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PROJECTIVE NORMALITY OF ABELIAN VARIETIES

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ABSTRACT. We show that ample line bundles L on a g-dimensional simple abelian variety A, satisfying $h^0(A,L) > 2^g \cdot g!$, give projective normal embeddings, for all $g \geq 1$.

1. INTRODUCTION

Let A be an abelian variety of dimension g defined over the field of complex numbers and let L be an ample line bundle on A. Consider the associated rational map $\phi_L : A \longrightarrow \mathbb{P}^{d-1} = \mathbb{P}H^0(L)$, where $d = \dim H^0(A,L)$. Suppose $L = M^n$ for some ample line bundle M on A. Then Koizumi has shown that L gives a projectively normal embedding if $n \geq 3$ (see [2]).

When $n = 2$, Ohbuchi (see [7]) has shown the following.

Theorem 1.1. *Suppose M is a symmetric ample line bundle on a g-dimensional abelian variety A. Then $L = M^2$ gives a projectively normal embedding of A if and only if the origin 0 of A is not contained in $Bs|M \otimes P_\alpha|$ for any $\alpha \in \hat{A}_2 = \{\alpha \in \hat{A} : 2\alpha = 0\}$, where $\hat{A}$ is the dual abelian variety of A, P is the Poincaré bundle on $A \times \hat{A}, P_\alpha = P_{|A\times\alpha}$ for $\alpha \in \hat{A}$ and $Bs|M \otimes P_\alpha|$ is the set of all base points of $M \otimes P_\alpha$.*

Suppose $L \neq M^n$ for any ample line bundle M on A and $n > 1$. When $g = 2$, Lazarsfeld (see [4]) has shown that if ϕ_L is birational onto its image, then ϕ_L gives a projectively normal embedding, for $d = 7, 9, 11$ and for $d \geq 13$. We showed that if the *Neron Severi group* $\mathrm{NS}(A)$ of A is $\mathbb{Z}$, generated by L and $d \geq 7$, then ϕ_L gives a projectively normal embedding (see [1]).

In this article, we show

Theorem 1.2. *Suppose L is an ample line bundle on a g-dimensional simple abelian variety A. If $d > 2^g \cdot g!$, then L gives a projectively normal embedding, for all $g \geq 1$. (Here $d = \dim H^0(A,L)$).*

We outline the proof of Theorem 1.2.

For a polarized abelian variety (A,L), consider the multiplication maps

$$\rho_r : Sym^r H^0(A,L) \longrightarrow H^0(A,L^r).$$

By definition, L gives a projectively normal embedding if ρ_r is surjective, for all $r \geq 1$. We first show that it suffices to show ρ_2 is surjective. More precisely, we show that ρ_2 surjective implies that the maps ρ_r are surjective, for $r \geq 3$ (see Prop. 2.1).

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To prove the surjectivity of the map ρ_2 we consider a finite isogeny $A \longrightarrow B = A/H$, where H is a maximal isotropic subgroup of the fixed group $K(L)$ of L. Then L descends down to a principal polarization M on B. Let $\hat{H}$ denote the group of characters on H. By associating to a character $\chi \in \hat{H}$ a degree 0 line bundle L_χ on B one can identify $\hat{H}$ as a subgroup of the dual abelian variety $\mathrm{Pic}^0(B)$ of B. The homomorphism $\psi_M : B \longrightarrow \mathrm{Pic}^0(B), b \mapsto t_b^* M \otimes M^{-1}$ is an isomorphism and we denote $H' = \psi_M^{-1}(\hat{H})$.

We then show that the surjectivity of the map ρ_2 is equivalent to showing that the subgroup H' of B generates the projective space $\mathbb{P}H^0(B, M^2)$ and its translates $\mathbb{P}H^0(t_\sigma^* M^2)$, where $\sigma \in B$ is such that $\psi_M(2\sigma) = L_\chi, L_\chi \in \hat{H}$, i.e., the images of points of H', under the morphism $B \stackrel{\phi_{t_\sigma^* M^2}}{\longrightarrow} \mathbb{P}H^0(t_\sigma^* M^2) \simeq |t_\sigma^* M^2|$, $b \mapsto t_b^*\theta + t_{-b+2\sigma}^*\theta$ (due to Wirtinger), have their linear span as $|t_\sigma^* M^2|$. (Here we assume that M is symmetric and that θ is the unique symmetric divisor in $|M|$.)

To see this, we show

Proposition 1.3. *Let $\mathcal{L}$ be an ample line bundle on a simple abelian variety Z of dimension g and consider the associated rational map $Z \stackrel{\phi_{\mathcal{L}}}{\longrightarrow} \mathbb{P}H^0(\mathcal{L})$. Then any finite subgroup G of Z of order strictly greater than $h^0(\mathcal{L}) \cdot g!$, generates the linear system $\mathbb{P}H^0(\mathcal{L})$. More precisely, the points $\phi_{\mathcal{L}}(h)$ where h runs over all elements of G not in the base locus of $\mathcal{L}$ span $\mathbb{P}H^0(\mathcal{L})$ (see Prop.* 3.4).

We then apply Proposition 1.3 to $\mathcal{L} = t_\sigma^* M^2$ to obtain bounds as asserted for a polarized abelian variety (A, L) in Theorem 1.2.

Notation. The varieties considered in this article are defined over the complex numbers.

Let $\mathcal{L}$ be an ample line bundle on an abelian variety Z of dimension g.

1. The *fixed group* of $\mathcal{L}$ is the group $K(\mathcal{L}) = \{z \in Z : \mathcal{L} \simeq t_z^*\mathcal{L}\}$, $t_z : Z \longrightarrow Z, x \mapsto z + x$.
2. The *theta group* of $\mathcal{L}$ is the group $\mathcal{G}(\mathcal{L}) = \{(z, \phi) : \mathcal{L} \stackrel{\phi}{\simeq} t_z^*\mathcal{L}\}$.
3. The *Weil form* $e^{\mathcal{L}} : K(\mathcal{L}) \times K(\mathcal{L}) \longrightarrow \mathbb{C}^*$ is the commutator map $(x, y) \mapsto x'y'x'^{-1}y'^{-1}$, for any lifts $x', y' \in \mathcal{G}(\mathcal{L})$ of $x, y \in K(\mathcal{L})$.
4. $h^0(\mathcal{L}) = \dim H^0(Z, \mathcal{L})$.
5. If G is a finite subgroup of Z, then $\mathrm{Card}(G) = \mathrm{order}(G)$.

2. Surjectivity of the maps ρ_r, $r \geq 3$

Suppose $\mathcal{L}$ is an ample line bundle on a g-dimensional abelian variety A. Consider the multiplication maps

$$H^0(\mathcal{L})^{\otimes r} \stackrel{\rho_r}{\longrightarrow} H^0(\mathcal{L}^r), \text{ for } r \geq 2.$$

The main result of this section is the following.

Proposition 2.1. *Suppose $\mathcal{L}$ is an ample line bundle on an abelian variety A. If the multiplication map ρ_2 is surjective, then ρ_r is surjective, for all $r \geq 3$.*

First, we recall

Proposition 2.2. *Suppose L and M are ample line bundles on an abelian variety A.*

1) *The multiplication map*

$$\sum_{\alpha\in U} H^0(L\otimes\alpha)\otimes H^0(M\otimes\alpha^{-1}) \longrightarrow H^0(L\otimes M)$$

is surjective, for any nonempty Zariski open subset U *of* $\mathrm{Pic}^0(A)$.

2) *If the multiplication map* $H^0(L)\otimes H^0(M)\longrightarrow H^0(L\otimes M)$ *is surjective, then the multiplication maps*

$$(a)\, H^0(L)\otimes H^0(M\otimes\alpha) \longrightarrow H^0(L\otimes M\otimes\alpha)$$

and

$$(b)\, H^0(L\otimes\alpha^{-1})\otimes H^0(M\otimes\alpha) \longrightarrow H^0(L\otimes M)$$

are also surjective, for α *in some nonempty Zariski open subset* U *of* $\mathrm{Pic}^0(A)$.

Proof. 1) See [3], 7.3.3.

2) The proof is standard. □

Proof of Proposition 2.1. We prove by induction on r. Suppose the multiplication map $\rho_r : H^0(\mathcal{L})^{\otimes r} \longrightarrow H^0(\mathcal{L}^r)$ is surjective, for some $r\geq 2$.

Consider the composed multiplication map

$$H^0(\mathcal{L})^{\otimes r+1} \xrightarrow{Id\otimes\rho_r} H^0(\mathcal{L})\otimes H^0(\mathcal{L}^r) \xrightarrow{\rho_{1,r}} H^0(\mathcal{L}^{r+1}).$$

To see the surjectivity of the map $\rho_{r+1}=\rho_{1,r}\circ(Id\otimes\rho_r)$ we need to show that the map $\rho_{1,r}$ is surjective.

Using Proposition 2.2 1), we can write

$$(*)\qquad H^0(\mathcal{L}).H^0(\mathcal{L}^r) = \sum_{\alpha\in U} H^0(\mathcal{L}).H^0(\mathcal{L}\otimes\alpha^{-1}).H^0(\mathcal{L}^{r-1}\otimes\alpha)$$

for any nonempty Zariski open subset U of $\mathrm{Pic}^0(A)$.

Since ρ_2 is surjective, by Proposition 2.2 2) (a), there exists a nonempty Zariski open subset U' of $\mathrm{Pic}^0(A)$, such that for $\alpha^{-1}\in U'$,

$$(**)\qquad H^0(\mathcal{L}).H^0(\mathcal{L}\otimes\alpha^{-1}) = H^0(\mathcal{L}^2\otimes\alpha^{-1})$$

Now in $(*)$, using $(**)$ and again applying Proposition 2.2 1), we obtain

$$\begin{aligned} H^0(\mathcal{L}).H^0(\mathcal{L}^r) &= \sum_{\alpha^{-1}\in U'} H^0(\mathcal{L}^2\otimes\alpha^{-1}).H^0(\mathcal{L}^{r-1}\otimes\alpha) \\ &= H^0(\mathcal{L}^{r+1}). \end{aligned}$$

□

3. Surjectivity of the map ρ_2

Let Z be a g-dimensional abelian variety and let D be an ample divisor on Z . We denote $M=\mathcal{O}(D)$ to be the ample line bundle on Z. Let G be a finite subgroup of Z. Consider the homomorphism $\psi_M : Z\longrightarrow \mathrm{Pic}^0(Z)$, $z\mapsto t_z^*(M)\otimes M^{-1}$. Let $G'\subset \mathrm{Pic}^0(Z)$ be the image of G under this homomorphism. Consider a finite subgroup $J\subset\mathrm{Pic}^0(Z)$ and containing the subgroup G'. Construct an étale cover $\pi : X\longrightarrow Z$ corresponding to J, which is of degree equal to Card$J$. Let $N=\mathcal{O}(\pi^{-1}D)$ be the ample line bundle on X.

Notice that if $h\in G\cap K(M)$, then $t_h^*M\simeq M$, and this implies that $D+h$ is linearly equivalent to D on Z. If $\psi_N : X\longrightarrow \mathrm{Pic}^0(X)$ is the map $x\mapsto t_x^*N\otimes N^{-1}$ and $\hat{\pi} : \mathrm{Pic}^0(Z)\longrightarrow\mathrm{Pic}^0(X)$ is the dual of the map π, then since $\hat{\pi}(J)=$

$\{0\}$, $\pi^{-1}G \subset K(N) = \mathrm{Ker}\psi_N$ (since $\psi_N = \hat{\pi} \circ \psi_M \circ \pi$). This means that the divisors $\pi^{-1}(D+h) \in |N|$, for all $h \in G$.

Choose the subgroup J such that N is base point free. (In fact, if J contains the subgroup of 3-torsion points of $\mathrm{Pic}^0(Z)$ and G', then, by the above discussion, $X_{[3]} \subset K(N)$, where $X_{[3]}$ is the subgroup of 3-torsion points of X. This implies, by [3] 2.5.6, that $N = K^3$, for some ample line bundle K on X and by a theorem of Lefschetz (see [3], 4.5.1), N is very ample.)

We will use the following.

Lemma 3.1. *Let V be a variety and $\mathcal{V} \subset \mathrm{Div}(V)$ be an irreducible family of effective Cartier divisors D_t on V. Suppose $W = \bigcap_{t\in\mathcal{V}} D_t \subset V$ and is nonempty and $r = \mathrm{codim}(W)$. Then there exist divisors D_j, $j = 1, 2, ..., r$, in $\mathcal{V}$ that intersect properly and $\dim W = \dim \bigcap_{i=1}^{r} D_i$.*

Proof. We use induction on j. Let $D_1, D_2, ..., D_j$ $(j < r)$ be chosen in $\mathcal{V}$ such that they intersect properly in V. Now write $D_1 \cap D_2 \cap ... \cap D_j = G_1 \cup G_2 \cup ... \cup G_s$, where $G_1, ..., G_s$ are irreducible components. Consider the closed subset $\mathcal{W}_i \subset \mathcal{V}$ parametrizing divisors that contain G_i for $i = 1, 2, ..., s$. (Note that $W_i \neq \mathcal{V}$, otherwise $G_i \subset W$, which is not possible since $\dim G_i > \dim W$.) Let U be the complement of $\bigcup_{i=1}^{s} W_i$ in $\mathcal{V}$, which is nonempty since $\mathcal{V}$ is irreducible. If $D_{j+1} \in U$, then $D_1 \cap ... \cap D_j \cap D_{j+1}$ has codimension $j+1$ (communicated to us by A. Hirschowitz). □

Remark 3.2. Suppose $D_1, D_2, ..., D_r$ are linearly equivalent effective divisors on a variety V, $W = \bigcap_{i=1}^{r} D_i$ and is nonempty and $r = \mathrm{codim}(W)$. If $\mathbb{P}^k$ denotes the span of the points D_i in the linear system $|D_1|$, then $W = \bigcap_{t\in\mathbb{P}^k} D_t$. Hence, by Lemma 3.1, there are r divisors $D_j \in \mathbb{P}^k$ that intersect properly.

With notation as above we have the following.

Proposition 3.3. *Let D be an ample divisor on a g-dimensional simple abelian variety Z. Let G be a finite subgroup of Z that is contained in D. Then* $\mathrm{Card}(G) \leq D^g$ *(which equals $h^0(\mathcal{O}(D)) \cdot g!$, by the Riemann-Roch Theorem).*

Proof. We prove this in several steps.

Step 1: We reduce to the case when the divisors D and $D+h$, for all $h \in G$, are linearly equivalent and $\mathcal{O}(D)$ is base point free. Indeed, by the above discussion, choose a triple (X, N, π), as above, corresponding to a subgroup $J \subset \mathrm{Pic}^0(Z)$ such that N is base point free and $\psi_M(G) \subset J$. This shows that the divisors $\pi^{-1}D$ and $\pi^{-1}(D+h)$, for all $h \in G$, are linearly equivalent. Then we have a morphism $\phi_N : X \longrightarrow \mathbb{P}H^0(N)$. Since π is a finite morphism of degree equal to $\mathrm{Card}(J)$, by the projection formula, one sees that $\deg(\pi^{-1}W) = \mathrm{Card}(J).\deg(W)$, for a subvariety W of Z. Since $(\pi^{-1}D)^g = \mathrm{Card}(J).D^g$, if $\mathrm{Card}(\pi^{-1}G) \leq (\pi^{-1}D)^g$, then $\mathrm{Card}(G) \leq D^g$.

Step 2: We can now assume that D is an ample divisor on X and that $G \subset D$ is a finite subgroup such that D is linearly equivalent to $D+h$ for all $h \in G$ and $N = \mathcal{O}(D)$ is base point free. Let $Y = \bigcap_{h\in G} D + h$ and $s = \dim(Y)$. By Lemma 3.2, $Y \subset \bigcap_{j=1}^{g-s} D_j$ for some $g-s$ divisors $D_j \in |N|$ that intersect properly. Now $\deg(Y) = [Y].[D^s]$ (here $\deg(Y) = \deg(S)$, where $S \subset Y$ is of pure dimension s). Since $Y \subset \bigcap_{j=1}^{g-s} D_j$ we see that $\deg(Y) \leq D^g$. In particular, when $s = 0$, since $G \subset Y$, we get $\mathrm{Card}(G) \leq D^g$.

Step 3: Suppose that $s > 0$. Let $Y = Y_1 \cup Y_2 \cup ... \cup Y_r$, where $Y_j, 1 \leq j \leq r$, are the irreducible components of Y such that $s = \dim Y_1 = \dim Y$. Then $\deg Y_1 \leq \deg Y$. Since Y is G-invariant, $\bigcup_{h\in G} Y_1 + h \subset Y$ and $\sum_{h\in \frac{G}{G_{Y_1}}} \deg(Y_1 + h) \leq \deg Y$, where $G_{Y_1} = \{h \in G : Y_1 + h = Y_1\}$ is a subgroup of G. Hence we get the inequalities $\mathrm{Card}(\frac{G}{G_{Y_1}}).\deg Y_1 \leq \deg Y \leq D^g$, i.e., $\mathrm{Card}(G) \leq \frac{\mathrm{Card}(G_{Y_1})}{\deg Y_1}.D^g$. To complete our proof, we need to show that $\mathrm{Card}(G_{Y_1}) \leq \deg Y_1$.

Step 4: Now $G_{Y_1} \subset \mathrm{Stab}(Y_1) = \{a \in X : Y_1 + a = Y_1\}$. Observe that $\mathrm{Stab}(Y_1) = \bigcap_{y\in Y_1} Y_1 - y$. Now for a point $y_0 \in Y_1$, $\mathrm{Stab}(Y_1) = (Y_1 - y_0)\bigcap_{y\in Y_1} Y_1 - y \subset (Y_1 - y_0)\bigcap_{h\in G, y\in Y_1} D + h - y$. Let $P = (Y_1 - y_0)\bigcap_{h\in G, y\in Y_1} D + h - y$. We proceed to show that $\deg(Stab(Y_1)) \leq \deg(P)$. This will be true if $\mathrm{Stab}(Y_1)$ and P have the same dimension. Now, we have

$$\begin{aligned}\bigcap_{h\in G, y\in Y_1} D + h - y &= \bigcap_{y\in Y_1} Y - y \\ &= \bigcap_{y\in Y_1} ((Y_1 \cup Y_2 \cup ... \cup Y_r) - y) \\ &= (\bigcap_{y\in Y_1} Y_1 - y) \cup (\bigcap_{y\in Y_1} Y_2 - y) \cup ... \cup (\bigcap_{y\in Y_1} Y_r - y).\end{aligned}$$

(To see the above last equality: if $x \in \bigcap_{y\in Y_1}(Y_1 \cup Y_2 \cup ... \cup Y_r) - y$, then $x + y \in Y_1 \cup Y_2 \cup ... \cup Y_r, \forall y \in Y_1$. Via the translation map $Y_1 \longrightarrow Y_1 \cup Y_2 \cup ... \cup Y_r, y \mapsto y + x$ and since Y_1 is irreducible, $x + y \in Y_j$, for some j and for all $y \in Y_1$, i.e., $x \in \bigcap_{y\in Y_1} Y_j - y$ showing one way inclusion, the other inclusion being obvious.)

We now see that if $j \neq 1$ and $x \in \bigcap_{y\in Y_1} Y_j - y$, then $Y_1 + x \subset Y_j$. If $\dim Y_j < \dim Y_1$, then this is absurd and so $\bigcap_{y\in Y_1} Y_j - y$ is empty. If $\dim Y_j \geq \dim Y_1$, since Y_1 is of maximal dimension in Y, $\dim Y_j = \dim Y_1$ and $Y_1 + x = Y_j$. This implies that $\bigcap_{y\in Y_1} Y_j - y = \bigcap_{y\in Y_1} Y_1 + x - y = \mathrm{Stab}(Y_1) + x$. Hence $\bigcap_{h\in G, y\in Y_1} D + h - y$, P and $\mathrm{Stab}(Y_1)$ are of equal dimension, say equal to m and $\deg(\mathrm{Stab}(Y_1)) \leq \deg P$.

Step 5: We proceed to show that $\deg(P) \leq \deg(Y_1)$. Consider the Poincaré line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^0(X)$. Let p_1 and p_2 denote the projections onto X and $\mathrm{Pic}^0(X)$ respectively from $X \times \mathrm{Pic}^0(X)$. Consider the sheaf $\mathcal{E} = p_{2*}(p_1^* N \otimes \mathcal{P})$ on $\mathrm{Pic}^0(X)$. Since the vector spaces $H^0(N \otimes \alpha)$ are of constant dimension for all $\alpha \in \mathrm{Pic}^0(X)$, by Grauert's theorem, $\mathcal{E}$ is a vector bundle on $\mathrm{Pic}^0(X)$. Let $\mathbb{P}(\mathcal{E})$ denote the associated projective bundle on $\mathrm{Pic}^0(X)$. Consider the natural morphism $p_2^*(\mathcal{E}) \longrightarrow p_1^* N \otimes \mathcal{P}$. This is surjective, since on any fibre $X \times \alpha$, $(p_1^* N \otimes \mathcal{P})_\alpha \simeq N \otimes \alpha$ which is globally generated (since N is globally generated) and $\mathcal{E}(\alpha) \simeq H^0(N \otimes \alpha)$. Hence this defines a morphism $\delta_N : X \times \mathrm{Pic}^0(X) \longrightarrow \mathbb{P}(\mathcal{E})$. Let $\mathbb{P}(\check{\mathcal{E}})$ denote the dual projective bundle over $\mathrm{Pic}^0(X)$. In general, the parameter space $\mathcal{V} \subset \mathbb{P}(\check{\mathcal{E}})$ of the family $\{D + h - y\}_{h\in G, y\in Y_1}$ may not form an irreducible variety (unless $G_{Y_1} = G$), but we construct an irreducible subvariety $\mathcal{F} \subset \mathbb{P}(\check{\mathcal{E}})$ such that $\mathcal{V} \subset \mathcal{F}$ and $\bigcap_{h\in G, y\in Y_1} D + h - y = \bigcap_{t\in\mathcal{F}} D_t$, where D_t denotes the divisor corresponding to t in $\mathbb{P}(\mathcal{E})(**)$.

Step 6: Construction of $\mathcal{F}$:

Consider the subspace T of $H^0(X, N)$ spanned by sections s_h, $h \in G$ such that the divisor of s_h is $D + h$. Consider the addition map $m : X \times X \longrightarrow X, (x, y) \mapsto x + y$. Recall the skew-Pontryagin product of the sheaves $\mathcal{O}_X$ and N, $N \hat{*} \mathcal{O}_X = (p_1)_*(m^* N)$ (see [8], p. 653), where p_1(resp. p_2) : $X \times X \longrightarrow X$

denotes the first (resp. second) projection. Then, by Grauert's theorem, $N\hat{*}\mathcal{O}_X$ forms a vector bundle on X with fibres $(N\hat{*}\mathcal{O}_X)_x \simeq H^0(t_x^*N)$. By [8], Remark 1.2, $N\hat{*}\mathcal{O}_X \simeq N * \mathcal{O}_X$ where $N * \mathcal{O}_X = m_*(p_1{}^*N)$ is the Pontryagin product and by [5], p. 161, there are isomorphisms $\mathcal{O}_X \otimes H^0(X,N) \overset{f}{\simeq} N\hat{*}\mathcal{O}_X \simeq \psi_N^*\mathcal{E} \otimes N$ ($\psi_N : X \longrightarrow \mathrm{Pic}^0(X)$ is the isogeny $x \mapsto t_x^*N \otimes N^{-1}$). Consider the image F under f of the trivial subbundle $\mathcal{O}_X \otimes T$ in $N\hat{*}\mathcal{O}_X$. Then the fibre of F at $x \in X$ is the vector subspace of $H^0(t_x^*N)$ spanned by the sections $t_x^*s_h$ whose divisor is $D+h-x$, for $h \in G$. Now $\mathbb{P}(F\check{})$ is a projective subbundle of $\mathbb{P}(\psi_N^*\mathcal{E} \otimes N\check{}) \simeq \mathbb{P}(\psi_N^*\mathcal{E}\check{})$ (since N is a line bundle). Since Y_1 is irreducible, the projective bundle $\mathbb{P}(F\check{})$ restricted to Y_1 is an irreducible subvariety, and let $\mathcal{F}$ be the image of this irreducible variety in $\mathbb{P}(\mathcal{E}\check{})$. Hence $\mathcal{F}$ is irreducible and, by construction, if $R \in |F_y|, y \in Y_1$, then $\bigcap_{h\in G} D + h - y \subset R$ and $\mathcal{F}$ satisfies property $(**)$.

Step 7: By Lemma 3.1, there exist divisors $D_1, D_2, ..., D_{g-m} \in \mathcal{F}$ such that $\bigcap_{h\in G, y\in Y_1} D + h - y \subset D_1 \cap D_2 \cap ... \cap D_{g-m}$. Hence $P \subset (Y_1 - y_0) \cap D_1 \cap D_2 \cap ... \cap D_{g-m} \subset D_1 \cap D_2 \cap ... \cap D_{g-m}$. This implies that $\deg(P) \leq \deg(Y_1 - y_0))$, and by Step 2 and Step 4, $\deg \mathrm{Stab}(Y_1) \leq \deg Y_1 \leq D^g$ (since by Step 2, $\deg(Y_1) \leq \deg(Y) \leq D^g$). Since X is simple, $\mathrm{Stab}(Y_1)$ is zero-dimensional and $G_{Y_1} \subset \mathrm{Stab}(Y_1)$ implies that $\mathrm{Card}(G_{Y_1}) \leq \deg(Y_1)$. Hence by Step 3, $\mathrm{Card}(G) \leq D^g$. This ends the proof. □

This is equivalent to the following.

Proposition 3.4. *Let $\mathcal{L}$ be an ample line bundle on a simple abelian variety Z and consider the associated rational map $Z \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}H^0(\mathcal{L})$. Then any finite subgroup G of Z, of order strictly greater than $h^0(\mathcal{L}) \cdot g!$, generates $\mathbb{P}H^0(\mathcal{L})$. More precisely, the points $\phi_{\mathcal{L}}(g)$ where g runs over all elements of G not in the base locus of $\mathcal{L}$ span $\mathbb{P}H^0(\mathcal{L})$.*

We recall the following result, which we will need in the proof of Theorem 1.2.

Proposition 3.5 (Wirtinger)**.** *Let (Z,Θ) be a principally polarized abelian variety and $\mathcal{L} = \mathcal{O}(\Theta)$ (here Θ is assumed to be a symmetric divisor). There is a nondegenerate inner product $R : H^0(\mathcal{L}^2) \otimes H^0(\mathcal{L}^2) \longrightarrow \mathbb{C}$ (which is symmetric or skew-symmetric depending on whether the multiplicity of the zero element 0 on Θ, $mult_0\Theta$, is even or odd) such that if R induces the isomorphism R',*

$$\mathbb{P}(H^0(\mathcal{L}^2)) \simeq \mathbb{P}(H^0(\mathcal{L}^2)^*) = |2\Theta|,$$

then the composed morphism

$$Z \xrightarrow{\phi_{\mathcal{L}^2}} \mathbb{P}(H^0(\mathcal{L}^2)) \xrightarrow{R'} |2\Theta|$$

is the morphism

$$\phi : Z \longrightarrow |2\Theta|, \quad x \mapsto \Theta_x + \Theta_{-x},$$

where Θ_x is the translate of Θ by x on Z.

Proof. See [6], Proposition, p. 335. □

Proof of Theorem 1.2. Consider a polarized simple abelian variety (A, L) of dimension g such that $h^0(L) > 2^g \cdot g!$.

Consider the multiplication map

$$H^0(L) \otimes H^0(L) \xrightarrow{\rho_2} H^0(L^2).$$

This map factors via

$$Sym^2 H^0(L) \xrightarrow{\rho_2} H^0(L^2).$$

Let $H \subset K(L)$ be a maximal isotropic subgroup for the Weil form e^L. Consider the isogeny $A \xrightarrow{\pi} B = \frac{A}{H}$. Then L descends down to a principal polarization M on B. We may assume that M is symmetric, i.e., $M \simeq i^*M$, $i(b) = -b, b \in B$. Using the fact that $\pi_*\mathcal{O}_A = \bigoplus_{\chi\in\hat{H}} L_\chi$, where L_χ denotes the degree 0 line bundle on B corresponding to the character χ on H, by the projection formula, $\pi_*L = \bigoplus_{\chi\in\hat{H}} M \otimes L_\chi$ and $\pi_*L^2 = \bigoplus_{\chi\in\hat{H}} M^2 \otimes L_\chi$. Hence we obtain the following decompositions:

$$H^0(L) = \bigoplus_{\chi\in\hat{H}} H^0(M \otimes L_\chi) H^0(L^2) = \bigoplus_{\chi\in\hat{H}} H^0(M^2 \otimes L_\chi).$$

Write $Sym^2H^0(L) = \sum_{\chi,\chi'\in\hat{H}} H^0(M \otimes L_{\chi'}).H^0(M \otimes L_{\chi.\chi'^{-1}})$. Consider the multiplication maps

$$\sum_{\chi'\in\hat{H}} H^0(M \otimes L_{\chi'}).H^0(M \otimes L_{\chi.\chi'^{-1}}) \xrightarrow{\rho_\chi} H^0(M^2 \otimes L_\chi).$$

Since $\rho_2 = \bigoplus_{\chi\in\hat{H}} \rho_\chi$, it will suffice to show the surjectivity of ρ_χ for each $\chi \in \hat{H}$.

Since the pair (B, M) is principally polarized, the homomorphism $\psi_M : B \longrightarrow \mathrm{Pic}^0(B)$ is an isomorphism. Let $H' = \psi_M^{-1}(\hat{H})$ and $\theta \in |M|$ be the unique symmetric divisor.

Case 1: Suppose χ is trivial.

We see that the surjectivity of the map ρ_{triv} is equivalent to showing that the reducible divisors $\theta_h + \theta_{-h}$ generate the linear system $|M^2|$, for $h \in H'$. By Proposition 3.5, using the morphism $\phi : B \longrightarrow |M^2|$, this is the same as saying that the image of the subgroup H' under the morphism ϕ_{M^2} generates the projective space $\mathbb{P}H^0(M^2)$.

Case 2: Suppose χ is nontrivial.

First, notice that if $b \in B$, then $\psi_{M^2}(b) = \psi_M(2b)$. Let $\sigma \in B$ be such that $\psi_{M^2}(\sigma) = L_\chi$, i.e., $\psi_M(2\sigma) = L_\chi$. Hence the map ρ_χ is surjective if the reducible divisors $\theta_h + \theta_{-h+2\sigma}$ span the linear system $|t_\sigma^*M^2|$ for $h \in H' = \psi_M^{-1}(\hat{H})$. Now if $b \in B$, then $\theta_b + \theta_{-b+2\sigma} = (\theta_\sigma)_{b-\sigma} + (\theta_\sigma)_{-b+\sigma}$, which is the image of the divisor $\theta_{b-\sigma} + \theta_{-b+\sigma}$ under the morphism $|M^2| \longrightarrow |t_\sigma^*M^2|$ given by the translation map $B \xrightarrow{t_\sigma} B$. Hence the morphism $\phi_\sigma : A \longrightarrow |t_\sigma^*M^2|$ is given as $b \mapsto \theta_b + \theta_{-x+2\sigma}$. This implies that ρ_χ is surjective if and only if the points in $\phi_\sigma(H')$ generate the linear system $|t_\sigma^*M^2|$.

Since the pair (A, L) is a simple polarized abelian variety with $h^0(L) = \mathrm{Card}(H')$ $> 2^g \cdot g! = h^0(t_\sigma^*M^2) \cdot g!$, by Proposition 3.4, ρ_χ is surjective for all $\chi \in \hat{H}$. Hence, by Proposition 2.1, our proof is now complete. □

Remark 3.6. 1) Suppose $g = 1$. Then any line bundle of degree strictly greater than 2 on an elliptic curve gives a projectively normal embedding. Hence the bound is sharp.

2) Suppose $g = 2$. If $L \simeq N^2$, where N is an ample symmetric line bundle with $h^0(N) = 2$ on an abelian surface A, then it follows that $h^0(L) = 8$ (in terms of "type" of an ample line bundle, N is of type $(1, 2)$ and hence L is of type $(2, 4)$ and $h^0(L) = 8$). By [3], 10.1.4, N has 4 base points, say x_1, x_2, x_3 and x_4, which are 4-torsion points on A and, moreover, $2x_i \in K(N) = \mathrm{Ker}\,\psi_N$ where

$\psi_N : A \longrightarrow \mathrm{Pic}^0(A), a \mapsto t_a^* N \otimes N^{-1}$. Let $\alpha_i = \psi_N(x_i)$, for $i = 1,2,3,4$. Now the points x_i are base points for N, for $i = 1,2,3,4$, is equivalent to saying that the origin $0 \in A$ is a base point for $N \otimes \alpha_i$, for $i = 1,2,3,4$. Also $2x_i \in K(N)$ implies that the points α_i are 2-torsion points in $\mathrm{Pic}^0(A)$. Hence by Ohbuchi's Theorem 1.1, L does not give a projectively normal embedding. So the bound is sharp.

3) Suppose $g = 3$. If $L \simeq N^3$, where N is a principal polarization on an abelian threefold A, then $h^0(L) = 27$. But by Koizumi's Theorem, L gives a projectively normal embedding. So the bound is not sharp in this case.

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References

[1] Iyer, J.: *Projective normality of abelian surfaces given by primitive line bundles*, Manuscripta Math., **98**, 139-153 (1999). MR **2000b:**14056

[2] Koizumi, S.: *Theta relations and projective normality of abelian varieties*, American Journal of Mathematics, **98**, 865-889 (1976). MR **58:**702

[3] Lange, H. and Birkenhake, Ch. : *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften, **302**, Springer-Verlag, Berlin, (1992). MR **94j:**14001

[4] Lazarsfeld, R.: *Projectivité normale des surfaces abéliennes*, Rédigé par O. Debarre. Prépublication No. **14**, Europroj- C.I.M.P.A., Nice, (1990).

[5] Mukai, S.: *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J., **81**, 153-175 (1981). MR **82f:**14036

[6] Mumford, D.: *Prym varieties I*, in: Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 325-350 (1974). MR **52:**415

[7] Ohbuchi, A.: *A note on the normal generation of ample line bundles on abelian varieties*, Proc. Japan Acad. Ser. A Math. Sci. **64**, 119-120 (1988). MR **90a:**14062a

[8] Pareschi, G.: *Syzygies of abelian varieties*, J. Amer. Math. Soc. **13**, 651-664 (2000). MR **2001f:**14086

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