TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 1, Pages 1–30
S 0002-9947(04)03688-8
Article electronically published on August 19, 2004

OPERATORS ON $C(K)$ SPACES PRESERVING COPIES OF SCHREIER SPACES

IOANNIS GASPARIS

Abstract. It is proved that an operator $T\colon C(K) \to X$, K compact metrizable, X a separable Banach space, for which the ϵ-Szlenk index of $T^*(B_{X^*})$ is greater than or equal to ω^ξ, $\xi < \omega_1$, is an isomorphism on a subspace of $C(K)$ isomorphic to X_ξ, the Schreier space of order ξ. As a corollary, one obtains that a complemented subspace of $C(K)$ with Szlenk index equal to $\omega^{\xi+1}$ contains a subspace isomorphic to X_ξ.

1. Introduction

It is an open question whether every infinite-dimensional complemented subspace of $C(K)$, K compact metrizable, is isomorphic to $C(L)$ for some compact metrizable space L ([36], [43]). Of course, $C(K)$ stands for the Banach space of scalar-valued functions continuous on K, under the supremum norm. A closed linear subspace of $C(K)$ is complemented if it is the range of a bounded linear idempotent operator on $C(K)$. The following list consists of papers closely connected to this conjecture: [1], [2], [3], [4], [5], [7], [8], [9], [15], [17], [18], [19], [20], [32], [33], [34], [36], [43], [44], [45], [47], [48], [55], [56].

It follows, by combining the results of [45], [15], [8] and [18], that in order to settle this conjecture in the affirmative, one needs to show that if E is complemented in $C(K)$ and E^* is separable, then E is isomorphic to $C(\omega^{\omega^\xi})$ for some countable ordinal ξ (in the sequel, for a given ordinal α, $C(\alpha)$ will denote the Banach space $C(K)$, where $K = [1, \alpha]$ is the interval of ordinals not exceeding α, endowed with the order topology). Note that E^* is isomorphic to ℓ_1 by the results of [32]. Also, the ordinal ξ in question can be computed in terms of the Szlenk index $\eta(E)$ of E, [52]. It is shown in [8], [16], [48] that $\eta(E) = \omega^{\xi+1}$. We shall next recall the definition of the Szlenk index of a Banach space, since it will play an important role in our considerations.

Given a w^*-compact subset B of X^*, the dual of a Banach space X, and a scalar $\epsilon > 0$, we define a transfinite sequence $(P_\alpha(\epsilon, B))_{\alpha<\omega_1}$ (ω_1 stands for the first uncountable ordinal) of subsets of B by transfinite induction: $P_0(\epsilon, B) = B$. Suppose that $\alpha < \omega_1$ and that $P_\beta(\epsilon, B)$ has been defined for all $\beta < \alpha$. Assume

Received by the editors February 4, 2003.
2000 *Mathematics Subject Classification.* Primary 46B03; Secondary 06A07, 03E02.
Key words and phrases. $C(K)$ space, Szlenk index, projection, Schreier sets.

©2004 American Mathematical Society

first that α is a successor ordinal, say $\alpha = \beta + 1$. Set

$$P_\alpha(\epsilon, B) = \{x^* \in B : \exists\,(x_n^*) \subset P_\beta(\epsilon, B),\, \lim_n x_n^* = x^*,\, (w^*),\, \exists\,(x_n) \subset B_X$$
$$\lim_n x_n = 0,\, (w),\, \inf_n |x_n^*(x_n)| \geq \epsilon\}$$

(B_X is the closed unit ball of X). If α is a limit ordinal, set

$$P_\alpha(\epsilon, B) = \bigcap_{\beta<\alpha} P_\beta(\epsilon, B).$$

Define the ϵ-Szlenk index of B as $\eta(\epsilon, B) = \sup\{\alpha < \omega_1 : P_\alpha(\epsilon, B) \neq \emptyset\}$. Then define the Szlenk index of B to be $\eta(B) = \sup\{\eta(\epsilon, B) : \epsilon > 0\}$. Finally, the Szlenk index $\eta(X)$ of X equals $\eta(B_{X^*})$. We shall discuss this index in detail in Section 3. We mention that $\eta(X)$ is a measurement for the norm-separability of the dual X^*. Szlenk [52] shows that if X^* is separable, then $\eta(X) < \omega_1$. It is also noted in [20] that the results of [51] imply that for a w^*-compact subset B of $C(K)^*$, $\eta(B) < \omega_1$ if, and only if, B is norm-separable.

Going back to E, a complemented subspace of $C(K)$ with $\eta(E) = \omega^{\xi+1}$, we remark that it is still unknown if $C(\omega^{\omega^\xi})$ is isomorphic to a subspace of E. A related problem that has been extensively studied is:

(P) : Suppose that X is a separable Banach space and $T\colon C(K) \to X$ is a bounded linear operator. Assume there exist $\epsilon > 0$ and $\xi < \omega_1$ such that $\eta(\epsilon, T^*(B_{X^*})) \geq \omega^\xi$. Does there exist a subspace Y of X isomorphic to $C(\omega^{\omega^\xi})$ and such that the restriction of T to Y is an isomorphism?

Note that an affirmative answer to *(P)* yields that a complemented subspace X of $C(K)$ with Szlenk index equal to $\omega^{\xi+1}$ must contain a subspace isomorphic (and also complemented in X, by the result of [44]) to $C(\omega^{\omega^\xi})$.

Pelczynski [43] showed *(P)* has an affirmative answer when $\xi = 0$. Rosenthal [45] proved that if $T^*(B_{X^*})$ is not norm-separable (which is equivalent to saying that, for some $\epsilon > 0$, the ϵ-Szlenk index of $T^*(B_{X^*})$ is ω_1), then T is an isomorphism on a subspace of $C(K)$ isometric to $C[0, 1]$ (necessarily K is uncountable and then $C(K)$ is isomorphic to $C[0, 1]$ by Miljutin's theorem [38]).

Alspach [1] settled *(P)* in the affirmative when $\xi = 1$. This result was crucial in Zippin's solution of the "separable extension" problem [56], as well as in Benyamini's characterization of the complemented subspaces of $C(\omega^\omega)$ [15]. Alspach also showed *(P)* has a negative answer when $\xi \neq \omega^\gamma$ for all $\gamma < \omega_1$ [3], [5].

Finally, Bourgain [20] established an affirmative answer for *(P)* when $\omega^\xi = \xi$.

The Banach spaces $C(\omega^{\omega^\xi})$, $\xi < \omega_1$, form a complete list of representatives for the isomorphic classes of the $C(K)$ spaces with K countable compact metrizable [18]. When $\xi = 0$, we obtain the familiar space c_0. Hence, $C(\omega^{\omega^\xi})$, $1 \leq \xi$, can be thought as the higher ordinal analogs of c_0. There is however one major difference between c_0 and its transfinite counterparts: Although c_0 possesses an unconditional basis, actually a symmetric one, $C(\omega^{\omega^\xi})$, $\xi \geq 1$, does not. In fact, it does not even embed in a quotient of a subspace of a space with an unconditional basis [46]. The purpose of the present paper is to show that this lack of unconditionality of $C(\omega^{\omega^\xi})$, $\xi \geq 1$, is a reason why *(P)* has, in general, a negative answer.

To explain our result we first note that there exist transfinite analogs of c_0 which do possess unconditional bases. These are the generalized *Schreier spaces* $\{X_\xi\}_{\xi<\omega_1}$, [6], [11]. The construction of these spaces, which has its origins in an example due to Schreier [50], is based on a method initiated by Tsirelson [54] and has many applications in modern Banach space theory [41]. It is shown in [11] that, given a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ containing the singletons and closed under restrictions to initial segments of $\mathbb{N}$ (we assume that $\emptyset \in \mathcal{F}$), one can define a norm $\|\cdot\|_{\mathcal{F}}$ on c_{00} (the linear space of ultimately vanishing scalar sequences) in the following manner:

$$\|x\|_{\mathcal{F}} = \sup\{\sum_{n\in F} |x(n)| : F \in \mathcal{F}\}.$$

Let $X_{\mathcal{F}}$ denote the completion of c_{00} under the norm $\|\cdot\|_{\mathcal{F}}$. It is not hard to see that the natural unit vector basis (e_n) of c_{00} becomes a normalized monotone Schauder basis for the Banach space $X_{\mathcal{F}}$, and that $\|\sum_n a_n e_n\|_{\mathcal{F}} = \sup\{\sum_{n\in F} |x(n)| : F \in \mathcal{F}\}$, for every finitely supported scalar sequence (a_n). In case $\mathcal{F}$ is assumed to be *hereditary* (this means $G \in \mathcal{F}$ whenever $G \subset F$ and $F \in \mathcal{F}$), then (e_n) is an unconditional basis for $X_{\mathcal{F}}$. If $\mathcal{F}$ is compact in the topology of pointwise convergence on the power set of $\mathbb{N}$ (by identifying sets with their indicator functions), then (e_n) is a normalized shrinking basis for $X_{\mathcal{F}}$. For example, taking $\mathcal{F} = \{F \subset \mathbb{N} : |F| \leq 1\}$, we obtain $X_{\mathcal{F}} = c_0$. If we take $\mathcal{F} = S_\xi$, the generalized ξ-th *Schreier family*, $\xi < \omega_1$, [6], [11], then $X_{\mathcal{F}}$ is the Schreier space X_ξ of order ξ. It is known that the Szlenk index of X_ξ is $\omega^{\xi+1}$ (the proof of this fact is implicit in the arguments of [10]; see also Corollary 3.4) and thus equals the Szlenk index of $C(\omega^{\omega^\xi})$. Therefore, in terms of the Szlenk index, X_ξ is comparable to $C(\omega^{\omega^\xi})$ and has an unconditional basis. This justifies our preceding remark that the Schreier spaces X_ξ stand as the transfinite unconditional analogs of c_0.

The Schreier families have played a key role in the recent developments in the geometry of Banach spaces [41]. We shall not describe S_ξ explicitly, because we would rather focus on certain properties these families satisfy which are also shared by other families of finite subsets of $\mathbb{N}$. This will allow us to extend the results stated in the abstract to a more general setting.

A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is *spreading*, if for every choice $\{m_1 <, \dots, < m_k\}$ and $\{n_1 <, \dots, < n_k\}$ of subsets of $\mathbb{N}$ such that $\{m_1, \dots, m_k\} \in \mathcal{F}$ and $m_i \leq n_i$, for all $i \leq k$, we have that $\{n_1, \dots, n_k\} \in \mathcal{F}$. We call $\mathcal{F}$ *regular* [41] if it is hereditary, compact in the topology of pointwise convergence and spreading. It is shown in [6] that the Schreier families $\{S_\xi\}_{\xi<\omega_1}$ are regular and that they exhaust the complexity of countable compact metric spaces. It turns out, [42], that for a regular family $\mathcal{F}$ there is a $\xi < \omega_1$ such that $\mathcal{F}^{(\xi)} = \{\emptyset\}$ (see Proposition 2.11 for a proof of this known fact) and hence $\mathcal{F}$ is homeomorphic to the ordinal interval $[1, \omega^\xi]$, by the Mazurkiewicz-Sierpinski theorem [37]. We shall then say that $\mathcal{F}$ is of *order* ξ. S_ξ is of order ω^ξ, as shown in [6]. We recall here that for a compact metrizable space K and $\alpha < \omega_1$, $K^{(\alpha)}$ denotes the α-th Cantor-Bendixson derivative of K.

A hereditary family $\mathcal{F}$ is *stable*, provided that $F \in \mathcal{F}$ is a maximal, under inclusion, member of $\mathcal{F}$ if, and only if, there exists an $n > \max F$ such that $F \cup \{n\} \notin \mathcal{F}$. It is easy to see that a stable family $\mathcal{F}$ with more than one element must contain all singletons. We also observe that for a non-maximal $F \in \mathcal{F}$, $F \cup \{n\} \in \mathcal{F}$ for all $n > \max F$. It is shown in [24] that S_ξ is stable. We are now ready to state our results.

Theorem 1.1. *Let K be a compact metrizable space, X a separable Banach space and $T\colon C(K) \to X$ a bounded linear operator. Suppose that $\eta(\lambda, T^*(B_{X^*})) \geq \xi$,* *for some $\lambda > 0$ and $\xi < \omega_1$. Let $\mathcal{F}$ be a regular and stable family of order ξ. Then there exists a subspace Y of $C(K)$ isomorphic to $X_{\mathcal{F}}$ and such that the restriction of T to Y is an isomorphism.*

We prove Theorem 1.1 in Section 5. Let us remark here that when $\xi = 1$, the conclusion of Theorem 1.1 is Pelczynski's aforementioned result [43], that a non-weakly compact operator on $C(K)$ is an isomorphism on a subspace of $C(K)$ isomorphic to c_0. Indeed, it is shown in [1] that $T\colon C(K) \to X$ is non-weakly compact if, and only if, $\eta(\lambda, T^*(B_{X^*})) \geq 1$, for some $\lambda > 0$ (recall that $X_{\mathcal{F}}$ is isometric to c_0 when $\mathcal{F}$ is regular of order 1). We have already mentioned that the Schreier families S_ξ are regular and stable of order ω^ξ, and that the corresponding Schreier spaces X_ξ are the natural transfinite unconditional analogs of c_0. From this point of view, Theorem 1.1 can be thought as a natural generalization of Pelczynski's result.

An immediate corollary to Theorem 1.1 is

Corollary 1.2. *Let X be a complemented subspace of $C(K)$ with $\eta(X) = \omega^{\xi+1}$ for some $\xi < \omega_1$. Given any regular and stable family $\mathcal{F}$ of order ω^ξ, the space $X_{\mathcal{F}}$ is isomorphic to a subspace of X.*

Theorem 1.1 is a consequence of our next two results. The first is, roughly speaking, a representation result for measures belonging to the ξ-th Szlenk set of a w^*-compact subset of $B_{C(K)^*}$, using regular families of order ξ. More precisely, in Section 3 we show

Theorem 1.3. *Let K be a totally disconnected, compact, metrizable space. Let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact and such that $\eta(\lambda, \mathcal{M}) \geq \xi$ for some $\lambda > 0$ and $\xi < \omega_1$. Let $\mathcal{F}$ be a regular family of order ξ and set $\mathcal{F}^* = \mathcal{F} \setminus \{\emptyset\}$. Then given $0 < \epsilon < \lambda/4$ there exist a family $(G_\alpha)_{\alpha \in \mathcal{F}^*}$ of clopen subsets of K, and a w^*-compact subset $(\mu_\alpha)_{\alpha \in \mathcal{F}}$ of $\mathcal{M}$ so that the following properties are satisfied:*

1. *The map $\alpha \to \mu_\alpha$ is a homeomorphic embedding of $\mathcal{F}$, equipped with the topology of pointwise convergence, into $\mathcal{M}$, equipped with the w^*-topology.*
2. *$|\mu_\beta(G_\alpha)| \geq \lambda/4 - \epsilon$, for all α and β in $\mathcal{F}^*$ with $\alpha \leqslant \beta$ (the latter meaning that α is an initial segment of β).*
3. *The set $\mathcal{N} = \mathrm{Cl}_{w^*}\{|\mu_\alpha| : \alpha \in \mathcal{F} \text{ is terminal}\}$ is countable (α is terminal if $\alpha = \beta$ for every $\beta \in \mathcal{F}$ such that $\alpha \leqslant \beta$).*
4. *$\lim_n \nu(G_{\alpha \cup \{n\}}) = 0$, for every non-terminal $\alpha \in \mathcal{F}$ and all $\nu \in \mathcal{N}$.*

To prove Theorem 1.1 we shall need Theorem 1.4 which is a refinement of Theorem 1.3.

Theorem 1.4. *Let K be a totally disconnected, compact, metrizable space. Let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact and such that $\eta(\lambda, \mathcal{M}) \geq \xi$ for some $\lambda > 0$ and $\xi < \omega_1$. Let $\mathcal{F}$ be a regular family of order ξ. Then given $0 < \epsilon < \lambda/4$ and $\delta > 0$, there exists a family $(G_\alpha)_{\alpha \in \mathcal{F}^*}$ of clopen subsets of K fulfilling the following property: For every $\alpha \in \mathcal{F}^*$ there exists a $\mu \in \mathcal{M}$ such that $|\mu(G_\beta)| \geq \lambda/4 - \epsilon$, for all $\beta \in \mathcal{F}^*$, $\beta \leqslant \alpha$, yet $\sum_{\beta \in \mathcal{F}^*, \beta \nleqslant \alpha} |\mu|(G_\beta) < \delta$.*

Theorem 1.4 is proved in Section 4. We shall actually prove a stronger result, Theorem 4.3. The latter is a combinatorial result based on the infinite Ramsey theorem [21] and it might be of independent interest. The precise statement, which is too technical to appear in this introductory section, is given in Section 4. Roughly speaking, Theorems 1.4, 4.3 are "tree" versions of results in [12], [14], [22], [39], concerning the detection of a subsequence of a weakly null sequence, such that all further subsequences satisfy a certain property, for instance being nearly unconditional [22], [39], or convexly unconditional [14].

We shall use standard Banach space facts and terminology as may be found in [35].

2. Trees

In this section we shall discuss some basic facts about trees and families of finite subsets of $\mathbb{N}$ which are going to be very useful to our considerations. A *tree* is a partially ordered set $(\mathcal{T}, \leqslant)$ such that for every $\alpha \in \mathcal{T}$ the subset $\{\beta \in \mathcal{T} : \beta \leqslant \alpha\}$ of $\mathcal{T}$ is well ordered. The elements of the tree are called *nodes*. We now fix a tree $(\mathcal{T}, \leqslant)$. For a node $\alpha \in \mathcal{T}$ we let $D_\alpha^{\mathcal{T}}$ denote the set of its immediate successors in $\mathcal{T}$. Thus, if $\beta \in D_\alpha^{\mathcal{T}}$ and $\alpha < \gamma \leqslant \beta$ ($\alpha < \gamma$ means that $\alpha \leqslant \gamma$ and $\alpha \neq \gamma$), then $\gamma = \beta$. A node $\alpha \in \mathcal{T}$ is *terminal* if $D_\alpha^{\mathcal{T}} = \emptyset$. It is called a *root* if it has no predecessors in $\mathcal{T}$. The tree is *rooted* if it has a unique root. It is *infinitely branching* if $D_\alpha^{\mathcal{T}}$ is infinite for every non-terminal node $\alpha \in \mathcal{T}$. A *branch* of $\mathcal{T}$ is a maximal, under inclusion, well-ordered subset. The tree is *well-founded* if it contains no infinite branches.

Any subset of the tree is itself a tree, under the partial ordering inherited by $\mathcal{T}$. A subset $\mathcal{S} \subset \mathcal{T}$ is a *subtree* of $\mathcal{T}$ if for every $\alpha \in \mathcal{S}$ and $\beta \in \mathcal{T}$ such that $\beta \leqslant \alpha$, we have $\beta \in \mathcal{S}$. If $\mathcal{S}$ is a subtree of $\mathcal{T}$, then, clearly, $D_\alpha^{\mathcal{S}} \subset D_\alpha^{\mathcal{T}}$ for all $\alpha \in \mathcal{S}$. In case $|D_\alpha^{\mathcal{S}}| = |D_\alpha^{\mathcal{T}}|$ for all $\alpha \in \mathcal{S}$, we call $\mathcal{S}$ a *full subtree* of $\mathcal{T}$ (a similar terminology was introduced in [26]).

Given trees $(\mathcal{T}_1, \leqslant_1)$ and $(\mathcal{T}_2, \leqslant_2)$, then a map $\theta \colon \mathcal{T}_1 \to \mathcal{T}_2$ is said to be an *order preserving injection* provided $\alpha \leqslant_1 \beta$ in $\mathcal{T}_1$ if, and only if, $\theta(\alpha) \leqslant_2 \theta(\beta)$ in $\mathcal{T}_2$. If, additionally, θ is surjective, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are called *order isomorphic*.

Tree topology. Let $(\mathcal{T}, \leqslant)$ be a tree. We shall discuss a natural topology on $\mathcal{T}$ introduced by H. Rosenthal (unpublished). Consider the family

$$\mathfrak{U}_{\mathcal{T}} = \{U_{\alpha,F} : \alpha \in \mathcal{T},\, F \subset D_\alpha^{\mathcal{T}} \text{ is finite}\},$$

where $U_{\alpha,F} = \{\beta \in \mathcal{T} : \alpha \leqslant \beta,\, \gamma \nleqslant \beta \ \forall \gamma \in F\}$, $F \subset D_\alpha^{\mathcal{T}}$. It is not hard to see that $\mathfrak{U}_{\mathcal{T}}$ is a basis for a Hausdorff topology $\mathfrak{T}_{\mathcal{T}}$ on $\mathcal{T}$. In fact, for every $\alpha \in \mathcal{T}$ the set $\{\beta \in \mathcal{T} : \alpha \leqslant \beta\}$ is clopen with respect to $\mathfrak{T}_{\mathcal{T}}$, and, moreover, the family $\{U_{\alpha,F} : F \subset D_\alpha^{\mathcal{T}} \text{ is finite }\}$ forms a basis (consisting of clopen sets) for the open neighborhoods of α in $\mathfrak{T}_{\mathcal{T}}$. We call $\mathfrak{T}_{\mathcal{T}}$ the tree topology on $\mathcal{T}$. This topology is separable and metrizable if, and only if, $\mathcal{T}$ is countable. If $\mathcal{S} \subset \mathcal{T}$, then we can consider two topologies on it. The first is $\mathfrak{T}_{\mathcal{S}}$, the tree topology induced by the partial ordering $\mathcal{S}$ inherits from $\mathcal{T}$. The second one is $\mathfrak{T}_{\mathcal{T}} \,|\, \mathcal{S}$, the relative $\mathfrak{T}_{\mathcal{T}}$ topology. One checks that $\mathfrak{T}_{\mathcal{S}} \subset \mathfrak{T}_{\mathcal{T}} \,|\, \mathcal{S}$. It follows that the two topologies coincide if $\mathcal{S}$ is a compact subset of $\mathcal{T}$ with respect to $\mathfrak{T}_{\mathcal{T}}$. We also observe that $\mathfrak{T}_{\mathcal{T}} \,|\, \mathcal{S} = \mathfrak{T}_{\mathcal{S}}$ whenever $\mathcal{S}$ is a subtree of $\mathcal{T}$. The next lemma is well known.

Lemma 2.1. *Assume $\mathcal{T}$ is a countable, well-founded and rooted tree. Then the tree topology is compact.*

Proof. As we have already observed, the tree topology is metrizable. Let (α_n) be a sequence in $\mathcal{T}$. We show (α_n) has a convergent subsequence. Define $D = \{\beta \in \mathcal{T} : \beta \leqslant \alpha_n \text{ for infinitely many } n\text{'s}\}$. Note that $\rho \in D$ where ρ denotes the root of

$\mathcal{T}$. Since $\mathcal{T}$ is well-founded there exists an $\alpha_0 \in D$, maximal with respect to the ordering of $\mathcal{T}$. Choose an infinite subset N of $\mathbb{N}$ such that $\alpha_0 \leqslant \alpha_n$ for all $n \in N$. We show $(\alpha_n)_{n\in N}$ converges to α_0. To this end, let $F \subset D^{\mathcal{T}}_{\alpha_0}$ be finite. Given $\beta \in D^{\mathcal{T}}_{\alpha_0}$, we know $\beta \notin D$. It follows that $\bigcup_{\beta\in F}\{n \in \mathbb{N} : \beta \leqslant \alpha_n\}$ is finite. Hence $\alpha_n \in U_{\alpha_0,F}$ for all but finitely many $n \in N$ and thus $\lim_{n\in N} \alpha_n = \alpha_0$. □

An ordinal index for well-founded trees. We shall next deal with an ordinal index known as the *order* of a well-founded tree [28]. Given a rooted, well-founded tree $\mathcal{T}$, we let $\max \mathcal{T}$ denote the set of its terminal nodes. We shall define a decreasing transfinite family $\{\mathcal{T}^{(\alpha)}\}_{\alpha<\omega_1}$ of subtrees of $\mathcal{T}$ as follows: Set $\mathcal{T}^{(0)} = \mathcal{T}$. Suppose $\mathcal{T}^{(\beta)}$ has been defined for all $\beta < \alpha$. If α is a successor, say $\alpha = \beta + 1$, set $\mathcal{T}^{(\alpha)} = \mathcal{T}^{(\beta)} \setminus \max \mathcal{T}^{(\beta)}$. In case α is a limit ordinal, put $\mathcal{T}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{T}^{(\beta)}$. It is easy to verify that $\mathcal{T}^{(\alpha)}$ is a subtree of $\mathcal{T}$ for all $\alpha < \omega_1$. We let $o(\mathcal{T}) = \sup\{\alpha < \omega_1 : \mathcal{T}^{(\alpha)} \neq \emptyset\}$ be the order of $\mathcal{T}$. Because $\mathcal{T}$ is rooted the sup above is actually a max and if $o(\mathcal{T}) = \xi$, then $\mathcal{T}^{(\xi)} = \{\rho\}$, where ρ stands for the root of $\mathcal{T}$. Note also that if $\mathcal{T}$ is countable, then $o(\mathcal{T}) < \omega_1$.

Remark 2.2. We must note here that the derivation $\mathcal{T}^{(\alpha)}$ introduced above should not be confused with the Cantor-Bendixson derivation on $\mathcal{T}$, with respect to the tree topology, although the same notation is used for both derivations. However, we shall see later that those derivations coincide on a class of special trees (see Definition 2.5 and parts (3), (4) of Lemma 2.6).

Notation. Given $t \in \mathcal{T}$ we set $\mathcal{T}_t = \{x \in \mathcal{T} : t \leqslant x\}$. We also define $o(t) = o(\mathcal{T}_t)$, the order of t. Clearly, $o(\rho) = o(\mathcal{T})$ if ρ is the root of $\mathcal{T}$, while $o(t) = 0$ for every terminal node $t \in \mathcal{T}$.

The next two lemmas describe elementary properties of $o(\mathcal{T})$.

Lemma 2.3. *Let $\mathcal{T}$ be rooted and well-founded. Then for every $t \in \mathcal{T}$ and all $\xi < \omega_1$, $\mathcal{T}_t^{(\xi)} = \mathcal{T}_t \cap \mathcal{T}^{(\xi)}$.*

Proof. We prove the lemma by transfinite induction on ξ, the case $\xi = 0$ being trivial. Suppose the assertion holds for all rooted and well-founded trees and all ordinals smaller than ξ. Let $\mathcal{T}$ be rooted and well-founded. Suppose first ξ is a limit ordinal. Then

$$\begin{aligned}\mathcal{T}_t^{(\xi)} &= \bigcap_{\alpha<\xi} \mathcal{T}_t^{(\alpha)} = \bigcap_{\alpha<\xi} (\mathcal{T}_t \cap \mathcal{T}^{(\alpha)}), \text{ by the induction hypothesis,}\\ &= \mathcal{T}_t \cap \Big(\bigcap_{\alpha<\xi} \mathcal{T}^{(\alpha)}\Big) = \mathcal{T}_t \cap \mathcal{T}^{(\xi)}\end{aligned}$$

which proves the assertion for the limit ordinal case. If $\xi = \zeta + 1$, let x be an arbitrary node in $\mathcal{T}_t^{(\zeta+1)}$. Then $x \in \mathcal{T}_t^{(\zeta)} \setminus \max(\mathcal{T}_t^{(\zeta)})$, and thus $x \in (\mathcal{T}_t \cap \mathcal{T}^{(\zeta)}) \setminus \max(\mathcal{T}_t \cap \mathcal{T}^{(\zeta)})$, by the induction hypothesis. But now, x cannot be a terminal node in $\mathcal{T}^{(\zeta)}$, or else x would be terminal in $\mathcal{T}_t \cap \mathcal{T}^{(\zeta)}$ as well, which is a contradiction. Therefore $x \notin \max(\mathcal{T}^{(\zeta)})$ and subsequently $x \in (\mathcal{T}_t \cap \mathcal{T}^{(\zeta)}) \setminus \max(\mathcal{T}^{(\zeta)})$. It follows that $x \in \mathcal{T}_t \cap \mathcal{T}^{(\zeta+1)}$, showing that $\mathcal{T}_t^{(\zeta+1)} \subset \mathcal{T}_t \cap \mathcal{T}^{(\zeta+1)}$.

Conversely, suppose $x \in \mathcal{T}_t \cap \mathcal{T}^{(\zeta+1)}$. It follows that $x \in (\mathcal{T}_t \cap \mathcal{T}^{(\zeta)}) \setminus \max(\mathcal{T}^{(\zeta)})$ and so $x \in \mathcal{T}_t^{(\zeta)} \setminus \max(\mathcal{T}^{(\zeta)})$, by the induction hypothesis. We now claim that $x \notin \max(\mathcal{T}_t^{(\zeta)})$. Indeed, otherwise, x would be terminal in $\mathcal{T}_t \cap \mathcal{T}^{(\zeta)}$ and thus also terminal in $\mathcal{T}^{(\zeta)}$. However, $x \notin \max(\mathcal{T}^{(\zeta)})$. Concluding, $\mathcal{T}_t \cap \mathcal{T}^{(\zeta+1)} \subset \mathcal{T}_t^{(\zeta+1)}$, completing the inductive step and the proof of the lemma. $\square$

Lemma 2.4. *Let $\mathcal{T}$ be rooted and well-founded. Let $t \in \mathcal{T}$. Then $o(t) = 0$ if t is terminal, while $o(t) = \sup\{o(s) + 1 : s \in D_t^{\mathcal{T}}\}$ otherwise.*

Proof. The first assertion is trivial. To prove the second one we make a few preliminary observations. Suppose that $\mathcal{T}$ is of order ξ and let ρ denote its root. Then $\mathcal{T}^{(\xi)} = \{\rho\}$. Indeed, $\mathcal{T}^{(\xi)}$ is a non-empty subtree of $\mathcal{T}$. This is immediate if ξ is a limit ordinal, as $\rho \in \mathcal{T}^{(\alpha)}$ for all $\alpha < \xi$. In case ξ is a successor, say $\xi = \zeta + 1$, then $\mathcal{T}^{(\zeta)} \neq \emptyset$ and if $\mathcal{T}^{(\zeta+1)} = \emptyset$ we would have $o(\mathcal{T}) = \zeta$, a contradiction. Hence, $\rho \in \mathcal{T}^{(\xi)}$. Finally, if $t \in \mathcal{T}^{(\xi)}$, $t \neq \rho$, then we must have $\rho < t$ and subsequently $\max \mathcal{T}^{(\xi)} \neq \mathcal{T}^{(\xi)}$. Therefore $\mathcal{T}^{(\xi+1)} \neq \emptyset$, a contradiction.

A second observation is that if $\mathcal{S} \subset \mathcal{T}$ is itself rooted, then $\mathcal{S}^{(\alpha)} \subset \mathcal{T}^{(\alpha)}$, for all $\alpha < \omega_1$. It follows that $o(\mathcal{S}) \leq o(\mathcal{T})$.

In order to complete the proof of the lemma we show that $o(t_2) < o(t_1)$ whenever $t_1 < t_2$ in $\mathcal{T}$. Indeed, since $\mathcal{T}_{t_2} \subset \mathcal{T}_{t_1}$, our preceding observation yields $o(t_2) \leq o(t_1)$. Put $o(t_2) = \xi_2$. Then

$$
\begin{aligned}
\mathcal{T}_{t_2}^{(\xi_2)} &= \{t_2\} = \mathcal{T}_{t_2} \cap \mathcal{T}^{(\xi_2)}, \text{ by Lemma 2.3,} \\
&= \mathcal{T}_{t_2} \cap \mathcal{T}_{t_1} \cap \mathcal{T}^{(\xi_2)} = \mathcal{T}_{t_2} \cap \mathcal{T}_{t_1}^{(\xi_2)}, \text{ by Lemma 2.3.}
\end{aligned}
$$

Hence $t_1 < t_2$ and both belong to $\mathcal{T}_{t_1}^{(\xi_2)}$. It follows that $\mathcal{T}_{t_1}^{(\xi_2+1)} \neq \emptyset$. Therefore, $o(t_2) + 1 \leq o(t_1)$. Next suppose that there exists a $t \in \mathcal{T}$ such that $\alpha < o(t)$, where we have set $\alpha = \sup\{o(s) + 1 : s \in D_t^{\mathcal{T}}\}$. Then $\alpha + 1 \leq o(t)$ and so $\mathcal{T}_t^{(\alpha+1)} \neq \emptyset$. Choose $s \in \mathcal{T}_t^{(\alpha)}$, $t < s$. Then choose $s_0 \in D_t^{\mathcal{T}}$ with $s_0 \leqslant s$. It follows that

$$
\begin{aligned}
s_0 \in \mathcal{T}_t^{(\alpha)} \cap \mathcal{T}_{s_0} &= \mathcal{T}_t \cap \mathcal{T}^{(\alpha)} \cap \mathcal{T}_{s_0}, \text{ by Lemma 2.3,} \\
&= \mathcal{T}^{(\alpha)} \cap \mathcal{T}_{s_0} = \mathcal{T}_{s_0}^{(\alpha)}, \text{ by Lemma 2.3.}
\end{aligned}
$$

Hence, $o(s_0) + 1 \leq \alpha \leq o(s_0)$, a contradiction. $\square$

Subtrees of $[\mathbb{N}]^{<\infty}$. If X is any set, $[X]^{<\infty}$ denotes the set of its finite subsets. $[\mathbb{N}]^{<\infty}$ can be naturally viewed as a tree under the following partial ordering: $\{m_1 <, \dots, < m_k\} \leqslant \{n_1 <, \dots, < n_l\}$ if, and only if, $k \leq l$ and $m_i = n_i$ for all $i \leq k$. We agree that $\emptyset$ is the root of $[\mathbb{N}]^{<\infty}$. Of course $[\mathbb{N}]^{<\infty}$ is infinitely branching and it is easy to see that the tree topology coincides with the topology of pointwise convergence on $[\mathbb{N}]^{<\infty}$ (for the latter topology, we identify sets with their indicator functions). It is not hard to see, by transfinite induction on the order of the tree, that every countable, well-founded rooted tree is order isomorphic to a subtree of $[\mathbb{N}]^{<\infty}$.

Definition 2.5. A countable tree $\mathcal{T}$ is blossomed if it is rooted, infinitely branching, well-founded and for every non-terminal node $t \in \mathcal{T}$ there exists an enumeration $(t_n)_{n=1}^{\infty}$ of $D_t^{\mathcal{T}}$ such that the sequence $(o(t_n))_{n=1}^{\infty}$ is non-decreasing. We make the convention that trees with only one node are blossomed.

Blossomed trees are variants of the so-called *replacement trees* introduced by Judd and Odell [27]. We refer to [10], [27] for an in-depth treatment of trees in Banach space theory.

Blossomed trees will be very useful to our considerations. The next lemmas contain a few permanence properties of such trees.

Lemma 2.6. (1) *Let $\mathcal{T}$ be blossomed and $t \in \mathcal{T}$. Then $\mathcal{T}_t$ is blossomed.*

(2) *Suppose $\mathcal{T}$ is an infinitely branching, countable, well-founded, rooted tree. Let ρ be the root of $\mathcal{T}$ and assume that there exists an enumeration (t_n) of $D_\rho^{\mathcal{T}}$, with $(o(t_n))$ non-decreasing, such that $\mathcal{T}_{t_n}$ is blossomed for all $n \in \mathbb{N}$. Then $\mathcal{T}$ is blossomed.*

(3) *If $\mathcal{T}$ is blossomed and $o(\mathcal{T}) = \xi$, then $\mathcal{T}$, equipped with the tree topology, is homeomorphic to $[1, \omega^\xi]$. Moreover, the ξ-th Cantor-Bendixson derived set of $\mathcal{T}$ is $\{\rho\}$, where ρ is the root of $\mathcal{T}$.*

(4) *Let $\mathcal{T}$ be a countable, rooted, infinitely branching, well-founded tree satisfying the following property: For every $t \in \mathcal{T}$ there exists $\xi_t < \omega_1$ such that the ξ_t-th Cantor-Bendixson derived set (with respect to the tree topology) of $\mathcal{T}_t$ is $\{t\}$. Assume that for every non-terminal node t of $\mathcal{T}$ there exists an enumeration (t_n) of $D_t^{\mathcal{T}}$, such that (ξ_{t_n}) is non-decreasing. Then $\mathcal{T}$ is blossomed and $o(t) = \xi_t$, for all $t \in \mathcal{T}$.*

Proof. The first two assertions are easily established. We first prove (3). This is done by transfinite induction on ξ. If $\xi = 0$ the assertion is trivial. Assume the assertion holds for blossomed trees of order less than ξ. Let (t_n) be an enumeration of $D_\rho^{\mathcal{T}}$, with $(o(t_n))$ non-decreasing. Put $\xi_n = o(t_n)$, $n \in \mathbb{N}$. We know that $\xi = \sup_n(\xi_n + 1)$, thanks to Lemma 2.4. The induction hypothesis now yields that $\mathcal{T}_{t_n}$ is homeomorphic to $[1, \omega^{\xi_n}]$ with t_n being the only element in the ξ_n-th Cantor-Bendixson derived set of $\mathcal{T}_{t_n}$. Since $\mathcal{T}_{t_n}$ is clopen in $\mathcal{T}$ the assertion follows from Lemma 2.10.

We now show that (4) holds. Let ρ be the root of $\mathcal{T}$ and set $\xi^{\mathcal{T}} = \xi_\rho$. We prove the assertion of the lemma by transfinite induction on $\xi^{\mathcal{T}}$. The case $\xi^{\mathcal{T}} = 0$ is trivial. Let $\xi < \omega_1$ and suppose the assertion holds for all trees $\mathcal{T}$ such that $\xi^{\mathcal{T}} < \xi$. Now let $\mathcal{T}$ be a tree such that $\xi^{\mathcal{T}} = \xi$. Let (t_n) be an enumeration of the immediate successors of the root of $\mathcal{T}$, with (ξ_{t_n}) non-decreasing and such that the ξ_{t_n}-th Cantor-Bendixson derived set of $\mathcal{T}_{t_n}$ is $\{t_n\}$, for all $n \in \mathbb{N}$. It follows by Lemma 2.10 that $\sup_n(\xi_{t_n} + 1) = \xi$. The induction hypothesis now implies that $\mathcal{T}_{t_n}$ is blossomed. Moreover, $o(t) = \xi_t$, for every node t such that $t_n \leqslant t$, and for all $n \in \mathbb{N}$. We deduce from part (2), above, combined with Lemma 2.4, that $\mathcal{T}$ is blossomed with $o(\mathcal{T}) = \xi$. $\square$

Lemma 2.7. *Let $\mathcal{T}$ and $\mathcal{S}$ be blossomed trees such that $o(\mathcal{S}) \leq o(\mathcal{T})$. Then $\mathcal{S}$ is order isomorphic to a subtree of $\mathcal{T}$.*

Proof. We prove the lemma by transfinite induction on $o(\mathcal{T}) = \xi$. If $\xi = 0$, the assertion is trivial. Assume $\xi \geq 1$ and that the assertion holds for blossomed trees of order smaller than ξ. Let $\mathcal{T}$ be blossomed of order ξ. Let $\mathcal{S}$ be a blossomed tree such that $o(\mathcal{S}) = \zeta \leq \xi$. Assume $\zeta \geq 1$ as well, for otherwise the assertion is again trivial. Let s_0 and t_0 denote the roots of $\mathcal{S}$ and $\mathcal{T}$, respectively. We may choose enumerations (s_n) of $D_{s_0}^{\mathcal{S}}$ and (t_n) of $D_{t_0}^{\mathcal{T}}$ such that the ordinal sequences $(o(s_n))$ and $(o(t_n))$ are both non-decreasing. Put $\zeta_n = o(s_n)$ and $\xi_n = o(t_n)$, for all $n \in \mathbb{N}$. Since $\zeta_n < \zeta \leq \xi$, for all $n \in \mathbb{N}$, and $\xi = \sup_n(\xi_n + 1)$, we may choose positive integers $m_1 < m_2 < \dots$ such that $\zeta_n < \xi_{m_n} + 1$, for all $n \in \mathbb{N}$. Hence $\zeta_n \leq \xi_{m_n}$ and $o(\mathcal{T}_{t_{m_n}}) = \xi_{m_n} < \xi$, for all $n \in \mathbb{N}$. Because $o(\mathcal{S}_{s_n}) = \zeta_n$, Lemma 2.6 and the induction hypothesis yield a subtree $\mathcal{S}_n$ of $\mathcal{T}_{t_{m_n}}$ which is order isomorphic to $\mathcal{S}_{s_n}$, for all $n \in \mathbb{N}$. It is easy to check that $\bigcup_n \mathcal{S}_n \cup \{t_0\}$ is a subtree of $\mathcal{T}$ order isomorphic to $\mathcal{S}$. $\square$

Lemma 2.8. *Let $\mathcal{T}$ be blossomed of order ξ and $\mathcal{S}$ a full subtree of $\mathcal{T}$. Then $\mathcal{S}$ is also blossomed of order ξ.*

Proof. We prove the lemma by transfinite induction on $o(\mathcal{T}) = \xi$, the case $\xi = 0$ being trivial. Suppose the assertion holds for blossomed trees of order less than ξ and let $\mathcal{T}$ be blossomed of order ξ. Let ρ be the root of $\mathcal{T}$, and let (t_n) be an enumeration of $D_\rho^{\mathcal{T}}$ such that $o(t_n) = \xi_n$ with (ξ_n) non-decreasing. We can write $\mathcal{T} = \bigcup_n \mathcal{T}_{t_n} \cup \{\rho\}$. Next let $\mathcal{S}$ be a full subtree of $\mathcal{T}$ and define $N = \{n \in \mathbb{N} : \mathcal{S} \cap \mathcal{T}_{t_n} \neq \emptyset\}$. It is clear that N is infinite and that $\mathcal{S} \cap \mathcal{T}_{t_n}$ is a full subtree of $\mathcal{T}_{t_n}$, for all $n \in N$. Part (1) of Lemma 2.6 now implies that $\mathcal{T}_{t_n}$ is blossomed of order ξ_n, for all $n \in \mathbb{N}$. We infer from the induction hypothesis that $\mathcal{S} \cap \mathcal{T}_{t_n}$ is blossomed of order ξ_n, for all $n \in N$. Since $\mathcal{S} = \bigcup_{n\in N} (\mathcal{S} \cap \mathcal{T}_{t_n}) \cup \{\rho\}$ (actually, $\mathcal{S}_{t_n} = \mathcal{S} \cap \mathcal{T}_{t_n}$ for all $n \in N$) and N is infinite, part (2) of Lemma 2.6 yields $\mathcal{S}$ is blossomed of order ξ. □

Families of finite subsets of $\mathbb{N}$. Let $\mathcal{F} \subset [\mathbb{N}]^{<\infty}$. For such a family, the terms hereditary, spreading, compact, regular and stable have already been given in Section 1. Every hereditary family $\mathcal{F}$ can be viewed as a subtree of $[\mathbb{N}]^{<\infty}$ (it suffices that $\mathcal{F}$ be closed under restrictions to initial segments). If $\mathcal{F}$ is hereditary and compact in the topology of pointwise convergence, then it is a well-founded subtree of $[\mathbb{N}]^{<\infty}$. In case $\mathcal{F}$ is additionally assumed to be spreading, then $\mathcal{F}$ is a well-founded, infinitely branching subtree of $[\mathbb{N}]^{<\infty}$. We remark that in this case the tree topology on $\mathcal{F}$ coincides with the topology of pointwise convergence. We are going to show that every hereditary, compact and spreading family is a blossomed subtree of $[\mathbb{N}]^{<\infty}$. This will be a consequence of the Mazurkiewicz-Sierpinski theorem [37]. We shall need two preparatory lemmas.

Lemma 2.9. *Let K be a countable compact metric space. Suppose there exist a sequence (K_n) of pairwise disjoint non-empty clopen subsets of K and $x_0 \in K \setminus \bigcup_n K_n$, so that $K = \bigcup_n K_n \cup \{x_0\}$. Assume further that K_n is homeomorphic to a subset of K_{n+1}, for all $n \in \mathbb{N}$. Then there exists a $\xi < \omega_1$ such that $K^{(\xi)} = \{x_0\}$.*

Proof. The Mazurkiewicz-Sierpinski theorem [37] implies the existence of a $\xi < \omega_1$ such that $K^{(\xi)}$ is finite non-empty. Assume there exists some $x \in K^{(\xi)} \setminus \{x_0\}$. Then choose $n_0 \in \mathbb{N}$ such that $x \in K_{n_0}$. Since K_{n_0} is clopen in K, we infer that $x \in K_{n_0}^{(\xi)}$. Since K_{n_0} is homeomorphic to a subset of K_n, for all $n \geq n_0$, we conclude that there exists $x_n \in K_n^{(\xi)}$, for all $n \geq n_0$. It follows that $(x_n)_{n\geq n_0}$ is an infinite sequence in $K^{(\xi)}$, contrary to our assumption. □

Lemma 2.10. *Let K be a countable compact metric space. Suppose there exist a sequence (K_n) of pairwise disjoint non-empty clopen subsets of K and $x_0 \in K \setminus \bigcup_n K_n$, so that $K = \bigcup_n K_n \cup \{x_0\}$. Assume further there exists a non-decreasing sequence (ξ_n) of countable ordinals, such that $K_n^{(\xi_n)}$ is a singleton, for all $n \in \mathbb{N}$. Then $K^{(\xi)} = \{x_0\}$, where $\xi = \sup_n(\xi_n + 1)$.*

Proof. The Mazurkiewicz-Sierpinski theorem [37] implies that K_n is homeomorphic to $[1, \omega^{\xi_n}]$, for all $n \in \mathbb{N}$. Since (ξ_n) is non-decreasing, we infer that K_n is homeomorphic to a subset of K_{n+1}, for all $n \in \mathbb{N}$. Lemma 2.9 yields a $\zeta < \omega_1$ such that $K^{(\zeta)} = \{x_0\}$. We show $\zeta = \sup_n(\xi_n + 1)$. Indeed, first observe that $\xi_n < \zeta$, for all $n \in \mathbb{N}$. Otherwise, $\zeta \leq \xi_{n_0}$ for some $n_0 \in \mathbb{N}$ and so $K_{n_0}^{(\zeta)} \neq \emptyset$. Since

$K_{n_0}^{(\zeta)} \subset K^{(\zeta)}$, we deduce that $x_0 \in K_{n_0}$, contrary to our assumptions. It follows that $\xi = \sup_n(\xi_n + 1) \le \zeta$.

Finally, suppose $\xi < \zeta$. Then $K^{(\xi)}$ is infinite and we may choose $x \in K^{(\xi)} \setminus \{x_0\}$. Next choose $m \in \mathbb{N}$ with $x \in K_m$. Because K_m is clopen we infer that $x \in K_m^{(\xi)}$. However, $K_m^{(\xi)}$ is empty as $K_m^{(\xi_m)}$ is a singleton and $\xi_m < \xi$. □

Proposition 2.11. *Let $\mathcal{F}$ be a regular family of finite subsets of $\mathbb{N}$. Then $\mathcal{F}$ is a blossomed tree.*

Proof. To avoid trivialities assume $\mathcal{F} \neq \{\emptyset\}$. From a previous discussion we have $\mathcal{F}$ is a well-founded, infinitely branching subtree of $[\mathbb{N}]^{<\infty}$. Given $\alpha \in \mathcal{F}$ we recall that $\mathcal{F}_\alpha = \{\beta \in \mathcal{F} : \alpha \leqslant \beta\}$, which is a clopen subset of $\mathcal{F}$ (relatively to the tree topology of $\mathcal{F}$ which of course coincides with the topology of pointwise convergence on $\mathcal{F}$). If α is non-terminal in $\mathcal{F}$, set $M_\alpha = \{n \in \mathbb{N} : \max \alpha < n, \, \alpha \cup \{n\} \in \mathcal{F}\}$. This is an infinite subset of $\mathbb{N}$ by the spreading property of $\mathcal{F}$. We certainly have that $\mathcal{F}_\alpha = \bigcup_{n \in M_\alpha} \mathcal{F}_{\alpha \cup \{n\}} \cup \{\alpha\}$. We also have that the sequence $(\mathcal{F}_{\alpha \cup \{n\}})_{n \in M_\alpha}$ consists of pairwise disjoint, non-empty clopen subsets of $\mathcal{F}$. We are going to show that $\mathcal{F}_{\alpha \cup \{n_1\}}$ is homeomorphic to a subset of $\mathcal{F}_{\alpha \cup \{n_2\}}$, for all $n_1 < n_2$ in M_α. To see this define a map $\phi \colon \mathcal{F}_{\alpha \cup \{n_1\}} \to \mathcal{F}_{\alpha \cup \{n_2\}}$ by the rule $\phi(\alpha \cup F) = \alpha \cup [(n_2 - n_1) + F]$, for every $F \in \mathcal{F}$ such that $\min F = n_1$ and $\alpha \cup F \in \mathcal{F}$. In the above, for an integer k we let $k + F$ denote the set $\{k + i : i \in F\}$. ϕ is well defined because $\mathcal{F}$ is spreading. It is easy to see that ϕ is injective and continuous and thus a homeomorphic embedding. Lemma 2.9 yields an ordinal ξ_α such that $\mathcal{F}_\alpha^{(\xi_\alpha)} = \{\alpha\}$. Since $\xi_{\alpha \cup \{n_1\}} \le \xi_{\alpha \cup \{n_2\}}$ whenever $n_1 < n_2$ in M_α, we deduce, using part (4) of Lemma 2.6, that $\mathcal{F}$ is blossomed. □

Remark 2.12. We must mention here that for every $\xi < \omega_1$ there exists a regular and stable family of order ξ. Indeed, when ξ is of power-type, say, ω^α, then the Schreier family S_α is regular of order ξ [6] and stable [24]. If ξ is any countable ordinal, then regular and stable families of order ξ were first considered in [53], and later in [13], [23], [49]. Such families are constructed by transfinite induction on ξ. We shall briefly sketch the construction. If $\mathcal{F}$ is a regular and stable family of order ξ, then the family $\{F \in [\mathbb{N}]^{<\infty} : F \setminus \{\min F\} \in \mathcal{F}\} \cup \{\emptyset\}$ is regular and stable of order $\xi + 1$. Next suppose ξ is a limit ordinal and let (ξ_n) be a sequence of ordinals strictly increasing to ξ. Assume $\mathcal{F}_n$ is regular and stable of order ξ_n, for all $n \in \mathbb{N}$. Then $\{F \in [\mathbb{N}]^{<\infty} : \exists\, n \le \min F, \, F \in \mathcal{F}_n\} \cup \{\emptyset\}$ is a regular and stable family of order ξ. It is easy to see that those families are hereditary and spreading, containing the singletons. Compactness follows by applying Lemma 2.10. Stability is proven using the argument in Lemma 3.1 of [24].

The final result in this section discusses the construction of some special universal subtrees of $[\mathbb{N}]^{<\infty}$. In the sequel, if X is any set, we let $[X]$ denote the set of all infinite subsets of X.

Lemma 2.13. *Let $M \in [\mathbb{N}]$. There exists a family $(M_\alpha)_{\alpha \in [\mathbb{N}]^{<\infty}}$ of pairwise disjoint infinite subsets of M (i.e., $M_\alpha \cap M_\beta = \emptyset$ whenever $\alpha \neq \beta$ in $[\mathbb{N}]^{<\infty}$) so that letting $(m_i^\alpha)_{i=1}^\infty$ be the increasing enumeration of M_α, we have that $m_i^{\alpha^-} < m_i^\alpha$, for all $i \in \mathbb{N}$ and $\alpha \in [\mathbb{N}]^{<\infty}$, $\alpha \neq \emptyset$ (α^- stands for the predecessor of α in $[\mathbb{N}]^{<\infty}$).*

Proof. Choose $M_\emptyset \in [M]$ so that $M \setminus M_\emptyset \in [M]$. Let $(m_i^\emptyset)$ be the increasing enumeration of $M_\emptyset$ and choose an infinite sequence (N_i) of pairwise disjoint, infinite subsets of $M \setminus M_\emptyset$ such that $M \setminus M_\emptyset \setminus \bigcup_i N_i \in [M]$. We can now choose $M_{\{k\}} \in [N_k]$ such that, if we let $(m_i^{\{k\}})_{i=1}^\infty$ be the increasing enumeration of $M_{\{k\}}$, then $m_i^\emptyset < m_i^{\{k\}}$ for all integers i and k.

Suppose that $n \in \mathbb{N}$ and that we have constructed a family $\{M_\alpha : \alpha \in [\mathbb{N}]^{<\infty}, |\alpha| \le n\}$ of pairwise disjoint infinite subsets of M such that

(1) $M \setminus \bigcup_{\alpha\colon |\alpha|\le n} M_\alpha$ is infinite.
(2) If $(m_i^\alpha)_{i=1}^\infty$ is the increasing enumeration of M_α, then $m_i^{\alpha^-} < m_i^\alpha$, for all $i \in \mathbb{N}$ and $\alpha \in [\mathbb{N}]^{<\infty}$, $\alpha \ne \emptyset$, with $|\alpha| \le n$.

Let (α_k) be an enumeration of $\{\alpha \in [\mathbb{N}]^{<\infty}, |\alpha| = n+1\}$ and choose an infinite sequence (N_{α_k}) of pairwise disjoint infinite subsets of $M \setminus \bigcup_{\alpha\colon |\alpha|\le n} M_\alpha$ so that $M \setminus \bigcup_{\alpha\colon |\alpha|\le n} M_\alpha \setminus \bigcup_k N_{\alpha_k}$ is infinite. Given $\alpha \in [\mathbb{N}]^{<\infty}$ with $|\alpha| = n+1$, choose $M_\alpha \in [N_\alpha]$ so that if we let $(m_i^\alpha)_{i=1}^\infty$ be the increasing enumeration of M_α, then $m_i^{\alpha^-} < m_i^\alpha$, for all $i \in \mathbb{N}$ (recall that $|\alpha^-| = n$ and so M_{α^-} has been constructed). It follows that (1) and (2) hold for $n+1$ and therefore the construction of the required family $(M_\alpha)_{\alpha\in[\mathbb{N}]^{<\infty}}$ is carried over by induction. □

Proposition 2.14. *Let $M \in [\mathbb{N}]$. There exists a subtree $\mathcal{T}_\infty^M$ of $[M]^{<\infty}$ with the following properties:*

(1) *If α, β are nodes of $\mathcal{T}_\infty^M \setminus \{\emptyset\}$ and $\max \beta \in \alpha$, then $\beta \leqslant \alpha$.*
(2) *There exists an order preserving injection $\sigma\colon [\mathbb{N}]^{<\infty} \to \mathcal{T}_\infty^M$ such that $\sigma(\mathcal{F})$ is a subtree of $\mathcal{T}_\infty^M \cap \mathcal{F}$, for every hereditary and spreading family $\mathcal{F}$.*

Proof. Let $(M_\alpha)_{\alpha\in[\mathbb{N}]^{<\infty}}$ be the family of infinite subsets of M constructed in Lemma 2.13. Put $D_0 = \{\emptyset\}$. Suppose that $n \in \mathbb{N} \cup \{0\}$ and that $D_n \subset \{\alpha \in [\mathbb{N}]^{<\infty}, |\alpha| = n\}$ has been defined. Set

$$D_{n+1} = \{\alpha \in [\mathbb{N}]^{<\infty} : |\alpha| = n+1,\, \alpha^- \in D_n,\, \max \alpha \in M_{\alpha^-}\}.$$

We set $\mathcal{T}_\infty^M = \bigcup_{n=0}^\infty D_n$. It follows immediately from the inductive construction that $\mathcal{T}_\infty^M$ is a subtree of $[M]^{<\infty}$.

We first show that (1) holds. We shall prove by induction on $n \in \mathbb{N} \cup \{0\}$ that if $\alpha \in D_n$ and $\beta \in \mathcal{T}_\infty^M \setminus \{\emptyset\}$ satisfy $\max \beta \in \alpha$, then $\beta \leqslant \alpha$. This assertion is trivial if $n = 0$. Assuming the assertion is true for n, let $\alpha \in D_{n+1}$ and $\beta \in \mathcal{T}_\infty^M \setminus \{\emptyset\}$ with $\max \beta \in \alpha$. In case $\max \beta \in \alpha^-$, since $\alpha^- \in D_n$, the induction hypothesis yields $\beta \leqslant \alpha^-$ and so $\beta \leqslant \alpha$ as well. If $\max \beta \notin \alpha^-$, then $\max \beta = \max \alpha$. Note that by the inductive construction of $\mathcal{T}_\infty^M$, we have $\max \gamma \in M_{\gamma^-}$, for all $\gamma \in \mathcal{T}_\infty^M \setminus \{\emptyset\}$. Hence $M_{\alpha^-} \cap M_{\beta^-} \ne \emptyset$. We deduce from this that $\alpha^- = \beta^-$ and so $\alpha = \beta$ in this case, completing the inductive step.

To show (2), define a map $\sigma\colon [\mathbb{N}]^{<\infty} \to \mathcal{T}_\infty^M$ in the following manner: $\sigma(\emptyset) = \emptyset$. If $i_1 < \cdots < i_k$ are in $\mathbb{N}$, then $\sigma(\{i_1, \dots, i_k\}) = \{d_1 <, \dots, < d_k\}$, where d_1 is the i_1-st element of $M_\emptyset$, while d_j is the i_j-th element of $M_{\{d_1,\dots,d_{j-1}\}}$ for $2 \le j \le k$. It is easy to verify that σ is a well-defined, order preserving injection, mapping $[\mathbb{N}]^{<\infty}$ onto a subtree of $\mathcal{T}_\infty^M$. This is a consequence of the fact that $m_i^\alpha < m_i^\beta$ for all $i \in \mathbb{N}$ and $\alpha < \beta$ in $[\mathbb{N}]^{<\infty}$ (recall that $(m_i^\alpha)_{i=1}^\infty$ is the increasing enumeration of M_α). We also obtain that $i_j \le d_j$ for all $j \le k$. It follows from this that if $\mathcal{F}$ is spreading and $\{i_1 <, \dots, < i_k\} \in \mathcal{F}$, then $\sigma(\{i_1, \dots, i_k\}) \in \mathcal{F}$. □

3. Trees and the Szlenk index

The present section is devoted to the proof of Theorem 1.3. We shall also sketch a proof for the calculation of the Szlenk index of the Schreier spaces X_ξ in Corollary 3.4 (cf. also [10]). To achieve these goals, we shall use trees in order to find suitable representations for elements in the ξ-th Szlenk set of a w^*-compact subset of B_{X^*} (X a separable Banach space). The recent papers [10], [26], [27], [30], [31] make systematic use of trees in the study of the Szlenk index as well as other related ordinal indices. The main ideas contained in [10], [26], [27] about tree representations of Szlenk sets are also employed in the results of this section.

The Szlenk sets $P_\alpha(\epsilon, B)$, $\alpha < \omega_1$, B a w^*-compact subset of X^*, $\epsilon > 0$, were described in Section 1. It is easy to verify that $P_\beta(\epsilon, B) \subset P_\alpha(\epsilon, B)$, for all $\alpha \leq \beta < \omega_1$. Szlenk [52] shows that $P_\alpha(\epsilon, B)$ is w^*-compact when X^* is separable. In the first part of this section we show that $\eta(X_\xi) = \omega^{\xi+1}$, for all $\xi < \omega_1$.

Notation. Given a Banach space X, $\alpha < \omega_1$, and $\lambda > 0$, we shall write $P_\alpha(\lambda)$ instead of $P_\alpha(\lambda, B_{X^*})$.

The next lemma is implicitly contained in [10], [26].

Lemma 3.1. *Let X be a Banach space, $\lambda > 0$ and ζ, ξ countable ordinals. Suppose that $x^* \in P_\xi(\lambda)$ and $y^* \in P_\zeta(\lambda)$. Then $(x^* + y^*)/2 \in P_{\xi+\zeta}(\lambda/2 - \delta)$, for all $0 < \delta < \lambda/2$.*

Proof. We first show by transfinite induction on ξ that $(x^* + y^*)/2 \in P_\xi(\lambda/2 - \delta)$ for all $x^* \in P_\xi(\lambda)$, $y^* \in B_{X^*}$ and $0 < \delta < \lambda/2$. This is trivial if $\xi = 0$. Assume the assertion true for ordinals smaller than ξ. Let $x^* \in P_\xi(\lambda)$, $y^* \in B_{X^*}$ and $0 < \delta < \lambda/2$.

We first consider the case of a successor ordinal $\xi = \alpha + 1$. Choose sequences (x_n^*) in $P_\alpha(\lambda)$ w^*-converging to x^*, and (x_n) in B_X weakly converging to 0, so that $|x_n^*(x_n)| \geq \lambda$ for all $n \in \mathbb{N}$. The induction hypothesis implies $(x_n^* + y^*)/2 \in P_\alpha(\lambda/2 - \delta)$ for all $n \in \mathbb{N}$. We can assume without loss of generality that $|y^*(x_n)| < \delta$, for all $n \in \mathbb{N}$. It is now clear that $(x^* + y^*)/2 \in P_\xi(\lambda/2 - \delta)$.

Next suppose that ξ is a limit ordinal. The induction hypothesis yields $(x^* + y^*)/2 \in P_\alpha(\lambda/2 - \delta)$ for all $\alpha < \xi$ and so $(x^* + y^*)/2 \in P_\xi(\lambda/2 - \delta)$ as required.

The proof of the lemma will be completed once we show that for fixed $\xi < \omega_1$ and $x^* \in P_\xi(\lambda)$ we have $(x^* + y^*)/2 \in P_{\xi+\zeta}(\lambda/2 - \delta)$, for all $\zeta < \omega_1$, all $y^* \in P_\zeta(\lambda)$ and all $0 < \delta < \lambda/2$. This is accomplished by transfinite induction on ζ. The case $\zeta = 0$ was settled in the preceding paragraphs. Assume the assertion holds for ordinals smaller than ζ and let $y^* \in P_\zeta(\lambda)$, $0 < \delta < \lambda/2$.

Suppose first that ζ is a successor, say $\zeta = \alpha + 1$. Choose sequences (y_n^*) in $P_\alpha(\lambda)$ w^*-converging to y^*, and (y_n) in B_X weakly converging to 0, so that $|y_n^*(y_n)| \geq \lambda$ for all $n \in \mathbb{N}$. The induction hypothesis implies $(x^* + y_n^*)/2 \in P_{\xi+\alpha}(\lambda/2 - \delta)$ for all $n \in \mathbb{N}$. We can assume without loss of generality that $|x^*(y_n)| < \delta$, for all $n \in \mathbb{N}$. It is now clear that $(x^* + y^*)/2 \in P_{\xi+\zeta}(\lambda/2 - \delta)$.

Finally, assume ζ is a limit ordinal. Then $\xi + \zeta$ is also a limit ordinal. In fact, $\xi + \zeta = \sup\{\xi + \alpha : \alpha < \zeta\}$. Since $y^* \in P_\alpha(\lambda)$, for all $\alpha < \zeta$, the induction hypothesis implies $(x^* + y^*)/2 \in P_{\xi+\alpha}(\lambda/2 - \delta)$ for all $\alpha < \zeta$ and so $(x^* + y^*)/2 \in P_{\xi+\zeta}(\lambda/2 - \delta)$. The proof of the lemma is now complete. $\square$

Corollary 3.2. *Let X be a Banach space such that $P_\xi(\lambda) \neq \emptyset$ for some $\xi < \omega_1$ and $\lambda > 0$. Then $\eta(X) \geq \xi \cdot \omega$.*

Proof. We can inductively select a sequence of positive scalars (δ_n), such that $\sum_{i=1}^n (\delta_i / 2^{n-i}) < \lambda/2^n$, for all $n \in \mathbb{N}$. Set $\epsilon_n = \lambda/2^n - \sum_{i=1}^n (\delta_i/2^{n-i})$, $n \in \mathbb{N}$. Let $x^* \in P_\xi(\lambda)$. Successive applications of Lemma 3.1 yield $x^* \in P_{\xi \cdot 2^n}(\epsilon_n)$, for all $n \in \mathbb{N}$. It follows now that $\eta(\epsilon_n, B_{X^*}) \geq \xi \cdot 2^n$ for all $n \in \mathbb{N}$, whence $\eta(X) \geq \xi \cdot \omega$. □

Notation. Given a weakly null sequence (x_n) in some Banach space X, $F \in [\mathbb{N}]^{<\infty}$, $F \neq \emptyset$, and $\lambda > 0$, we set $K_{F,\lambda} = \{x^* \in B_{X^*} : |x^*(x_n)| \geq \lambda, \forall\, n \in F\}$.

Proposition 3.3. *Let X be a separable Banach space, (x_n) a normalized weakly null sequence in X and $\lambda > 0$. Suppose $\mathcal{T}$ is a blossomed subtree of $[\mathbb{N}]^{<\infty}$ for which there exists a collection $(L_t)_{t \in \mathcal{T}}$ of non-empty w^*-closed subsets of B_{X^*}, such that $L_t \subset K_{t,\lambda}$, for all $t \in \mathcal{T} \setminus \{\emptyset\}$, while $L_{t_2} \subset L_{t_1}$, whenever $t_1 \leqslant t_2$ in $\mathcal{T}$. Then $L_t \cap P_{o(t)}(\lambda) \neq \emptyset$, for all $t \in \mathcal{T}$.*

Proof. We prove the assertion of the proposition by transfinite induction on the order of $\mathcal{T}$. The assertion is trivial for subtrees of order 0. Let $1 \leq \xi < \omega_1$ and assume the assertion is true for blossomed subtrees of $[\mathbb{N}]^{<\infty}$, of order smaller than ξ. Consider now a blossomed subtree $\mathcal{T}$ of $[\mathbb{N}]^{<\infty}$, of order ξ. Let (t_n) be an enumeration of $D_\emptyset^{\mathcal{T}}$ such that $(o(t_n))$ is non-decreasing.

Fix $n \in \mathbb{N}$ and let $j_n \colon \mathcal{T}_{t_n} \to [\mathbb{N}]^{<\infty}$ be the order preserving injection given by $j_n(t) = t \setminus \{\min t_n\}$. Put $\mathcal{T}_n = j_n \mathcal{T}_{t_n}$ which is a blossomed subtree of $[\mathbb{N}]^{<\infty}$, of order $o(t_n) < \xi$ (see Lemma 2.6). Set $H_r = L_{j_n^{-1} r}$, for all $r \in \mathcal{T}_n$. Note in particular that $H_\emptyset = L_{t_n}$. We can now apply the induction hypothesis for $\mathcal{T}_n$ and the collection $(H_r)_{r \in \mathcal{T}_n}$, to conclude that $H_r \cap P_{o(r)}(\lambda) \neq \emptyset$, for all $r \in \mathcal{T}_n$. It follows that $L_t \cap P_{o(t)}(\lambda) \neq \emptyset$, for all $t \in \mathcal{T}_{t_n}$.

The proof will be completed once we show $L_\emptyset \cap P_\xi(\lambda) \neq \emptyset$. To this end, choose $x_n^* \in L_{t_n} \cap P_{o(t_n)}(\lambda)$ for all $n \in \mathbb{N}$. Set $m_n = \min t_n$ and note that the m_n's are pairwise distinct. Put $y_n = x_{m_n}$, $n \in \mathbb{N}$. It is clear that (y_n) is a normalized, weakly null sequence in X satisfying $|x_n^*(y_n)| \geq \lambda$, for all $n \in \mathbb{N}$, since $x_n^* \in L_{t_n} \subset K_{t_n,\lambda}$ and $m_n \in t_n$. Finally, let x^* be a w^*-cluster point of (x_n^*). Clearly, $x^* \in P_{o(t_n)+1}(\lambda)$, for all $n \in \mathbb{N}$, whence $x^* \in P_\xi(\lambda)$, as $\xi = \sup_n \bigl(o(t_n) + 1\bigr)$, by Lemma 2.4. We are done since our hypotheses yield $x^* \in L_\emptyset$. □

Corollary 3.4. *Let $\mathcal{F}$ be a regular family of order ξ, containing all singletons. Then $\eta(X_{\mathcal{F}}) = \xi \cdot \omega$.*

Proof. Let (e_n) be the natural Schauder basis of $X_{\mathcal{F}}$. Then (e_n) is normalized weakly null. Denote by (e_n^*) the sequence of functionals biorthogonal to (e_n). It is clear that $\sum_{i \in F} e_i^* \in K_{F,1}$, for every non-empty $F \in \mathcal{F}$. Proposition 2.11 tells us that $\mathcal{F}$ is a blossomed tree of order ξ. We can therefore apply Proposition 3.3, with $L_t = K_{t,1}$ for $t \neq \emptyset$ and $L_\emptyset = B_{X_{\mathcal{F}}^*}$, to conclude that $P_\xi(1) \neq \emptyset$. Corollary 3.2 now implies that $\eta(X_{\mathcal{F}}) \geq \xi \cdot \omega$. To obtain equality, we note that $X_{\mathcal{F}}$ is isometric to a subspace of $C(\mathcal{F})$ (see [11]), and thus $\eta(X_{\mathcal{F}}) \leq \eta(C(\omega^\xi)) = \xi \cdot \omega$, by [8], [48]. □

A variation of the Szlenk index. It will be more convenient for us to work with a variant of $P_\alpha(\epsilon, B)$, which we term $Q_\alpha(\epsilon, B)$. The definition is done by transfinite induction. Given a *separable* Banach space X, a w^*-compact subset B of B_{X^*} and $\epsilon > 0$, set $Q_0(\epsilon, B) = B$. Suppose that $\alpha < \omega_1$ and that $Q_\beta(\epsilon, B)$ has been defined for all $\beta < \alpha$. Assume first that α is a successor ordinal, say $\alpha = \beta + 1$. Set

$$Q_\alpha(\epsilon, B) = \{x^* \in B : \exists\,(x_n^*) \subset Q_\beta(\epsilon, B),\ \lim_n x_n^* = x^*,\ (w^*),\ \exists\,(x_n) \subset B_X$$
$$\lim_n x_n = 0,\ (w),\ \inf_n |x_n^*(x_n)| \geq \epsilon\}.$$

If α is a limit ordinal, set

$$Q_\alpha(\epsilon, B) = \{x^* \in B : \exists\,(\alpha_n) \text{ strictly increasing to } \alpha,$$
$$\exists\, x_n^* \in Q_{\alpha_n+1}(\epsilon, B),\ (n \in \mathbb{N}), \lim_n x_n^* = x^*,\ (w^*)\}.$$

It is easy to see that $P_\alpha(\epsilon, B) \subset Q_\alpha(\epsilon, B)$, for all $\alpha < \omega_1$. We also observe that if $\eta(\epsilon, B) \geq \xi$, then $Q_\xi(\epsilon, B) \neq \emptyset$. To see this first note that $P_\beta(\epsilon, B) \subset P_\alpha(\epsilon, B)$, for all $\alpha \leq \beta < \omega_1$. It follows from this that if $\xi < \eta(\epsilon, B)$, then $P_\xi(\epsilon, B) \neq \emptyset$ and thus $Q_\xi(\epsilon, B) \neq \emptyset$ as well. When $\xi = \eta(\epsilon, B)$, we distinguish two cases. The first case is that of a successor ordinal ξ. Necessarily, $P_\xi(\epsilon, B) \neq \emptyset$ in this case and so, again, $Q_\xi(\epsilon, B) \neq \emptyset$. If ξ is a limit ordinal, then $P_\alpha(\epsilon, B) \neq \emptyset$, for all $\alpha < \xi$. Let (ξ_n) be a strictly increasing sequence of ordinals tending to ξ. We can choose $x_n^* \in P_{\xi_n+1}(\epsilon, B)$, for all $n \in \mathbb{N}$. It follows that $x_n^* \in Q_{\xi_n+1}(\epsilon, B)$, for all $n \in \mathbb{N}$. Finally, let x^* be a w^*-cluster point of (x_n^*). Of course, $x^* \in Q_\xi(\epsilon, B)$.

Lemma 3.5. *Let X be a separable Banach space, B a w^*-compact subset of X^* and $\epsilon > 0$. Let $\xi < \omega_1$ be a limit ordinal and (ξ_n) a non-decreasing sequence of ordinals smaller than ξ such that $\xi = \lim_n \xi_n$. Assume that $x_n^* \in Q_{\xi_n}(\epsilon, B)$, for all $n \in \mathbb{N}$, and that $\lim_n x_n^* = x^*$, (w^*). Then $x^* \in Q_\xi(\epsilon, B)$.*

Proof. Assume without loss of generality that (ξ_n) is strictly increasing. If ξ_n is a successor for infinitely many n's, then the assertion of the lemma is trivial. We can thus assume, without loss of generality, that ξ_n is a limit ordinal for all $n \in \mathbb{N}$. We next choose, for all $n \in \mathbb{N}$, a strictly increasing sequence of ordinals $(\xi_{nk})_{k=1}^\infty$ tending to ξ_n, and a sequence $(x_{nk}^*)_{k=1}^\infty$ with $x_{nk}^* \in Q_{\xi_{nk}+1}(\epsilon, B)$, for all k, such that $\lim_k x_{nk}^* = x_n^*$ (w^*). Let d be a metric compatible to the relative w^*-topology on B. We can certainly assume without loss of generality that $d(x_{nk}^*, x_n^*) < 1/n$, for all integers n, k. Moreover, we can assume without loss of generality that $\xi_n < \xi_{n+1,k}$, for all integers n, k. It follows now that $\lim_n x_{nn}^* = x^*$, (w^*), and that (ξ_{nn}) is a strictly increasing sequence of ordinals tending to ξ. Of course $x_{nn}^* \in Q_{\xi_{nn}+1}(\epsilon, B)$ for all $n \in \mathbb{N}$, whence $x^* \in Q_\xi(\epsilon, B)$. □

In what follows, K is a totally disconnected, compact metrizable space. Subsets of $B_{C(K)^*}$ will be endowed with the w^*-topology, while trees will be given the tree topology.

Definition 3.6. Suppose that $\mathcal{M} \subset B_{C(K)^*}$, $\mu \in \mathcal{M}$ and $\xi < \omega_1$. A ξ-condensation of μ in $\mathcal{M}$ is a pair $(\mathcal{T}, \phi)$, consisting of a blossomed tree $\mathcal{T}$ of order ξ and a homeomorphic embedding $\phi\colon \mathcal{T} \to \mathcal{M}$ ($\mathcal{T}$ and $\mathcal{M}$ are endowed with the tree and w^* topologies, respectively), so that $\phi(\rho) = \mu$, where ρ is the root of $\mathcal{T}$. The ξ-condensation $(\mathcal{T}, \phi)$ of μ in $\mathcal{M}$ is λ-Szlenk, for some $\lambda > 0$, if $\mu \in Q_\xi(\lambda, \phi(\mathcal{S}))$, for every full subtree $\mathcal{S}$ of $\mathcal{T}$.

To prove Theorem 1.3 we shall need a few technical lemmas. The next lemma gives conditions that enable us to glue together Szlenk-type condensations to obtain a Szlenk condensation of higher order.

Notation. Let $(\mathcal{T}_n)$ be a sequence of blossomed trees such that the sequence $(o(\mathcal{T}_n))$ is non-decreasing. We denote by $\bigotimes_n \mathcal{T}_n$ the blossomed tree $\bigcup_n(\{n\} \times \mathcal{T}_n) \cup \{\emptyset\}$ having $\emptyset$ as its root under the ordering $(n_1, \alpha_1) \leqslant (n_2, \alpha_2)$ if, and only if, $n_1 = n_2$ and $\alpha_1 \leqslant \alpha_2$ in $\mathcal{T}_{n_1}$. It is clear that the order of this tree is equal to $\sup_n(o(\mathcal{T}_n)+1)$ (see Lemmas 2.4, 2.6). Assume further that we had mappings $\phi_n \colon \mathcal{T}_n \to E$, for some set E, for all $n \in \mathbb{N}$. Then given $e \in E$, $\bigotimes_n^e \phi_n$ is the unique mapping $\phi \colon \bigotimes_n \mathcal{T}_n \to E$ extending each ϕ_n and such that $\phi(\emptyset) = e$.

Lemma 3.7. *Let (ξ_n) be a non-decreasing sequence of countable ordinals and set $\xi = \sup_n(\xi_n + 1)$. Let (μ_n) be a sequence in $B_{C(K)^*}$, w^*-converging to μ. Suppose there exists a sequence (U_n) of pairwise disjoint relatively w^*-open subsets of $B_{C(K)^*}$ so that*

1. *$\mu_n \in U_n$, $\mu \notin U_n$ for all $n \in \mathbb{N}$.*
2. *μ_n admits a λ-Szlenk, ξ_n-condensation $(\mathcal{T}_n, \phi_n)$ in U_n for all $n \in \mathbb{N}$.*
3. *If $n_1 < n_2 < \dots$ and $\tau_i \in U_{n_i}$, $i \in \mathbb{N}$, then $\lim_i \tau_i = \mu$, (w^*).*
4. *If ξ is a successor, there exists a weakly null sequence (f_n) in $B_{C(K)}$ such that $|\mu_n(f_n)| \geq \lambda$, for all $n \in \mathbb{N}$.*

Then $(\bigotimes_n \mathcal{T}_n, \bigotimes_n^\mu \phi_n)$ is a λ-Szlenk, ξ-condensation of μ in $\bigcup_n \phi_n(\mathcal{T}_n) \cup \{\mu\}$.

Proof. It follows immediately from the definitions and conditions (1), (2), (3) that $(\bigotimes_n \mathcal{T}_n, \bigotimes_n^\mu \phi_n)$ is a ξ-condensation of μ in $\bigcup_n \phi_n(\mathcal{T}_n) \cup \{\mu\}$. We need only show it is λ-Szlenk. To see this, let $\mathcal{S}$ be a full subtree of $\bigotimes_n \mathcal{T}_n$. Let ρ_n be the root of $\mathcal{T}_n$ and choose an increasing sequence (k_n) in $\mathbb{N}$ so that $\{(k_n, \rho_{k_n}) : n \in \mathbb{N}\} = D_\emptyset^{\mathcal{S}}$. Put $\mathcal{S}_n = \{\alpha \in \mathcal{T}_{k_n} : (k_n, \alpha) \in \mathcal{S}\}$. This is a full subtree of $\mathcal{T}_{k_n}$. Condition (2) yields that $\mu_{k_n} \in Q_{\xi_{k_n}}\big(\lambda, \phi_{k_n}(\mathcal{S}_n)\big)$. Since $\phi_{k_n}(\mathcal{S}_n) = \bigotimes_m^\mu \phi_m(\{k_n\} \times \mathcal{S}_n)$, we infer that $\mu_{k_n} \in Q_{\xi_{k_n}}\big(\lambda, \bigotimes_m^\mu \phi_m(\mathcal{S})\big)$, for all $n \in \mathbb{N}$. It follows now that $\mu \in Q_\xi\big(\lambda, \bigotimes_m^\mu \phi_m(\mathcal{S})\big)$. Indeed, this is immediate when ξ is a successor, because of condition (4) in the hypothesis. When ξ is a limit ordinal, it follows from Lemma 3.5. $\square$

Lemma 3.8. *Let $\mathcal{M}$ be a w^*-compact subset of $B_{C(K)^*}$. Suppose $\mu \in Q_\xi(\lambda, \mathcal{M})$ for some $\lambda > 0$ and $1 \leq \xi < \omega_1$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be w^*-dense in $\mathcal{M}$, and let U be a relatively w^*-open neighborhood of μ in $B_{C(K)^*}$. Given $0 < \epsilon < \lambda$ and a countable subset B of $B_{C(K)^*}$, there exist a $(\lambda - \epsilon)$-Szlenk, ξ-condensation $(\mathcal{T}, \phi)$ of μ in $\mathcal{M} \cap U$ and a collection $(f_\alpha)_{\alpha \in \mathcal{T}^*}$ in $B_{C(K)}$ ($\mathcal{T}^* = \mathcal{T}^* \setminus \{\rho\}$, ρ being the root of $\mathcal{T}$) so that the following properties are satisfied:*

1. *$\phi(\alpha) \in \mathcal{M}_0$, for every terminal node $\alpha \in \mathcal{T}$.*
2. *$|\phi(\alpha)\big(f_\alpha\big)| > \lambda - \epsilon$, for all $\alpha \in \mathcal{T}^*$.*
3. *$\lim_{\beta \in D_\alpha^{\mathcal{T}}} \nu(|f_\beta|) = 0$, for all non-terminal $\alpha \in \mathcal{T}$ and all $\nu \in B$.*

Proof. We prove the lemma by transfinite induction on ξ. When $\xi = 1$, choose a sequence $(\mu_n) \subset \mathcal{M} \cap U$, w^*-convergent to μ, and a weakly null sequence (f_n) in $B_{C(K)}$ so that $|\mu_n(f_n)| \geq \lambda$, for all $n \in \mathbb{N}$. There is no loss of generality in assuming that $|\mu|(|f_n|) < \lambda - \epsilon$ for all $n \in \mathbb{N}$. Let d be a metric compatible with the relative w^*-topology in $B_{C(K)^*}$. Fix n and define $W_n = \{\nu \in U : d(\nu, \mu_n) < 1/n, |\nu(f_n)| > \lambda - \epsilon\}$, which is a relatively w^*-open neighborhood of μ_n in $B_{C(K)^*}$. Since $\mu \notin W_n$, for all $n \in \mathbb{N}$, there exist a subsequence (μ_{k_n}) of (μ_n) and pairwise disjoint relatively w^*-open subsets U_n of $B_{C(K)^*}$, such that $\mu_{k_n} \in U_n \subset W_{k_n}$, for all $n \in \mathbb{N}$. We can now choose $\nu_n \in \mathcal{M}_0 \cap U_n$, and set $\mathcal{T} = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}$. Define $\phi \colon \mathcal{T} \to \mathcal{M} \cap U$ by the rule $\phi(\{n\}) = \nu_n$, for all $n \in \mathbb{N}$ and $\phi(\emptyset) = \mu$. Also put $f_{\{n\}} = f_{k_n}$, $n \in \mathbb{N}$, and it is easy to verify that $(\mathcal{T}, \phi)$ and $(f_\alpha)_{\alpha\in\mathcal{T}\setminus\{\emptyset\}}$ satisfy the requirements of the lemma for $\xi = 1$.

Assume the assertion of the lemma holds for all ordinals smaller than ξ and let $\mu \in Q_\xi(\lambda, \mathcal{M})$. It follows that there exist a non-decreasing sequence of ordinals (ξ_n) with $\sup_n(\xi_n + 1) = \xi$, and a sequence (μ^n), w^*-converging to μ and such that $\mu^n \in Q_{\xi_n+1}(\lambda, \mathcal{M})$ for all $n \in \mathbb{N}$. We can now choose sequences $(\mu_{ni})_{i=1}^\infty$ in $Q_{\xi_n}(\lambda, \mathcal{M})$, w^*-convergent to μ^n, and weakly null sequences $(f_{ni})_{i=1}^\infty$ in $B_{C(K)}$ such that $|\mu_{ni}(f_{ni})| \geq \lambda$, for all integers n, i. We can clearly assume that $\lim_k \mu_{n_k,i_k} = \mu$, (w^*), for all $n_1 < n_2 < \dots$ and all choices $i_k \in \mathbb{N}$. Now let B be a countable subset of $B_{C(K)^*}$. Since $(f_{ni})_{i=1}^\infty$ is weakly null, we can select indices $i_1 < i_2 < \dots$ so that $\lim_n \nu(|f_{n,i_n}|) = 0$, for all $\nu \in B \cup \{|\mu|\}$. Set $\mu_n = \mu_{n,i_n}$ and $f_n = f_{n,i_n}$, for all $n \in \mathbb{N}$. We can assume that $|\mu|(|f_n|) < \lambda - \epsilon$, for all $n \in \mathbb{N}$. Note also that in case ξ is a successor, (f_n) can be chosen to be weakly null (this is so since we simply take $\mu^n = \mu$ for all n).

Arguing as in case $\xi = 1$, we can assume without loss of generality, by passing to subsequences, that there exist pairwise disjoint relatively w^*-open subsets U_n of U such that $\mu_n \in U_n \subset W_n$, where $W_n = \{\nu \in U : d(\nu, \mu_n) < 1/n, |\nu(f_n)| > \lambda - \epsilon\}$, and $\mu \notin U_n$, for all $n \in \mathbb{N}$. Since $\mu_n \in Q_{\xi_n}(\lambda, \mathcal{M})$, the induction hypothesis applied on μ_n, U_n and B yields a $(\lambda-\epsilon)$-Szlenk, ξ_n-condensation $(\mathcal{T}_n, \phi_n)$ of μ_n in $\mathcal{M}\cap U_n$, and a subset $(f^n_\alpha)_{\alpha\in\mathcal{T}^*_n}$ of $B_{C(K)}$, satisfying all three requirements of the lemma for the ordinal ξ_n. Let $\mathcal{T} = \bigotimes_n \mathcal{T}_n$ and $\phi = \bigotimes^\mu_n \phi_n$. Lemma 3.7 yields that $(\mathcal{T}, \phi)$ is a $(\lambda - \epsilon)$-Szlenk, ξ-condensation of μ in $\mathcal{M} \cap U$, satisfying (1). Let ρ_n be the root of $\mathcal{T}_n$. Define $f_{(n,\rho_n)} = f_n$ and $f_{(n,\alpha)} = f^n_\alpha$ when $\alpha \in \mathcal{T}^*_n$, for all $n \in \mathbb{N}$. It is easy to check that $(f_\alpha)_{\alpha\in\mathcal{T}^*}$ satisfies (2) and (3) for ξ. $\square$

Lemma 3.9. *Let $\mathcal{M}$ be a w^*-compact subset of $B_{C(K)^*}$. Suppose $\mu \in Q_\xi(\lambda, \mathcal{M})$ for some $\lambda > 0$ and $1 \leq \xi < \omega_1$. Let U be a relatively w^*-open neighborhood of μ in $B_{C(K)^*}$. Then there exist a non-negative measure $\nu \in B_{C(K)^*}$, ξ-condensations $(\mathcal{T}, \phi)$ for μ in $\mathcal{M} \cap U$ and $(\mathcal{T}, \theta)$ for ν in $B_{C(K)^*}$, so that*

(1) *$\theta(\alpha) = |\phi(\alpha)|$, for every terminal node $\alpha \in \mathcal{T}$.*
(2) *$(\mathcal{T}, \phi)$ is λ-Szlenk.*

Proof. Note (1) implies that $\tau \geq 0$, for all $\tau \in \theta(\mathcal{T})$. We prove the assertion of the lemma by transfinite induction on ξ. When $\xi = 1$ choose a sequence (μ_n) in $\mathcal{M}\cap U$, w^*-converging to μ, and a weakly null sequence (f_n) in $B_{C(K)}$ so that $|\mu_n(f_n)| \geq \lambda$, for all $n \in \mathbb{N}$. We can assume without loss of generality that $\lim_n |\mu_n| = \nu$, (w^*), for some non-negative measure $\nu \in B_{C(K)^*}$. Since $|\mu_n|(|f_n|) \geq \lambda$, arguing as in case $\xi = 1$ of Lemma 3.7, we can assume that $\mu_n \neq \mu$, $|\mu_n| \neq \nu$ for all $n \in \mathbb{N}$, and that the μ_n's as well as the $|\mu_n|$'s are pairwise different. Set $\mathcal{T} = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}$. Define mappings $\phi\colon \mathcal{T} \to \mathcal{M} \cap U$ and $\theta\colon \mathcal{T} \to B_{C(K)^*}$, by the rules $\phi(\{n\}) = \mu_n$, $\theta(\{n\}) = |\mu_n|$, for all $n \in \mathbb{N}$, $\phi(\emptyset) = \mu$, $\theta(\emptyset) = \nu$. It is easily seen that $(\mathcal{T}, \phi)$, ν and $(\mathcal{T}, \theta)$ satisfy (1), (2) for $\xi = 1$.

Assume the assertion of the lemma is true for ordinals smaller than ξ and let $\mu \in Q_\xi(\lambda, \mathcal{M})$. It follows that there exist a non-decreasing sequence of ordinals (ξ_n) with $\sup_n(\xi_n + 1) = \xi$, and a sequence (μ^n), w^*-converging to μ and such that $\mu^n \in Q_{\xi_n+1}(\lambda, \mathcal{M})$ for all $n \in \mathbb{N}$. We can now choose sequences $(\mu_{ni})_{i=1}^\infty$ in $Q_{\xi_n}(\lambda, \mathcal{M})$, w^*-convergent to μ^n, and weakly null sequences $(f_{ni})_{i=1}^\infty$ in $B_{C(K)}$ such that $|\mu_{ni}(f_{ni})| \geq \lambda$, for all integers n, i. We can clearly assume that $\lim_k \mu_{n_k,i_k} = \mu$, (w^*), for all $n_1 < n_2 < \dots$ and all choices $i_k \in \mathbb{N}$. Let d be a metric compatible to the w^*-topology in $B_{C(K)^*}$. Choose some $0 < \epsilon < \lambda/2$ and set $O_{ni} = \{\tau \in U : d(\tau, \mu_{ni}) < 1/n,\, |\tau(f_{ni})| > \lambda - \epsilon\}$, for all integers i, n.

The induction hypothesis applied on μ_{ni} and the neighborhood O_{ni}, yields non-negative measures ν_{ni} in $B_{C(K)^*}$, ξ_n-condensations $(\mathcal{T}_{ni}, \phi_{ni})$ for μ_{ni} in $\mathcal{M} \cap O_{ni}$ and $(\mathcal{T}_{ni}, \theta_{ni})$ for ν_{ni} in $B_{C(K)^*}$, fulfilling conditions (1), (2) of the induction hypothesis for ξ_n.

By passing to subsequences and relabeling, we can assume that the w^*-closure of $\{\nu_{ni} : n, i \in \mathbb{N}\}$ is countable, and that there exists a non-negative measure ν in $B_{C(K)^*}$ such that $\lim_k \nu_{n_k, i_k} = \nu$, (w^*), for all $n_1 < n_2 < \dots$ and all choices $i_k \in \mathbb{N}$. Since $(f_{ni})_{i=1}^\infty$ is weakly null, we can choose indices $i_1 < i_2 < \dots$ so that $|\mu|(|f_{n,i_n}|) < \lambda - 2\epsilon$ and $\nu(|f_{n,i_n}|) < \lambda - 2\epsilon$, for all $n \in \mathbb{N}$. Now set $f_n = f_{n,i_n}$, $\nu_n = \nu_{n,i_n}$, $\mu_n = \mu_{n,i_n}$, $\mathcal{T}_n = \mathcal{T}_{n,i_n}$, $\phi_n = \phi_{n,i_n}$, $\theta_n = \theta_{n,i_n}$, and $O_n = O_{n,i_n}$, for all $n \in \mathbb{N}$. Note that in case ξ is a successor, (f_n) can be chosen to be weakly null (same comment as in the proof of Lemma 3.8).

It is crucial to observe here that condition (1) of the induction hypothesis implies that $\nu_n(|f_n|) \geq \lambda - \epsilon$, for all $n \in \mathbb{N}$. Indeed, fixing n, we have $|\tau(f_n)| > \lambda - \epsilon$ for every $\tau \in \phi_n(\mathcal{T}_n) \subset O_n$. It follows now by (1) of the induction hypothesis for $(\mathcal{T}_n, \phi_n)$ and $(\mathcal{T}_n, \theta_n)$ that $\theta_n(\alpha)\big(|f_n|\big) > \lambda - \epsilon$, for every terminal node $\alpha \in \mathcal{T}_n$. Hence $\nu_n(|f_n|) \geq \lambda - \epsilon$.

Next put $R_n = \{\tau \in B_{C(K)^*} : d(\tau, \nu_n) < 1/n,\, \tau(|f_n|) > \lambda - 2\epsilon\}$, for all $n \in \mathbb{N}$. Since $\lim_n \mu_n = \mu$, $\lim_n \nu_n = \nu$, (w^*), and $\mu \notin O_n$, $\nu \notin R_n$ for all $n \in \mathbb{N}$, we can assume without loss of generality after passing to subsequences that there exist relatively w^*-open subsets U_n and V_n of $B_{C(K)^*}$ with $\mu_n \in U_n \subset O_n$, $\nu_n \in V_n \subset R_n$, and so that each one of the sequences (U_n) and (V_n) consists of pairwise disjoint sets.

Observe now that $\phi_n^{-1}[\phi_n(\mathcal{T}_n) \cap U_n]$ and $\theta_n^{-1}[\theta_n(\mathcal{T}_n) \cap V_n]$ are open neighborhoods of the root of $\mathcal{T}_n$, and therefore their intersection must contain a full subtree of $\mathcal{T}_n$. Since the desired properties (1), (2) are preserved by restrictions to full subtrees, there will be no loss of generality in assuming that $\phi_n(\mathcal{T}_n) \subset U_n$ and $\theta_n(\mathcal{T}_n) \subset V_n$, for all $n \in \mathbb{N}$. Let $\mathcal{T} = \bigotimes_n \mathcal{T}_n$, $\phi = \bigotimes_n^\mu \phi_n$ and $\theta = \bigotimes_n^\nu \theta_n$. Lemma 3.7, combined with the induction hypothesis, readily imply that ν and the ξ-condensations $(\mathcal{T}, \phi)$, $(\mathcal{T}, \theta)$ satisfy (1), (2) for ξ. $\square$

An immediate consequence of Lemma 3.9 is the next

Corollary 3.10. *Let $\mathcal{P} \subset B_{C(K)^*}$ be w^*-compact such that $\eta(\lambda, \mathcal{P}) \geq \xi$ for some $\lambda > 0$ and $\xi < \omega_1$. Then for every $\mu \in Q_\xi(\lambda, \mathcal{P})$ there exist a countable, w^*-compact subset $\mathcal{M}$ of $\mathcal{P}$ with $\mu \in Q_\xi(\lambda, \mathcal{M})$, and a subset $\mathcal{M}_0$ of $\mathcal{M}$ w^*-dense in $\mathcal{M}$ and such that the w^*-closure of $\{|\nu| : \nu \in \mathcal{M}_0\}$ is countable.*

Corollary 3.11. *Let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact such that $\eta(\lambda, \mathcal{M}) \geq \xi$ for some $\lambda > 0$ and $\xi < \omega_1$. Given $0 < \epsilon < \lambda$, there exist a blossomed tree $\mathcal{T}$ of order ξ, a subset $(\mu_t)_{t \in \mathcal{T}}$ of $\mathcal{M}$ and a collection $(f_t)_{t \in \mathcal{T}^*}$ ($\mathcal{T}^* = \mathcal{T} \setminus \{\rho\}$ with ρ denoting the root of $\mathcal{T}$) in $B_{C(K)}$ so that the following are satisfied:*

1. *The map $\phi\colon \mathcal{T} \to \mathcal{M}$ given by $\phi(t) = \mu_t$ is a homeomorphic embedding.*
2. *$\mu_\rho \in Q_\xi(\lambda - \epsilon, \{\mu_t : t \in \mathcal{S}\})$, for every full subtree $\mathcal{S}$ of $\mathcal{T}$.*
3. *$|\mu_t(f_t)| > \lambda - \epsilon$, for all $t \in \mathcal{T}^*$.*
4. *$B = \mathrm{Cl}_{w^*} \{|\mu_t| : t \in \mathcal{T} \text{ is terminal}\}$ is countable.*
5. *$\lim_{s \in D_t^{\mathcal{T}}} \nu(|f_s|) = 0$, for every non-terminal node $t \in \mathcal{T}$ and all $\nu \in B$.*

Proof. Apply Corollary 3.10 to obtain subsets $\mathcal{M}_1$, $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M}_1$ is countable w^*-compact with $Q_\xi(\lambda, \mathcal{M}_1) \neq \emptyset$, while $\mathcal{M}_0 \subset \mathcal{M}_1$ is w^*-dense in $\mathcal{M}_1$ and such that $B_1 = \mathrm{Cl}_{w^*}\{|\nu| : \nu \in \mathcal{M}_0\}$ is countable. Next apply Lemma 3.8 for $\mathcal{M}_1$, the given ϵ, the w^*-dense subset $\mathcal{M}_0$ of $\mathcal{M}_1$ and the countable subset B_1 of $B_{C(K)^*}$, to obtain a blossomed tree $\mathcal{T}$ of order ξ, a homeomorphic embedding $\phi\colon \mathcal{T} \to \mathcal{M}_1$ and a collection $(f_t)_{t\in\mathcal{T}^*}$, satisfying the conclusion of Lemma 3.8. Set $\phi(t) = \mu_t$ for all $t \in \mathcal{T}$ to obtain the desired conclusion. □

Lemma 3.12. *Let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact. Let $\mathcal{T}$, $(\mu_t)_{t\in\mathcal{T}}$, $(f_t)_{t\in\mathcal{T}^*}$ and B satisfy conditions* (1), (3), (4) *and* (5) *of Corollary 3.11 for some* $0 < \epsilon < \lambda/4$. *Then there exists a family $(G_t)_{t\in\mathcal{T}^*}$ of clopen subsets of K such that*

(1) $|\mu_t(G_t)| > \lambda/4 - \epsilon$, *for all $t \in \mathcal{T}^*$.*
(2) $\lim_{s\in D_t^{\mathcal{T}}} \nu(G_s) = 0$, *for every non-terminal node $t \in \mathcal{T}$ and all $\nu \in B$.*

Proof. Fix some $t \in \mathcal{T}^*$ and put $W_t = \{x \in K : |f_t(x)| > \epsilon\}$. Then $|\mu_t|(W_t) > \lambda - 2\epsilon$. Choose a Borel subset B_t of W_t such that $|\mu_t(B_t)| > \lambda/4 - \epsilon/2$. Next choose a closed subset F_t of B_t such that $|\mu_t(F_t)| > \lambda/4 - \epsilon/2$. Finally, choose a clopen set G_t with $F_t \subset G_t \subset W_t$ and such that $|\mu_t(G_t)| > \lambda/4 - \epsilon/2$. Let $\nu \in B$. Because $G_t \subset W_t$ we deduce that $\epsilon\nu(G_t) \leq \int_{G_t} |f_t| \,\mathrm{d}\nu \leq \nu(|f_t|)$. It is clear that $(G_t)_{t\in\mathcal{T}^*}$ is the required family. □

Lemma 3.13. *Let $\mathcal{T}$, $(\mu_t)_{t\in\mathcal{T}}$, $(G_t)_{t\in\mathcal{T}^*}$ be as in the conclusion of Lemma 3.12. Then there exists a full subtree $\mathcal{S}$ of $\mathcal{T}$ such that $|\mu_t(G_s)| > \lambda/4 - \epsilon$, for all $s \leqslant t$ in $\mathcal{S}^*$.*

Proof. We prove the lemma by transfinite induction on $o(\mathcal{T}) = \xi$. When $\xi = 1$, the assertion is trivial. Assume the assertion holds for trees of order less than ξ and consider a blossomed tree $\mathcal{T}$ of order $\xi > 1$. Let (t_n) be an enumeration of $D_\rho^{\mathcal{T}}$ so that $(o(t_n))$ is non-decreasing. Fix n and note that the set $\{t \in \mathcal{T}_{t_n} : |\mu_t(G_{t_n})| > \lambda/4 - \epsilon\}$ is an open neighborhood of t_n in $\mathcal{T}_{t_n}$; therefore it contains a full subtree $\mathcal{R}_n$ of $\mathcal{T}_{t_n}$. Since $o(t_n) < \xi$, the induction hypothesis applied on $\mathcal{R}_n$, $(\mu_t)_{t\in\mathcal{R}_n}$ and $(G_t)_{t\in\mathcal{R}_n^*}$ yields a full subtree $\mathcal{S}_n$ of $\mathcal{R}_n$ such that $|\mu_t(G_s)| > \lambda/4 - \epsilon$, for all $s \leqslant t$ in $\mathcal{S}_n \setminus \{t_n\}$. Set $\mathcal{S} = \bigcup_n \mathcal{S}_n \cup \{\rho\}$. This is the required full subtree of $\mathcal{T}$. □

The next corollary follows directly from Corollary 3.11 combined with Lemmas 3.12 and 3.13.

Corollary 3.14. *Let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact such that $\eta(\lambda, \mathcal{M}) \geq \xi$ for some $\lambda > 0$ and $\xi < \omega_1$. Given $0 < \epsilon < \lambda/4$, there exist a blossomed tree $\mathcal{T}$ of order ξ, a subset $(\mu_t)_{t\in\mathcal{T}}$ of $\mathcal{M}$ and a collection $(G_t)_{t\in\mathcal{T}^*}$ ($\mathcal{T}^* = \mathcal{T} \setminus \{\rho\}$ with ρ denoting the root of $\mathcal{T}$) of clopen subsets of K so that the following are satisfied:*

(1) *The map $\phi\colon \mathcal{T} \to \mathcal{M}$ given by $\phi(t) = \mu_t$ is a homeomorphic embedding such that $\mu_\rho \in Q_\xi(\lambda - \epsilon, \{\mu_t : t \in \mathcal{S}\})$, for every full subtree $\mathcal{S}$ of $\mathcal{T}$.*
(2) $|\mu_t(G_s)| > \lambda/4 - \epsilon$, *for all $s \leqslant t$ in $\mathcal{T}^*$.*
(3) $B = \mathrm{Cl}_{w^*}\{|\mu_t| : t \in \mathcal{T}$ *is terminal*$\}$ *is countable.*
(4) $\lim_{s\in D_t^{\mathcal{T}}} \nu(G_s) = 0$, *for every non-terminal node $t \in \mathcal{T}$ and all $\nu \in B$.*

We are now ready for the

Proof of Theorem 1.3. We apply Corollary 3.14 to obtain a blossomed tree $\mathcal{T}$ of order ξ, a subset $(\nu_t)_{t\in\mathcal{T}}$ of $\mathcal{M}$, and a collection of clopen subsets $(G_t)_{t\in\mathcal{T}^*}$ of K so that

(1.1) The map $\theta\colon \mathcal{T} \to \mathcal{M}$ given by $\theta(t) = \nu_t$ is a homeomorphic embedding.
(1.2) $|\nu_t(G_s)| > \lambda/4 - \epsilon$, for all $s \leqslant t$ in $\mathcal{T}^*$.
(1.3) $B = \mathrm{Cl}_{w^*}\{|\nu_t| : t \in \mathcal{T} \text{ is terminal}\}$ is countable.
(1.4) $\lim_{s\in D_t^{\mathcal{T}}} \tau(G_s) = 0$, for every non-terminal node $t \in \mathcal{T}$ and all $\tau \in B$.

Now let $\mathcal{F}$ be a regular family of order ξ. Proposition 2.11 yields that $\mathcal{F}$ is a blossomed tree of order ξ. We infer from Lemma 2.7 that $\mathcal{F}$ is order isomorphic to a subtree $\mathcal{R}$ of $\mathcal{T}$. Let (r_n) be an enumeration of the terminal nodes of $\mathcal{R}$. For each $n \in \mathbb{N}$, choose a terminal node t_n of $\mathcal{T}$ such that $r_n \leqslant t_n$. Define a map $\psi\colon \mathcal{R} \to \mathcal{T}$ as follows: $\psi(r) = \begin{cases} r, & \text{if } r \notin \{r_n : n \in \mathbb{N}\}; \\ t_n, & \text{if } r = r_n, \text{ for some } n \in \mathbb{N}. \end{cases}$ It is easy to see that ψ is a homeomorphic embedding with respect to the tree topology. We remark that $r \leqslant \psi(r)$, for all $r \in \mathcal{R}$. Moreover, r is terminal in $\mathcal{R}$ if, and only if, $\psi(r)$ is terminal in $\mathcal{T}$.

Let $j\colon \mathcal{F} \to \mathcal{R}$ be an order isomorphism and define $G_\alpha = G_{j(\alpha)}$ for all $\alpha \in \mathcal{F}^*$. Finally define $\phi\colon \mathcal{F} \to \mathcal{M}$ with $\phi = \theta \circ \psi \circ j$. ϕ is a homeomorphic embedding and thus, setting $\mu_\alpha = \phi(\alpha)$, for $\alpha \in \mathcal{F}$, we infer that (1) holds.

Since the terminal nodes of $\psi(\mathcal{R})$ are contained among the terminal nodes of $\mathcal{T}$, (1.3) implies that $\mathcal{N} \subset B$ and thus (3) holds. On the other hand, if $\alpha \leqslant \beta$ in $\mathcal{F}^*$, then

$$\begin{aligned} |\mu_\beta(G_\alpha)| &= |\theta(\psi \circ j(\beta))\big(G_{j(\alpha)}\big)| \\ &= |\nu_{\psi\circ j(\beta)}(G_{j(\alpha)})| > \lambda/4 - \epsilon \end{aligned}$$

by (1.2), since $j(\alpha) \leqslant j(\beta) \leqslant \psi \circ j(\beta)$ in $\mathcal{T}$. Hence (2) holds. Because $\mathcal{N} \subset B$ and $G_\alpha = G_{j(\alpha)}$ for $\alpha \in \mathcal{F}^*$, we deduce from (1.4) that (4) holds. $\square$

Remark 3.15. Note that for a spreading family $\mathcal{F}$ and a non-terminal $\alpha \in \mathcal{F}$, there exists $n_0 \in \mathbb{N}$ such that $\alpha \cup \{n\} \in \mathcal{F}$, for every $n \geq n_0$. Hence the limit appearing in (4) of Theorem 1.3 makes sense.

4. An application of Ramsey's theorem

This section is devoted to the proof of Theorem 1.4. We shall actually prove a stronger result, Theorem 4.3, which is of a combinatorial nature and its proof requires the infinite Ramsey theorem [21] that we now recall.

Theorem 4.1. *Let $\mathcal{A} \subset [\mathbb{N}]$ be analytic in the topology of pointwise convergence. Then for every $N \in [\mathbb{N}]$ there exists $M \in [N]$ such that either $[M] \subset \mathcal{A}$ or $[M] \cap \mathcal{A} = \emptyset$.*

For applications of Ramsey's theorem in Banach space theory, we refer to [39]. We shall next introduce some definitions and terminology that are necessary in the statement of Theorem 4.3. In what follows, K is a compact metrizable space.

Definition 4.2. Let $\mathcal{T}$ be a rooted tree and $(\psi_\alpha)_{\alpha\in\mathcal{T}^*}$ a collection of mappings with $\psi_\alpha\colon B_{C(K)^*} \to c_{00}$ (recall that $\mathcal{T}^* = \mathcal{T} \setminus \{\rho\}$, where ρ is the root of $\mathcal{T}$). Let $B \subset c_{00}$. A subset $\mathcal{M}$ of $B_{C(K)^*}$ is said to satisfy a B-lattice property with respect to $(\psi_\alpha)_{\alpha\in\mathcal{T}^*}$ provided the following condition is fulfilled: Let A be a finite, well-ordered subset of $\mathcal{T}^*$ such that for every $\alpha \in A$ there exists a $\mu_\alpha \in \mathcal{M} \cap \psi_\alpha^{-1}(B)$. Then there exists $\alpha_0 \in A$ such that $\mu_{\alpha_0} \in \psi_\alpha^{-1}(B)$, for all $\alpha \in A$.

Terminology 1. A triple $\big(\mathcal{F},(f_\alpha)_{\alpha\in\mathcal{F}^*},\mathcal{M}\big)$ on K consists of a hereditary and spreading family $\mathcal{F}$, a collection $(f_\alpha)_{\alpha\in\mathcal{F}^*}$ of functions in $B_{C(K)}$ and a subset $\mathcal{M}$ of $B_{C(K)^*}$ so that $\lim_n \nu(|f_{\alpha\cup\{n\}}|)=0$, for every non-terminal $\alpha\in\mathcal{F}$ and all $\nu\in \mathrm{Cl}_{w^*}\{|\mu| :\, \mu\in\mathcal{M}\}$ (see Remark 3.15).

Theorem 4.3. *Let $\big(\mathcal{F},(f_\alpha)_{\alpha\in\mathcal{F}^*},\mathcal{M}\big)$ be a triple on K. Let $B\subset c_{00}$ with $\|b\|_\infty\le 1$, for all $b\in B$. Let $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$ be a collection of mappings with $\psi_\alpha\colon B_{C(K)^*}\to c_{00}$, such that $\psi_\alpha^{-1}(B)$ is w^*-closed for all $\alpha\in\mathcal{F}^*$. Assume $\mathcal{M}$ satisfies a B-lattice property with respect to $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$. Then for every $\epsilon>0$ there exists $M\in[\mathbb{N}]$ with the following property: Suppose $\alpha\in\mathcal{F}^*[M]$ is such that $\mathcal{M}\cap\psi_\alpha^{-1}(B)\neq\emptyset$. Then there exists $\mu\in\psi_\alpha^{-1}(B)\cap\mathrm{Cl}_{w^*}\mathcal{M}$ such that $\sum_{\beta\in\mathcal{F}^*[M],\,\max\beta\notin\alpha}|\mu|(|f_\beta|)<\epsilon$.*

In the above, $\mathcal{F}[M]=\mathcal{F}\cap[M]^{<\infty}$ and $\mathcal{F}^*[M]=\mathcal{F}[M]\setminus\{\emptyset\}$. To prove this result we need to introduce some more terminology. In the sequel, $\mathcal{F}$, $(f_\alpha)_{\alpha\in\mathcal{F}^*}$, $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$, B and $\mathcal{M}$ are as in the statement of Theorem 4.3. Choose first a sequence of positive scalars $(\epsilon_i)_{i=0}^\infty$ such that $\sum_{i=0}^\infty\epsilon_i<\epsilon$.

Notation 4.4. Given G_1, G_2 finite subsets of $\mathbb{N}$, we set

$$[G_1,G_2]_{\mathcal{F}}=\{\alpha\in\mathcal{F}^*,\,\alpha\subset G_2,\,\max\alpha\notin G_1\}.$$

Terminology 2. Let $F_1\subset F_2$ be finite subsets of $\mathbb{N}$ and $L\in[\mathbb{N}]$ with $\max F_2<\min L$ ($\max\emptyset=0$). We say that L is (F_1,F_2)-*admissible* if for every $\alpha\in\mathcal{F}^*$ which is an initial segment of $F_1\cup(L\setminus\{\min L\})$ with $\mathcal{M}\cap\psi_\alpha^{-1}(B)\neq\emptyset$, there exists $\mu\in\mathcal{M}\cap\psi_\alpha^{-1}(B)$ such that

$$\sum_{\beta\in[F_1,F_2\cup\{\min L\}]_{\mathcal{F}}}|\mu|(|f_\beta|)<\sum_{i=0}^{|F_2|}\epsilon_i.$$

Given $F_0\in[\mathbb{N}]^{<\infty}$ and $L\in[\mathbb{N}]$ with $\max F_0<\min L$, we say that L is F_0-*admissible* if it is (F,F_0)-admissible for every $F\subset F_0$.

Remark 4.5. Saying L is not (F_1,F_2)-admissible means that there is an initial segment $\alpha\in\mathcal{F}^*$ of $F_1\cup(L\setminus\{\min L\})$ with $\mathcal{M}\cap\psi_\alpha^{-1}(B)\neq\emptyset$, and such that

$$\sum_{\beta\in[F_1,F_2\cup\{\min L\}]_{\mathcal{F}}}|\mu|(|f_\beta|)\ge\sum_{i=0}^{|F_2|}\epsilon_i,$$

for all $\mu\in\mathcal{M}\cap\psi_\alpha^{-1}(B)$.

Lemma 4.6. *There exists $M_0\in[\mathbb{N}]$ such that L is $\emptyset$-admissible for every $L\in[M_0]$.*

Proof. Let $\mathfrak{D}=\{L\in[\mathbb{N}] :\, L \text{ is } \emptyset\text{-admissible}\}$. Clearly, $\mathfrak{D}$ is closed in the topology of pointwise convergence in $[\mathbb{N}]$. The infinite Ramsey theorem yields $M_0\in[\mathbb{N}]$ such that either $[M_0]\subset\mathfrak{D}$, or $[M_0]\cap\mathfrak{D}=\emptyset$. If the former, we are done. If the latter, we shall derive a contradiction as follows:

First observe that if $L\in[M_0]$, then $L\notin\mathfrak{D}$. Therefore (see Remark 4.5), there exists an initial segment $\alpha_L\in\mathcal{F}^*$ of $L\setminus\{\min L\}$ such that $\psi_{\alpha_L}^{-1}(B)\cap\mathcal{M}\neq\emptyset$ and $|\mu|(|f_{\{\min L\}}|)\ge\epsilon_0$, for all $\mu\in\psi_{\alpha_L}^{-1}(B)\cap\mathcal{M}$.

Next choose $l_1<l_2<\dots$ in M_0 and fix a $k\in\mathbb{N}$. Set $L_i=\{l_i\}\cup\{l_j :\, j>k\}$, for all $i\le k$. Our preceding comment yields, for all $i\le k$, an initial segment $\alpha_{L_i}\in\mathcal{F}^*$ of $\{l_j :\, j>k\}$ with $\psi_{\alpha_{L_i}}^{-1}(B)\cap\mathcal{M}\neq\emptyset$ and such that $|\mu|(|f_{\{l_i\}}|)\ge\epsilon_0$, for all $\mu\in\psi_{\alpha_{L_i}}^{-1}(B)\cap\mathcal{M}$.

Since $\{\alpha_{L_i} : i \leq k\}$ is a finite, well-ordered subset of $\mathcal{F}^*$ (its elements are initial segments of $\{l_j : j > k\}$) and $\mathcal{M}$ satisfies a B-lattice property with respect to $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$ (see Definition 4.2), we infer that there exists $\mu_k \in \mathcal{M}$ such that $\mu_k \in \psi^{-1}_{\alpha_{L_i}}(B)$, for all $i \leq k$.

It follows now that $|\mu_k|(|f_{\{l_i\}}|) \geq \epsilon_0$, for all $i \leq k$. But if ν is a w^*-cluster point of the sequence $(|\mu_k|)$, we must have that $\nu(|f_{\{l_i\}}|) \geq \epsilon_0$, for all $i \in \mathbb{N}$. This is a contradiction because $\lim_i \nu(|f_{\{l_i\}}|) = 0$ as $(\mathcal{F}, (f_\alpha)_{\alpha\in\mathcal{F}^*}, \mathcal{M})$ is a triple on K (see Terminology 1). $\square$

Lemma 4.7. *Suppose $F_0 \in [\mathbb{N}]^{<\infty}$ and $P \in [\mathbb{N}]$ are so that L is F_0-admissible for every $L \in [P]$. Let $p_0 \in P \setminus \{\min P\}$. Then for every $Q \in [P]$ there exists $R \in [Q]$ such that L is $F_0 \cup \{p_0\}$-admissible, for every $L \in [R]$.*

Proof. Choose $Q_0 \in [Q]$ with $p_0 < \min Q_0$. Given $F \subset F_0 \cup \{p_0\}$ define

$$\mathfrak{D}_F = \{L \in [Q_0] : L \text{ is } (F, F_0 \cup \{p_0\})\text{-admissible}\}.$$

We are going to show that there exists $Q_F \in [Q_0]$ such that $[Q_F] \subset \mathfrak{D}_F$. Once this is accomplished, letting $(F_i)_{i=1}^k$ be an enumeration of the subsets of $F_0 \cup \{p_0\}$, we may choose infinite subsets $Q_{F_1} \supset \cdots \supset Q_{F_k}$ of Q_0 such that $[Q_{F_i}] \subset \mathfrak{D}_{F_i}$ for all $i \leq k$. Setting $R = Q_{F_k}$, we see that $[R] \subset \mathfrak{D}_F$, for all $F \subset F_0 \cup \{p_0\}$. Hence R is desired.

We now fix some $F \subset F_0 \cup \{p_0\}$ and show there exists $Q_F \in [Q_0]$ such that $[Q_F] \subset \mathfrak{D}_F$. Since $\mathfrak{D}_F$ is closed in the topology of pointwise convergence of $[\mathbb{N}]$, the infinite Ramsey theorem implies the existence of $Q_F \in [Q_0]$ such that either $[Q_F] \subset \mathfrak{D}_F$ or $[Q_F] \cap \mathfrak{D}_F = \emptyset$. We show the second alternative cannot occur and thus complete the proof of the lemma.

Suppose instead that $[Q_F] \cap \mathfrak{D}_F = \emptyset$, for some $Q_F \in [Q_0]$. By the spreading property of $\mathcal{F}$, there exists $n_0 \in \mathbb{N}$ such that $\alpha \cup \{n\} \in \mathcal{F}$ for every non-terminal $\alpha \in \mathcal{F}$ with $\alpha \subset F_0 \cup \{p_0\}$, and all $n \geq n_0$. We then choose $q_1 < q_2 < \dots$ in Q_F with $n_0 < q_1$. Fix a $k \in \mathbb{N}$ and define $L_k \in [P]$ as follows: Set $L_k = \{\min P, p_0\} \cup \{q_j : j > k\}$, if $p_0 \in F$; otherwise, that is, when $p_0 \notin F$, set $L_k = \{p_0\} \cup \{q_j : j > k\}$.

We observe that $[F, F_0 \cup \{p_0\}]_{\mathcal{F}} \subset [F \setminus \{p_0\}, F_0 \cup \{\min L_k\}]_{\mathcal{F}}$. This is so since $\beta \subset F_0 \cup \{p_0\}$ implies that either $\beta \subset F_0$ or $p_0 \in \beta$. If the former then, clearly, $\beta \subset F_0 \cup \{\min L_k\}$ and if $\max \beta \notin F$, then also $\max \beta \notin F \setminus \{p_0\}$. If the latter, that is, $p_0 \in \beta$, then $p_0 = \max \beta$. Hence, if $\max \beta \notin F$, then $p_0 \notin F$ and so $\min L_k = p_0$. Again, $\beta \subset F_0 \cup \{\min L_k\}$ and $\max \beta \notin F \setminus \{p_0\}$. This observation leads us to

$$\sum_{\beta\in[F,F_0\cup\{p_0\}]_{\mathcal{F}}} |\mu|(|f_\beta|) \leq \sum_{\beta\in[F\setminus\{p_0\},F_0\cup\{\min L_k\}]_{\mathcal{F}}} |\mu|(|f_\beta|),\ \forall\, \mu \in B_{C(K)^*}. \tag{4.1}$$

We next define sets $R_i \in [Q_F]$ with $R_i = \{q_i\} \cup \{q_j : j > k\}$, $i \leq k$. Since each $R_i \notin \mathfrak{D}_F$ and $F \cup (R_i \setminus \{\min R_i\}) = F \cup \{q_j : j > k\}$, there exists an initial segment $\alpha_{R_i} \in \mathcal{F}^*$ of $F \cup \{q_j : j > k\}$ with $\psi^{-1}_{\alpha_{R_i}}(B) \cap \mathcal{M} \neq \emptyset$ and such that

$$\sum_{\beta\in[F,F_0\cup\{p_0,q_i\}]_{\mathcal{F}}} |\mu|(|f_\beta|) \geq \sum_{j=0}^{|F_0|+1} \epsilon_j,\ \forall\, \mu \in \psi^{-1}_{\alpha_{R_i}}(B) \cap \mathcal{M},\ \forall\, i \leq k. \tag{4.2}$$

By assumption, L_k is F_0-admissible and thus also $(F \setminus \{p_0\}, F_0)$-admissible. Note also that $(F \setminus \{p_0\}) \cup (L_k \setminus \{\min L_k\}) = F \cup \{q_j : j > k\}$, by the definition of L_k. Therefore, for all $i \leq k$, $\alpha_{R_i} \in \mathcal{F}^*$ is an initial segment of $(F\setminus\{p_0\})\cup(L_k\setminus\{\min L_k\})$

satisfying $\psi_{\alpha_{R_i}}^{-1}(B) \cap \mathcal{M} \neq \emptyset$. It follows now that there exists, for all $i \leq k$, $\mu_{R_i} \in \psi_{\alpha_{R_i}}^{-1}(B) \cap \mathcal{M}$ such that

$$\sum_{\beta \in [F \setminus \{p_0\}, F_0 \cup \{\min L_k\}]_{\mathcal{F}}} |\mu_{R_i}|(|f_\beta|) < \sum_{j=0}^{|F_0|} \epsilon_j.$$

(4.1) now implies

$$\sum_{\beta \in [F, F_0 \cup \{p_0\}]_{\mathcal{F}}} |\mu_{R_i}|(|f_\beta|) < \sum_{j=0}^{|F_0|} \epsilon_j, \, \forall \, i \leq k. \tag{4.3}$$

We next observe that $\{\alpha_{R_i} : i \leq k\}$ is well ordered in $\mathcal{F}^*$ (its elements are initial segments of $F \cup \{q_j : j > k\}$). Moreover, $\mu_{R_i} \in \psi_{\alpha_{R_i}}^{-1}(B) \cap \mathcal{M}$, for all $i \leq k$. Since $\mathcal{M}$ satisfies a B-lattice property with respect to $(\psi_\alpha)_{\alpha \in \mathcal{F}^*}$, there exists $\mu_k \in \{\mu_{R_i} : i \leq k\}$ so that $\mu_k \in \psi_{\alpha_{R_i}}^{-1}(B) \cap \mathcal{M}$ for all $i \leq k$. We deduce from (4.2) that

$$\sum_{\beta \in [F, F_0 \cup \{p_0, q_i\}]_{\mathcal{F}}} |\mu_k|(|f_\beta|) \geq \sum_{j=0}^{|F_0|+1} \epsilon_j, \, \forall \, i \leq k. \tag{4.4}$$

Taking in account (4.3) we obtain

$$\sum_{\beta \in [F, F_0 \cup \{p_0\}]_{\mathcal{F}}} |\mu_k|(|f_\beta|) < \sum_{j=0}^{|F_0|} \epsilon_j,$$

and thus (4.4) implies

$$\sum_{\beta \in [F, F_0 \cup \{p_0, q_i\}]_{\mathcal{F}}, \, \max \beta = q_i} |\mu_k|(|f_\beta|) \geq \epsilon_{|F_0|+1}, \, \forall \, i \leq k.$$

Equivalently,

$$\sum_{\beta \subset F_0 \cup \{p_0\}, \, \beta \in \mathcal{F}, \text{ non-terminal}} |\mu_k|(|f_{\beta \cup \{q_i\}}|) \geq \epsilon_{|F_0|+1}, \, \forall \, i \leq k.$$

Finally let ν be a w^*-cluster point of the sequence $(|\mu_k|)$. It follows that

$$\sum_{\beta \subset F_0 \cup \{p_0\}, \, \beta \in \mathcal{F}, \text{ non-terminal}} \nu(|f_{\beta \cup \{q_i\}}|) \geq \epsilon_{|F_0|+1}, \, \forall \, i \in \mathbb{N}.$$

This contradicts our assumption that $(\mathcal{F}, (f_\alpha)_{\alpha \in \mathcal{F}^*}, \mathcal{M})$ is a triple on K. $\square$

Proof of Theorem 4.3. Choose $M_0 \in [\mathbb{N}]$ according to Lemma 4.6. Then choose $m_1 \in M_0$, $\min M_0 < m_1$ and apply Lemma 4.7 to obtain $M_1 \in [M_0]$ such that every $L \in [M_1]$ is $\{m_1\}$-admissible. Choose $m_2 \in M_1$, $\min M_1 < m_2$. Successive applications of Lemma 4.7 yield a sequence $M_1 \supset M_2 \supset \dots$ of infinite subsets of M_0 and integers $m_1 < m_2 < \dots$ such that for all $i \in \mathbb{N}$ we have $m_i \in M_{i-1}$, $\min M_{i-1} < m_i$ and every $L \in [M_i]$ is $\{m_1, \dots, m_i\}$-admissible. We show $M = \{m_i : i \in \mathbb{N}\}$ satisfies the assertion of the theorem. Indeed, let $\alpha \in \mathcal{F}[M]^*$ such that $\psi_\alpha^{-1}(B) \cap \mathcal{M} \neq \emptyset$. Put $A_n = [\alpha, \{m_1, \dots, m_n\}]_{\mathcal{F}}$, for every $n \in \mathbb{N}$, $\max \alpha < m_n$ (see Notation 4.4). Since M_n is $\{m_1, \dots, m_n\}$-admissible, it is also $(\alpha, \{m_1, \dots, m_n\})$-admissible. It follows that there exists $\mu_n \in \psi_\alpha^{-1}(B) \cap \mathcal{M}$ such that $\sum_{\beta \in A_n} |\mu_n|(|f_\beta|) < \sum_{i=0}^n \epsilon_i$. Let $\mu \in \mathrm{Cl}_{w^*} \mathcal{M}$ be a w^*-cluster point of (μ_n). Our hypotheses yield $\mu \in \psi_\alpha^{-1}(B)$

and, of course, $\sum_{\beta\in A_n}|\mu|(|f_\beta|)\le\sum_{i=0}^\infty \epsilon_i$, for every $n\in\mathbb{N}$ with $\max\alpha<m_n$. We conclude that $\sum_{\beta\in\mathcal{F}^*[M],\,\max\beta\notin\alpha}|\mu|(|f_\beta|)<\epsilon$. $\square$

Corollary 4.8. *Let $\mathcal{M}\subset B_{C(K)^*}$ (K a totally disconnected, compact, metrizable space) be w^*-compact. Suppose $\lambda>0$ and $\xi<\omega_1$ are such that $\eta(\lambda,\mathcal{M})\ge\xi$. Let $0<\epsilon<\lambda/4$, and let $\mathcal{F}$ be a regular family of order ξ. Assume that $(\mu_\alpha)_{\alpha\in\mathcal{F}}$, $(G_\alpha)_{\alpha\in\mathcal{F}^*}$, $\mathcal{N}$ satisfy the conclusion of Theorem 1.3. Then for every $\delta>0$ there exists $M\in[\mathbb{N}]$ with the following property: For every $\alpha\in\mathcal{F}^*[M]$ there exists $\mu\in\mathcal{M}$ such that $|\mu(G_\beta)|\ge\lambda/4-\epsilon$ for all $\beta\leqslant\alpha$ in $\mathcal{F}^*[M]$, yet $\sum_{\beta\in\mathcal{F}^*[M],\,\max\beta\notin\alpha}|\mu|(G_\beta)<\delta$.*

Proof. Put $\mathcal{M}_0=\{\mu_\alpha:\alpha\in\mathcal{F}\text{ is terminal}\}$. We know that $\mathrm{Cl}_{w^*}\{|\mu|:\mu\in\mathcal{M}_0\}=\mathcal{N}$ and thus $\lim_n\nu(G_{\beta\cup\{n\}})=0$, for every non-terminal $\beta\in\mathcal{F}$ and all $\nu\in\mathrm{Cl}_{w^*}\{|\mu|:\mu\in\mathcal{M}_0\}$, because of (4) of Theorem 1.3. It follows that $(\mathcal{F},(\chi_{G_\alpha})_{\alpha\in\mathcal{F}^*},\mathcal{M}_0)$ is a triple on K.

We now set $B=\{x\in c_{00}\setminus\{0\}:1\ge|x(n)|\ge\lambda/4-\epsilon,\ \forall n\in\operatorname{supp}x\}$. Define mappings $\psi_\alpha\colon B_{C(K)^*}\to c_{00}$, $\alpha\in\mathcal{F}^*$, by

$$\psi_\alpha(\tau)=\begin{cases}\sum_{\beta\in\mathcal{F}^*,\,\beta\leqslant\alpha}\tau(G_\beta)\chi_{\{\max\beta\}}, & \text{if } \tau(G_\beta)\ne0,\ \forall\,\beta\leqslant\alpha;\\ 0, & \text{otherwise.}\end{cases}$$

It is easily verified that $\psi_\alpha^{-1}(B)$ is w^*-closed for all $\alpha\in\mathcal{F}^*$. We check that $B_{C(K)^*}$ (and thus also $\mathcal{M}_0$) satisfies a B-lattice property with respect to $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$. Indeed, suppose that $\alpha_1<\dots<\alpha_k$ in $\mathcal{F}^*$ and that $\mu_{\alpha_i}\in\psi_{\alpha_i}^{-1}(B)$, for all $i\le k$. It follows from the definitions that $\mu_{\alpha_k}\in\psi_{\alpha_i}^{-1}(B)$, for all $i\le k$.

We now apply Theorem 4.3 for the triple $(\mathcal{F},(\chi_{G_\alpha})_{\alpha\in\mathcal{F}^*},\mathcal{M}_0)$, the subset B of c_{00}, the mappings $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$ defined above, and the given $\delta>0$ to obtain $M\in[\mathbb{N}]$ satisfying the conclusion of Theorem 4.3.

Finally, let $\alpha\in\mathcal{F}^*[M]$. Choose α_0 terminal in $\mathcal{F}$ so that $\alpha\leqslant\alpha_0$. We infer from (2) of Theorem 1.3 that $|\mu_{\alpha_0}(G_\beta)|\ge\lambda/4-\epsilon$, for all $\beta\leqslant\alpha_0$ in $\mathcal{F}^*$. It follows that $\mu_{\alpha_0}\in\psi_\alpha^{-1}(B)\cap\mathcal{M}_0$. Theorem 4.3 now yields $\mu\in\psi_\alpha^{-1}(B)\cap\mathcal{M}$ (and hence $|\mu(G_\beta)|\ge\lambda/4-\epsilon$ for all $\beta\leqslant\alpha$ in $\mathcal{F}^*[M]$) such that $\sum_{\beta\in\mathcal{F}^*[M],\,\max\beta\notin\alpha}|\mu|(G_\beta)<\delta$, completing the proof of the corollary. $\square$

The next corollary is also obtained in [14], [12].

Corollary 4.9. *Let K be a compact metrizable space and (f_n) a weakly null sequence in $B_{C(K)}$. Let $\mathcal{M}\subset B_{C(K)^*}$ be w^*-compact and $\lambda\in(0,1]$. Given $\epsilon>0$, there exists $M\in[\mathbb{N}]$ with the following property: Suppose that $F\in[M]^{<\infty}$ and $\mu\in\mathcal{M}$ are so that $|\mu(f_i)|\ge\lambda$, for all $i\in F$, and the scalars $\mu(f_i)$, $i\in F$, are all of the same sign. Then there exists $\nu\in\mathcal{M}$ so that $|\nu(f_i)|\ge\lambda$, for all $i\in F$, the scalars $\nu(f_i)$, $i\in F$, are all of the same sign and $\sum_{i\in M\setminus F}|\nu|(|f_i|)<\epsilon$.*

Proof. Put $\mathcal{F}=[\mathbb{N}]^{<\infty}$ and $f_\alpha=f_{\max\alpha}$, for $\alpha\in\mathcal{F}^*$. Because (f_n) is weakly null, it is clear that $(\mathcal{F},(f_\alpha)_{\alpha\in\mathcal{F}^*},\mathcal{M})$ is a triple on K. Set $B_+=\{x\in c_{00}\setminus\{0\}:1\ge x(n)\ge\lambda,\ \forall n\in\operatorname{supp}x\}$ and $B_-=\{x\in c_{00}\setminus\{0\}:-1\le x(n)\le-\lambda,\ \forall n\in\operatorname{supp}x\}$. Define mappings $\psi_\alpha\colon B_{C(K)^*}\to c_{00}$, $\alpha\in\mathcal{F}^*$, by

$$\psi_\alpha(\tau)=\begin{cases}\sum_{\beta\in\mathcal{F}^*,\,\beta\leqslant\alpha}\tau(f_\beta)\chi_{\{\max\beta\}}, & \text{if } \tau(f_\beta)\ne0,\ \forall\,\beta\leqslant\alpha;\\ 0, & \text{otherwise.}\end{cases}$$

Arguing as in Corollary 4.8, we see that $\psi_\alpha^{-1}(B_+)$ and $\psi_\alpha^{-1}(B_-)$ are w^*-closed for all $\alpha \in \mathcal{F}^*$, and that $\mathcal{M}$ satisfies a B_+ (resp. B_-)-lattice property with respect to $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$.

Note that we can replace $\mathbb{N}$ in the statement of Theorem 4.3 by any of its infinite subsets, say N, and derive the same conclusion, with the resulting set $M \in [N]$ (observe that in Lemma 4.6 we can take $M_0 \in [N]$). We can therefore apply Theorem 4.3, consecutively, for the triple $(\mathcal{F}, (f_\alpha)_{\alpha\in\mathcal{F}^*}, \mathcal{M})$, the subsets B_+ and B_- of c_{00}, the mappings $(\psi_\alpha)_{\alpha\in\mathcal{F}^*}$ defined above, and the given $\epsilon > 0$ to obtain $M \in [\mathbb{N}]$ satisfying the conclusion of Theorem 4.3 simultaneously for B_+ and B_-.

It follows now that if $F \in [M]^{<\infty}$, $\mu \in \mathcal{M}$ are so that $|\mu(f_i)| \geq \lambda$ for all $i \in F$, and the scalars $\mu(f_i)$, $i \in F$, are all of the same sign, say they are all positive, then $\mu \in \psi_F^{-1}(B_+) \cap \mathcal{M}$. Therefore there exists $\nu \in \psi_F^{-1}(B_+) \cap \mathcal{M}$ such that $\sum_{\beta\in\mathcal{F}^*[M],\, \max\beta\notin F} |\nu|(|f_\beta|) < \epsilon$. Because $f_\beta = f_{\max\beta}$, we deduce that $\sum_{i\in M\setminus F} |\nu|(|f_i|) < \epsilon$. Since $\nu \in \psi_F^{-1}(B_+)$, we infer that $\nu(f_i) \geq \lambda$, for all $i \in F$.

If the scalars $\mu(f_i)$, $i \in F$, were all negative, then we would have a similar argument using B_-. The proof is now complete. □

We conclude this section with the

Proof of Theorem 1.4. Apply Corollary 4.8 to obtain a family $(H_\alpha)_{\alpha\in\mathcal{F}^*}$ of clopen subsets of K and $M \in [\mathbb{N}]$ with the following property: For every $\alpha \in \mathcal{F}^*[M]$ there exists $\mu \in \mathcal{M}$ such that $|\mu(H_\beta)| \geq \lambda/4 - \epsilon$ for all $\beta \leqslant \alpha$ in $\mathcal{F}^*[M]$, yet $\sum_{\beta\in\mathcal{F}^*[M],\, \max\beta\notin\alpha} |\mu|(H_\beta) < \delta$.

Let $\sigma\colon [\mathbb{N}]^{<\infty} \to \mathcal{T}_\infty^M$ be the order preserving injection given in (2) of Proposition 2.14. Then $\sigma(\mathcal{F})$ is a subtree of $\mathcal{F}[M] \cap \mathcal{T}_\infty^M$. Set $G_\alpha = H_{\sigma(\alpha)}$, for all $\alpha \in \mathcal{F}^*$. Fix some $\alpha \in \mathcal{F}^*$ and choose $\mu \in \mathcal{M}$ so that $|\mu(H_\gamma)| \geq \lambda/4 - \epsilon$ for all $\gamma \leqslant \sigma(\alpha)$ in $\mathcal{F}^*$, yet

$$\sum_{\gamma\in\mathcal{F}^*[M],\, \max\gamma\notin\sigma(\alpha)} |\mu|(H_\gamma) < \delta.$$

It follows now, since σ is an order preserving injection, that $|\mu(G_\beta)| \geq \lambda/4 - \epsilon$, for all $\beta \leqslant \alpha$ in $\mathcal{F}^*$.

On the other hand, if $\beta \in \mathcal{F}^*$ satisfies $\beta \nleqslant \alpha$, then $\sigma(\beta) \nleqslant \sigma(\alpha)$ in $\mathcal{T}_\infty^M$, again because σ is an order preserving injection. Hence, $\max\sigma(\beta) \notin \sigma(\alpha)$, by (1) of Proposition 2.14. We deduce from this that

$$\{\sigma(\beta) :\, \beta \in \mathcal{F}^*,\, \beta \nleqslant \alpha\} \subset \{\gamma \in \mathcal{F}^*[M] :\, \max\gamma \notin \sigma(\alpha)\}.$$

Therefore,

$$\begin{aligned} \sum_{\beta\in\mathcal{F}^*,\, \beta\nleqslant\alpha} |\mu|(G_\beta) &= \sum_{\beta\in\mathcal{F}^*,\, \beta\nleqslant\alpha} |\mu|(H_{\sigma(\beta)}) \\ &\leq \sum_{\gamma\in\mathcal{F}^*[M],\, \max\gamma\notin\sigma(\alpha)} |\mu|(H_\gamma) < \delta. \end{aligned}$$

□

5. Proof of Theorem 1.1

First note that in proving Theorem 1.1 there will be no loss of generality if we assume K is totally disconnected, in view of Miljutin's theorem [38]. Theorem 1.1 is an immediate consequence of the next more precise theorem.

Theorem 5.1. *Let K be a compact, totally disconnected and metrizable space. Let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact. Assume that $\eta(\lambda, \mathcal{M}) \geq \xi$ for some $\lambda > 0$ and $1 \leq \xi < \omega_1$. Let $\mathcal{F}$ be a regular and stable family of order ξ and $0 < \epsilon < \lambda/16$. Then there exists a sequence (V_n) of non-empty clopen subsets of K with (χ_{V_n}) equivalent to the natural basis of $X_{\mathcal{F}}$ and such that the closed linear span $[\chi_{V_n}, n \in \mathbb{N}]$ of (χ_{V_n}) in $C(K)$ is normed by $\mathcal{M}$. Moreover, in the above, the equivalence constant may be chosen not to exceed $(\lambda/16 - \epsilon)^{-1}$, while the norming constant may be chosen no less than $\lambda/16 - \epsilon$.*

We recall that a hereditary family $\mathcal{F}$ is stable provided that $F \in \mathcal{F}$ is maximal, under inclusion, in $\mathcal{F}$ if, and only if, there exists $n > \max F$ such that $F \cup \{n\} \notin \mathcal{F}$. We also recall that $\mathcal{M}$ *norms* a subspace X of $C(K)$ if there exists a scalar $c > 0$ such that $\sup_{\mu \in \mathcal{M}} |\mu(f)| \geq c\|f\|$, for all $f \in X$. We then call c the *norming constant* of $\mathcal{M}$. Finally two Schauder basic sequences (x_n) and (y_n) are *equivalent* if there exist positive scalars c_1, c_2 such that $c_1\| \sum_{i=1}^n a_i x_i\| \leq \| \sum_{i=1}^n a_i y_i\| \leq c_2\| \sum_{i=1}^n a_i x_i\|$, for all $n \in \mathbb{N}$, and all choices of scalars $(a_i)_{i=1}^n$. We then call c_2/c_1 the *equivalence constant* of (x_n) and (y_n).

Theorem 5.1 will follow from

Proposition 5.2. *Let K be a compact, totally disconnected, metrizable space, and let $\mathcal{M} \subset B_{C(K)^*}$ be w^*-compact. Let $\mathcal{F}$ be a regular and stable family containing the singletons, and let (U_n) be a sequence of non-empty clopen subsets of K. Suppose there exist positive constants $\rho < \lambda$ such that for every $F \in \mathcal{F}^*$ there exists $\mu \in \mathcal{M}$ satisfying $|\mu(U_n)| \geq \lambda$ for all $n \in F$ and $|\mu|(\bigcup_{n \notin F} U_n) < \rho$. Then there exists a sequence (V_n) of non-empty clopen subsets of K with (χ_{V_n}) equivalent to the natural basis of $X_{\mathcal{F}}$ and such that the closed linear span $[\chi_{V_n}, n \in \mathbb{N}]$ of (χ_{V_n}) in $C(K)$ is normed by $\mathcal{M}$. Moreover, in the above, given $\delta > 0$, $\delta < (\lambda - \rho)/4$, (V_n) may be chosen so that the equivalence constant does not exceed $\big((\lambda - \rho)/4 - \delta\big)^{-1}$, while the norming constant is no less than $(\lambda - \rho)/4 - \delta$.*

We postpone the proof of Proposition 5.2 in order to give the

Proof of Theorem 5.1. Choose a positive scalar $\delta < \min\{\lambda/28, \epsilon/2\}$. Apply Theorem 1.4 to obtain a family $(G_\alpha)_{\alpha \in \mathcal{F}^*}$ of (necessarily non-empty) clopen subsets of K such that for every $\alpha \in \mathcal{F}^*$ there exists $\mu \in \mathcal{M}$ such that $|\mu(G_\beta)| \geq \lambda/4 - \delta$ for all $\beta \leqslant \alpha$ in $\mathcal{F}^*$, yet $\sum_{\beta \in \mathcal{F}^*, \beta \nleqslant \alpha} |\mu|(G_\beta) < \delta$.

Put $U_n = \bigcup_{\alpha \in \mathcal{F}^*, \max \alpha = n} G_\alpha$, for all $n \in \mathbb{N}$. Clearly, (U_n) is a sequence of non-empty clopen subsets of K. Let $\{n_1, \dots, n_k\} \in \mathcal{F}^*$ with $n_1 < \dots < n_k$. Set $\alpha_i = \{n_1, \dots, n_i\}$, for $i \leq k$. Choose $\mu \in \mathcal{M}$ so that $|\mu(G_{\alpha_i})| \geq \lambda/4 - \delta$ for all $i \leq k$, and $\sum_{\alpha \in \mathcal{F}^*, \alpha \nleqslant \alpha_k} |\mu|(G_\alpha) < \delta$. Observe that for all $i \leq k$,

$$U_{n_i} \setminus G_{\alpha_i} \subset \bigcup_{\alpha \in \mathcal{F}^* \setminus \{\alpha_i\}, \max \alpha = n_i} G_\alpha \subset \bigcup_{\alpha \in \mathcal{F}^*, \alpha \nleqslant \alpha_k} G_\alpha$$

and so

$$\begin{aligned} |\mu|(U_{n_i} \setminus G_{\alpha_i}) &\leq |\mu|\Big(\bigcup_{\alpha \in \mathcal{F}^*, \alpha \nleqslant \alpha_k} G_\alpha\Big) \\ &\leq \sum_{\alpha \in \mathcal{F}^*, \alpha \nleqslant \alpha_k} |\mu|(G_\alpha) < \delta. \end{aligned}$$

Thus, $|\mu(U_{n_i})| \geq \lambda/4 - 2\delta$, for all $i \leq k$. Arguing similarly, we obtain that

$$\bigcup_{n \notin \{n_1, \dots, n_k\}} U_n \subset \bigcup_{\alpha \in \mathcal{F}^*,\, \alpha \nleq \alpha_k} G_\alpha$$

and so again

$$|\mu|(\bigcup_{n \notin \{n_1, \dots, n_k\}} U_n) \leq \sum_{\alpha \in \mathcal{F}^*,\, \alpha \nleq \alpha_k} |\mu|(G_\alpha) < \delta.$$

Proposition 5.2 (with "λ"$= \lambda/4 - 2\delta$, $\rho = \delta$ and the chosen δ) yields a sequence (V_n) of non-empty clopen subsets of K such that (χ_{V_n}) is $(\lambda/16 - \epsilon)^{-1}$ equivalent to the natural basis of $X_{\mathcal{F}}$, while $[\chi_{V_n}, n \in \mathbb{N}]$ is $\lambda/16 - \epsilon$-normed by $\mathcal{M}$. The proof of the theorem is now complete. □

The proof of Proposition 5.2 requires the following key lemma.

Lemma 5.3. *Let (U_n) be a sequence of clopen subsets of K, and let $\mathcal{F}$ be a regular and stable family containing the singletons. Then there exists a sequence (W_n) of clopen subsets of K with $W_n \subset U_n$ for all $n \in \mathbb{N}$, and such that*

1. *If $\bigcap_{n \in I} W_n \neq \emptyset$ for some $I \subset \mathbb{N}$, then $I \in \mathcal{F}$.*
2. *For every $I \in \mathcal{F}$ and all $n \in I$ we have $U_n \setminus W_n \subset \bigcup_{i \notin I} U_i$.*

Proof. Given $n \in \mathbb{N}$, set

$$\mathcal{F}_n = \{F \in \mathcal{F} : \max F < n \text{ and } F \cup \{n\} \notin \mathcal{F}\}.$$

The stability of $\mathcal{F}$ yields that every $F \in \mathcal{F}_n$ is a maximal, under inclusion, member of $\mathcal{F}$. Define a sequence (W_n) of subsets of K as follows:

$$W_n = \begin{cases} U_n, & \text{if } \mathcal{F}_n = \emptyset; \\ U_n \setminus \bigcup_{F \in \mathcal{F}_n} (\bigcap_{i \in F} U_i), & \text{if } \mathcal{F}_n \neq \emptyset. \end{cases}$$

Clearly, each W_n is a clopen subset of U_n. Suppose that $I \subset \mathbb{N}$ and $\bigcap_{n \in I} W_n \neq \emptyset$. We show $I \in \mathcal{F}$.

If that were not the case, then there would exist a finite subset J of I such that $J \notin \mathcal{F}$ (otherwise, I would be infinite, say $I = \{i_n : n \in \mathbb{N}\}$ with $i_1 < i_2 < \dots$ and $\{i_1, \dots, i_n\} \in \mathcal{F}$ for all $n \in \mathbb{N}$, contradicting the compactness of $\mathcal{F}$). We can thus select $n_1 < \dots < n_k$ in I so that $\{n_i : i \leq k\} \notin \mathcal{F}$. Since $\mathcal{F}$ contains the singletons, there exists $k_0 < k$ which is the largest $j < k$ such that $\{n_i : i \leq j\} \in \mathcal{F}$. Then $F = \{n_i : i \leq k_0\} \in \mathcal{F}$, yet $F \cup \{n_{k_0+1}\} \notin \mathcal{F}$. It follows that F is a maximal member of $\mathcal{F}$ and that $F \in \mathcal{F}_{n_k}$.

Now let $t \in \bigcap_{n \in I} W_n$. Since $t \in W_{n_k}$ and $F \in \mathcal{F}_{n_k}$, we infer that $t \notin \bigcap_{m \in F} U_m$. However, $t \in W_{n_i} \subset U_{n_i}$, for all $i \leq k_0$ as $n_i \in I$ for all $i \leq k$. Thus $t \in \bigcap_{i \leq k_0} U_{n_i} = \bigcap_{m \in F} U_m$, which is a contradiction. Hence $I \in \mathcal{F}$, which proves (1).

To show (2) holds, let $I \in \mathcal{F}$ and $n \in I$. Suppose $t \in U_n \setminus W_n$. It follows that $t \in \bigcup_{F \in \mathcal{F}_n} (\bigcap_{j \in F} U_j)$. Choose $F \in \mathcal{F}_n$ with $t \in \bigcap_{j \in F} U_j$. We claim that $F \setminus I \neq \emptyset$. Indeed, were $F \subset I$ we would have $F = I$ by the maximality of F in $\mathcal{F}$. But then $n \in F$, contradicting $\max F < n$. Finally choose $j \in F \setminus I$. Then $t \in U_j \subset \bigcup_{i \notin I} U_i$. □

Proof of Proposition 5.2. Choose a sequence (W_n) of clopen subsets of K according to Lemma 5.3, applied on the sequence (U_n) and the family $\mathcal{F}$. Since $\bigcap_{n \in I} W_n \neq \emptyset$ implies $I \in \mathcal{F}$, we infer that (χ_{W_n}) is a weakly null sequence in $C(K)$. Let $F \in \mathcal{F}^*$

and choose $\mu \in \mathcal{M}$ such that $|\mu(U_n)| \geq \lambda$, for all $n \in F$, yet $|\mu|(\bigcup_{n\notin F} U_n) < \rho$. Condition (2) in Lemma 5.3 now yields

$$U_n \setminus W_n \subset \bigcup_{i\notin F} U_i, \text{ for all } n \in F,$$

and thus $|\mu|(U_n \setminus W_n) \leq |\mu|(\bigcup_{i\notin F} U_i) < \rho$. Summarizing,

(5.1) For every $F \in \mathcal{F}^*$ there exists $\mu \in \mathcal{M}$ such that $|\mu(W_n)| \geq \lambda - \rho$, for all $n \in F$.

Now let $\delta > 0$ and apply Corollary 4.9 for the weakly null sequence (χ_{W_n}) and the set of measures $\mathcal{M}$, to obtain a subsequence $(\chi_{W_{m_n}})$ with the following property:

(5.2) Let $I \in [\mathbb{N}]^{<\infty}$ and $\mu \in \mathcal{M}$ satisfy $|\mu(W_{m_i})| \geq \lambda - \rho$ for all $i \in I$ and the scalars $\mu(W_{m_i})$, $i \in I$, are all of the same sign. Then there exists $\nu \in \mathcal{M}$ such that $|\nu(W_{m_i})| \geq \lambda - \rho$ for all $i \in I$, the scalars $\nu(W_{m_i})$, $i \in I$, are all of the same sign, and $\sum_{i\notin I} |\nu|(W_{m_i}) < \delta$.

We now apply Lemma 5.3 for the sequence of clopen sets (W_{m_n}) and the family $\mathcal{F}$ to obtain a sequence (V_n) of clopen subsets of K satisfying (1) and (2) of Lemma 5.3. We are going to show that (V_n) is the desired sequence by establishing the following

(5.3) $\|\sum_n a_n\chi_{V_n}\| \leq \|\sum_n a_n e_n\|_{\mathcal{F}}$ for all $(a_n) \in c_{00}$.

(5.4) If $(a_n) \in c_{00}$ satisfies $\|\sum_n a_n e_n\|_{\mathcal{F}} = 1$, then there exists $\nu \in \mathcal{M}$ such that $|\nu(\sum_n a_n\chi_{V_n})| \geq (\lambda - \rho - \delta)/4 - \delta$.

The assertion of the proposition will then follow if we take δ sufficiently small. Note in particular that (5.4) implies $V_n \neq \emptyset$, for all $n \in \mathbb{N}$. (5.3) is easily verified since $\bigcap_{n\in I} V_n \neq \emptyset$ implies $I \in \mathcal{F}$, by (1) of Lemma 5.3.

To show (5.4), let $(a_n) \in c_{00}$ such that $\|\sum_n a_n e_n\|_{\mathcal{F}} = 1$, and choose $F_1 \in \mathcal{F}$ with $\sum_{n\in F_1} |a_n| = 1$. Next choose $F_2 \subset F_1$ such that

$$\sum_{n\in F_2} |a_n| \geq 1/2 \text{ and the scalars } a_n,\, n \in F_2, \text{ are all of the same sign.}$$

Since $\mathcal{F}$ is spreading we infer that $\{m_i : i \in F_2\} \in \mathcal{F}$ and thus (5.1) yields $\mu \in \mathcal{M}$ such that $|\mu(W_{m_i})| \geq \lambda - \rho$, for all $i \in F_2$. We next choose $F \subset F_2$ such that

$$\sum_{i\in F} |a_i| \geq 1/4 \text{ and the scalars } \mu(W_{m_i}),\, i \in F, \text{ are all of the same sign.}$$

Applying (5.2), we find $\nu \in \mathcal{M}$ such that

$$\begin{aligned} &|\nu(W_{m_i})| \geq \lambda - \rho, \text{ for all } i \in F, \\ &\text{the scalars } \nu(W_{m_i}),\, i \in F, \text{ are all of the same sign,} \\ &\sum_{i\notin F} |\nu|(W_{m_i}) < \delta. \end{aligned}$$

We next have, by (2) of Lemma 5.3, that $W_{m_i} \setminus V_i \subset \bigcup_{n\notin F} W_{m_n}$, for all $i \in F$, and so

$$|\nu|(W_{m_i} \setminus V_i) \leq |\nu|(\bigcup_{n\notin F} W_{m_n}) \leq \sum_{n\notin F} |\nu|(W_{m_n}) < \delta, \text{ for all } i \in F.$$

Hence,

$$\begin{aligned}&|\nu(V_i)| \geq \lambda - \rho - \delta, \text{ for all } i \in F,\\&\text{the scalars } \nu(V_i),\ i \in F, \text{ are all of the same sign},\\&\sum_{i \notin F} |\nu|(V_i) < \delta \text{ (because } V_i \subset W_{m_i}).\end{aligned}$$

Concluding,

$$\begin{aligned}&|\nu(\sum_i a_i \chi_{V_i})| = |\nu(\sum_{i\in F} a_i \chi_{V_i}) + \nu(\sum_{i\notin F} a_i \chi_{V_i})|\\&\geq |\sum_{i\in F} a_i \nu(V_i)| - \sum_{i\notin F} |\nu|(V_i)\\&= \sum_{i\in F} |a_i||\nu(V_i)| - \sum_{i\notin F} |\nu|(V_i)\\&> (\lambda - \rho - \delta)\sum_{i\in F} |a_i| - \delta \geq (\lambda - \rho - \delta)/4 - \delta.\end{aligned}$$

Therefore (5.4) holds. This completes the entire proof. □

Acknowledgment

I would like to thank Dale Alspach for many useful discussions concerning operators on $C(K)$ spaces and their relation to the Szlenk and other ordinal indices. The possibility that an operator on $C(K)$ with Szlenk index exceeding ω^ξ could be an isomorphism on a subspace isomorphic to the Schreier space of order ξ (Theorem 1.1) emerged at a discussion we had during my visit at Oklahoma State University.

References

[1] D. Alspach, *Quotients of $C[0,1]$ with separable dual*, Israel J. Math. **29** (1978), 361–384. MR0491925 (80b:46039)

[2] D. Alspach, *A quotient of $C(\omega^\omega)$ which is not isomorphic to $C(\alpha)$, $\alpha < \omega_1$*, Israel J. Math. **35** (1980), 49–60. MR0576461 (81g:54012)

[3] D. Alspach, *$C(K)$ norming subsets of $C[0,1]$*, Studia Math. **70** (1981), 27–61. MR0646959 (83h:46032)

[4] D. Alspach, *$C(\alpha)$ preserving operators on separable Banach spaces*, J. Funct. Anal. **45** (1982), 139–168. MR0647068 (83h:46033)

[5] D. Alspach, *Operators on $C(\omega^\alpha)$ which do not preserve $C(\omega^\alpha)$*, Fund. Math. **153** (1997), 81–98. MR1450997 (98c:46016)

[6] D. Alspach and S.A. Argyros, *Complexity of weakly null sequences*, Dissertationes Mathematicae **321** (1992), 1–44. MR1191024 (93j:46014)

[7] D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math. **27** (1977), 64–92. MR0440349 (55:13224)

[8] D. Alspach and Y. Benyamini, *$C(K)$ quotients of separable $\mathcal{L}_\infty$ spaces*, Israel J. Math. **32** (1979), 145–160. MR0531258 (82j:46017)

[9] D. Alspach and Y. Benyamini, *A geometrical property of $C(K)$ spaces*, Israel J. Math. **64** (1988), 179–194. MR0986146 (91b:46023)

[10] D. Alspach, R. Judd and E. Odell, *The Szlenk index and local ℓ_1 indices*, Positivity (to appear).

[11] D. Alspach and E. Odell, *Averaging weakly null sequences*, Lecture Notes in Math. **1332**, Springer, Berlin (1988), 126–144. MR0967092 (89j:46014)

[12] S.A. Argyros and I. Gasparis, *Unconditional structures of weakly null sequences*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 2019–2058. MR1813606 (2002b:46012)

[13] S.A. Argyros and V. Kanellopoulos, *Optimal sequences of continuous functions converging to a Baire-1 function*, Math. Ann. **324** (2002), no. 4, 689–729. MR1942246 (2004d:46028)
[14] S.A. Argyros, S. Mercourakis and A. Tsarpalias, *Convex unconditionality and summability of weakly null sequences*, Israel J. Math. **107** (1998), 157–193. MR1658551 (99m:46021)
[15] Y. Benyamini, *An extension theorem for separable Banach spaces*, Israel J. Math. **29** (1978), 24–30. MR0482075 (58:2163)
[16] Y. Benyamini, *Private communication*.
[17] Y. Benyamini and J. Lindenstrauss, *A predual of ℓ_1 which is not isomorphic to a $C(K)$ space*, Israel J. Math. **13** (1972), 246–254. MR0331013 (48:9348)
[18] C. Bessaga and A. Pelczynski, *Spaces of continuous functions IV*, Studia Math. **19** (1960), 53–62. MR0113132 (22:3971)
[19] P. Billard, *Sur la primarite des espaces $C(\alpha)$*, Studia Math. **62** (1978), 143–162. MR0499961 (81f:46036)
[20] J. Bourgain, *The Szlenk index and operators on $C(K)$ spaces*, Bull. Soc. Math. Belg. Ser. B **31** (1979), 87–117. MR0592664 (83j:46027)
[21] E.E. Ellentuck, *A new proof that analytic sets are Ramsey*, J. Symbolic Logic **39** (1974), 163–165. MR0349393 (50:1887)
[22] J. Elton, *Thesis*, Yale University (1978).
[23] V. Farmaki, *Ramsey dichotomies with ordinal index*, preprint.
[24] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proc. Amer. Math. Soc. **129** (2001), 759–764. MR1707146 (2001f:03089)
[25] I. Gasparis and D.H. Leung, *On the complemented subspaces of the Schreier spaces*, Studia Math. **141** (2000), 273–300. MR1784674 (2001j:46009)
[26] G. Godefroy, N.J. Kalton and G. Lancien, *Szlenk indices and uniform homeomorphisms*, Trans. Amer. Math. Soc. **353** (2001), 3895–3918. MR1837213 (2003c:46023)
[27] R. Judd and E. Odell, *Concerning Bourgain's ℓ_1-index of a Banach space*, Israel J. Math. **108** (1998), 145–171. MR1669388 (2000k:46013)
[28] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Vol. 156, Springer Verlag, 1994. MR1321597 (96e:03057)
[29] H. Knaust, E. Odell and T. Schlumprecht, *On asymptotic structure, the Szlenk index and UKK properties in Banach spaces*, Positivity **3** (1999), 173–199. MR1702641 (2001f:46011)
[30] D.H. Leung and Wei-Kee Tang, *The ℓ_1-indices of Tsirelson type spaces*, Proc. Amer. Math. Soc., in press.
[31] D.H. Leung and Wei-Kee Tang, *The Bourgain ℓ_1-index of mixed Tsirelson spaces*, preprint.
[32] D.R. Lewis and C. Stegall, *Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$*, J. Funct. Anal. **12** (1973), 177–187. MR0342987 (49:7731)
[33] J. Lindenstrauss and A. Pelczynski, *Contributions to the theory of the classical Banach spaces*, J. Funct. Anal. **8** (1971), 225–249. MR0291772 (45:863)
[34] J. Lindenstrauss and H.P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349. MR0270119 (42:5012)
[35] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, New York (1977). MR0500056 (58:17766)
[36] J. Lindenstrauss and D.E. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Funct. Anal. **4** (1969), 332–349. MR0250033 (40:3274)
[37] S. Mazurkiewicz and W. Sierpinski, *Contributions a la topologie des ensembles denomrables*, Fund. Math. **1** (1920), 17–27.
[38] A.A. Miljutin, *On spaces of continuous functions*, Dissertation, Moscow State University (1952) (Russian).
[39] E. Odell, *Applications of Ramsey theorems to Banach space theory*, Notes in Banach spaces, (H.E. Lacey, ed.), Univ. Texas Press (1980), 379–404. MR0606226 (83g:46018)
[40] E. Odell, *On quotients of Banach spaces having shrinking unconditional bases*, Ill. J. Math. **36** (1992), 681–695. MR1215801 (94f:46015)
[41] E. Odell, *On subspaces, asymptotic structure, and distortion of Banach spaces; Connections with logic*, Analysis and Logic (Mons, 1997), 189–267, London Math. Soc. Lecture Note Ser., 262, Cambridge Univ. Press, Cambridge (2002). MR1967836 (2004b:46016)
[42] E. Odell, N. Tomczak-Jaegermann and R. Wagner, *Proximity to ℓ_1 and distortion in asymptotic ℓ_1 spaces*, J. Funct. Anal. **150** (1997), 101–145. MR1473628 (2000c:46027)

[43] A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228. MR0126145 (23:A3441)
[44] A. Pelczynski, *On $C(S)$-subspaces of separable Banach spaces*, Studia Math. **31** (1968), 513–522. MR0234261 (38:2578)
[45] H.P. Rosenthal, *On factors of $C[0,1]$ with non-separable dual*, Israel J. Math. **13** (1972), 361–378. MR0388063 (52:8900)
[46] H.P. Rosenthal, *Some aspects of the subspace structure of infinite-dimensional Banach spaces*, Approximation Theory and Functional Analysis, Academic Press, Boston, MA (1991), 151–176. MR1090543 (92g:41001)
[47] C. Samuel, *Sur certains espaces $C_\sigma(S)$ et sur les sous-espaces complementes de $C(S)$*, Bull. Sci. Math. (2) **95** (1971), 65–82. MR0285898 (44:3115)
[48] C. Samuel, *The Szlenk index of $C(K)$ (K a countable compact topological space)*, Publ. Math. Univ. Paris VII **18** (Seminar on the geometry of Banach spaces Vol. I, II) (1984), 81–91. MR0781569 (87b:46026)
[49] T. Schlumprecht, *How many operators do there exist on a Banach space*, preprint.
[50] J. Schreier, *Ein Gegenbeispiel zur theorie der schwachen konvergenz*, Studia Math. **2** (1930), 58–62.
[51] C. Stegall, *The Radon-Nicodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213–223. MR0374381 (51:10581)
[52] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. **30** (1968), 53–61. MR0227743 (37:3327)
[53] N. Tomczak-Jaegermann, *Banach spaces of type p have arbitrarily distortable subspaces*, Geom. Funct. Anal. **6** (1996), 1074–1082. MR1421875 (98g:46020)
[54] B. S. Tsirelson, *Not every Banach space contains ℓ_p or c_0*, Funct. Anal. Appl. **8** (1974), 138–141.
[55] J. Wolfe, *$C(\alpha)$-preserving operators on $C(K)$ spaces*, Trans. Amer. Math. Soc. **273** (1982), 705–719. MR0667169 (84a:46055)
[56] M. Zippin, *The separable extension problem*, Israel J. Math. **26** (1977), 372–387. MR0442649 (56:1030)

Department of Mathematics, University of Crete, Knossou Avenue, P.O. Box 2208, Heracleion 71409, Greece

E-mail address: ioagaspa@math.uoc.gr

Current address: Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece

E-mail address: iogaspa@auth.gr