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THE STRINGY E-FUNCTION OF THE MODULI SPACE OF RANK 2 BUNDLES OVER A RIEMANN SURFACE OF GENUS 3

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Abstract. We compute the stringy E-function (or the motivic integral) of the moduli space of rank 2 bundles over a Riemann surface of genus 3. In doing so, we answer a question of Batyrev about the stringy E-functions of the GIT quotients of linear representations.

1. Statement of the main result

The stringy E-function is an invariant for singular varieties, due to Kontsevich, Batyrev, Denef and Loeser, which retains useful information about the singularities (see [Bat], [DL1], [DL2], [Cra], [Loo]).

Let X be a variety with at worst *log-terminal* singularities, i.e.,

- X is $\mathbb{Q}$-Gorenstein, and
- for a resolution of singularities $\rho : Y \to X$ such that the exceptional locus of ρ is a divisor D whose irreducible components $D_1, \cdots, D_r$ are smooth divisors with only normal crossings, we have
$$K_Y = \rho^* K_X + \sum_{i=1}^{r} a_i D_i$$
with $a_i > -1$ for all i, where D_i runs over all irreducible components of D. The divisor $K_Y - \rho^* K_X$ is called the *discrepancy divisor*.

For each subset $J \subset I = \{1, 2, \cdots, r\}$, define $D_J = \bigcap_{j \in J} D_j$, $D_\emptyset = Y$ and $D_J^0 = D_J \setminus \bigcup_{j \in I \setminus J} D_j$. Then the stringy E-function of X is defined by

$$E_{st}(X; u, v) = \sum_{J \subset I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1}, \tag{1}$$

where
$$E(Z) = \sum_{p,q} \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(Z; \mathbb{C})) u^p v^q$$
is the Hodge-Deligne polynomial for a variety Z.

The "change of variable formula" (Theorem 6.27 in [Bat], Lemma 3.3 in [DL1]) implies that the function E_{st} is independent of the choice of a resolution. In particular, if ρ is a crepant resolution (i.e., $\rho^* K_X = K_Y$), then $E_{st}(X; u, v) = E(Y; u, v)$.

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A projective $\mathbb{Q}$-Gorenstein algebraic variety of dimension d with at worst log-terminal singularities has the Poincaré duality

$$E_{st}(X;u,v) = (uv)^d\, E_{st}(X;u^{-1},v^{-1}) \tag{2}$$

with $E_{st}(X;0,0)=1$ (Theorem 3.7 in [Bat]).

In this paper, we compute the stringy E-function of the moduli space $\mathcal{N}$ of rank 2 bundles of even degree over a Riemann surface of genus 3 with fixed determinant.[1] Our main result is the following:

Theorem 1.1.

$$\begin{aligned} E_{st}(\mathcal{N}) = {} & \frac{(1-u^2v)^3(1-uv^2)^3-(uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)} \\ & - \frac{(uv)^2}{2}\Big(\frac{(1-u)^3(1-v)^3}{1-uv} - \frac{(1+u)^3(1+v)^3}{1+uv}\Big) \\ & + 2^6(uv)^5(1+uv+(uv)^2)(1+(uv)^2)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2. \end{aligned}$$

The stringy Euler number is

$$e_{st}(\mathcal{N}) = \lim_{u,v\to 1} E_{st}(\mathcal{N}) = 31\frac{9}{25}.$$

The deepest singularities in the moduli space are the geometric invariant theory (GIT) quotient $sl(2)^3/\!\!/SL(2)$, where the action is the diagonal adjoint action. This is a hypersurface singularity, and that makes the genus 3 case special. Batyrev asked (Question 5.5 in [Bat]) the following:

Question (Batyrev): Let X be a GIT quotient of $\mathbb{C}^n$ modulo a linear action of $G \subset SL(n)$. Is it true that $E_{st}(X;u,v)$ is a polynomial?

He showed that this is true when G is abelian or finite. A corollary of our computation is that the answer is NO in general.

Corollary 1.2.

$$\begin{aligned} E_{st}(\mathbb{C}^9/\!\!/SL(2)) = {} & E([\mathbb{C}^9/\!\!/SL(2)]^s) + \frac{(uv)^3(1+uv+(uv)^2)}{1+uv} \\ & + (uv)^5(1+uv+(uv)^2)(1+(uv)^2)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2, \end{aligned}$$

where $[\mathbb{C}^9/\!\!/SL(2)]^s$ *denotes the smooth part of* $\mathbb{C}^9/\!\!/SL(2)$.

Since $E([\mathbb{C}^9/\!\!/SL(2)]^s)$ is a polynomial, we deduce that the stringy E-function of $\mathbb{C}^9/\!\!/SL(2)$ is *not* a polynomial.

When the genus of the Riemann surface is 2, the moduli space is isomorphic to $\mathbb{P}^3$ and thus the E-function is $1+uv+(uv)^2+(uv)^3$. When the genus is greater than 3, the deepest singularities are no longer hypersurface singularities and it doesn't seem possible to find the discrepancy divisor by explicit computation as in this paper.

In §2, we study the singularities of the moduli space $\mathcal{N}$. In §§3, 4, 5, we work out the blow-ups to get a desingularization of $\mathcal{N}$. We compute the discrepancy divisor in §6, and we prove Theorem 1.1 and Corollary 1.2 in §7. We conclude this paper with a formula for the stringy E-function of the moduli space $\mathcal{M}$ of rank 2 bundles of even degree, without fixing determinant, over a Riemann surface of genus 3.

[1] See [New], [Ses] for general facts about the moduli space.

2. The moduli space

The moduli space $\mathcal{N}$ of rank 2 semistable bundles of degree 0 with trivial determinant over a Riemann surface of genus $g = 3$ is a singular projective variety of complex dimension 6. The singularities are Gorenstein by Theorem A of [DN] and log-terminal as we will see in §6. We refer to [New], [Ses], [Ki1] for general results on the moduli space.

The singular locus in $\mathcal{N}$ is the Kummer variety $\mathcal{K}$, which corresponds to those rank 2 bundles $L \oplus L^{-1}$ for some line bundle L of degree 0. The involution $L \to L^{-1}$ gives us a $\mathbb{Z}_2$ action on the Jacobian Jac_0, and the Kummer variety $\mathcal{K}$ is identified with $Jac_0/\mathbb{Z}_2$. There are 2^{2g} fixed points $\mathbb{Z}_2^{2g} = \{[L \oplus L^{-1}] : L \cong L^{-1}\}$. Thus we have a stratification

$$\mathcal{N} = \mathcal{N}^s \sqcup (\mathcal{K} - \mathbb{Z}_2^{2g}) \sqcup \mathbb{Z}_2^{2g}.$$

The moduli space $\mathcal{N}$ is constructed as the GIT quotient of a smooth quasi-projective variety $\mathfrak{R}$, which is a subset of the space of holomorphic maps from the Riemann surface to the Grassmannian $Gr(2, p)$ of 2-dimensional quotients of $\mathbb{C}^p$, where p is a large even number, by the action of $G = SL(p)$. By deformation theory, the slice at a point $h \in \mathfrak{R}$, which represents $L \oplus L^{-1}$ where $L \cong L^{-1}$, is

$$H^1(\mathrm{End}_0(L \oplus L^{-1})) \cong H^1(\mathcal{O}) \otimes sl(2),$$

where the subscript 0 denotes the trace-free part. According to Luna's slice theorem, there is a neighborhood of the point $[L \oplus L^{-1}]$ with $L \cong L^{-1}$, isomorphic to $H^1(\mathcal{O}) \otimes sl(2)/\!\!/SL(2)$ since the stabilizer of the point is $SL(2)$ ([Ki1] (3.3)). Because $\dim H^1(\mathcal{O}) = g$, the deepest singularities are just

$$sl(2)^g/\!\!/SL(2) = \mathrm{Spec}\,\mathbb{C}[z_1, \cdots, z_{3g}]^{SL(2)}.$$

By the classical invariant theory (see [Wey], or more precisely [Hue], 5.1), there is an explicit description of the generators and relations of the invariant subring $\mathbb{C}[z_1, \cdots, z_{3g}]^{SL(2)}$. The special feature of the case $g = 3$ is that the quotient $X := sl(2)^g/\!\!/SL(2)$ is a hypersurface: For each $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3) \in sl(2)^3$, let $x_1 = \overrightarrow{u}_1 \cdot \overrightarrow{u}_1, x_2 = \overrightarrow{u}_2 \cdot \overrightarrow{u}_2, x_3 = \overrightarrow{u}_3 \cdot \overrightarrow{u}_3, x_4 = \overrightarrow{u}_1 \cdot \overrightarrow{u}_2, x_5 = \overrightarrow{u}_1 \cdot \overrightarrow{u}_3, x_6 = \overrightarrow{u}_2 \cdot \overrightarrow{u}_3, x_7 = \det(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$. Then $\mathbb{C}^9/\!\!/SL(2)$ is the hypersurface of $\mathbb{C}^7 = \mathrm{Spec}\,\mathbb{C}[x_1, x_2, \cdots, x_6, x_7]$ given by the equation

$$f(x_1, \cdots, x_7) = x_1x_2x_3 + 2x_4x_5x_6 - x_1x_6^2 - x_2x_5^2 - x_3x_4^2 - x_7^2.$$

The locus of $\mathcal{K}$ in this neighborhood, as a set, is given by

$$x_7 = 0, \quad x_1x_2 - x_4^2 = 0, \quad x_1x_3 - x_5^2 = 0, \quad x_1x_6 - x_4x_5 = 0, \tag{3}$$

because a point in $\mathcal{K}_X := \mathcal{K} \cap X$ can be represented by the $\mathbb{C}^*$-fixed points $(\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3)$ with $\overrightarrow{u}_i \in \{\mathrm{diag}(a, -a) : a \in \mathbb{C}\}$, $i = 1, 2, 3$.

Next, we consider the middle stratum $\mathcal{K} - \mathbb{Z}_2^{2g}$. Once again, if we consider a point $h \in \mathfrak{R}$ representing $L \oplus L^{-1}$ with $L \not\cong L^{-1}$, the slice to the orbit is isomorphic to

$$H^1(\mathrm{End}_0(L \oplus L^{-1})) \cong H^1(\mathcal{O}) \oplus H^1(L^2) \oplus H^1(L^{-2}). \tag{4}$$

The stabilizer $\mathbb{C}^*$ acts with weights $0, 2, -2$ respectively on the components. Hence, there is a neighborhood of the point $[L \oplus L^{-1}] \in \mathcal{K} - \mathbb{Z}_2^{2g}$ in $\mathcal{N}$, isomorphic to

$$H^1(\mathcal{O}) \oplus \big(H^1(L^2) \oplus H^1(L^{-2})/\!\!/\mathbb{C}^*\big).$$

Notice that $H^1(\mathcal{O})$ is the tangent space to $\mathcal{K}$, and hence

$$H^1(L^2) \oplus H^1(L^{-2})/\!\!/\mathbb{C}^* = \mathbb{C}^{2g-2}/\!\!/\mathbb{C}^*$$

is the normal cone. The GIT quotient of the projectivization $\mathbb{P}\mathbb{C}^{2g-2}$ by the induced $\mathbb{C}^*$ action is $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$, and the normal cone $\mathbb{C}^{2g-2}/\!\!/\mathbb{C}^*$ is obtained by collapsing the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}}(-1,-1)$.

3. First blow-up

We will desingularize the moduli space $\mathcal{N}$ by blowing up three times. In this section, we describe the first blow-up.

Let $\mathcal{N}_1$ be the blow-up of $\mathcal{N}$ along the deepest strata $\mathbb{Z}_2^{2g}$, and let D_1' be the exceptional divisor. Since the deepest singularities are all $X := \mathbb{C}^9/\!\!/SL(2)$, we consider only one of them. The GIT quotient X is the hypersurface of $\mathbb{C}^7$ with the equation

$$f(x_1,\cdots,x_7) = x_1x_2x_3 + 2x_4x_5x_6 - x_1x_6^2 - x_2x_5^2 - x_3x_4^2 - x_7^2.$$

We blow up at the origin and denote the exceptional divisor also by D_1'. In terms of a local chart, the blow-up map is

$$(y_1,\cdots,y_7) \to (y_1, y_1y_2,\cdots,y_1y_7). \tag{5}$$

We have $f(x_1,\cdots,x_7) = y_1^2 g_1(y_1,\cdots,y_7)$, where

$$g_1(y_1,\cdots,y_7) = y_1(y_2y_3 + 2y_4y_5y_6 - y_6^2 - y_2y_5^2 - y_3y_4^2) - y_7^2. \tag{6}$$

Hence, the blow-up X_1 is the hypersurface given by g_1, and the exceptional divisor D_1' is the subset $y_1 = 0, y_7 = 0$ in the local chart. Let $\tilde{\mathcal{K}}_X$ be the proper transform of $\mathcal{K}_X$.

The singular set of X_1 in this chart is, by solving $\nabla g_1 = 0$, the union of

$$y_1 = 0, \quad y_7 = 0, \quad y_2y_3 + 2y_4y_5y_6 - y_6^2 - y_2y_5^2 - y_3y_4^2 = 0 \tag{7}$$

and

$$y_7 = 0, \quad y_2 - y_4^2 = 0, \quad y_3 - y_5^2 = 0, \quad y_6 - y_4y_5 = 0. \tag{8}$$

Notice that the second component of the singular set is just the proper transform $\tilde{\mathcal{K}}_X$ in view of (3).

Now we switch to other charts. Since x_1, x_2, x_3 are symmetric, we consider, for instance,

$$(y_1,\cdots,y_7) \to (y_5y_1, y_5y_2, y_5y_3, y_5y_4, y_5, y_5y_6, y_5y_7). \tag{9}$$

In this chart, X_1 is given by the equation

$$g_5(y_1,\cdots,y_7) = y_5(y_1y_2y_3 + 2y_4y_6 - y_1y_6^2 - y_2 - y_3y_4^2) - y_7^2. \tag{10}$$

and D_1' by $y_5 = 0, y_7 = 0$. The singular locus in this chart is the union of

$$y_5 = 0, \quad y_7 = 0, \quad y_1y_2y_3 + 2y_4y_6 - y_1y_6^2 - y_2 - y_3y_4^2 = 0 \tag{11}$$

and

$$y_7 = 0, \quad y_1y_2 - y_4^2 = 0, \quad y_1y_3 - 1 = 0, \quad y_1y_6 - y_4 = 0. \tag{12}$$

Again the second component is $\tilde{\mathcal{K}}_X$ by comparing with (3).[2]

From the local descriptions (7), (11), we see that the first component of the singular set is the subvariety

$$\Delta_X = \{(y_1 : \cdots : y_7) \,|\, y_7 = 0, y_1y_2y_3 + 2y_4y_5y_6 - y_1y_6^2 - y_2y_5^2 - y_3y_4^2 = 0\}$$

[2]There is one more chart $(y_1,\cdots,y_7) \to (y_1y_7,\cdots,y_6y_7,y_7)$, but it doesn't intersect with the exceptional divisor.

of the projective space $\mathbb{P}^6$.

Lemma 3.1. $\Delta_X \cong \mathbb{P}^2 \times \mathbb{P}^2/\mathbb{Z}_2$.

Proof. Define a morphism $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^6$ by

$$\big((x:y:z),(p:q:r)\big) \to (2xp : 2yq : 2zr : xq+yp : xr+zp : yr+zq : 0).$$

In fact, this came from the identity

$$\begin{aligned}&(xt+ys+zu)(pt+qs+ru)\\ &\qquad = 2xp\frac{t^2}{2} + 2yq\frac{s^2}{2} + 2zr\frac{u^2}{2} + (xq+yp)ts + (xr+zp)tu + (yr+zq)su.\end{aligned}$$

Since $\mathbb{C}[t,s,u]$ is a UFD, the morphism is a 2:1 map whose image is precisely Δ_X, as one can easily check. □

The singular locus of X_1 is thus $\Delta_X \cup \tilde{\mathcal{K}}_X$. By direct computation, the singular locus $\mathbb{P}^2$ of Δ_X is the intersection $\Delta_X \cap \tilde{\mathcal{K}}_X$, which is the exceptional divisor of the proper transform $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$. For instance, in terms of the local chart of (5), the singular locus of (7) is given by the equations of (7) and (8).

We denote by Δ the disjoint union of 2^6 Δ_X's in the exceptional divisor in $\mathcal{N}_1$ having 2^6 components. Then $\mathcal{N}_1$ is smooth away from $\Delta \cup \tilde{\mathcal{K}}$.

4. Second blow-up

In this section, we consider the second blow-up. Namely, we blow up $\mathcal{N}_1$ along the proper transform $\tilde{\mathcal{K}}$ of $\mathcal{K}$. This is particularly important because it is the partial desingularization of $\mathcal{N}$, defined in [Ki3].

Let $\mathcal{N}_2$ be the blow-up of $\mathcal{N}_1$ along $\tilde{\mathcal{K}}$. Let D_2' be the exceptional divisor and $\tilde{D}_1'$ the proper transform of D_1' having 2^{2g} connected components. We will describe $\mathcal{N}_2$ as the partial desingularization of $\mathcal{N}$. For more details on partial desingularization, we refer to [Ki1] and [Ki3].

Let H be a reductive subgroup of $G = SL(p)$ and define Z_H^{ss} as the set of semistable points in $\mathfrak{R}$ fixed by H. Let $\mathfrak{R}_1$ be the blow-up of $\mathfrak{R}^{ss}$ along the smooth subvariety $GZ_{SL(2)}^{ss}$. Then by Lemma 3.11 in [Ki3], the GIT quotient $\mathfrak{R}_1^{ss}/\!\!/G$ is the first blow-up $\mathcal{N}_1$. The $\mathbb{C}^*$-fixed point set in $\mathfrak{R}_1^{ss}$ is the proper transform $\tilde{Z}_{\mathbb{C}^*}^{ss}$ of $Z_{\mathbb{C}^*}^{ss}$ and the quotient of $G\tilde{Z}_{\mathbb{C}^*}^{ss}$ by G is $\tilde{\mathcal{K}}$. If we denote by $\mathfrak{R}_2$ the blow-up of $\mathfrak{R}_1^{ss}$ along the smooth subvariety $G\tilde{Z}_{\mathbb{C}^*}^{ss} = G \times_{N^{\mathbb{C}^*}} \tilde{Z}_{\mathbb{C}^*}^{ss}$, where $N^{\mathbb{C}^*}$ is the normalizer of $\mathbb{C}^*$, then the GIT quotient $\mathfrak{R}_2^{ss}/\!\!/G$ is our second blow-up $\mathcal{N}_2$, again by Lemma 3.11 of [Ki3]. This is Kirwan's partial desingularization of $\mathcal{N}$ (see §3 in [Ki1]), which is an orbifold.

By applying the algorithm for Betti numbers described in [Ki3], the Poincaré series $P(\mathcal{N}_2) = \sum_{k\geq 0} t^k \dim H^k(\mathcal{N}_2)$ can be computed as follows. By [Ki2], the equivariant Poincaré series $P^G(\mathfrak{R}^{ss}) = \sum_{k\geq 0} t^k \dim H_G^k(\mathfrak{R}^{ss})$ is given by the gauge-theoretic computation of Atiyah and Bott in §11 of [AB], and we get

$$P^G(\mathfrak{R}^{ss}) = \frac{(1+t^3)^6 - t^8(1+t)^6}{(1-t^2)(1-t^4)}.$$

In order to get $\mathfrak{R}_1^{ss}$ we blow up $\mathfrak{R}^{ss}$ along $GZ_{SL(2)}^{ss}$ and delete the unstable strata. So we get

$$P^G(\mathfrak{R}_1^{ss}) = P^G(\mathfrak{R}^{ss}) + 2^6\Big(\frac{t^2+t^4+\cdots+t^{16}}{1-t^4} - \frac{t^{10}(1+t^2+t^4)}{1-t^2}\Big).$$

Now $\mathfrak{R}_2^{ss}$ is obtained by blowing up $\mathfrak{R}_1^{ss}$ along $G\tilde{Z}^{ss}_{\mathbb{C}^*}$ and deleting the unstable strata. Thus we have

$$\begin{aligned} P^G(\mathfrak{R}_2^{ss}) = P^G(\mathfrak{R}_1^{ss}) + (t^2+t^4+t^6)\big(\frac{1}{2}\frac{(1+t)^6}{1-t^2} + \frac{1}{2}\frac{(1-t)^6}{1+t^2} + 2^6\frac{t^2+t^4}{1-t^4}\big) \\ - \frac{t^4(1+t^2)}{1-t^2}\big((1+t)^6 + 2^6(t^2+t^4)\big). \end{aligned} \tag{13}$$

Because the stabilizers of the G action on $\mathfrak{R}_2^{ss}$ are all finite, we have

$$H_G^*(\mathfrak{R}_2^{ss}) \cong H^*(\mathfrak{R}_2^{ss}/G) = H^*(\mathcal{N}_2),$$

and hence we deduce that

$$\begin{aligned} P(\mathcal{N}_2) = & \frac{(1+t^3)^6 - t^8(1+t)^6}{(1-t^2)(1-t^4)} \\ & + 2^6\big(\frac{t^2+t^4+\cdots+t^{16}}{1-t^4} - \frac{t^{10}(1+t^2+t^4)}{1-t^2}\big) \\ & + (t^2+t^4+t^6)\big(\frac{1}{2}\frac{(1+t)^6}{1-t^2} + \frac{1}{2}\frac{(1-t)^6}{1+t^2} + 2^6\frac{t^2+t^4}{1-t^4}\big) \\ & - \frac{t^4(1+t^2)}{1-t^2}\big((1+t)^6 + 2^6(t^2+t^4)\big). \end{aligned} \tag{14}$$

See [Ki1] for the Betti number computation of the partial desingularization of $\mathcal{M}$, the moduli space without fixing determinant.

Furthermore, we can refine the above computation to get the Hodge-Deligne polynomial for $\mathcal{N}_2$, since the observation in §14 of [Ki4] tells us that the morphisms involved in the above Betti number computation are strictly compatible with the mixed Hodge structures. By the gauge-theoretic computation of [AB], the Hodge-Deligne series for the equivariant cohomology $H_G^*(\mathfrak{R}^{ss})$ is[3]

$$\frac{(1-u^2v)^3(1-uv^2)^3 - (uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)}.$$

Blowing up along $GZ^{ss}_{SL(2)}$ and deleting the unstable part amounts to adding

$$2^6\big(\frac{uv+(uv)^2+\cdots+(uv)^8}{1-(uv)^2} - \frac{(uv)^5(1+uv+(uv)^2)}{1-uv}\big),$$

and blowing up along $G\tilde{Z}^{ss}_{\mathbb{C}^*}$ and deleting unstable points amounts to adding

$$\begin{aligned} & (uv+(uv)^2+(uv)^3)\big(\tfrac{1}{2}\tfrac{(1-u)^3(1-v)^3}{1-uv} + \tfrac{1}{2}\tfrac{(1+u)^3(1+v)^3}{1+uv} + 2^6\tfrac{uv+(uv)^2}{1-(uv)^2}\big) \\ & - \tfrac{(uv)^2(1+uv)}{1-uv}\big((1-u)^3(1-v)^3 + 2^6(uv+(uv)^2)\big). \end{aligned}$$

[3]The recent article [EK] by Earl and Kirwan contains detailed arguments for the Hodge number computation of the equivariant cohomology.

Therefore, we get

(15)
$$\begin{aligned}E(\mathcal{N}_2) &= \frac{(1-u^2v)^3(1-uv^2)^3-(uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)}\\ &+2^6\big(\frac{uv+(uv)^2+\cdots+(uv)^8}{1-(uv)^2}-\frac{(uv)^5(1+uv+(uv)^2)}{1-uv}\big)\\ &+(uv+(uv)^2+(uv)^3)\big(\frac{1}{2}\frac{(1-u)^3(1-v)^3}{1-uv}+\frac{1}{2}\frac{(1+u)^3(1+v)^3}{1+uv}+2^6\frac{uv+(uv)^2}{1-(uv)^2}\big)\\ &-\frac{(uv)^2(1+uv)}{1-uv}\big((1-u)^3(1-v)^3+2^6(uv+(uv)^2)\big).\end{aligned}$$

Notice that (15) reduces to (14) if we put $u=v=-t$.

In this context, D_1' is the disjoint union of 2^6 copies of $\mathbb{P}(sl(2)^3)/\!\!/SL(2)$ and $\tilde{D}_1'$ is its partial desingularization. The algorithm in [Ki3] gives us

$$E(\tilde{D}_1') = 2^6(1+uv+(uv)^2)(1+uv+(uv)^2+(uv)^3). \tag{16}$$

The normal bundle to $G\tilde{Z}^{ss}_{\mathbb{C}^*}$ has rank $2g-2=4$, as we saw in (4). As $G\tilde{Z}^{ss}_{\mathbb{C}^*}\cong G\times_{N^{\mathbb{C}^*}}\tilde{Z}^{ss}_{\mathbb{C}^*}$ from [Ki3] and the normal bundle can be written similarly, the quotient of the normal bundle by G is the quotient of its restriction to $\tilde{Z}^{ss}_{\mathbb{C}^*}$ by the action of $N^{\mathbb{C}^*}$. If we first take the quotient by the identity component $N_0^{\mathbb{C}^*}$ of $N^{\mathbb{C}^*}$, we get a $\mathbb{C}^4/\!\!/\mathbb{C}^*$-bundle over $\widetilde{Jac}$, the blow-up of Jac along $\mathbb{Z}_2^6$, since $\widetilde{Z}_{\mathbb{C}^*}/\!\!/N_0^{\mathbb{C}^*}\cong\widetilde{Jac}$. Hence there is a neighborhood of $\tilde{\mathcal{K}}$ in $\mathcal{N}_1$, which is isomorphic to the $\mathbb{Z}_2$-quotient of the $\mathbb{C}^4/\!\!/\mathbb{C}^*$-bundle over $\tilde{Jac}$ because $\pi_0(N^{\mathbb{C}^*})=\mathbb{Z}_2$. As we mentioned at the end of §2, the normal cone $\mathbb{C}^4/\!\!/\mathbb{C}^*$ is obtained by collapsing the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$, and thus the exceptional divisor D_2' is the $\mathbb{Z}_2$ quotient of the $\mathbb{P}^1\times\mathbb{P}^1$ bundle over $\widetilde{Jac}$. Hence, the E-polynomial of D_2' is

$$\begin{aligned}E(D_2') &= [\big((1-u)^3(1-v)^3+2^6(uv+(uv)^2)\big)(1+uv)^2]^{\mathbb{Z}_2}\\ &= \big(\frac{1}{2}(1-u)^3(1-v)^3+\frac{1}{2}(1+u)^3(1+v)^3+2^6(uv+(uv)^2)\big)\\ &\qquad\times(1+uv+(uv)^2)\\ &+\big(\frac{1}{2}(1-u)^3(1-v)^3-\frac{1}{2}(1+u)^3(1+v)^3\big)(uv),\end{aligned} \tag{17}$$

where $[\cdot]^{\mathbb{Z}_2}$ denotes the $\mathbb{Z}_2$-invariant part. The intersection of the two divisors D_2' and $\tilde{D}_1'$ has 2^6 components, each of which is isomorphic to a bundle over $\mathbb{P}^2$ with fiber $\mathbb{P}^2=\mathbb{P}^1\times_{\mathbb{Z}_2}\mathbb{P}^1$.

Now, we can compute the E-function of the smooth part $\mathcal{N}^s=\mathcal{N}-\mathcal{K}=\mathcal{N}_2-D_2'-\tilde{D}_1'$. The E-polynomials of $\mathcal{N}_2$ and $\tilde{D}_1'$ are (15) and (16), respectively. The E-polynomial of $D_2'-\tilde{D}_1'$ is

$$\begin{aligned}&\big(\tfrac{1}{2}(1-u)^3(1-v)^3+\tfrac{1}{2}(1+u)^3(1+v)^3-2^6\big)(1+uv+(uv)^2)\\ &+\big(\tfrac{1}{2}(1-u)^3(1-v)^3-\tfrac{1}{2}(1+u)^3(1+v)^3\big)(uv),\end{aligned}$$

by subtracting $E(D_2' \cap \tilde{D}_1') = 2^6(1+uv+(uv)^2)^2$ from (17). Therefore, the E-polynomial of $\mathcal{N}^s$ is

$$\begin{aligned} E(\mathcal{N}^s) &= E(\mathcal{N}_2) - E(\tilde{D}_1') - E(D_2' - \tilde{D}_1') \\ &= \frac{(1-u^2v)^3(1-uv^2)^3 - (uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)} \\ &\quad - \frac{1}{2}\Big(\frac{(1-u)^3(1-v)^3}{1-uv} + \frac{(1+u)^3(1+v)^3}{1+uv}\Big). \end{aligned} \tag{18}$$

To end this section, we consider the singular locus of $\mathcal{N}_2$. At a point in $D_2' \setminus \tilde{D}_1'$, $\mathcal{N}_2$ looks like a line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ times $\mathbb{C}^3$, and hence is smooth. The singular locus thus lies in $\tilde{D}_1'$, and so we restrict our concern to X_1, the blow-up of $X = sl(2)^3/\!\!/SL(2)$. We know from the previous section that X_1 is smooth at points in $D_1' \setminus \Delta_X$. Hence, the singular locus of $\mathcal{N}_2$ lies over Δ. We claim that the proper transform $\tilde{\Delta}$ of Δ is precisely the singular locus in $\mathcal{N}_2$. To verify our claim, we return to the local chart description.

In terms of the local chart (5), X_1 is given by the equation (6) and $\tilde{\mathcal{K}}_X$ is given by (8). We introduce new coordinates

$$\begin{aligned} &w_1 = y_1,\ w_2 = y_2 - y_4^2,\ w_3 = y_3 - y_5^2, \\ &w_4 = y_4,\ w_5 = y_5,\ w_6 = y_6 - y_4y_5, w_7 = y_7. \end{aligned}$$

Then the equation of X_1 is $w_1(w_2w_3 - w_6^2) - w_7^2$ and $\tilde{\mathcal{K}}_X$ is given by $w_2 = w_3 = w_6 = w_7 = 0$. The blow-up along $\tilde{\mathcal{K}}_X$ can now be described locally as

$$(t_1, \cdots, t_7) \to (t_1, t_2, t_2t_3, t_4, t_5, t_2t_6, t_2t_7). \tag{19}$$

Since $w_1(w_2w_3 - w_6^2) - w_7^2 = t_2^2\big(t_1(t_3 - t_6^2) - t_7^2\big)$ in this chart, X_2 is given by the equation

$$g_{12}(t_1, \cdots, t_7) = t_1(t_3 - t_6^2) - t_7^2. \tag{20}$$

The singular locus is, from $\nabla g_{12} = 0$,

$$t_1 = 0, \quad t_7 = 0, \quad t_3 - t_6^2 = 0, \tag{21}$$

which is the proper transform of Δ_X in view of the fact that Δ_X is $w_1 = w_7 = 0,\ w_2w_3 - w_6^2 = 0$ from (7). Similarly, one can use other charts for the second blow-up to check that the proper transform $\tilde{\Delta}_X$ of Δ_X is the singular locus over the local chart (5).

In the local chart (9), X_1 is given by (10) and $\tilde{\mathcal{K}}_X$ by (12), while Δ_X is given by (11). Since we are interested in a neighborhood of $\Delta_X \cap \tilde{\mathcal{K}}_X$ where $y_1 \neq 0$, we may assume that $y_1 \neq 0$. We introduce new coordinates

$$\begin{aligned} &w_1 = y_1,\ w_2 = y_2 - y_4^2/y_1,\ w_3 = y_3 - 1/y_1,\ w_4 = y_4, \\ &w_5 = y_5, \quad w_6 = y_6 - y_4/y_1, \quad w_7 = y_7. \end{aligned}$$

In terms of w-coordinates, X_1 is just $w_1w_5(w_2w_3 - w_6^2) - w_7^2$ and $\tilde{\mathcal{K}}_X$ is $w_2 = w_3 = w_6 = w_7 = 0$. The blow-up map along $\tilde{\mathcal{K}}_X$ can be written locally as (19), for instance. One can check again that the singular locus of $\mathcal{N}_2$ over the local chart (9) is precisely $\tilde{\Delta}_X$.

By a similar computation for each local chart for X_1, we deduce that the singular locus of $\mathcal{N}_2$ is $\tilde{\Delta}$, as claimed. Observe from the above that $\tilde{\Delta}$ and $\tilde{D}_1'$ are smooth.

5. Third blow-up

To obtain a desingularization $\tilde{\mathcal{N}}$ of $\mathcal{N}$, we blow up $\mathcal{N}_2$ along $\tilde{\Delta}$. Let D_3 be the exceptional divisor of this third blow-up, and let D_1, D_2 denote the proper transforms of $\tilde{D}_1', D_2'$ respectively.

In terms of the t-coordinates (19) of $\mathcal{N}_2$, one can readily deduce from (20) and (21) that the singularity along $\tilde{\Delta}$ is just the $(xy = z^2)$-singularity in $\mathbb{C}^3$, and by blowing up along $\tilde{\Delta}$ we get a smooth variety. As one can check, the same is true for each local chart of $\mathcal{N}_2$. Hence, $\tilde{\mathcal{N}}$ is smooth.

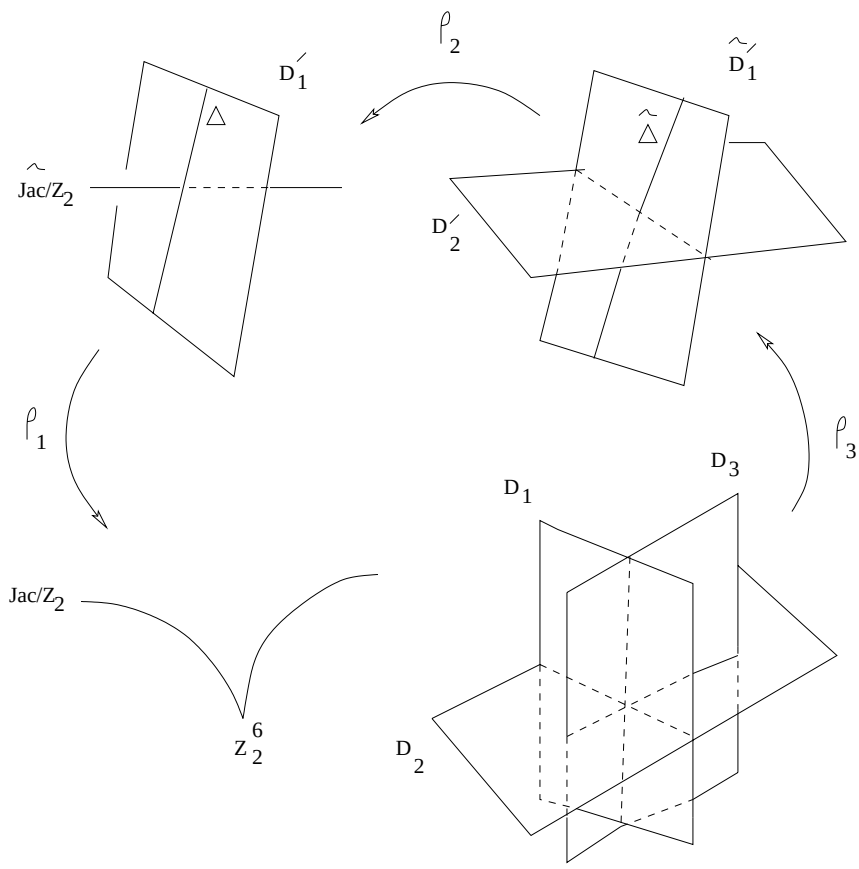

Figure 1.

One can also explicitly check in terms of local coordinates that the divisors D_1, D_2, D_3 are smooth divisors with only normal crossings. For instance, consider the t-coordinates (19) for $\mathcal{N}_2$ again. Before blowing up, we introduce new coordinates $r_1 = t_1, r_2 = t_2, r_3 = t_3 - t_6^2, r_4 = t_4, r_5 = t_5, r_6 = t_6, r_7 = t_7$. Then $\mathcal{N}_2$ is given by $r_1 r_3 - r_7^2 = 0$ and the blow-up center is $r_1 = r_3 = r_7 = 0$. If we consider the local description of the third blow-up, for instance,

$$(\alpha_1, \cdots, \alpha_6, \alpha_7) \to (\alpha_3\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_3\alpha_7),$$

then $\tilde{\mathcal{N}}$ is $\alpha_1 - \alpha_7^2 = 0$, D_1 is $\alpha_1 = \alpha_7 = 0$, D_2 is $\alpha_2 = 0 = \alpha_1 - \alpha_7^2$ and D_3 is $\alpha_3 = 0 = \alpha_1 - \alpha_7^2$. By repeating a similar computation for each chart, we see that the divisors have only normal crossings.

The desingularization process we described can be schematically summarized in Figure 1.

6. Canonical divisors

The purpose of this section is to prove the following.

Proposition 6.1. *If $\rho : \tilde{\mathcal{N}} \to \mathcal{N}$ is the desingularization described above, then $K_{\tilde{\mathcal{N}}} = \rho^* K_{\mathcal{N}} + 4D_1 + D_2 + 4D_3$.*

We consider a differential

$$s = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_6 \wedge dx_7}{\partial f / \partial x_4}$$

on $X = sl(2)^3 /\!\!/ SL(2)$. On the smooth part of X, s is not vanishing and thus the divisor of s is zero. (See (1.7) in [Rei].) In terms of local coordinates, the first blow-up map ρ_1 is given by $(y_1, \cdots, y_6, y_7) \to (y_1, y_1y_2, \cdots, y_1y_6, y_1y_7)$, and we have a rational differential on X_1,

$$s = y_1^4 \frac{dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_5 \wedge dy_6 \wedge dy_7}{\partial g_1 / \partial y_4},$$

where $f(x_1, \cdots, x_7) = y_1^2 g_1(y_1, \cdots, y_7)$. Hence, $K_{\mathcal{N}_1} = \rho_1^* K_{\mathcal{N}} + 4D_1'$.

Now we switch to the w-coordinates $w_1 = y_1, w_2 = y_2 - y_4^2, w_3 = y_3 - y_5^2, w_4 = y_4, w_5 = y_5, w_6 = y_6 - y_4y_5$. Then $g_1 = w_1(w_2w_3 - w_6^2) - w_7^2$. The second blow-up, in terms of local coordinates, is $(t_1, \cdots, t_7) \to (t_1, t_2, t_2t_3, t_4, t_5, t_2t_6, t_2t_7)$, and we get a rational differential on X_2:

$$\begin{aligned} s &= w_1^4 \frac{dw_1 \wedge dw_2 \wedge dw_3 \wedge dw_5 \wedge dw_6 \wedge dw_7}{\partial g_1 / \partial w_3} \\ &= t_1^4 t_2 \frac{dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_5 \wedge dt_6 \wedge dt_7}{\partial g_{12} / \partial t_3}, \end{aligned}$$

where $g_1(y_1, \cdots, y_7) = t_2^2 g_{12}(t_1, \cdots, t_7)$. Hence,

$$K_{\mathcal{N}_2} = \rho_2^* \rho_1^* K_{\mathcal{N}} + 4\tilde{D}_1' + D_2'.$$

We next use the r-coordinates $r_1 = t_1, r_2 = t_2, r_3 = t_3 - t_6^2, r_4 = t_4, r_5 = t_5, r_6 = t_6, r_7 = t_7$. Then $g_{12} = r_1r_3 - r_7^2$. Finally, we blow up along $r_1 = r_3 = r_7 = 0$. In terms of local coordinates, the blow-up is $(\alpha_1, \cdots, \alpha_7) \to (\alpha_3\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_3\alpha_7)$ and $g_{12} = \alpha_3^2(\alpha_1 - \alpha_7^2)$. The equation for $\tilde{\mathcal{N}}$ in the α-coordinates is thus $g_{123} = \alpha_1 - \alpha_7^2$, and we have a rational differential on $\tilde{X}$:

$$s = \alpha_1^4 \alpha_2 \alpha_3^4 \frac{d\alpha_2 \wedge d\alpha_3 \wedge \cdots \wedge d\alpha_7}{\partial g_{123} / \partial \alpha_1}.$$

By a similar computation for each chart, we deduce that

$$K_{\tilde{\mathcal{N}}} = \rho^* K_{\mathcal{N}} + 4D_1 + D_2 + 4D_3.$$

7. The stringy E-function

We can now compute the stringy E-function of the moduli space $\mathcal{N}$.

The E-function of the smooth part is from §4:

$$\begin{aligned} E(\mathcal{N}^s) &= E(\mathcal{N}_2) - E(\tilde{D}_1') - E(D_2' - \tilde{D}_1') \\ &= \frac{(1-u^2v)^3(1-uv^2)^3 - (uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)} \\ &\quad - \frac{1}{2}\Big(\frac{(1-u)^3(1-v)^3}{1-uv} + \frac{(1+u)^3(1+v)^3}{1+uv}\Big). \end{aligned}$$

Next, $D_1^0 = D_1 - (D_2 \cup D_3)$ is $\tilde{D}_1' - \tilde{\Delta} - \tilde{D}_1' \cap D_2'$. Since each component of $\tilde{D}_1' \cap D_2'$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$ and each component of $\tilde{\Delta} - D_2'$ is $\mathbb{P}^2 \times_{\mathbb{Z}_2} \mathbb{P}^2 - \mathbb{P}^2$, $E(D_1^0)$ is (16) minus $E(\mathbb{P}^2 \times \mathbb{P}^2)$ and $2^6[(1+uv+(uv)^2)^2]^{\mathbb{Z}_2} - 2^6(1+uv+(uv)^2)$. Hence,

$$E(D_1^0)\frac{uv-1}{(uv)^5-1} = 2^6((uv)^5 - (uv)^2)\frac{uv-1}{(uv)^5-1}.$$

Since $D_2^0 = D_2 - (D_1 \cup D_3) = D_2' - \tilde{D}_1'$, the E-function of D_2^0 is (17) minus the E-function of $D_2' \cap \tilde{D}_1'$, $2^6(1+uv+(uv)^2)^2$. Hence,

$$\begin{aligned} &E(D_2^0)\frac{uv-1}{(uv)^2-1} \\ &= \big(\frac{1}{2}(1-u)^3(1-v)^3 + \frac{1}{2}(1+u)^3(1+v)^3 - 2^6\big)(1+uv+(uv)^2)\frac{uv-1}{(uv)^2-1} \\ &+ \ \big(\frac{1}{2}(1-u)^3(1-v)^3 - \frac{1}{2}(1+u^3(1+v)^3\big)(uv)\frac{uv-1}{(uv)^2-1}. \end{aligned}$$

As $D_3 \cap D_1$ is isomorphic to $\tilde{\Delta}$ and a component of $D_3 \cap D_2$ is a $\mathbb{P}^1$-bundle over $\tilde{\Delta} \cap D_2'$, we see that the E-function of D_3^0 is 2^6 times the E-function of a $\mathbb{P}^1$-bundle over $\tilde{\Delta}$ minus $E(\tilde{\Delta})$ and 2^6 times $E((\mathbb{P}^1 - pt) \times \mathbb{P}^1 \times \mathbb{P}^2)$. Hence,

$$E(D_3^0)\frac{uv-1}{(uv)^5-1} = 2^6((uv)^3 + (uv)^4 + (uv)^5)\frac{uv-1}{(uv)^5-1}.$$

Notice that $D_{12}^0 = D_1 \cap D_2 - D_3 = \tilde{D}_1' \cap D_2' - \tilde{\Delta}$ is the disjoint union of 2^6 copies of a $(\mathbb{P}^2 - \mathbb{P}^1)$-bundle over $\mathbb{P}^2$. Hence,

$$E(D_{12}^0)\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1} = 2^6((uv)^2 + (uv)^3 + (uv)^4)\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1}.$$

Also, $D_{13}^0 = D_1 \cap D_3 - D_2$ is $\tilde{\Delta}$ minus 2^6 $\mathbb{P}^1$-bundles over $\mathbb{P}^2$. Hence,

$$E(D_{13}^0)\big(\frac{uv-1}{(uv)^5-1}\big)^2 = 2^6((uv)^2 + (uv)^3 + (uv)^4)\big(\frac{uv-1}{(uv)^5-1}\big)^2.$$

Finally, a component of $D_{23}^0 = D_2 \cap D_3 - D_1$ is a $(\mathbb{P}^1 - pt)$-bundle over a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, and a component of $D_{123}^0 = D_1 \cap D_2 \cap D_3$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. Therefore,

$$E(D_{23}^0)\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1} = 2^6(uv + (uv)^2 + (uv)^3)\frac{uv-1}{(uv)^5-1}$$

and

$$E(D_{123}^0)\big(\frac{uv-1}{(uv)^5-1}\big)^2\frac{uv-1}{(uv)^2-1} = 2^6(1+uv+(uv)^2)\big(\frac{uv-1}{(uv)^5-1}\big)^2.$$

Putting together all the pieces above, we get from formula (1) that

$$\begin{aligned} E_{st}(\mathcal{N}) = & \frac{(1-u^2v)^3(1-uv^2)^3-(uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)} \\ & - \frac{(uv)^2}{2}\Big(\frac{(1-u)^3(1-v)^3}{1-uv} - \frac{(1+u)^3(1+v)^3}{1+uv}\Big) \\ & + 2^6(uv)^5(1+uv+(uv)^2)(1+(uv)^2)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2. \end{aligned}$$

This satisfies the Poincaré duality (2), which serves as a check for our result. Notice that it is *not* a polynomial.

To prove Corollary 1.2, let $D_{j,X}$ be the divisors in $\tilde{X}$ corresponding to D_j. Then, from the above, we have

$$E(D^0_{1,X})\frac{uv-1}{(uv)^5-1} = ((uv)^5-(uv)^2)\frac{uv-1}{(uv)^5-1},$$

$$E(D^0_{2,X})\frac{uv-1}{(uv)^2-1} = ((uv)^3-1)(1+uv+(uv)^2)\frac{uv-1}{(uv)^2-1},$$

$$E(D^0_{3,X})\frac{uv-1}{(uv)^5-1} = ((uv)^3+(uv)^4+(uv)^5)\frac{uv-1}{(uv)^5-1},$$

$$E(D^0_{12,X})\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1} = ((uv)^2+(uv)^3+(uv)^4)\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1},$$

$$E(D^0_{13,X})\Big(\frac{uv-1}{(uv)^5-1}\Big)^2 = ((uv)^2+(uv)^3+(uv)^4)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2,$$

$$E(D^0_{23,X})\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1} = (uv+(uv)^2+(uv)^3)\frac{uv-1}{(uv)^5-1},$$

and

$$E(D^0_{123,X})\Big(\frac{uv-1}{(uv)^5-1}\Big)^2\frac{uv-1}{(uv)^2-1} = (1+uv+(uv)^2)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2.$$

By putting them together, we get

$$\begin{aligned} E_{st}(\mathbb{C}^9/\!\!/SL(2)) = E([\mathbb{C}^9/\!\!/SL(2)]^s) + & \frac{(uv)^3(1+uv+(uv)^2)}{1+uv} \\ & + (uv)^5(1+uv+(uv)^2)(1+(uv)^2)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2, \end{aligned}$$

where $[\mathbb{C}^9/\!\!/SL(2)]^s$ denotes the smooth part of $\mathbb{C}^9/\!\!/SL(2)$.

Remark 7.1. If we denote by $\mathcal{M}$ the moduli space of rank 2 semistable bundles of even degree over a Riemann surface of genus 3 (without fixing the determinant), the stringy E-function is

$$\begin{aligned} E_{st}(\mathcal{M}) = (1-u)^3(1-v)^3\Big\{ & \frac{(1-u^2v)^3(1-uv^2)^3-(uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)} \\ & - \frac{(uv)^2}{2}\Big(\frac{(1-u)^3(1-v)^3}{1-uv} - \frac{(1+u)^3(1+v)^3}{1+uv}\Big) \\ & + (uv)^5(1+uv+(uv)^2)(1+(uv)^2)\Big(\frac{uv-1}{(uv)^5-1}\Big)^2\Big\}. \end{aligned} \tag{22}$$

We just sketch the computation, and leave the details to the reader. The determinant map $det : \mathcal{M} \to Jac$ is a fibration with fiber $\mathcal{N}$, and $\mathcal{M}$ has the same

singularities as $\mathcal{N}$. So we need three blow-ups, exactly as in §§3, 4, 5, and the discrepancy divisor is given as in Proposition 6.1. It is now easy to modify the computation to get

$$E(\mathcal{M}^s) = (1-u)^3(1-v)^3[\frac{(1-u^2v)^3(1-uv^2)^3-(uv)^4(1-u)^3(1-v)^3}{(1-uv)(1-(uv)^2)} - \frac{1}{2}(\frac{(1-u)^3(1-v^3}{1-uv}+\frac{(1+u)^3(1+v)^3}{1+uv})],$$

$$E(D^0_{1,\mathcal{M}})\frac{uv-1}{(uv)^5-1} = (1-u)^3(1-v)^3((uv)^5-(uv)^2)\frac{uv-1}{(uv)^5-1},$$

$$\begin{aligned} &E(D^0_{2,\mathcal{M}})\frac{uv-1}{(uv)^2-1} \\ &= (1-u)^3(1-v)^3 \\ &\quad\times[\big(\frac{1}{2}(1-u)^3(1-v)^3+\frac{1}{2}(1+u)^3(1+v)^3-1\big)(1+uv+(uv)^2) \\ &\quad+\big(\frac{1}{2}(1-u)^3(1-v)^3-\frac{1}{2}(1+u)^3(1+v)^3\big)(uv)]\frac{uv-1}{(uv)^2-1}, \end{aligned}$$

$$E(D^0_{3,\mathcal{M}})\frac{uv-1}{(uv)^5-1} = (1-u)^3(1-v)^3((uv)^3+(uv)^4+(uv)^5)\frac{uv-1}{(uv)^5-1},$$

$$\begin{aligned} &E(D^0_{12,\mathcal{M}})\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1} \\ &\quad= (1-u)^3(1-v)^3((uv)^2+(uv)^3+(uv)^4)\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1}, \end{aligned}$$

$$E(D^0_{13,\mathcal{M}})\big(\frac{uv-1}{(uv)^5-1}\big)^2 = (1-u)^3(1-v)^3((uv)^2+(uv)^3+(uv)^4)\big(\frac{uv-1}{(uv)^5-1}\big)^2,$$

$$E(D^0_{23,\mathcal{M}})\frac{uv-1}{(uv)^5-1}\frac{uv-1}{(uv)^2-1} = (1-u)^3(1-v)^3(uv+(uv)^2+(uv)^3)\frac{uv-1}{(uv)^5-1},$$

and

$$\begin{aligned} &E(D^0_{123,\mathcal{M}})\big(\frac{uv-1}{(uv)^5-1}\big)^2\frac{uv-1}{(uv)^2-1} \\ &\quad= (1-u)^3(1-v)^3(1+uv+(uv)^2)\big(\frac{uv-1}{(uv)^5-1}\big)^2. \end{aligned}$$

Combining these, we get (22).

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