SELF-TILTING COMPLEXES YIELD UNSTABLE MODULES

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ABSTRACT. Let G be a group and R a commutative ring. Let $TrPic_R(RG)$ be the group of isomorphism classes of standard self-equivalences of the derived category of bounded complexes of RG-modules. The subgroup $HD_R(G)$ of $TrPic_R(RG)$ consisting of self-equivalences fixing the trivial RG-module acts on the cohomology ring $H^*(G, R)$. The action is functorial with respect to R. The self-equivalences which are 'splendid' in a sense defined by J. Rickard act naturally with respect to transfer and restriction to centralizers of p-subgroups in case R is a field of characteristic p. In the present paper we prove that this action of self-equivalences on $H^*(G, R)$ commutes with the action of the Steenrod algebra, and study the behaviour of the action of splendid self-equivalences with respect to Lannes' T-functor.

Introduction

In an earlier joint paper [9] with Raphaël Rouquier I defined $TrPic_R(A)$, the group of standard self-equivalences of a derived module category $D^b(A)$ for an R-algebra A which is projective as an R-module. For any A-module M let $HD_M(A)$ be the subgroup of $TrPic_R(A)$ which is formed by the self-equivalences mapping M to an isomorphic copy. Then, in an earlier paper [11] I showed that, under a certain hypothesis on M, the group $HD_M(A)$ acts in a natural way on the Ext-algebra $Ext_A^*(M, M)$. When A is a group algebra RG, with R being a field of characteristic p and G being a finite group, J. Rickard defined in [8] a *splendid equivalence* by some technical conditions, basically by asking that the homogeneous components of a tilting complex be p-permutation modules induced from diagonal p-subgroups, and by an invertibility condition in the homotopy category. These splendid equivalences induce self-equivalences of the derived categories of centralizers of p-subgroups by the Brauer construction. In [12] I showed that then, for $M = R$ being the trivial module, the actions of those splendid equivalences commute with transfer and restriction from and to the cohomology rings of centralizers of p-subgroups. In the present paper we enlarge these properties still further. The action of self-equivalences of the bounded derived category $D^b(RG)$ on $H^*(G, R)$ commutes with the action of the Steenrod algebra on $H^*(G, R)$ for any prime p.

As a consequence of the above statements, any cohomology ring $H^*(G, \mathbb{F}_p)$ defines a functor $H^*(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} HD_{\mathbb{F}_p}(B_0(\mathbb{F}_pG)) -$ from the modules over the group of derived self-equivalences of the principal block of the group ring $\mathbb{F}_p HD_{\mathbb{F}_p}(\mathbb{F}_pG)$ fixing the trivial module to the category of unstable modules $\mathfrak{U}_p$ and similarly in the opposite direction. Obviously, we may restrict to splendid self-equivalences.

We shall describe the composition of Lannes' T-functor with the first functor and the image of free unstable modules by the second. Moreover, by a result of Lannes $T_V(H^*(G, \mathbb{F}_p))$ decomposes into a direct product of cohomology rings as unstable modules. We shall prove that this decomposition is also a decomposition as modules over the action of splendid self-equivalences. This will give evidence that splendid equivalences are the correct objects to study in this context.

The paper is organized as follows. Section 1 recalls the necessary definitions and properties of Evens' norm map as it is used here and the definition of the Steenrod operation. In Section 2 it is shown that the normalized part of the outer automorphism group of the group ring behaves well with respect to the norm map $norm_G^{G \times C_p}$ on the cohomology ring of G. Section 3 proves that splendid self-equivalences of the derived category of a group ring fixing the trivial module act equivariantly with respect to the Steenrod operation on the cohomology ring of the group. Finally, in Section 4 we work on the relation of unstable modules and the action of splendid self-equivalences of the derived category of a group ring.

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1. Preliminary definitions and known results

1.1. **Equivalences between derived categories of group rings.** The notation concerning derived categories and derived equivalences we use here and throughout the whole article is the one in [5]. Let R be a commutative ring and let A and B be R-algebras. A complex T is a *tilting complex* if it is a bounded complex with finitely generated projective homogeneous components so that $Hom_{D^b(A)}(T, T[n]) = 0$ for $n \neq 0$ and so that the smallest triangulated subcategory of $D^b(A)$ which is closed under taking direct summands of finite direct sums and which contains T also contains the rank 1 free module A. The fundamental result of Rickard [7] shows that the derived categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories if and only if there is a tilting complex T in $D^b(A)$ so that $End_{D^b(A)}(T) \simeq B$.

If A is flat as an R-module, then for any tilting complex T there is a complex X in $D^b(A \otimes_R B^{op})$ so that $X \otimes_B^{\mathbb{L}} -$ is an equivalence which maps B to T. A complex X in $D^b(A \otimes_R B^{op})$ so that $X \otimes_B^{\mathbb{L}} -$ is an equivalence is called a *two-sided tilting complex*. Suppose $A = B$ and A is projective as an R-module. Then (cf. [9]), the set $TrPic_R(A)$ of isomorphism classes of two-sided tilting complexes in $D^b(A \otimes_R A^{op})$ is a group with group law $- \otimes_A^{\mathbb{L}} -$.

For any A-module M set

$$HD_M(A) := \{[X] \in TrPic_R(A) \mid X \otimes_A^{\mathbb{L}} M \simeq M\}.$$

If any automorphism of M is given by multiplication with a unit in the centre of A, then in [11] it is shown that $Ext_A^*(M, M)$ is an $R\, HD_M(A)$-module by the action

$$Hom_{D^b(A)}(M, M[n]) \longrightarrow Hom_{D^b(A)}(X \otimes_A^{\mathbb{L}} M, X \otimes_A^{\mathbb{L}} M[n])$$

$$\overset{(\alpha_X^{-1})^*(\alpha_X)_*}{\longrightarrow} Hom_{D^b(A)}(M, M[n])$$

where $\alpha_X : X \otimes_A^{\mathbb{L}} M \longrightarrow M$ is an isomorphism.

In case $A = RG$ is a group ring for a finite group G, a more subtle notion is useful. Let R be a field of characteristic p or a complete discrete valuation ring of characteristic 0 with residue field of characteristic p. Let B be the principal block of RG and let P be a Sylow p-subgroup of G. Let Δ be the diagonal embedding of G into $G \times G$. A two-sided tilting complex X in $D^b(B \otimes B^{op})$ is a *splendid complex* (cf. Rickard [8]) if X is a complex of modules whose homogeneous components are projective as right RG-modules and projective as left RG-modules, so that $Hom^{\bullet}_{RG}(X, X)$ is homotopy equivalent to B as a complex of left and as a complex of right modules, and so that each of the homogeneous components of X are $\Delta(P)$-projective p-permutation modules. Let $SplenPic_R(G)$ be the set of homotopy equivalence classes (X) of splendid complexes X whose isomorphism class is in $TrPic_R(B)$. Let

$$HSplen_R(G) := \{(X) \in SplenPic_R(G) \mid [X] \in HD_R(B)\}.$$

Suppose that R is a field of characteristic p. Then, for any p-subgroup Q of G, denote by b the principal block of $RC_G(Q)$. The Brauer functor

$$B \otimes B^{op} - mod \ni N \mapsto N(\Delta Q) := N^{\Delta Q} / \sum_{S < Q;\ S \neq Q} N^{\Delta S} \in b \otimes b^{op} - mod$$

induces a homomorphism

$$SplenPic_R(G) \longrightarrow SplenPic_R(C_G(Q)).$$

In [12] it is shown that the action of an X in $HSplen_R(G)$ so that $X(\Delta(Q)) \in HSplen_R(C_G(Q))$ on $H^*(G, R)$ commutes with restriction $res^G_{C_G(Q)}$ and transfer $tr^G_{C_G(Q)}$ between $H^*(G, R)$ and $H^*(C_G(Q), R)$.

1.2. Steenrod operations. Let G be a group and let H be a subgroup of G of finite index. Then, $G \leq H \wr \mathfrak{S}_{|G:H|}$ via the action of G on the cosets modulo H. Suppose R is a commutative, hereditary coefficient ring.

We recall below the definitions of Evens' norm map and Steenrod operations from Benson's book [1].

Let $\mathbf{P} = (P_i, d_i)_{i \in \mathbb{N}}$ be a projective resolution of R as RH-module. Then, $\bigotimes_{i=1}^{|G:H|} \mathbf{P}$ is a (not necessarily projective) resolution of R as $R(H \wr \mathfrak{S}_{|G:H|})$-module. Let $\mathbf{Q} = (Q_i, \partial_i)_{i \in \mathbb{N}}$ be a projective resolution of R as $R(H \wr \mathfrak{S}_{|G:H|})$-module. Then, there is a quasiisomorphism of $R(H \wr \mathfrak{S}_{|G:H|})$-module complexes $\tau : \mathbf{Q} \longrightarrow \bigotimes_{i=1}^{|G:H|} \mathbf{P}$.

We denote by R_s the sign representation of $\mathfrak{S}_{|G:H|}$, and by $R_s^{(r)}$ the r-fold tensor product of R_s with itself.

An element $x \in H^n(H, M)$ is given as an equivalence class of homomorphisms with representative $x : P_n \longrightarrow M$, and this defines a homomorphism $\bigotimes_{i=1}^{|G:H|} P_n \longrightarrow \bigotimes_{i=1}^{|G:H|} M$ of $R(H \wr \mathfrak{S}_{|G:H|})$-modules. As $\bigotimes_{i=1}^{|G:H|} P_n$ is a direct factor of the degree $|G:H| \cdot n$-homogeneous component of $\bigotimes_{i=1}^{|G:H|} \mathbf{P}$, the cocycle x defines a homomorphism

$$Q_{n \cdot |G:H|} \longrightarrow (\bigotimes_{i=1}^{|G:H|} \mathbf{P})_{n \cdot |G:H|} \longrightarrow \bigotimes_{i=1}^{|G:H|} P_n \longrightarrow \bigotimes_{i=1}^{|G:H|} M \otimes R_s^{(n)}\ .$$

We have to tensor with $R_s^{(n)}$ in order to get the mapping well defined with respect to the sign convention for tensor products of complexes. Observe that for even degree cohomology this is the tensor product with the identity. The restriction to

$G \subseteq (H \wr \mathfrak{S}_{|G:H|})$ defines an element $norm_H^G(x)$ in $H^{n|G:H|}(G, \bigotimes_{i=1}^{|G:H|} M \otimes_R R_s^{(n)})$. However, the sign of $norm_H^G(x)$ depends on the embedding of G into $H \wr \mathfrak{S}_{|G:H|}$. Since later on the embedding will be fixed once and for all, this ambiguity will not be of importance for us.

The Steenrod operations are defined by the following procedure.

Observe that, when p is odd, any p-cycle is of even parity, and so one can forget the sign representation which had to be introduced earlier in order to get the norm map well defined. We take

$$norm_G^{G \times C_p} : H^n(G, \mathbb{F}_p) \longrightarrow H^{n \cdot p}(G \times C_p, \mathbb{F}_p) \simeq \bigoplus_{i=0}^{np} H^{np-i}(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^i(C_p, \mathbb{F}_p)$$

and express

$$norm_G^{G \times C_p}(x) = \bigoplus_{i=0}^{np} D^{np-i}(x) \otimes a_i,$$

where $a_{2i} = a_1^{2i}$ and $a_{2i+1} = \beta(a_1)a_{2i}$. We define

$$Sq^i := D^{n-i} : H^n(G, \mathbb{F}_2) \longrightarrow H^{n+i}(G, \mathbb{F}_2)$$

if $p = 2$ and

$$P^i := (-1)^{i+mn(n+1)/2}(m!)^{-n} D^{(p-1)(n-2i)} : H^n(G, \mathbb{F}_p) \longrightarrow H^{n+2(p-1)i}(G, \mathbb{F}_p)$$

with $m = \frac{p-1}{2}$ for p odd.

We will have to use a more diagrammatic method to express the above in order to be able to prove that the Steenrod operations commute with the action of self-equivalences of the derived category.

As above, let $\mathbf{P}$ be a projective resolution of R as RG-module and let $\mathbf{Q}$ be a projective resolution of R as $RG \wr \mathfrak{S}_p$-module. As usual, there is a mapping, unique up to homotopy, $\mathbf{Q} \xrightarrow{\tau} \bigotimes_{1 \leq i \leq p} \mathbf{P}$, lifting the identity on R. Then,

$$Hom_{R(G^p)}(\bigotimes_{1 \leq i \leq p} \mathbf{P}, (\bigotimes_{1 \leq i \leq p} M)[p \cdot n]) \simeq Hom_{R(G^p)}(\bigotimes_{1 \leq i \leq p} \mathbf{P}, \bigotimes_{1 \leq i \leq p} M[n])$$

$$\simeq \bigotimes_{1 \leq i \leq p} Hom_{RG}(\mathbf{P}, M[n])$$

with obvious isomorphisms. Observe that since the action of $\mathfrak{S}_p$ does not enter the picture yet, the sign representation does not play a rôle here.

With the diagonal Δ we get

$$Hom_{RG}(\mathbf{P}, M[n]) \xrightarrow{\Delta} \bigotimes_{1 \leq i \leq p} Hom_{RG}(\mathbf{P}, M[n]),$$

and we observe that the image of Δ under the isomorphism

$$Hom_{R(G^p)}(\bigotimes_{1 \leq i \leq p} \mathbf{P}, \bigotimes_{1 \leq i \leq p} M[n]) \simeq \bigotimes_{1 \leq i \leq p} Hom_{RG}(\mathbf{P}, M[n])$$

is in $Hom_{R(G \wr \mathfrak{S}_p)}(\bigotimes_{1 \leq i \leq p} \mathbf{P}, (\bigotimes_{1 \leq i \leq p} M[n]) \otimes_R R_s^{(n)})$. Now, the restriction of the sign representation of $\mathfrak{S}_p$ to C_p always is the trivial representation, for $p = 2$, since then $1 = -1$, and for p an odd prime, since then any p-cycle has even parity. Therefore, Δ factors through $res_{G^p}^{G \wr \mathfrak{S}_p}$ by a mapping Δ'. So, the norm map

$norm_G^{G \times C_p}$ can be seen as left composition of the left vertical homomorphisms in the following commutative diagram:

$$\begin{array}{ccc}
Hom_{RG}(\mathbf{P}, M[n]) & \xrightarrow{\ \Delta\ } & Hom_{R(G^p)}(\bigotimes_{i=1}^p \mathbf{P}, \bigotimes_{i=1}^p M[n]) \\
\downarrow{\scriptstyle \Delta'} & & \| \\
Hom_{R(G \wr \mathfrak{S}_p)}(\bigotimes_{i=1}^p \mathbf{P}, (\bigotimes_{i=1}^p M[n]) \otimes_R R_s^{(n)}) & \xrightarrow{res_{G^p}^{G \wr \mathfrak{S}_p}} & Hom_{R(G^p)}(\bigotimes_{i=1}^p \mathbf{P}, \bigotimes_{i=1}^p M[n]) \\
\downarrow & & \\
Hom_{R(G \wr \mathfrak{S}_p)}(\mathbf{Q}, (\bigotimes_{i=1}^p M[n]) \otimes_R R_s^{(n)}) & & \\
\downarrow{\scriptstyle res_{G \times C_p}^{G \wr \mathfrak{S}_p}} & & \\
Hom_{R(G \times C_p)}(\mathbf{Q}, \bigotimes_{i=1}^p M[n]) & &
\end{array}$$

One should observe that Δ is not linear over the base ring unless this base ring R is $\mathbb{F}_p$.

2. Ring automorphisms

The group $Out_R(RG) = Aut_{R,1}(RG)/Inn(RG)$ of outer augmented R-algebra automorphisms acts on $H^*(G, R)$ in the obvious way. We shall prove that this action is compatible with Steenrod operations.

Proposition 2.1. *Let $R = \mathbb{F}_p$. The action of $Out_{\mathbb{F}_p}(\mathbb{F}_p G)$ on $H^*(G, \mathbb{F}_p)$ commutes with the action of mod p Steenrod operations on $H^*(G, \mathbb{F}_p)$.*

To prove this, we shall show that for any automorphism α of $\mathbb{F}_p G$ there is an automorphism $\hat{\alpha}$ of $\mathbb{F}_p(G \wr \mathfrak{S}_p)$ so that the diagram

$$\begin{array}{ccc}
{}^{\hat{\alpha}}\mathbf{Q} & \xrightarrow{\ \tau\ } & \bigotimes_1^p {}^{\alpha}\mathbf{P} \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \otimes\sigma} \\
\mathbf{Q} & \xrightarrow{\ \tau\ } & \bigotimes_1^p \mathbf{P}
\end{array}$$

commutes. Here, the vertical homomorphisms are chain maps which lift the identity on $\mathbb{F}_p$; the complex $\mathbf{Q}$ is a projective resolution of $\mathbb{F}_p$, and a mapping ρ lifting the identity on $\mathbb{F}_p$ exists by the universal property of $\mathbf{Q}$ being a projective resolution. $\mathbf{P}$ is a projective resolution of $\mathbb{F}_p$ as G-module. Then, we use the tensor product of p copies of the lifting σ of the identity on $\mathbb{F}_p$ onto $\mathbf{P}$ as a complex of $\mathbb{F}_p G$-modules. This tensor product is easily seen to be $\mathbb{F}_p(G \wr \mathfrak{S}_p)$-linear.

In order to define $\hat{\alpha}$ we observe the well known fact that

$$\mathbb{F}_p(G \wr \mathfrak{S}_p) \simeq \mathbb{F}_p \underbrace{(G \times G \times \cdots \times G)}_{p \text{ factors}} \otimes_{\mathbb{F}_p} \mathbb{F}_p \mathfrak{S}_p$$

is isomorphic to the skew tensor product of the first factor with the quotient group acted upon: $\mathbb{F}_p(G \wr \mathfrak{S}_p)$ is the ordinary tensor product as an $\mathbb{F}_p$-module, and the multiplication is defined via a torsion by $\mathfrak{S}_p$:

$$(g_1 \otimes \sigma_1) \cdot (g_2 \otimes \sigma_2) = g_1 \sigma_1(g_2) \otimes \sigma_1 \sigma_2 \ .$$

Since $\mathbb{F}_p \underbrace{(G \times G \times \cdots \times G)}_{p \text{ factors}} \simeq \bigotimes_{i=1}^p \mathbb{F}_p G$, we may take $\hat{\alpha} := (\bigotimes_{i=1}^p \alpha) \otimes id_{\mathfrak{S}_p}$, and we verify readily that $\hat{\alpha}$ is a ring automorphism of $\mathbb{F}_p(G \wr \mathfrak{S}_p)$. Moreover, a projective resolution of $\bigotimes_{i=1}^p \mathbb{F}_p$ as an $\mathbb{F}_p(G \wr \mathfrak{S}_p)$-module is $\mathbf{Q} := (\bigotimes_{i=1}^p \mathbf{P}) \otimes_{\mathbb{F}_p} \mathbf{R}$, where $\mathbf{P}$ is a projective resolution of $\mathbb{F}_p$ as an $\mathbb{F}_p G$-module and $\mathbf{R}$ is a projective resolution of $\mathbb{F}_p$ as an $\mathbb{F}_p \mathfrak{S}_p$-module (cf. e.g. [1, Chapter 4.1]). Note that since the Sylow p-subgroup of $\mathfrak{S}_p$ is cyclic, the homogeneous components of $\mathbf{R}$ are indecomposable projective $\mathbb{F}_p \mathfrak{S}_p$-modules.

We shall have to compare the diagram

$$
\begin{array}{ccc}
Hom_{\mathbb{F}_pG}(\mathbf{P}, \mathbb{F}_p[n]) & \xrightarrow{\quad \Delta \quad} & Hom_{\mathbb{F}_p(G^p)}(\bigotimes_{i=1}^p \mathbf{P}, \bigotimes_{i=1}^p \mathbb{F}_p[n]) \\
\downarrow{\scriptstyle \Delta'} & & \| \\
Hom_{\mathbb{F}_p(G \wr \mathfrak{S}_p)}(\bigotimes_{i=1}^p \mathbf{P}, (\bigotimes_{i=1}^p \mathbb{F}_p[n]) \otimes_{\mathbb{F}_p} \mathbb{F}_{p_s}^{(n)}) & \xrightarrow{res_{G^p}^{G \wr \mathfrak{S}_p}} & Hom_{\mathbb{F}_p(G^p)}(\bigotimes_{i=1}^p \mathbf{P}, \bigotimes_{i=1}^p \mathbb{F}_p[n]) \\
\downarrow & & \\
Hom_{\mathbb{F}_p(G \wr \mathfrak{S}_p)}(\mathbf{Q}, (\bigotimes_{i=1}^p \mathbb{F}_p[n]) \otimes_{\mathbb{F}_p} \mathbb{F}_{p_s}^{(n)}) & & \\
\downarrow{\scriptstyle res_{G \times C_p}^{G \wr \mathfrak{S}_p}} & & \\
Hom_{\mathbb{F}_p(G \times C_p)}(\mathbf{Q}, \bigotimes_{i=1}^p \mathbb{F}_p[n]) & &
\end{array}
$$

with the diagram corresponding to the twisted resolutions

$$
\begin{array}{ccc}
Hom_{\mathbb{F}_pG}({}^\alpha\mathbf{P}, \mathbb{F}_p[n]) & \xrightarrow{\quad \Delta \quad} & Hom_{\mathbb{F}_p(G^p)}(\bigotimes_{i=1}^p {}^{\otimes\alpha}\mathbf{P}, \bigotimes_{i=1}^p \mathbb{F}_p[n]) \\
\downarrow{\scriptstyle \Delta'} & & \| \\
Hom_{\mathbb{F}_p(G \wr \mathfrak{S}_p)}({}^{\hat{\alpha}}\bigotimes_{i=1}^p \mathbf{P}, (\bigotimes_{i=1}^p \mathbb{F}_p[n]) \otimes_{\mathbb{F}_p} \mathbb{F}_{p_s}^{(n)}) & \xrightarrow{res_{G^p}^{G \wr \mathfrak{S}_p}} & Hom_{\mathbb{F}_p(G^p)}({}^{\otimes\alpha}\bigotimes_{i=1}^p \mathbf{P}, \bigotimes_{i=1}^p \mathbb{F}_p[n]) \\
\downarrow & & \\
Hom_{\mathbb{F}_p(G \wr \mathfrak{S}_p)}({}^{\hat{\alpha}}\mathbf{Q}, (\bigotimes_{i=1}^p \mathbb{F}_p[n]) \otimes_{\mathbb{F}_p} \mathbb{F}_{p_s}^{(n)}) & & \\
\downarrow{\scriptstyle res_{G \times C_p}^{G \wr \mathfrak{S}_p}} & & \\
Hom_{\mathbb{F}_p(G \times C_p)}({}^{\hat{\alpha}}\mathbf{Q}, \bigotimes_{i=1}^p \mathbb{F}_p[n]) & &
\end{array}
$$

and we shall have to define mappings from the morphism groups of the bottom diagram to the corresponding morphism sets of the top diagram in such a way that the squares which are created this way are commutative. Since $\mathbf{P}$ as well as ${}^\alpha\mathbf{P}$ are projective resolutions of the trivial module, there is a chain homotopy isomorphism $\sigma : \mathbf{P} \longrightarrow^\alpha \mathbf{P}$ and this isomorphism induces a chain homotopy isomorphism $\bigotimes_1^p \sigma : \bigotimes_{i=1}^p \mathbf{P} \longrightarrow \bigotimes_{i=1}^p {}^\alpha\mathbf{P}$ and a chain homotopy isomorphism $\rho : \mathbf{Q} \longrightarrow {}^{\hat{\alpha}}\mathbf{Q}$ defined by taking the identity on the factors corresponding to $\mathbf{R}$. It is then straightforward to see that all squares which occur are commutative.

In order to compute

$$norm_G^{G \times C_p} : H^n(G, \mathbb{F}_p) \longrightarrow H^{pn}(G \times C_p, \mathbb{F}_p)$$

we have to map a cocycle by $res_{G \times C_p}^{G \wr \mathfrak{S}_p} \circ \tau^* \circ \Delta'$.

Since $res_{G \times C_p}^{G \wr \mathfrak{S}_p} = \delta^*$ for δ being the diagonal embedding of G into G^p, and since $\hat\alpha \circ \delta = \delta \circ (\alpha \times id_{C_p})$,

$$
\begin{aligned}
norm_G^{G \times C_p} \circ \alpha^* &= res_{G \times C_p}^{G \wr \mathfrak{S}_p} \circ \tau^* \circ \Delta' \circ \alpha^* \\
&= res_{G \times C_p}^{G \wr \mathfrak{S}_p} \circ \hat\alpha^* \circ \tau^* \circ \Delta' \\
&= (\alpha \times id_{C_p})^* \circ res_{G \times C_p}^{G \wr \mathfrak{S}_p} \circ \tau^* \circ \Delta' \\
&= (\alpha \times id_{C_p})^* \circ norm_G^{G \times C_p}.
\end{aligned}
$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. Derived self-equivalences

Let R be a field of characteristic p, let $B(RG)$ be the principal block of RG and let X be a two-sided tilting complex in $D^b(B(RG) \otimes_R B(RG)^{op})$. Starting from X, we shall construct a two-sided tilting complex in $D^b(B(RG \wr \mathfrak{S}_p) \otimes_R B(RG \wr \mathfrak{S}_p)^{op})$.

It should be noted that Andrei Marcus constructed in [6] a tilting complex in $B(RG \wr \mathfrak{S}_n)$ if p does not divide n. However, in order to apply this construction to the Steenrod operation we are obliged to pass to $n = p$.

We consider the complex

$$X^{\otimes p} := X \otimes_R X \otimes_R \cdots \otimes_R X$$

with p factors as an object in $D^b(B(RG^p) \otimes_R B(RG^p)^{op})$. Of course, this is an invertible complex. As for the endomorphism complex, we get

$$Hom_{RG^p}(X^{\otimes p}, X^{\otimes p}) \simeq \bigotimes_{i=1}^{p} Hom_{RG}(X, X) \simeq \bigotimes_{i=1}^{p} B(RG)$$

in $D^b(RG^p \otimes_R (RG^p)^{op})$.

We consider now the complex of $R(G \wr \mathfrak{S}_p)$-right modules

$$X^{G \wr \mathfrak{S}_p} := X^{\otimes p} \otimes_{RG^p} R(G \wr \mathfrak{S}_p).$$

It is immediate that $X^{G \wr \mathfrak{S}_p} \simeq X^{\otimes p} \otimes_R R\mathfrak{S}_p$ as $R(G^p)$-left modules, where the p factors of $G \times G \times \ldots \cdots \times G$ act on the respective copies of $X \otimes X \otimes \cdots \otimes X$ respectively and each acts trivially on $R\mathfrak{S}_p$.

We shall now define a left action of $R(G \wr \mathfrak{S}_p)$ on $X^{G \wr \mathfrak{S}_p}$ which commutes with the right action of $G \wr \mathfrak{S}_p$ (see [1, Chapter 4.1]).

Let σ be an element of $\mathfrak{S}_p$. The degree n homogeneous component of $X^{\otimes p}$ is

$$(X^{\otimes p})_n = \bigoplus_{k_1 + \cdots + k_p = n} X_{k_1} \otimes_R X_{k_2} \otimes_R \cdots \otimes_R X_{k_p}.$$

The differential δ_n on the tensor product complex is

$$\sum_{k_1 + \cdots + k_p = n} d \otimes id \otimes \cdots \otimes id + (-1)^{k_1} id \otimes d_{n-k} \otimes id \otimes \cdots \otimes id$$

$$+ \cdots + (-1)^{k_1 + \cdots + k_{p-1}} id \otimes \cdots \otimes id \otimes d.$$

Now, σ acts on the tensor product complex permuting the factors

$$x_1 \otimes \cdots \otimes x_p \in X_{k_1} \otimes_R X_{k_2} \otimes_R \cdots \otimes_R X_{k_p}$$

accordingly as σ acts on the index set $\{1, \ldots, p\}$ multiplied by a sign $(-1)^{\nu}$, where

$$\nu = \sum_{j<k;\ \sigma(j)>\sigma(k)} deg(x_j)deg(x_k)\ .$$

A more standard definition of $X^{G \wr \mathfrak{S}_p}$, by tensor induction followed by ordinary induction, can be done using the fact that X itself is a complex of $G \times G$-modules.

Let H be a finite group and let Y be a complex of RH-modules. Then, the tensor induced complex $Y^{\otimes \uparrow_H^{H \times C_p}}$ is isomorphic to $\underbrace{Y \otimes_R Y \otimes_R \cdots \otimes_R Y}_{p \text{ factors}}$ as a complex of

$R(H \times C_p)$-modules. But actually, $Y^{\otimes \uparrow_H^{H \times C_p}}$ is a complex of $R(H \wr \mathfrak{S}_p)$-modules, as is seen easily from the definition (cf. e.g. [1]).

We will not use the two following lemmas, though we cite them since they might become important in later studies.

Lemma 3.1. *Let S be a p-subgroup of H. If Y is a direct sum of relatively S-projective H-modules, then $Y^{\otimes \uparrow_H^{H \times C_p}}$ is relatively $S \wr C_p$-projective.*

Proof. Let U be an RS-module, so that $Y \mid U \uparrow_S^H$. Let $\pi : U \uparrow_S^H \longrightarrow Y$ and $\iota : Y \longrightarrow U \uparrow_S^H$ with $\pi \circ \iota = id_Y$. Then, the monomorphism

$$\underbrace{Y \otimes_R Y \otimes_R \cdots \otimes_R Y}_{p \text{ factors}} \overset{\iota \otimes \cdots \otimes \iota}{\longrightarrow} \underbrace{U \uparrow_S^H \otimes_R U \uparrow_S^H \otimes_R \cdots \otimes_R U \uparrow_S^H}_{p \text{ factors}}$$

is split by $\pi \otimes \pi \otimes \cdots \otimes \pi$, and both morphisms are clearly morphisms of $R(H \wr \mathfrak{S}_p)$-modules.

Moreover,

$$\underbrace{U \uparrow_S^H \otimes_R U \uparrow_S^H \otimes_R \cdots \otimes_R U \uparrow_S^H}_{p \text{ factors}} \simeq \underbrace{(U \otimes_R U \otimes_R \cdots \otimes_R U)}_{p \text{ factors}} \uparrow_{S \times S \times \cdots \times S}^{H \times H \times \cdots \times H}$$

as $R(H \times H \times \cdots \times H)$-modules. Nevertheless, the isomorphism consists of re-identifying the order of the tensors, and this isomorphism is clearly $H \wr \mathfrak{S}_p$-linear. But now, the isomorphism induces an isomorphism

$$\underbrace{U \uparrow_S^H \otimes_R U \uparrow_S^H \otimes_R \cdots \otimes_R U \uparrow_S^H}_{p \text{ factors}} \simeq \underbrace{(U \otimes_R U \otimes_R \cdots \otimes_R U)}_{p \text{ factors}} \uparrow_{S \wr \mathfrak{S}_p}^{H \wr \mathfrak{S}_p}\ .$$

Since Y is a direct factor of this induced module, and since C_p is the Sylow p-subgroup of $\mathfrak{S}_p$, the result follows.   □

We observe that

$$X^{G \wr \mathfrak{S}_p} := \left(X^{\otimes \uparrow_{G \times G}^{(G \times G) \times C_p}} \right) \uparrow_{(G \times G) \wr \mathfrak{S}_p}^{(G \wr \mathfrak{S}_p) \times (G \wr \mathfrak{S}_p)}$$

Indeed, $\left(X^{\otimes \uparrow_{G \times G}^{(G \times G) \times C_p}} \right)$ is a tensor product of p copies of X, the direct product of p copies of G acting from the left and from the right on the different copies of the p-fold tensor product, and on which C_p acts from the left and from the right by permuting the factors of the tensor product. Moreover, $(G \times G) \wr \mathfrak{S}_p$ embeds into $(G \wr \mathfrak{S}_p) \times (G \wr \mathfrak{S}_p)$ via the mapping $(g_1, g_2, c) \mapsto (g_1, c) \times (g_2, c)$, where $g_1, g_2 \in G^p$. It is immediate that this construction gives the one defined above in a more elementary manner. However, we have not looked at the differentials yet. Here, either we deal

with multicomplexes and take the total complex of these multicomplexes, or we have to go back to the elementary definition above.

Lemma 3.2. *Let P be a Sylow p-subgroup of G. If X is a direct sum of $\Delta(P)$-projective modules, then $X^{G \wr \mathfrak{S}_p}$ is a direct sum of $\Delta(P \wr C_p)$-projective modules.*

Proof. Let P be a p-Sylow subgroup of G. Then, P^p is a p-Sylow subgroup of G^p and $P \wr C_p$ is a p-Sylow subgroup of $G \wr \mathfrak{S}_p$. Let $\Delta(P)$ be the diagonal of P in $G \times G$. Since by Lemma 3.1 $\left(X^{\otimes \uparrow_{G \times G}^{(G \times G) \times C_p}} \right)$ is $\Delta(P^p) \rtimes C_p$-projective, $X^{G \wr \mathfrak{S}_p}$ is $\Delta(P \wr C_p)$-projective. Indeed, C_p in $\Delta(P^p) \rtimes C_p$ is diagonally embedded in $(G \wr \mathfrak{S}_p) \times (G \wr \mathfrak{S}_p)$ and $X^{G \wr \mathfrak{S}_p}$ is induced from $(G \times G)^p \rtimes C_p$. So, if $\left(X^{\otimes \uparrow_{G \times G}^{(G \times G) \times C_p}} \right)$ is $\Delta(P^p) \rtimes C_p$-projective, $X^{G \wr \mathfrak{S}_p}$ is $\Delta(P \wr C_p)$-projective.

The result follows from this. $\qquad \square$

Proposition 3.3. *Let R be a field of characteristic p, and G a finite group with Sylow p subgroup P. Then, for any X with isomorphism class in $TrPic_R(B(RG))$, the complex $X^{G \wr \mathfrak{S}_p}$ is a two-sided tilting complex in $TrPic_R(B(R(G \wr \mathfrak{S}_p)))$.*

If X is splendid, then $X^{G \wr \mathfrak{S}_p}$ is a two-sided tilting complex with homogeneous components which are $\Delta(P)$-projective p-permutation modules.

Remark 3.4. The complex $X^{G \wr \mathfrak{S}_p}$ is in general not splendid even if X is splendid. This surprising fact can be seen if one wants to apply Brauer functors with respect to the Sylow p-subgroup of $\mathfrak{S}_p$. The complex $X^{G \wr \mathfrak{S}_p}$ will then be transformed into a complex which is not a tilting complex anymore.

Proof of Proposition 3.3. We will show that the above defined complex $X^{G \wr \mathfrak{S}_p}$ is a two-sided tilting complex in $D^b(B(R(G \wr \mathfrak{S}_p)) \otimes_R B(R(G \wr \mathfrak{S}_p))^{op})$.

We get isomorphisms of (the total complexes of) double complexes

$$
\begin{aligned}
Hom^{\bullet}_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p}) \quad &\simeq \quad Hom^{\bullet}_{R(G^p)}(X^{\otimes p}, X^{G \wr \mathfrak{S}_p} \downarrow_{G^p}) \\
&\simeq \quad \bigoplus_{\sigma \in \mathfrak{S}_p} Hom^{\bullet}_{R(G^p)}(X^{\otimes p}, X^{\otimes p}) \otimes \sigma \\
&\simeq \quad \bigoplus_{\sigma \in \mathfrak{S}_p} \left(\bigotimes_{i=1}^{p} Hom^{\bullet}_{RG}(X, X) \right) \otimes \sigma
\end{aligned}
$$

Moreover, we have to recover the action of $R(G \wr \mathfrak{S}_p)$ coming from the left action of $R(G \wr \mathfrak{S}_p)$ on the first and on the second variable $X^{G \wr \mathfrak{S}_p}$. The action on the first variable is going to become the right action of $R(G \wr \mathfrak{S}_p)$ on $Hom^{\bullet}_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p})$, and the action on the second variable is going to become the left action on this Hom-complex.

The action of the subgroup G^p of $G \wr \mathfrak{S}_p$ trivially is factorwise on the tensor products, on each of the direct summands. The action of $\mathfrak{S}_p$ from the right on the complex $X^{G \wr \mathfrak{S}_p}$ is given by the induction

$$
X^{\otimes p} \uparrow_{G^2 \wr \mathfrak{S}_p}^{(G \wr \mathfrak{S}_p)^2} \simeq X^{\otimes p} \otimes_R R\mathfrak{S}_p.
$$

The action from the right on the $Hom^{\bullet}$-complex comes from the action on the second variable, which is, after applying Frobenius reciprocity, just the above. Hence

the action of $\mathfrak{S}_p$ from the right permutes the direct summands of

$$\bigoplus_{\sigma \in \mathfrak{S}_p} \left(\bigotimes_{i=1}^{p} Hom^{\bullet}_{RG}(X,X) \right) \otimes \sigma .$$

We shall study the left action of $G \wr \mathfrak{S}_p$ on the *Hom*-complex. The action of $G^p \leq G \wr \mathfrak{S}_p$ on the left trivially is again the factorwise action on the two factors of the tensor products, in each of the direct summands. Now, the action of $\sigma \in \mathfrak{S}_p$ from the left permutes the factors in the tensor product on the level of the modules in each degree, and applies a sign according to the above formula, and henceforth also the factors in the tensor product

$$\bigoplus_{\sigma \in \mathfrak{S}_p} \left(\bigotimes_{i=1}^{p} Hom^{\bullet}_{RG}(X,X) \right) \otimes \sigma .$$

Now, we may assume the homogeneous components of X are free as $B(RG) \otimes_R B(RG)^{op}$-modules, except the leftmost one, which is projective as an RG-right module and projective as an RG-left module. Then, using the Künneth formula,

$$H^0(Hom^{\bullet}_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p})) = \left(\bigotimes_{i=1}^{p} Hom_{K(RG)}(X,X) \right) \otimes_R R\mathfrak{S}_p,$$

and $H^n(Hom^{\bullet}_{RG}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p})) = 0$ for $n \neq 0$, again by the Künneth formula. The Künneth formula applies without problems since R is a field. Then, at the final step we use that X is a tilting complex from either side. So, $Hom^{\bullet}_{RG}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p})$ is quasi-isomorphic to its degree 0 homology as left module and as right module. Therefore, by ordinary Morita theory (see e.g. Curtis-Reiner [2, (3.54)]) $Hom^{\bullet}_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p})$ is an invertible $B(R(G \wr \mathfrak{S}_p))$-bimodule. This implies that $X^{G \wr \mathfrak{S}_p}$ is a two-sided tilting complex in $TrPic_R(B(R(G \wr \mathfrak{S}_p)))$. As a consequence, since $R(G \wr \mathfrak{S}_p)$ is a symmetric algebra,

$$Hom^{\bullet}_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p}) \simeq H^0(Hom^{\bullet}_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p})) \simeq B(R(G \wr \mathfrak{S}_p))$$

in the derived category of $R(G \wr \mathfrak{S}_p) \otimes_R R(G \wr \mathfrak{S}_p)^{op}$-modules.

By Lemma 3.2, if X is a complex of p-permutation modules over $G \times G$ which are induced from the diagonal of a fixed p-Sylow subgroup of G, then $X^{G \wr \mathfrak{S}_p}$ also is a complex of p-permutation modules over $(G \wr \mathfrak{S}_p) \times (G \wr \mathfrak{S}_p)$, induced from a fixed Sylow p-subgroup of $G \wr \mathfrak{S}_p$.

Proposition 3.3 follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.5. It is straightforward to generalize the statement and the proof of Proposition 3.3 to the case where X is a two-sided tilting complex between two blocks B and b of RG, and RH for two finite groups G and H, and R is a field of finite characteristic. Moreover, it is an easy task to pass to any wreath product by a fixed permutation group Π.

We now study first properties of $X^{G \wr \mathfrak{S}_p}$ with respect to the action on *Ext*-algebras [11].

Lemma 3.6. *Let M be an RG-module for a finite group G and let X be a tilting complex with isomorphism class in $HD_M(G)$. Let R_s be the sign representation of $\mathfrak{S}_p$. Then, the isomorphism class of $X^{G \wr \mathfrak{S}_p}$ is in $HD_{(\otimes_{i=1}^p M)}(G \wr \mathfrak{S}_p) \cap HD_{((\otimes_{i=1}^p M) \otimes_R R_s)}(G \wr \mathfrak{S}_p)$.*

Proof. We have

$$
X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} (\bigotimes_{i=1}^{p} M) \ \simeq \ (X^{\otimes p} \otimes_R R\mathfrak{S}_p) \otimes_{G^p \rtimes \mathfrak{S}_p} (\bigotimes_{i=1}^{p} M)
$$

$$
\simeq \ (X^{\otimes p}) \otimes_{G^p} (\bigotimes_{i=1}^{p} M)
$$

$$
\simeq \ \bigotimes_{i=1}^{p} (X \otimes_G M)
$$

$$
\simeq \ \bigotimes_{i=1}^{p} M,
$$

where the last is an isomorphism in the derived category. Moreover,

$$
X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} ((\bigotimes_{i=1}^{p} M) \otimes_R R_s) \ \simeq \ (X^{\otimes p} \otimes_R R\mathfrak{S}_p) \otimes_{G^p \rtimes \mathfrak{S}_p} ((\bigotimes_{i=1}^{p} M) \otimes_R R_s)
$$

$$
\simeq \ \left((X^{\otimes p}) \otimes_{G^p} (\bigotimes_{i=1}^{p} M) \right) \otimes_R (R\mathfrak{S}_p \otimes_{\mathfrak{S}_p} R_s)
$$

$$
\simeq \ \left(\bigotimes_{i=1}^{p} (X \otimes_G M) \right) \otimes_R R_s
$$

$$
\simeq \ \left(\bigotimes_{i=1}^{p} M \right) \otimes_R R_s
$$

is a similar computation. This proves the statement. $\qquad\square$

We are now able to prove that the mod 2 Steenrod square Sq and the mod p Steenrod operation P commute with the action of $HD_R(G)$.

Theorem 1. *Let G be a finite group, and let R be a field of characteristic p. Then, for any X with isomorphism class in $HD_R(G)$, the following squares are commutative. If $p = 2$, then*

$$
\begin{array}{ccc}
H^n(G,R) & \xrightarrow{X \otimes_G -} & H^n(G,R) \\
{\scriptstyle Sq_n^i} \downarrow & & \downarrow {\scriptstyle Sq_n^i} \\
H^{n+i}(G,R) & \xrightarrow{X \otimes_G -} & H^{n+i}(G,R)
\end{array}
$$

is commutative, and if $p \geq 3$, then

$$
\begin{array}{ccc}
H^n(G,R) & \xrightarrow{X \otimes_G -} & H^n(G,R) \\
{\scriptstyle P_n^i} \downarrow & & \downarrow {\scriptstyle P_n^i} \\
H^{n+2(p-1)i}(G,R) & \xrightarrow{X \otimes_G -} & H^{n+2(p-1)i}(G,R)
\end{array}
$$

is commutative.

Proof. We use the interpretation of the Steenrod operation from section 1.2. We have to apply $X \otimes_G -$ or $X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} -$ or $(X \otimes_R RC_p) \otimes_{G \times C_p} -$ to the various

mappings in the diagram

$$
\begin{array}{ccc}
Hom_{RG}(\mathbf{P}, R[n]) & \xrightarrow{\;\;\Delta\;\;} & \bigotimes_{i=1}^{p} Hom_{RG}(\mathbf{P}, R[n]) \\[2mm]
\downarrow{\scriptstyle \Delta'} & & \| \\[2mm]
Hom_{R(G\wr\mathfrak{S}_p)}(\bigotimes_{i=1}^{p}\mathbf{P}, (\bigotimes_{i=1}^{p} R[n])\otimes_R R_s^{(n)}) & \xrightarrow{\;res_{GP}^{G\wr\mathfrak{S}_p}\;} & Hom_{R(G^p)}(\bigotimes_{i=1}^{p}\mathbf{P}, \bigotimes_{i=1}^{p} R[n]) \\[2mm]
\downarrow & & \\[2mm]
Hom_{R(G\wr\mathfrak{S}_p)}(\mathbf{Q}, (\bigotimes_{i=1}^{p} R[n])\otimes_R R_s^{(n)}) & & \\[2mm]
\downarrow{\scriptstyle res_{G\times C_p}^{G\wr C_p}} & & \\[2mm]
Hom_{R(G\times C_p)}(\mathbf{Q}, \bigotimes_{i=1}^{p} R) & &
\end{array}
$$

and shall prove that the squares which are induced by this are commutative in each case.

The square

$$
\begin{array}{ccc}
Hom_{RG}(\mathbf{P}, R[n]) & \xrightarrow{\;\;\Delta\;\;} & \bigotimes_{i=1}^{p} Hom_{RG}(\mathbf{P}, R[n]) \\[2mm]
\downarrow{\scriptstyle X\otimes_G -} & & \downarrow{\scriptstyle \bigotimes_{i=1}^{p} X\otimes_G -} \\[2mm]
Hom_{RG}(X\otimes_G \mathbf{P}, X\otimes_G R[n]) & \xrightarrow{\;\;\Delta\;\;} & \bigotimes_{i=1}^{p} Hom_{RG}(X\otimes_G \mathbf{P}, X\otimes_G R[n])
\end{array}
$$

is obviously commutative. Moreover, the square

$$
\begin{array}{ccc}
\bigotimes_{i=1}^{p} Hom_{RG}(\mathbf{P}, R[n]) & \simeq & Hom_{RG^p}(\bigotimes_{i=1}^{p}\mathbf{P}, \bigotimes_{i=1}^{p} R[n]) \\[3mm]
\bigotimes_{i=1}^{p}(X\otimes_G -)\downarrow & & \downarrow{\scriptstyle X^{\otimes p}\otimes_{G^p} -} \\[3mm]
\bigotimes_{i=1}^{p} Hom_{RG}(X\otimes_G \mathbf{P}, X\otimes_G R[n]) & \simeq & Hom_{RG^p}(\bigotimes_{i=1}^{p}\mathbf{P}, \bigotimes_{i=1}^{p} R[n])
\end{array}
$$

is commutative as well. This proves the announced commutativity of the diagrams with respect to the mappings Δ and Δ'.

The square

$$
\begin{array}{ccc}
Hom_{R(G\wr\mathfrak{S}_p)}(\bigotimes_{i=1}^{p}\mathbf{P}, \bigotimes_{i=1}^{p} R[n]\otimes_R R_s^{(n)}) & \xrightarrow{\;res_{GP}^{G\wr C_p}\;} & Hom_{R(G^p)}(\bigotimes_{i=1}^{p}\mathbf{P}, \bigotimes_{i=1}^{p} R[n]) \\[3mm]
\downarrow{\scriptstyle X^{G\wr\mathfrak{S}_p}\otimes_{G\wr\mathfrak{S}_p} -} & & \downarrow{\scriptstyle X^{\otimes p}\otimes_{G^p} -} \\[3mm]
Hom_{R(G\wr\mathfrak{S}_p)}(X^{G\wr\mathfrak{S}_p}\otimes_{G\wr\mathfrak{S}_p}\bigotimes_{i=1}^{p}\mathbf{P}, & \xrightarrow{\;res_{GP}^{G\wr\mathfrak{S}_p}\;} & Hom_{R(G^p)}\left(X^{\otimes p}\otimes_{G^p}\bigotimes_{i=1}^{p}\mathbf{P},\right. \\
\quad X^{G\wr\mathfrak{S}_p}\otimes_{G\wr\mathfrak{S}_p}\bigotimes_{i=1}^{p} R[n]\otimes_R R_s^{(n)}) & & \left.\quad X^{\otimes p}\otimes_{G^p}\bigotimes_{i=1}^{p} R[n]\right)
\end{array}
$$

again is commutative. In fact the complex $X^{G\wr\mathfrak{S}_p}$ is an induced complex. And so, $X^{G\wr\mathfrak{S}_p}\otimes_{G\wr\mathfrak{S}_p} -$ is the same as $X^{\otimes p}\otimes_{G^p} -$. This proves that applying the various two-sided tilting complexes derived from X to the upper square of the original diagram yields a parallel square which, together with the mappings of the corresponding corners, forms a commutative cube.

Next, the square

$$Hom_{R(G \wr \mathfrak{S}_p)}(\mathbf{Q}, \bigotimes_{i=1}^p R[n] \otimes_R R_s^{(n)}) \longrightarrow Hom_{R(G^p)}(\bigotimes_{i=1}^p \mathbf{P}, \bigotimes_{i=1}^p R[n])$$

$$\downarrow X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} - \qquad\qquad \downarrow X^{\otimes p} \otimes_{G^p} -$$

$$Hom_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \mathbf{Q}, \longrightarrow Hom_{R(G^p)}(X^{\otimes p} \otimes_{G^p} \bigotimes_{i=1}^p \mathbf{P},$$
$$X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \bigotimes_{i=1}^p R[n] \otimes_R R_s^{(n)}) \qquad X^{\otimes p} \otimes_{G^p} \bigotimes_{i=1}^p R[n])$$

is commutative, for the same trivial reasons.

The only part which remains to be proven is that the square

$(*)$

$$Hom_{R(G \wr \mathfrak{S}_p)}(\mathbf{Q}, \bigotimes_{i=1}^p R[n] \otimes_R R_s^{(n)}) \overset{res_{G \times C_p}^{G \wr \mathfrak{S}_p}}{\longrightarrow} Hom_{R(G \times \mathfrak{S}_p)}(\mathbf{Q}, \bigotimes_{i=1}^p R[n])$$

$$\downarrow X^{G \wr C_p} \otimes_{G \wr C_p} - \qquad\qquad (X \otimes_R RC_p) \otimes_{G \times C_p} - \downarrow$$

$$Hom_{R(G \wr \mathfrak{S}_p)}(X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \mathbf{Q}, \overset{res_{G \times C_p}^{G \wr \mathfrak{S}_p}}{\longrightarrow} Hom_{R(G \times C_p)}((X \otimes_R RC_p) \otimes_{G \times C_p} \mathbf{Q},$$
$$X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \bigotimes_{i=1}^p R[n] \otimes_R R_s^{(n)}) \qquad (X \otimes_R RC_p) \otimes_{G \times C_p} \bigotimes_{i=1}^p R[n])$$

is commutative.

In order to prove this fact we replace R by a fixed projective resolution. Let $\mathbf{P}$ be a projective resolution of R as an RG-module and let $\mathbf{Q}$ be a minimal projective resolution of R as an $R\mathfrak{S}_p$-module. Since $R\mathfrak{S}_p$ has cyclic defect group, each homogeneous component of $\mathbf{Q}$ is indecomposable. A projective resolution $\mathbf{P}^{G \wr \mathfrak{S}_p}$ of R as an $R(G \wr \mathfrak{S}_p)$-module is then $\bigotimes_{i=1}^p \mathbf{P} \otimes_R \mathbf{Q}$, and a projective resolution of $R_s^{(n)}$ as an $R(G \wr \mathfrak{S}_p)$-module is $\bigotimes_{i=1}^p \mathbf{P} \otimes_R \left(\mathbf{Q} \otimes_R R_s^{(n)}\right)$. Note that $\mathbf{Q}_s := \mathbf{Q} \otimes_R R_s^{(n)}$. We shall replace R by this resolution $\mathbf{P}^{G \wr \mathfrak{S}_p}$ and $R_s^{(n)}$ by the resolution $\mathbf{P}^{G \wr \mathfrak{S}_p} \otimes_R R_s^{(n)}$.

Now,

$$X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \mathbf{P}^{G \wr \mathfrak{S}_p} \simeq X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \left(\bigotimes_{i=1}^p \mathbf{P} \otimes_R \mathbf{Q}\right) \simeq \left(\bigotimes_{i=1}^p (X \otimes_G \mathbf{P})\right) \otimes_R \mathbf{Q}$$

and so, using Künneth formulas,

$$Hom_{K(RG \wr \mathfrak{S}_p)}\left(X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \mathbf{P}^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \left(\mathbf{P}^{G \wr \mathfrak{S}_p} \otimes_R R_s^{(n)}\right)[n]\right)$$

$$\simeq Hom_{K(RG \wr \mathfrak{S}_p)}\left(\left(\bigotimes_{i=1}^p (X \otimes_G \mathbf{P})\right) \otimes_R \mathbf{Q}, \left(\bigotimes_{i=1}^p (X \otimes_G \mathbf{P})\right) \otimes_R \mathbf{Q}_s[n]\right)$$

$$\simeq \bigoplus_{i+j=n} Hom_{K(RG^p)}\left(\left(\bigotimes_{i=1}^p (X \otimes_G \mathbf{P})\right), \left(\bigotimes_{i=1}^p (X \otimes_G \mathbf{P})\right)[i]\right)$$
$$\otimes_R Hom_{K(R\mathfrak{S}_p)}(\mathbf{Q}, \mathbf{Q}_s[j])$$

$$\simeq \bigoplus_{i+j=n} \left(\bigotimes_{\sum_{l=1}^p i_l = i} Hom_{K(RG)}(X \otimes_G \mathbf{P}, X \otimes_G \mathbf{P}[i_l])\right)$$
$$\otimes_R Hom_{K(R\mathfrak{S}_p)}(\mathbf{Q}, \mathbf{Q}_s[j])$$

Similarly,

$$Hom_{K(RG \wr \mathfrak{S}_p)}\left(\mathbf{P}^{G \wr \mathfrak{S}_p}, \ \mathbf{P}^{G \wr \mathfrak{S}_p} \otimes_R R_s^{(n)}[n]\right)$$

$$\simeq Hom_{K(RG \wr \mathfrak{S}_p)}\left(\left(\bigotimes_{i=1}^{p} \mathbf{P}\right) \otimes_R \mathbf{Q}, \left(\bigotimes_{i=1}^{p} \mathbf{P}\right) \otimes_R \mathbf{Q}_s[n]\right)$$

$$\simeq \bigoplus_{i+j=n} Hom_{K(RG^p)}\left(\bigotimes_{i=1}^{p} \mathbf{P}, \left(\bigotimes_{i=1}^{p} \mathbf{P}\right)[i]\right) \otimes_R Hom_{K(R\mathfrak{S}_p)}(\mathbf{Q}, \mathbf{Q}_s[j])$$

$$\simeq \bigoplus_{i+j=n} \left(\bigotimes_{\sum_{l=1}^{p} i_l = i} Hom_{K(RG)}\left(\mathbf{P}, \mathbf{P}[i_l]\right)\right) \otimes_R Hom_{K(R\mathfrak{S}_p)}(\mathbf{Q}, \mathbf{Q}_s[j]),$$

with the same isomorphisms.

Now, we shall have to study how the restriction $res_{G \times C_p}^{G \wr \mathfrak{S}_p}$ is interpreted in this context. Actually,

$$res_{G \times C_p}^{G \wr \mathfrak{S}_p}\left(Hom_{K(RG \wr \mathfrak{S}_p)}\left(\mathbf{P}^{G \wr \mathfrak{S}_p}, \mathbf{P}^{G \wr \mathfrak{S}_p} \otimes_R R_s^{(n)}[n]\right)\right)$$

$$\simeq res_{G \times C_p}^{G \wr \mathfrak{S}_p}\left(\bigoplus_{i+j=n} \left(\bigotimes_{\sum_{l=1}^{p} i_l = i} Hom_{K(RG)}\left(\mathbf{P}, \mathbf{P}[i_l]\right)\right) \otimes_R Hom_{K(R\mathfrak{S}_p)}(\mathbf{Q}, \mathbf{Q}_s[j])\right)$$

$$\simeq \bigoplus_{i+j=n} Hom_{K(RG)}\left(\mathbf{P}, \mathbf{P}[i]\right) \otimes_R Hom_{K(RC_p)}\left(\mathbf{Q}, \mathbf{Q}[j]\right)),$$

since $Hom_R^{\bullet}(\mathbf{P}, \mathbf{P}) \simeq R$.

Similarly,

$$res_{G \times C_p}^{G \wr \mathfrak{S}_p}\left(Hom_{K(RG \wr \mathfrak{S}_p)}\left(X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \mathbf{P}^{G \wr \mathfrak{S}_p}, X^{G \wr \mathfrak{S}_p} \otimes_{G \wr \mathfrak{S}_p} \mathbf{P}^{G \wr \mathfrak{S}_p} \otimes_R R_s^{(n)}[n]\right)\right)$$

$$\simeq res_{G \times C_p}^{G \wr \mathfrak{S}_p}\left(\bigoplus_{i+j=n} \left(\bigotimes_{\sum_{l=1}^{p} i_l = i} Hom_{K(RG)}(X \otimes_G \mathbf{P}, X \otimes_G \mathbf{P}[i_l])\right)\right.$$

$$\left. \otimes_R Hom_{K(R\mathfrak{S}_p)}(\mathbf{Q}, \mathbf{Q}_s[j])\right)$$

$$\simeq \bigoplus_{i+j=n} Hom_{K(RG)}\left(X \otimes_G \mathbf{P}, X \otimes_G \mathbf{P}[i]\right) \otimes_R Hom_{K(RC_p)}\left(\mathbf{Q}, \mathbf{Q}[j]\right)$$

by the same isomorphisms. This proves that the square (*) is commutative.

It is now direct to see that the action of X on the Ext-algebra commutes with mod p Steenrod operations. In fact, the left square of the diagram

$$\begin{array}{ccccc}
H^n(G, \mathbb{F}_p) & \xrightarrow{norm_G^{G \times C_p}} & H^{pn}(G \times C_p, \mathbb{F}_p) & \simeq & \bigoplus_{i=0}^{np} H^i(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^{np-i}(C_p, \mathbb{F}_p) \\[2mm]
X \otimes - \downarrow & & (X \otimes id) \otimes_{G \times C_p} - \downarrow & & \bigoplus_{i=0}^{np}(X \otimes id) \otimes - \downarrow \\[2mm]
H^n(G, \mathbb{F}_p) & \xrightarrow{norm_G^{G \times C_p}} & H^{pn}(G \times C_p, \mathbb{F}_p) & \simeq & \bigoplus_{i=0}^{np} H^i(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^{np-i}(C_p, \mathbb{F}_p)
\end{array}$$

is commutative, as is shown above. The right square is clearly commutative for $p = 2$, since the cohomology of a cyclic group of order 2 s $\mathbb{F}_2$ in each degree. For odd p the right square is commutative up to a scalar, which is easily seen to be the identity.

The proof of the theorem is finished. $\qquad \square$

4. CONSEQUENCES

Let G be a finite group. As in [12] we denote by $HSplen_{\mathbb{F}_p}(G)$ the subgroup of homotopy equivalence classes of splendid tilting complexes in $TrPic_{\mathbb{F}_p}(\mathbb{F}_p G)$ which in addition map the trivial $\mathbb{F}_p G$-module to an (in the derived category of $\mathbb{F}_p G$-modules) isomorphic copy.

By Theorem 1 the action of the mod p Steenrod algebra $\mathcal{A}_p$ and the action of $HD_{\mathbb{F}_p}(G)$ on $H^*(G, \mathbb{F}_p)$ commute. Hence the cohomology ring $H^*(G, \mathbb{F}_p)$ is an $\mathcal{A}_p$-$HD_{\mathbb{F}_p}(G)$-bimodule.

Let $\mathfrak{U}_p$ be the category of unstable modules over the mod p Steenrod algebra (cf. e.g., Schwartz [10]). Then, for any group G we get a functor $\mathcal{D}_p(G)$ defined by

$$H^*(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p \, HD_{\mathbb{F}_p}(G)} - \; : \; \mathbb{F}_p \, HD_{\mathbb{F}_p}(G)\text{-mod} \longrightarrow \mathfrak{U}_p$$

By the usual adjointness formula

$$Hom_{\mathbb{F}_p \, HD_{\mathbb{F}_p}(G)} \left(-, Hom_{\mathfrak{U}_p}(H^*(G, \mathbb{F}_p), ?) \right)$$
$$\simeq Hom_{\mathfrak{U}_p} \left(H^*(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p \, HD_{\mathbb{F}_p}(G)} -, ? \right),$$

as bifunctors $\mathbb{F}_p \, HD_{\mathbb{F}_p}(G)\text{-mod} \times \mathfrak{U}_p \longrightarrow \mathcal{E}ns$ the functor $\mathcal{D}_p(G)$ has a right adjoint $\mathcal{E}_p(G) := Hom_{\mathfrak{U}_p}(H^*(G, \mathbb{F}_p), -)$.

Of course there is a dual version of the above defined by $\mathcal{E}_p^{Hom}(G)$:

$$- \otimes_{\mathcal{A}_p} H^*(G, \mathbb{F}_p) : \; \mathfrak{U}_p \longrightarrow mod\text{-}(\mathbb{F}_p \, HD_{\mathbb{F}_p}(G))$$

which has a right adjoint $\mathcal{D}_p^{Hom}(G) := Hom_{\mathbb{F}_p \, HD_{\mathbb{F}_p}(G)}(H^*(G, \mathbb{F}_p), -)$.

We get a system of functors

$$
\begin{array}{lccc}
\mathcal{D}_p(G): & \mathbb{F}_p \, HD_p(G)\text{-mod} & \longrightarrow & \mathfrak{U}_p, \\
\mathcal{D}_p^{Hom}(G): & \mathbb{F}_p \, HD_p(G)\text{-mod} & \longrightarrow & \mathfrak{U}_p, \\
\mathcal{E}_p(G): & \mathbb{F}_p \, HD_p(G)\text{-mod} & \longleftarrow & \mathfrak{U}_p, \\
\mathcal{E}_p^{\otimes}(G): & \mathbb{F}_p \, HD_p(G)\text{-mod} & \longleftarrow & \mathfrak{U}_p,
\end{array}
$$

where $(\mathcal{D}_p(G), \mathcal{E}_p(G))$ and $(\mathcal{D}_p^{Hom}(G), \mathcal{E}_p^{\otimes}(G))$ are adjoint pairs and where we abbreviate $HD_p(G)$ for $HD_{\mathbb{F}_p}(G)$.

First, not very astonishing, we are able to find the image of the free unstable modules $F(n)$ by $\mathcal{E}_p^{\otimes}(G)$. Recall that $F(n)$ are determined as representing object to the functor $M \mapsto M^n$ for any $n \in \mathbb{N}$:

$$Hom_{\mathfrak{U}_p}(F(n), -) \simeq (-)^n .$$

Lemma 4.1. $\mathcal{E}_p^{\otimes}(G)(F(n)) \simeq H^*(G)^n$ as $\mathbb{F}_p \, HD_p(G)$-modules.

Proof. We apply Yoneda's lemma to the equation

$$
\begin{array}{rcl}
Hom_{\mathbb{F}_p \, HD_p(G)}(F(n) \otimes_{\mathfrak{U}_p} H^*(G), -) & \simeq & Hom_{\mathfrak{U}_p}(F(n), Hom_{HD_p(G)}(H^*(G), -)) \\
& \simeq & Hom_{\mathbb{F}_p \, HD_p(G)}(H^*(G), -)^n \\
& \simeq & Hom_{\mathbb{F}_p \, HD_p(G)}(H^*(G)^n, -)
\end{array}
$$

to get the result. $\qquad \square$

We shall study the compatibility with Lannes' T-functor (cf. e.g., [10]). For this, we observe that by functoriality, if N is an $\mathcal{A}_p$-$HD_k(G)$-bimodule, so is $T_V(N)$.

Proposition 4.2. *Let V be an elementary abelian p-group and M an $HD_{\mathbb{F}_p}(G)$-module. Then,*

$$(T_V \circ (\mathcal{D}_p(G))) \simeq T_V(H^*(G)) \otimes_{\mathbb{F}_p \ HD_{\mathbb{F}_p}(G)} -.$$

Proof. We use the different adjointness formulas:

$$
\begin{aligned}
Hom_{\mathfrak{U}_p}\left(T_V(\mathcal{D}_p(G)M), -\right) &\simeq Hom_{\mathfrak{U}_p}\left(\mathcal{D}_p(G)(M), H^*(V) \otimes_{\mathbb{F}_p} -\right) \\
&\simeq Hom_{HD_p(G)}\left(M, Hom_{\mathfrak{U}_p}(H^*(G), H^*(V) \otimes_{\mathbb{F}_p} -)\right) \\
&\simeq Hom_{HD_p(G)}\left(M, Hom_{\mathfrak{U}_p}(T_V(H^*(G), -)\right) \\
&\simeq Hom_{\mathfrak{U}_p}(T_V(H^*(G)) \otimes_{HD_p(G)} M, -)
\end{aligned}
$$

The rest follows by Yoneda's lemma. $\square$

We shall elaborate more on Proposition 4.2. As stated in Henn, Lannes and Schwartz [4, Theorem 5.2] (see [3] for a proof), Lannes found that

$$T_V(H^*(G, \mathbb{F}_p)) \simeq \prod_{\rho \in Hom(V,G)/\text{conjugation}} H^*(C_G(\rho(V)), \mathbb{F}_p).$$

Let us recall how the mapping is constructed. For any fixed $\rho \in Hom(V, G)$ one gets

$$
\begin{aligned}
V \times C_G(im\ \rho) &\xrightarrow{\mu_\phi} C_G(im\ \rho) \hookrightarrow G, \\
(v, c) &\mapsto \rho(v)c\ .
\end{aligned}
$$

Taking cohomology, this induces a mapping

$$H^*(G, \mathbb{F}_p) \xrightarrow{res^G_{C_G(im\ \rho)}} H^*(C_G(im\ \rho), \mathbb{F}_p) \longrightarrow H^*(V, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(C_G(im\ \rho), \mathbb{F}_p).$$

Now, T_V is left adjoint to $H^*(V, \mathbb{F}_p) \otimes_{\mathbb{F}_p} -$, and therefore, this mapping induces a mapping

$$T_V(H^*(G, \mathbb{F}_p)) \xrightarrow{ad(\mu_\phi^* \circ res_C^G)} H^*(C_G(im\ \rho))\ .$$

Taking the product of these mappings gives the required isomorphism.

In [12] we obtained that the Brauer construction with respect to the diagonal of a p-subgroup of G in $G \times G$ yields a group homomorphism

$$SplenPic_{\mathbb{F}_p}(G) \longrightarrow SplenPic_{\mathbb{F}_p}(C_G(Q))\ .$$

In [12] we were not able to prove that under this homomorphism the property of fixing the trivial module is preserved. Nevertheless, we conjecture that this is true.

Conjecture 4.3. *The mapping $SplenPic_{\mathbb{F}_p}(G) \longrightarrow SplenPic_{\mathbb{F}_p}(C_G(Q))$ induces a mapping $HSplen_{\mathbb{F}_p}(G) \longrightarrow HSplen_{\mathbb{F}_p}(C_G(Q))$.*

Supposing Conjecture 4.3 is true, we shall show that the decomposition

$$T_V(H^*(G, \mathbb{F}_p)) = \prod_\rho H^*(C_G(im\ \rho), \mathbb{F}_p)$$

is compatible under the action of $HSplen_{\mathbb{F}_p}(G)$ from the right.

Actually, the mapping

$$HSplen_{\mathbb{F}_p}(G) \to HSplen_{\mathbb{F}_p}(C_G(Q))$$

then gives an $\mathbb{F}_p \; HSplen_{\mathbb{F}_p}(G)$-module structure on $H^*(C_G(im \; \rho), \mathbb{F}_p)$ for any $\rho \in Hom(V, G)$.

Proposition 4.4. *Suppose that Conjecture 4.3 is true. Then, the mappings above induce a decomposition*

$$T_V(H^*(G, \mathbb{F}_p)) = \prod_\rho H^*(C_G(im \; \rho), \mathbb{F}_p)$$

as $\mathcal{A}_p$-$\mathbb{F}_p \; HSplen_{\mathbb{F}_p}(G)$-bimodules.

Proof. Let X be a two-sided tilting complex in $HSplen_{\mathbb{F}_p}(G)$. In [12] it is proved that the diagram

$$
\begin{array}{ccc}
H^*(G, \mathbb{F}_p) & \xrightarrow{res^G_{C_G(im \; \rho)}} & H^*(C_G(im \; \rho), \mathbb{F}_p) \\
{\scriptstyle X \otimes_G -} \Big\downarrow & & \Big\downarrow {\scriptstyle X(\Delta(im \; \rho)) \otimes_{C_G(im \; \rho)} -} \\
H^*(G, \mathbb{F}_p) & \xrightarrow{res^G_{C_G(im \; \rho)}} & H^*(C_G(im \; \rho), \mathbb{F}_p)
\end{array}
$$

is commutative.

Now, V is abelian. So, $im \; \rho \subseteq C_G(im \; \rho)$ for any $\rho \in Hom(V, G)$. Let $\mathbf{Q}$ be a *free* resolution of the trivial $\mathbb{F}_p V$-module and let $\mathbf{P}$ be a projective resolution of the trivial $\mathbb{F}_p C_G(im \; \rho)$-module. Then, $\mathbf{Q} \otimes_{\mathbb{F}_p} \mathbf{P}$ is a projective resolution of the trivial $\mathbb{F}_p(V \times C_G(im \; \rho))$-module. In order to describe the mapping $H^*(C_G(im \; \rho), \mathbb{F}_p) \longrightarrow H^*(V, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(C_G(im \; \rho), \mathbb{F}_p)$ we have to determine a mapping τ which makes the diagram of $\mathbb{F}_p(V \times C_G(im \; \rho))$-modules

$$
\begin{array}{ccc}
{}^\mu \mathbf{P} & \xrightarrow{\hspace{1cm}} & \mathbb{F}_p \\
{\scriptstyle \tau} \Big\uparrow & & \Big\| \\
\mathbf{Q} \otimes_{\mathbb{F}_p} \mathbf{P} & \xrightarrow{\hspace{1cm}} & \mathbb{F}_p
\end{array}
$$

commutative. Since $\mathbf{Q}$ is a free resolution, and since $im \; \rho \subseteq C_G(im \; \rho)$, the resolution $\mathbf{P}$ is a $C_G(im \; \rho)$-module, and the mapping τ can simply be chosen to be multiplication via ρ. It is obvious that this makes the diagram commutative.

Denote for the moment $X' := X(\Delta(C_G(im \; \rho)))$. Then, choosing these particular resolutions and mappings, it is easily seen that

$$
\begin{array}{ccc}
H^*(C_G(im \; \rho), \mathbb{F}_p) & \xrightarrow{\mu^*} & H^*(C_G(im \; \rho), \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(C_G(im \; \rho), \mathbb{F}_p) \\
{\scriptstyle X' \otimes_G -} \Big\downarrow & & \Big\downarrow {\scriptstyle (\mathbb{F}_p V \otimes_{\mathbb{F}_p} X') \otimes_{V \times C_G(im \; \rho)} -} \\
H^*(C_G(im \; \rho), \mathbb{F}_p) & \xrightarrow{\hspace{1cm}} & H^*(C_G(im \; \rho), \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(C_G(im \; \rho), \mathbb{F}_p)
\end{array}
$$

is commutative.

Now, it is sufficient to consider the adjoint mappings to the above. These yield commutative diagrams

$$
\begin{array}{ccc}
T_V(H^*(G, \mathbb{F}_p)) & \xrightarrow{\hspace{1cm}} & H^*(C_G(im \; \rho), \mathbb{F}_p) \\
{\scriptstyle X \otimes_G -} \Big\downarrow & & \Big\downarrow {\scriptstyle X(\Delta(im \; \rho)) \otimes_{C_G(im \; \rho)} -} \\
T_V(H^*(G, \mathbb{F}_p)) & \xrightarrow{\hspace{1cm}} & H^*(C_G(im \; \rho), \mathbb{F}_p)
\end{array}
$$

by the functoriality of the adjointness formulas. This finishes the proof. $\qquad\square$

Remark 4.5. If Conjecture 4.3 is not true, Proposition 4.4 remains true for the action of

$$\bigcap_{\rho \in Hom(V,G)} \left(-(\Delta(C_G(im\ \rho))) \right)^{-1} \left(HSplen_{\mathbb{F}_p}(C_G(im\ \rho)) \right)$$

on $T_V(H^*(G,\mathbb{F}_p))$.

Remark 4.6. Observe that we really needed to have splendid equivalences in the proof of Proposition 4.4. Moreover, Lemma 4.2 does not use any hypothesis on the nature of the equivalences. Since the decomposition of $T_V(H^*(G,\mathbb{F}_p))$ as an unstable module holds in general, one would like to have an interpretation of this decomposition as a module over the bigger group of self-equivalences. This seems to include a generalization of [12].

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