) \rtimes_{\alpha^{\mathrm{dec}}} G,$$

where the decomposition actions $(\tau \otimes \beta)^{\mathrm{dec}}$ and α^{dec} are defined on the dense subalgebras $C_c(G, C_0(\mathcal{P}, \mathcal{K})) \subseteq C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\alpha^F} G$ and $C_c(\widehat{H}_{\mathrm{ab}}, C_0(\mathcal{P}, \mathcal{K})) \subseteq C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}}$ by the formulas

$$\big((\tau \otimes \beta)^{\mathrm{dec}}_\chi(f)\big)(s) = (\tau \otimes \beta)_\chi(f(s)) \quad \text{and} \quad \big(\alpha^{\mathrm{dec}}_s(g)\big)(\chi) = \alpha^F_s(g(\chi)). \tag{5.2}$$

The following proposition shows that the above crossed products are all $C_0(X)$-linearly isomorphic to $(\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}) \rtimes_\alpha G$.

Proposition 5.6. *There exists a $C_0(X)$-linear isomorphism between the crossed products $(\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}) \rtimes_\alpha G$ and $\big(C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{ab}\big) \rtimes_{\alpha^{dec}} G$.*

Proof. We actually show that there is a $C_0(X)$-Morita equivalence between the systems $(\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}, G, \alpha)$ and $\big(C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}}, G, \alpha^{\mathrm{dec}}\big)$. Since the algebras $\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$ and $C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}}$ are both separable and stable, it will follow from [CKRW, Lemma 3.1] that α and α^{dec} are outer conjugate, i.e., there exists a $C_0(X)$-linear isomorphism $\Phi : C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}} \to \operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$ which transports α^{dec} to an action which is exterior equivalent to α. But this will imply that there exists a $C_0(X)$-linear isomorphism between the crossed products.

For the construction of the $C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}} - \operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$ imprimitivity bimodule we follow the construction given in [RW, Theorem 2.2]. We write A_0 for the dense subalgebra $C_c(\widehat{H}_{\mathrm{ab}}, C_0(\mathcal{P}, \mathcal{K}))$ of $C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}}$, and set $\mathcal{X}_0 = C_c(\widehat{H}_{\mathrm{ab}} \times \mathcal{P}, \mathcal{K})$. We define left and right module actions of A_0 and $\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$ and A_0- and $\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$-valued inner products on $\mathcal{X}_0$ by

$$
\begin{aligned}
f \cdot \xi(y) &= \int_{\widehat{H}_{\mathrm{ab}}} f(\chi, y)\beta_\chi(\xi(\chi^{-1} \cdot y))\, d\chi,\\
\xi \cdot \psi(y) &= \xi(y)\psi(y),\\
\langle \xi, \eta\rangle_{\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}}(y) &= \int_{\widehat{H}_{\mathrm{ab}}} \beta_\chi\big(\xi(\chi^{-1} \cdot y)^*\eta(\chi^{-1} \cdot y)\big)\, d\chi,\\
{}_{A_0}\langle \xi, \eta\rangle(\chi, y) &= \xi(y)\beta_\chi\big(\eta(\chi^{-1} \cdot y)^*\big),
\end{aligned}
\tag{5.3}
$$

where $\xi, \eta \in \mathcal{X}_0$, $f \in A_0$ and $\psi \in \operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$. Then it follows from [RW, Theorem 2.2] that the completion $\mathcal{X}$ of $\mathcal{X}_0$ is a $C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}} - \operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$ imprimitivity bimodule. It follows directly from the above formulas, that

$$(f \cdot g) \cdot (\xi \cdot \psi) = (f \cdot \xi) \cdot (g \cdot \psi)$$

for $g \in C_0(X)$, $\xi \in \mathcal{X}_0$, $f \in A_0$ and $\psi \in \operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}$, so that $\mathcal{X}$ is a actually a $C_0(X)$-Morita equivalence.

For $s \in G$ and $\xi \in \mathcal{X}_0$ define $\gamma_s(\xi)(y) = (\alpha^F)^y_s(\xi(y))$, where $(\alpha^F)^y$ denotes the evaluation of α^F at $y \in \mathcal{P}$. Then some routine checking shows that

1. the map $s \mapsto \gamma_s(\xi)$ is inductive limit continuous for all $\xi \in \mathcal{X}_0$;
2. $\gamma_s\big(\xi \cdot {}_{A_0}\langle \eta, \zeta\rangle\big) = \gamma_s(\xi) \cdot {}_{A_0}\langle \gamma_s(\eta), \gamma_s(\zeta)\rangle$ for all $\xi, \eta, \zeta \in \mathcal{X}_0$;
3. ${}_{A_0}\langle \gamma_s(\xi), \gamma_s(\eta)\rangle = \alpha_s^{\mathrm{dec}}\big({}_{A_0}\langle \xi, \eta\rangle\big)$ and $\langle \gamma_s(\xi), \gamma_s(\eta)\rangle_{\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}} = \alpha_s\big(\langle \xi, \eta\rangle_{\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}}\big)$ for all $\xi, \eta \in \mathcal{X}_0$.

(3) implies in particular, that γ_s is norm preserving, so it extends to a linear map on $\mathcal{X}$ which by (2) is an automorphism of $\mathcal{X}$ in the sense of [C]. It follows from the formula for the A_0-valued inner product on $\mathcal{X}_0$ that the inductive limit topology on $\mathcal{X}_0$ is stronger than the norm topology. Thus (1) implies that $s \mapsto \gamma_s(\xi)$ is norm continuous for all $\xi \in \mathcal{X}$. But then (3) implies that $(\mathcal{X}, G, \gamma)$ is a Morita equivalence for $\big(\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}, G, \alpha\big)$ and $\big(C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\tau\otimes\beta} \widehat{H}_{\mathrm{ab}}, G, \alpha^{\mathrm{dec}}\big)$. This completes the proof. ∎

Corollary 5.7. *There exist $C_0(X)$-linear isomorphisms between the crossed products $(\operatorname{Ind}_\beta^{\mathcal{P}} \mathcal{K}) \rtimes_\alpha G$ and $\big(C_0(\mathcal{P}, \mathcal{K}) \rtimes_{\alpha^F} G\big) \rtimes_{(\tau\otimes\beta)^{dec}} \widehat{H}_{ab}$.*

Proof. By the discussion preceding the proposition there exists a $C_0(X)$-linear isomorphism between $\big(C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F} G\big)\rtimes_{(\tau\otimes\beta)^{\mathrm{dec}}}\widehat{H}_{\mathrm{ab}}$ and $\big(C_0(\mathcal{P},\mathcal{K})\rtimes_{\tau\otimes\beta}\widehat{H}_{\mathrm{ab}}\big)\rtimes_{\alpha^{\mathrm{dec}}} G$. But the proposition implies that the latter is $C_0(X)$-linearly isomorphic to $(\mathrm{Ind}_\beta^{\mathcal{P}}\mathcal{K})\rtimes_\alpha G$. □

The last ingredient we need for the proof of the theorem is a very precise version of the Packer-Raeburn stabilization trick for central extensions of groups.

Lemma 5.8. *Assume that $1\to Z\to H\to G\to 1$ is a central extension of second countable groups. Let $c: G\to H$ be a Borel cross-section and let $\mu\in Z^2(G, C(\widehat{Z},\mathbb{T}))$ be given by $\mu(s,t)(\chi)=\chi(c(s)c(t)c(st)^{-1})$. Further, let $\alpha^\mu: G\to \operatorname{Aut} C_0(\widehat{Z},\mathcal{K})$, $\mathcal{K}=\mathcal{K}(L^2(G))$, denote the inner action implemented by the Borel map $L_\mu: G\to C(\widehat{Z},\mathcal{U}(L^2(G)))$ as in (2.2), and let $\phi: C_0(\widehat{Z})\to M(C^*(H))$ denote the canonical embedding as in the discussion preceding Lemma 5.2. Define*

$$\Phi: C_0(\widehat{Z})\otimes\mathcal{K}\to M(C^*(H)\otimes\mathcal{K}),\quad v: G\to \mathcal{U}M(C^*(H)\otimes\mathcal{K})$$

by

$$\Phi=\phi\otimes\mathrm{id}_{\mathcal{K}}\quad\text{and}\quad v_s=\Phi(L_\mu(s))\cdot\big(i_H(c(s))\otimes 1\big). \tag{5.4}$$

Then (Φ, v) is a covariant homomorphism of $\big(C_0(\widehat{Z},\mathcal{K}), G, \alpha^\mu\big)$ into $M(C^*(H)\otimes\mathcal{K})$ such that its integrated form $\Phi\times v$ is a $C_0(\widehat{Z})$-linear isomorphism between $C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu} G$ and $C^*(H)\otimes\mathcal{K}$.

Proof. This is basically a consequence of the Packer-Raeburn stabilization trick (see [PR1, Theorem 3.4 and Theorem 4.1]). But in order to get the specific structure of the isomorphism we have to do some work. By [EW2, Proposition 4.6] we know that we have an isomorphism

$$(1\otimes_{\widehat{Z}}\mathrm{id})\times(i_G\otimes_{\widehat{Z}} L_\mu): C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu} G\to C^*(G,\widehat{Z},\overline{\mu})\otimes_{C_0(\widehat{Z})} C_0(\widehat{Z},\mathcal{K})$$

(we refer to [EW2] for the relevant notations) and by [EW2, Lemma 6.3] we get an isomorphism

$$\phi\times(i_H\circ c): C^*(G,\widehat{Z},\overline{\mu})\to C^*(H),$$

where $\phi: C_0(\widehat{Z})\to M(C^*(H))$ is the map considered above (note that we have to use $\overline{\mu}$ instead of μ, which was used in [EW2, Lemma 6.3], by our convention for the Fourier-transform). Combining these isomorphisms, we get an isomorphism

$$\big((\phi\times(i_H\circ c))\otimes_{\widehat{Z}}\mathrm{id}\big)\circ\big((1\otimes_{\widehat{Z}}\mathrm{id})\times(i_G\otimes_{\widehat{Z}} L_\mu)\big)$$

from $C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu} G$ to $C^*(H)\otimes_{C_0(\widehat{Z})} C_0(\widehat{Z},\mathcal{K})\cong C^*(H)\otimes\mathcal{K}$. Following closely what it does to the canonical images of $C_0(\widehat{Z},\mathcal{K})$ and G in $M\big(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu} G\big)$, we see that this isomorphism coincides with the integrated form $\Phi\times v$ of (Φ, v) as defined in the lemma. Since Φ is $C_0(\widehat{Z})$-linear, the same is true for $\Phi\times v$. □

Proof of Theorem 5.4. Using Corollary 5.7, the proof follows if we can show that $\big(C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F} G\big)\rtimes_{(\tau\otimes\beta)^{\mathrm{dec}}}\widehat{H}_{\mathrm{ab}}$ is stably $C_0(X)$-isomorphic to $\big(C_0(\mathcal{P})\otimes_F C^*(H)\big)\rtimes_{\tau\otimes_{\widehat{Z}}\gamma}\widehat{H}_{\mathrm{ab}}$.

As a first step we show that $C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F} G$ is $C_0(X)$-linearly isomorphic to $\big(C_0(\mathcal{P})\otimes_F C^*(H)\big)\otimes\mathcal{K}$, with $\mathcal{K}=\mathcal{K}(L^2(G))$. For this let $\Phi\times v: C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu} G\to$

$C^*(H)\otimes\mathcal{K}$ be the isomorphism of Lemma 5.8. Since $\Phi\times v$ is $C_0(\widehat{Z})$-linear, it follows that

$$\big(C_0(\mathcal{P})\otimes_F C^*(H)\big)\otimes\mathcal{K}\cong C_0(\mathcal{P})\otimes_F\big(C^*(H)\otimes\mathcal{K}\big)$$

is naturally isomorphic to the pull-back $C_0(\mathcal{P})\otimes_F\big(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\big)$ as a $C_0(\mathcal{P})$-algebra.

The action $\alpha^F:G\to\operatorname{Aut}(C_0(\mathcal{P},\mathcal{K}))$ is by definition the pull-back of the action $\alpha^\mu:G\to\operatorname{Aut}C_0(\widehat{Z},\mathcal{K})$, i.e., it can be identified with the diagonal action $\mathrm{id}\otimes_{\widehat{Z}}\alpha^\mu$ on $C_0(\mathcal{P})\otimes_F C_0(\widehat{Z},\mathcal{K})\cong C_0(\mathcal{P},\mathcal{K})$. Thus we can apply [EW2, Proposition 4.3] to see that $C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F}G$ is isomorphic to $C_0(\mathcal{P})\otimes_F\big(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\big)$ as a $C_0(\mathcal{P})$-algebra (since the C^*-algebra part of the isomorphism given in [EW2, Proposition 4.3] is $C_0(\mathcal{P})$-linear). Combining this with the above result we get a $C_0(\mathcal{P})$-linear (and hence also a $C_0(X)$-linear) isomorphism between $C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F}G$ and $\big(C_0(\mathcal{P})\otimes_F C^*(H)\big)\otimes\mathcal{K}$.

For the second step we claim that the above isomorphism carries the action $(\tau\otimes\beta)^{\mathrm{dec}}$ of $\widehat{H}_{\mathrm{ab}}$ on the first algebra to the action $(\tau\otimes_{\widehat{Z}}\gamma)\otimes\beta$ on the second algebra. For this recall that we have an action $\delta:\widehat{H}_{\mathrm{ab}}\to\operatorname{Aut}C_0(\widehat{Z})$ defined by $\big(\delta_\chi\psi\big)(\nu)=\psi(\overline{\chi}|_Z\cdot\nu)$. The same arguments as used in Lemma 4.5 show that the action $\delta\otimes\beta$ of $\widehat{H}_{\mathrm{ab}}$ on $C_0(\mathcal{P},\mathcal{K})$ commutes with α^μ. Thus $\delta\otimes\beta$ induces the decomposition action $(\delta\otimes\beta)^{\mathrm{dec}}$ of $\widehat{H}_{\mathrm{ab}}$ on $C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G$. The isomorphism between $C_0(\mathcal{P})\otimes_F\big(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\big)$ and $C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F}G$ is given by first identifying $C_0(\mathcal{P})\otimes_F\big(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\big)$ with $\big(C_0(\mathcal{P})\otimes_F C_0(\widehat{Z},\mathcal{K})\big)\rtimes_{\mathrm{id}\otimes_{\widehat{Z}}\alpha^\mu}G$ and then identifying the latter with $C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F}G$ via the isomorphism of the crossed products which is induced by the canonical $\mathrm{id}\otimes_{\widehat{Z}}\alpha^\mu-\alpha^F$-equivariant isomorphism

$$C_0(\mathcal{P})\otimes_F C_0(\widehat{Z},\mathcal{K})\cong\big(C_0(\mathcal{P})\otimes_F C_0(\widehat{Z})\big)\otimes\mathcal{K}\cong C_0(\mathcal{P})\otimes\mathcal{K}.$$

(The isomorphism $C_0(\mathcal{P})\otimes_F C_0(\widehat{Z})\to C_0(\mathcal{P})$ is given by $f\otimes g\mapsto f\cdot g$.) Having this picture, it is straightforward to check that the isomorphism $C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F}G\cong C_0(\mathcal{P})\otimes_F\big(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\big)$ carries $(\tau\otimes\beta)^{\mathrm{dec}}$ to the action $\tau\otimes_{\widehat{Z}}(\delta\otimes\beta)^{\mathrm{dec}}$.

Furthermore, the claim will follow if we can show that $\Phi\times v:C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\to C^*(H)\otimes\mathcal{K}$ carries $(\delta\otimes\beta)^{\mathrm{dec}}$ to $\gamma\otimes\beta$. For this it suffices to show that

(1) $(\gamma\otimes\beta)_\chi(\Phi(\psi))=\Phi\big((\delta\otimes\beta)_\chi(\psi)\big)$ for all $\psi\in C_0(\widehat{Z},\mathcal{K})$, and
(2) $(\gamma\otimes\beta)_\chi(v_s)=v_s$ for all $s\in G$.

(The second requirement comes from the equation

$$\Phi\times v\big((\delta\otimes\beta)^{\mathrm{dec}}_\chi(i_G(s))\big)=\Phi\otimes v(i_G(s))=v_s,$$

where $i_G:G\to M\big(C_0(C_0(\widehat{Z},\mathcal{K})\rtimes_{\alpha^\mu}G\big)$ denotes the canonical map.)

But (1) follows directly from the definition of Φ and the $\delta-\gamma$ equivariance of the embedding $\phi:C_0(\widehat{Z})\to M(C^*(H))$ (see the proof of Lemma 5.2). In order to check (2) we first note that it follows from Lemma 2.3 and the computations given in the proof of Lemma 4.5 that for $s\in G$, $\chi\in\widehat{H}_{\mathrm{ab}}$ and $\sigma\in\widehat{Z}$ we get

$$(\delta\otimes\beta)_\chi(L_\mu(s))(\sigma)=V_\chi\big(L_{\mu(\overline{\chi}|_Z\cdot\sigma)}(s)\big)V_\chi^*=\overline{\chi}(c(s))L_\mu(s)(\sigma).\tag{5.5}$$

Using this, the equivariance of Φ, and the definition of γ (see Lemma 5.2) we get

$$\begin{aligned}(\gamma\otimes\beta)_\chi(v_s) &= (\gamma\otimes\beta)_\chi\big(\Phi(L_\mu(s))\cdot(i_H(c(s))\otimes 1)\big)\\ &= \Phi\big((\delta\otimes\beta)_\chi(L_\mu(s))\big)\cdot(\gamma\otimes\beta)_\chi(i_H(c(s))\otimes 1)\\ &\overset{(5.5)}{=} \Phi(\overline{\chi}(c(s))L_\mu(s))\cdot\chi(c(s))(i_H(c(s))\otimes 1)\\ &= \Phi(L_\mu(s))\cdot(i_H(c(s))\otimes 1) = v_s.\end{aligned}$$

This proves the claim.

It follows now that $\big(C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F} G\big)\rtimes_{(\tau\otimes\beta)^{\mathrm{dec}}}\widehat{H}_{\mathrm{ab}}$ is $C_0(X)$-linearly isomorphic to

$$\big((C_0(\mathcal{P})\otimes_F C^*(H))\otimes\mathcal{K}\big)\rtimes_{(\tau\otimes_{\widehat{Z}}\gamma)\otimes\beta}\widehat{H}_{\mathrm{ab}}.$$

Since β is implemented by a strongly continuous homomorphism $V:\widehat{H}_{\mathrm{ab}}\to\mathcal{U}(L^2(G))$, it follows that it is exterior equivalent to the trivial action $\mathrm{id}_{\mathcal{K}}$. But this implies that $(\tau\otimes_{\widehat{Z}}\gamma)\otimes\beta$ is exterior equivalent to $(\tau\otimes_{\widehat{Z}}\gamma)\otimes\mathrm{id}_{\mathcal{K}}$. Since exterior equivalent $C_0(X)$-actions have isomorphic $C_0(X)$-crossed products it follows that

$$\begin{aligned}&\big(C_0(\mathcal{P},\mathcal{K})\rtimes_{\alpha^F} G\big)\rtimes_{(\tau\otimes\beta)^{\mathrm{dec}}}\widehat{H}_{\mathrm{ab}}\\ &\qquad\cong \big((C_0(\mathcal{P})\otimes_F C^*(H))\otimes\mathcal{K}\big)\rtimes_{(\tau\otimes_{\widehat{Z}}\gamma)\otimes\beta}\widehat{H}_{\mathrm{ab}}\\ &\qquad\cong \big((C_0(\mathcal{P})\otimes_F C^*(H))\otimes\mathcal{K}\big)\rtimes_{(\tau\otimes_{\widehat{Z}}\gamma)\otimes\mathrm{id}_{\mathcal{K}}}\widehat{H}_{\mathrm{ab}}\\ &\qquad\cong \big((C_0(\mathcal{P})\otimes_F C^*(H))\rtimes_{\tau\otimes_{\widehat{Z}}\gamma}\widehat{H}_{\mathrm{ab}}\big)\otimes\mathcal{K},\end{aligned}$$

where all isomorphism are $C_0(X)$-linear. This finishes the proof. □

If A is a separable and stable continuous-trace C^*-algebra with spectrum X, and $\alpha: G\to\operatorname{Aut}A$ is $C_0(X)$-linear, then it follows from the canonical splitting $\mathrm{Br}_G(X)=H^3(X,\mathbb{Z})\oplus\mathcal{E}_G(X)$ (e.g., see the discussions preceding [EW1, Corollary 5.5]) that there exists an element $[\beta]\in\mathcal{E}_G(X)$ such that α is exterior equivalent to the action $\mathrm{id}\otimes_X\beta$ of G on $A\otimes_{C_0(X)}C_0(X,\mathcal{K})\cong A$. It follows then from [EW2, Proposition 4.3] that $A\rtimes_\alpha G$ is isomorphic to $A\otimes_{C_0(X)}\big(C_0(X,\mathcal{K})\rtimes_\beta G\big)$. Thus we get

Corollary 5.9. *Assume that G has a splitting extension $1\to Z\to H\to G\to 1$ such that $\widehat{H}_{ab}$ is compactly generated. Let $\alpha: G\to\operatorname{Aut}A$ be a $C_0(X)$-linear action of G on the continuous-trace algebra A with spectrum $\widehat{A}=X$ and let $[\beta]\in\mathcal{E}_G(X)$ such that α is exterior equivalent to $\mathrm{id}\otimes_X\beta$. Then $A\rtimes_\alpha G$ is isomorphic to*

$$A\otimes_{C_0(X)}\big((C_0(\mathcal{P})\otimes_F C^*(H))\rtimes_{\tau\otimes_{\widehat{Z}}\gamma}\widehat{H}_{ab}\big),$$

where $(\mathcal{P},F)$ is an equivariant pair corresponding to $[\beta]$.

We are now going to investigate the dual space of

$$C_0(X,\mathcal{K})\rtimes_\alpha G\cong\big(C_0(\mathcal{P})\otimes_F C^*(H)\big)\rtimes_{\tau\otimes_{\widehat{Z}}\gamma}\widehat{H}_{\mathrm{ab}}.$$

Recall that if A is a C^*-algebra, then the *dual space* $\widehat{A}$ of A is the set of equivalence classes of irreducible representations of A equipped with the Jacobsen topology (see [D, Chapter 3]). If A is a $C_0(Y)$-algebra, then each irreducible representation factors through an irreducible representation of some unique fibre A_y and the resulting projection $q:\widehat{A}\to Y$ is continuous. As we shall see, the following general result is a consequence of Green's version of the Mackey machine [G, Theorem 17] for crossed products.

Proposition 5.10. *Suppose that (A, G, α) is a C^*-dynamical system such that A is a $C_0(Y)$-algebra for some free and proper G-space Y. Suppose further that the embedding $\phi : C_0(Y) \to \mathcal{Z}M(A)$ is G-equivariant. Then $(A \rtimes_\alpha G)^\wedge$ is homeomorphic to $\widehat{A}/G$, where the action of G on $\widehat{A}$ is given by $s \cdot \pi = \pi \circ \alpha_{s^{-1}}$, $s \in G, \pi \in \widehat{A}$.*

Proof. Let $\pi \in \widehat{A}$. Then we claim that the induced representation $\operatorname{Ind}_{\{e\}}^G \pi$ of $A \rtimes_\alpha G$ (in the sense of Green [G]) is irreducible, and that the resulting map

$$\operatorname{Ind} : \widehat{A} \to (A \rtimes_\alpha G)^\wedge; \pi \mapsto \operatorname{Ind}_{\{e\}}^G \pi$$

is surjective.

For this let $X = Y/G$. Then A becomes a $C_0(X)$-algebra in a canonical way such that the action $\alpha : G \to \operatorname{Aut} A$ is $C_0(X)$-linear. Thus each irreducible representation of A factors through a unique fibre A_x and it follows from [G, Proposition 12] that the induced representation $\operatorname{Ind}_{\{e\}}^G \pi$ factors through the corresponding induced representation of $A_x \rtimes_{\alpha^x} G$. Thus we may as well assume that $Y = G$ with action of G on Y given by translation. Let A_e be the fibre of A over the unit $e \in G$. Then it follows from [G, Theorem 17] that induction of representations of A_e to $A \rtimes_\alpha G$ gives a homeomorphism between $\widehat{A}_e$ and $(A \rtimes_\alpha G)^\wedge$. In particular, $\operatorname{Ind}_{\{e\}}^G \pi$ is irreducible for all $\pi \in \widehat{A_e}$. Since each irreducible representation of A is conjugate to a (unique) irreducible representation of A_e, the claim follows from the fact that conjugate representations induce to equivalent representations of $A \rtimes_\alpha G$.

The above reasoning also implies that the map $\operatorname{Ind} : \widehat{A} \to (A \rtimes_\alpha G)^\wedge$ factors through a continuous bijection $I : \widehat{A}/G \to (A \rtimes_\alpha G)^\wedge$. The openness then follows from the continuity of restricting representations (e.g., combine [E, Proposition 1.4.6] with [E, Proposition 1.4.8]). $\square$

If $(\mathcal{P}, F)$ is a covariant pair corresponding to a $C_0(X)$-linear action $\beta : G \to \operatorname{Aut} C_0(X, \mathcal{K})$, then the natural embedding of $C_0(\mathcal{P})$ into $M(C_0(\mathcal{P}) \otimes_F C^*(H))$ is clearly $\widehat{H}_{\mathrm{ab}}$ equivariant with respect to the diagonal action $\tau \otimes_{\widehat{Z}} \gamma$ of $\widehat{H}_{\mathrm{ab}}$ on $C_0(\mathcal{P}) \otimes_F C^*(H)$. Thus the above proposition implies that the spectrum $\big((C_0(\mathcal{P}) \otimes_F C^*(H)) \rtimes_{\tau \otimes_{\widehat{Z}} \gamma} \widehat{H}_{\mathrm{ab}}\big)^\wedge$ is homeomorphic to $\big(C_0(\mathcal{P}) \otimes_F C^*(H)\big)^\wedge / \widehat{H}_{\mathrm{ab}}$. But it is shown in [RW] that $\big(C_0(\mathcal{P}) \otimes_F C^*(H)\big)^\wedge$ is homeomorphic to the set

$$\mathcal{P} \times_F \widehat{H} = \{(z, U) \in \mathcal{P} \times \widehat{H} : F(z) = q(U)\},$$

where $q : \widehat{H} \to \widehat{Z}$ denotes the projection corresponding to our embedding $\phi : C_0(\widehat{Z}) \to \mathcal{Z}M(C^*(H))$ (here, and in the rest of the paper, $\widehat{H} \cong C^*(H)^\wedge$ denotes the set of equivalence classes of irreducible unitary representations of H). By our convention on the Fourier transform on Z, we get $q(U) = \operatorname{res}(U)^{-1}$, where $\operatorname{res} : \widehat{H} \to \widehat{Z}$ is the restriction map (i.e., $\operatorname{res} U$ denotes the unique character χ of Z which satisfies the equation $U(z) = \chi(z)U(e)$ for all $z \in Z$). Thus we get

$$\mathcal{P} \times_F \widehat{H} = \{(z, U) \in \mathcal{P} \times \widehat{H} : F(z)^{-1} = \operatorname{res} U\},$$

Moreover, since $U \circ \gamma_{\overline{\chi}}(f) = (\overline{\chi} \otimes U)(f)$, the corresponding action of $\widehat{H}_{\mathrm{ab}}$ on $\mathcal{P} \times_F \widehat{H}$ is given by

$$\chi \cdot (z, U) = (\chi \cdot z, \overline{\chi} \otimes U).$$

Thus, as a consequence of Theorem 5.4 and Proposition 5.10 we get

Corollary 5.11. *Assume that* $1 \to Z \to H \to G \to 1$ *is a splitting extension of* G *such that* $\widehat{H}_{ab}$ *is compactly generated. Assume that* $(\mathcal{P}, F)$ *is an equivariant pair corresponding to a* $C_0(X)$*-linear action* $\alpha : G \to \operatorname{Aut} C_0(X, \mathcal{K})$. *Then* $\big(C_0(X, \mathcal{K}) \rtimes_\beta G\big)^\wedge$ *is homeomorphic to* $(\mathcal{P} \times_F \widehat{H})/\widehat{H}_{ab}$.

Of course, in view of Corollary 5.9 we get an analogous result for any $C_0(X)$-linear action of G on a separable continuous trace algebra with spectrum X: If $\alpha : G \to \operatorname{Aut} A$ is such action, then by passing to the action $\alpha \otimes \mathrm{id}$ of G on $A \otimes \mathcal{K}$ we may assume without loss of generality that A is stable. We then use the fact that

$$(A \otimes_{C_0(X)} B)^\wedge \cong \widehat{A} \times_X \widehat{B} = X \times_X \widehat{B} = \widehat{B}$$

for any $C_0(X)$-algebra B (see [EW2, Remark 2.5]).

6. Appendix: Locally inner actions on $\mathcal{CR}(X)$-algebras

In [EW1, EW2] the authors considered locally inner actions on $\mathcal{CR}(X)$-algebras, i.e., on C^*-algebras whose primitive ideal space $\mathrm{Prim}(A)$ has complete regularization X, where X is a given second countable locally compact Hausdorff space (we refer to [EW1] for the details). Let us denote by $\mathcal{LI}_G(A)$ the set of exterior equivalence classes of locally inner G-actions on a given $\mathcal{CR}(X)$-algebra A. If A is stable, $\beta \in \mathcal{E}_G^{li}(X)$, and $\alpha \in \mathcal{LI}_G(A)$, then $\beta \otimes_X \alpha$ is again a locally inner action of G on $C_0(X, \mathcal{K}) \otimes_{C_0(X)} A \cong A$, and it is a consequence of [EW1, Lemma 3.5] that this induces a well-defined map

$$\mathcal{E}_G^{li}(X) \times \mathcal{LI}_G(A) \to \mathcal{LI}_G(A); \quad ([\beta], [\alpha]) \mapsto [\beta \otimes_X \alpha].$$

The following result generalizes [EW1, Theorem 6.3] (at least if A is stable).

Proposition 6.1. *Suppose that* A *is a stable* $\mathcal{CR}(X)$*-algebra. Then the map* $\mathcal{E}_G^{li}(X) \to \mathcal{LI}_G(A), [\beta] \mapsto [\beta \otimes_X \mathrm{id}]$ *is a bijection.*

Proof. If A is a $\mathcal{CR}(X)$-algebra and $\alpha : G \to \operatorname{Aut} A$ is locally inner (in the sense of [EW2, Definition 2.8]), then α is locally implemented by Borel maps $\pi_i : G \to UM(A_{U_i})$ and the pair $(\check{\partial}\pi, \partial_G\pi)$ gives a cocycle in $\boldsymbol{Z}^2(G, X)$ (using Lemma 2.6 and Lemma 3.1 of [EW2], this follows from the same arguments as used for the special case $A = C_0(X, \mathcal{K})$ as considered in §2). Moreover, the same arguments as used in the proof of Proposition 2.2 show that the class $[(\check{\partial}\pi, \partial_G\pi)] \in \mathbb{H}^2(G, X)$ only depends on the exterior equivalence class of α, that α is exterior equivalent to the trivial action if and only if $[(\check{\partial}\pi, \partial_G\pi)]$ is trivial, and that the class corresponding to $[\beta \otimes_X \alpha]$ is the product of the classes corresponding to $[\beta] \in \mathcal{E}_G^{li}(X)$ and $[\alpha] \in \mathcal{LI}_G(A)$. From this and Proposition 2.2 one can easily deduce that the map $[\beta] \mapsto [\beta \otimes_X \mathrm{id}]$ is injective.

To see surjectivity, let $[\alpha] \in \mathcal{LI}_G(A)$ be given. Then it follows from Proposition 2.2 and Proposition 2.4 that there exists a unique class $[\beta] \in \mathcal{E}_G^{li}(X)$ such that $[\beta]$ and $[\alpha]$ determine the same class in $\mathbb{H}^2(G, X)$. Let β^{-1} denote a representative of $[\beta]^{-1} \in \mathcal{E}_G^{li}(X)$. Then the class in $\mathbb{H}^2(G, X)$ corresponding to $[\beta^{-1} \otimes_X \alpha]$ is trivial, and therefore $\beta^{-1} \otimes_X \alpha$ is exterior equivalent to the trivial action. Thus we get

$$[\beta \otimes_X \mathrm{id}] = [\beta \otimes_X (\beta^{-1} \otimes_X \alpha)] = [(\beta \otimes_X \beta^{-1}) \otimes_X \alpha] = [\mathrm{id} \otimes_X \alpha] = [\alpha].$$

$\square$

If G has a splitting extension $1 \to Z \to H \to G \to 1$ such that H_{ab} is compactly generated, the above proposition combined with Theorem 5.4 provides the following description of the crossed product $A \rtimes_\alpha G$ in terms of equivariant pairs.

Proposition 6.2. *Assume that G is as above, let A be a (stable) $\mathcal{CR}(X)$-algebra and assume that $\alpha : G \to \operatorname{Aut} A$ is locally inner. Let $[\beta] \in \mathcal{E}_G(X)$ such that α is exterior equivalent to $\mathrm{id}_A \otimes_X \beta$ and let $(\mathcal{P}, F)$ be an equivariant pair corresponding to $[\beta]$. Then $A \rtimes_\alpha G$ is $C_0(X)$-isomorphic to*

$$A \otimes_{C_0(X)} \big((C_0(\mathcal{P}) \otimes_F C^*(H)) \rtimes_{\tau \otimes_{\widehat{Z}} \gamma} \widehat{H}_{ab}\big).$$

Proof. Using the results mentioned in the discussion above, the proof follows from Theorem 5.4 and

$$A \rtimes_\alpha G \cong \big(A \otimes_{C_0(X)} C_0(X, \mathcal{K})\big) \rtimes_{\mathrm{id} \otimes_X \beta} G \cong A \otimes_{C_0(X)} \big(C_0(X, \mathcal{K}) \rtimes_\beta G\big),$$

where all isomorphisms are $C_0(X)$-linear. □

Of course, if A is a $\mathcal{CR}(X)$-algebra which is non-stable, we can stabilize on both sides to get a similar result.

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