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ELEMENTS OF SPECIFIED ORDER IN SIMPLE ALGEBRAIC GROUPS

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ABSTRACT. In this paper we let G be a simple algebraic group and r be a natural number, and consider the codimension in G of the variety of elements $g \in G$ satisfying $g^r = 1$. We shall obtain a lower bound for this codimension which is independent of characteristic, and show that it is attained if G is of adjoint type.

Let G be a simple algebraic group over an algebraically closed field K of characteristic p; let Φ be the root system of G, and take $r \in \mathbb{N}$. Define

$$G_{[r]} = \{g \in G : g^r = 1\} \qquad \text{and} \qquad G_{(r)} = \{g \in G : o(g) = r\}$$

(where $o(g)$ denotes the order of g); then $G_{[r]}$ and $G_{(r)}$ are both subvarieties of G, and $G_{[r]}$ is the disjoint union of those $G_{(r')}$ with r' dividing r. Our attention here is on the codimension in G of these varieties (if they are non-empty; clearly $G_{[r]} \neq \emptyset$, but the example of $G = SL_2(K)$, $p = 2$ and $r = 4$ shows that $G_{(r)}$ may be empty). It is immediate that if $G_{(r)} \neq \emptyset$ we have $\operatorname{codim} G_{(r)} \geq \operatorname{codim} G_{[r]}$. Our main result may be stated as follows.

Theorem 1. *Given G, Φ and r as above, there is a number $d_{\Phi,r}$, depending only on Φ and r and satisfying $d_{\Phi,r} \geq |\Phi|/r$, with the property that* $\operatorname{codim} G_{[r]} \geq d_{\Phi,r}$*; if G is of adjoint type we in fact have* $\operatorname{codim} G_{[r]} = d_{\Phi,r}$*, and if in addition $G_{(r)} \neq \emptyset$, then* $\operatorname{codim} G_{(r)} = d_{\Phi,r}$.

Statements equivalent to the inequality $\operatorname{codim} G_{(r)} \geq |\Phi|/r$ are already known in certain cases. If $r = 2$ and $p \neq 2$, the equivalent statement that, if $g \in G$ is an involution, then $\dim C_G(g) \geq \dim(G/B)$ (where B is a Borel subgroup), is well known; the stronger statement that $C_G(g)$ is then spherical, i.e., it has finitely many orbits on the flag variety G/B, was proved by Matsuki in [13] for $K = \mathbb{C}$, and by Springer in [19] for p odd—recently Seitz gave an alternative proof of Springer's result in [16]. In the case $r = 3$ and $p \neq 3$, the result follows from work of Liebeck and Shalev in [10]; this case and that with $r = 2$ are used in work of Liebeck, Seitz and the author concerning dimensions of fixed point spaces in [9]. More generally, for r an odd prime a result in this direction appears in further work of Liebeck and Shalev in [11], while the results proved here find application in [12] to homomorphisms from Fuchsian groups to finite simple groups.

Notice in particular that the statement of Theorem 1 is independent of the characteristic p. The author is grateful to Martin Liebeck for the initial observation

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that there appeared to be independence of p for r small and prime, which helped to motivate the present work.

Some calculations similar to certain ones here appear in a recent paper [2] of Carlson, Lin, Nakano and Parshall, concerning elements in a restricted Lie algebra in characteristic p which satisfy $x^{[p]} = 0$.

The organization of this paper is as follows. In section 1 we give the values $d_{\Phi,r}$ and prove some results about them. Next in section 2 we consider unipotent elements, and calculate the minimal dimensions of centralizers of unipotent elements having a prescribed power equal to the identity. Finally we apply these results in section 3 to consider arbitrary elements, and prove the various statements involved in Theorem 1.

1. The values $d_{\Phi,r}$

We begin with some notation. Given $x \in \mathbb{Z}$, we set $\epsilon_x = 1$ or 0 according to whether x is odd or even. For $y \in \mathbb{R}$ we write $\lceil y \rceil$ for the least integer greater than or equal to y.

Let Φ be a simple root system of rank ℓ, and let $h = \frac{|\Phi|}{\ell}$ be the Coxeter number of Φ; take $r \in \mathbb{N}$. For Φ of classical type, write

$$h = zr + e \qquad \text{with } z \in \mathbb{Z},\ 0 \leq e < r,$$

and set

$$\begin{aligned} d_{A_\ell,r} &= z^2 r + e(2z+1) - 1, \\ d_{B_\ell,r} &= \tfrac{1}{2}(z^2 r + e(2z+1)) + \epsilon_r \lceil \tfrac{z}{2} \rceil, \\ d_{C_\ell,r} &= \tfrac{1}{2}(z^2 r + e(2z+1)) + \epsilon_r \lceil \tfrac{z}{2} \rceil, \\ d_{D_\ell,r} &= \tfrac{1}{2}(z^2 r + e(2z+1)) + \epsilon_r \lceil \tfrac{z}{2} \rceil + z + 1 - \epsilon_z. \end{aligned}$$

For Φ of exceptional type, write

$$\epsilon_{\Phi,r} = \begin{cases} 1 & \text{if } (\Phi,r) = (F_4,5), (E_6,2), (E_6,4), (E_8,11), \\ 0 & \text{otherwise;} \end{cases}$$

then set

$$d_{\Phi,r} = \ell + 2\max\left(0, \left\lceil \tfrac{1}{2}(|\Phi|/r - \ell(1-\delta_{1r})) \right\rceil + \epsilon_{\Phi,r}\right),$$

where δ_{**} is the Kronecker delta. For convenience we list the values of $d_{\Phi,r}$ for $r < h$ and Φ exceptional in the following table:

r	G_2	F_4	E_6	E_7	E_8	r	F_4	E_6	E_7	E_8	r	E_7	E_8	r	E_8	r	E_8
						6	8	12	21	40	12	11	20	18	14	24	10
1	14	52	78	133	248	7	8	12	19	36	13	11	20	19	14	25	10
2	6	24	38	63	120	8	6	10	17	30	14	9	18	20	12	26	10
3	4	16	24	43	80	9	6	8	15	28	15	9	16	21	12	27	10
4	4	12	20	33	60	10	6	8	13	24	16	9	16	22	12	28	10
5	4	12	16	27	48	11	6	8	13	24	17	9	16	23	12	29	10

Our first result shows why it suffices to list $d_{\Phi,r}$ for $r < h$.

Lemma 1.1. *If $r \geq h$, then $d_{\Phi,r} = \ell$.*

Proof. First consider Φ of classical type. If $r = h$, then $z = 1$ and $e = 0$, so that $z^2 r + e(2z+1) = h$; since $h = \ell+1, 2\ell, 2\ell, 2\ell-2$ for $\Phi = A_\ell, B_\ell, C_\ell, D_\ell$ respectively, in each case we obtain $d_{\Phi,r} = \ell$. If instead $r > h$, then $z = 0$ and $e = h$, and the result again follows. For Φ of exceptional type, if $r \geq h$ we have $\frac{|\Phi|}{r} - \ell \leq 0$ (and $\delta_{1r} = \epsilon_{\Phi,r} = 0$), and the result is immediate. $\square$

Our next result establishes lower bounds on the values $d_{\Phi,r}$. Recall that a prime p_1 is said to be very good for Φ if either $\Phi = A_\ell$ and p_1 does not divide $\ell+1$, or Φ is not of type A and p_1 is good for Φ (so that $p_1 \neq 2$ if $\Phi = B_\ell, C_\ell, D_\ell$, $p_1 \neq 2,3$ if $\Phi = G_2, F_4, E_6, E_7$, and $p_1 \neq 2,3,5$ if $\Phi = E_8$).

Lemma 1.2. *We have $d_{\Phi,r} \geq \frac{|\Phi|}{r}$; moreover if r is a product of very good primes, then $d_{\Phi,r} \geq \frac{|\Phi|+\ell}{r}$.*

Proof. Begin with the first inequality. For Φ of exceptional type, this is clear from the definition of $d_{\Phi,r}$. For Φ of classical type we find that we have

$$rd_{\Phi,r} - |\Phi| = \begin{cases} zr + (e-1)(r-e) & \text{if } \Phi = A_\ell, \\ \frac{1}{2}e(r-e) + r\epsilon_r\lceil\frac{z}{2}\rceil & \text{if } \Phi = B_\ell \text{ or } C_\ell, \\ \frac{1}{2}(e+2)(r-e) + r(\epsilon_r\lceil\frac{z}{2}\rceil - \epsilon_z) & \text{if } \Phi = D_\ell; \end{cases}$$

in each case the expression given is non-negative, as required (note that if r is even the last may be written as $\frac{1}{2}e(r-e-2) + r(1-\epsilon_z)$, and we have $e \leq r-2$ because the equation $2\ell - 2 = h = zr + e$ forces e to be even).

Now assume that r is a product of very good primes. For Φ of exceptional type the second inequality is clear by inspection of the above table. For Φ of classical type we subtract ℓ from the expressions in the previous paragraph to obtain

$$rd_{\Phi,r} - |\Phi| - \ell = \begin{cases} (e-1)(r-e-1) & \text{if } \Phi = A_\ell, \\ \frac{1}{2}e(r-e-1) + r(\epsilon_r\lceil\frac{z}{2}\rceil - \frac{z}{2}) & \text{if } \Phi = B_\ell \text{ or } C_\ell, \\ \frac{1}{2}(e+2)(r-e-1) + r(\epsilon_r\lceil\frac{z}{2}\rceil - \frac{z}{2} - \epsilon_z) & \text{if } \Phi = D_\ell. \end{cases}$$

For $\Phi = A_\ell$ the condition that r is a product of very good primes implies that $e > 0$, so that the expression above is non-negative. For the other classical types r must be odd and so $\epsilon_r = 1$; this makes the expressions for B_ℓ and C_ℓ non-negative, while for D_ℓ we need only consider the case where z is odd. Here the equation $2\ell - 2 = h = zr + e$ forces e to be odd as well, so that $e \leq r-2$, and then we have

$$rd_{\Phi,r} - |\Phi| - \ell = \tfrac{1}{2}(e+2)(r-e-1) - \tfrac{1}{2}r = \tfrac{1}{2}(e+1)(r-e-2) \geq 0$$

as required. □

We shall also require the following lemma.

Lemma 1.3. *With the notation established, if $m > 1$, then $d_{\Phi,r} \geq d_{\Phi,mr}$, with equality if and only if $d_{\Phi,r} = \ell$.*

Proof. For Φ of exceptional type this is immediate by inspection of the table above. For Φ of classical type we write $z = z_1 m + t$ with $0 \leq t < m$; then $h = z_1(mr) + (tr + e)$ and $0 \leq tr + e < mr$. We then calculate

$$\begin{aligned} d_{A_\ell,r} - d_{A_\ell,mr} &= z^2 r + e(2z+1) - 1 - {z_1}^2 mr - (tr+e)(2z_1+1) + 1 \\ &= z_1(m-1)(z_1 mr + 2tr + 2e) + tr(t-1) + 2te \\ &\geq 0, \end{aligned}$$

with equality requiring $z_1 = 0$, which forces $mr > h$ and $d_{A_\ell,r} = d_{A_\ell,mr} = \ell$. Similarly

$$\begin{aligned} d_{B_\ell,r} - d_{B_\ell,mr} &= \tfrac{1}{2}z^2 r + \tfrac{1}{2}e(2z+1) + \epsilon_r \lceil \tfrac{z}{2} \rceil \\ &\quad - \tfrac{1}{2}{z_1}^2 mr - \tfrac{1}{2}(tr+e)(2z_1+1) - \epsilon_{mr} \lceil \tfrac{z_1}{2} \rceil \\ &= \tfrac{1}{2}(z_1(m-1)(z_1 mr + 2tr + 2e) + tr(t-1) + 2te) \\ &\quad + (\epsilon_r \lceil \tfrac{z_1 m + t}{2} \rceil - \epsilon_{mr} \lceil \tfrac{z_1}{2} \rceil) \\ &\geq 0, \end{aligned}$$

with equality again requiring $z_1 = 0$ and so $d_{B_\ell,r} = d_{B_\ell,mr} = \ell$; the calculation for $d_{C_\ell,r}$ is identical. Finally we have

$$\begin{aligned} d_{D_\ell,r} - d_{D_\ell,mr} &= \tfrac{1}{2}z^2 r + \tfrac{1}{2}e(2z+1) + \epsilon_r \lceil \tfrac{z}{2} \rceil + z + 1 - \epsilon_z \\ &\quad - \tfrac{1}{2}{z_1}^2 mr - \tfrac{1}{2}(tr+e)(2z_1+1) - \epsilon_{mr} \lceil \tfrac{z_1}{2} \rceil - z_1 - 1 + \epsilon_{z_1} \\ &= \tfrac{1}{2}(z_1(m-1)(z_1 mr + 2tr + 2e) + tr(t-1) + 2te) \\ &\quad + (\epsilon_r \lceil \tfrac{z_1 m + t}{2} \rceil - \epsilon_{mr} \lceil \tfrac{z_1}{2} \rceil) + z_1(m-1) + (t + \epsilon_{z_1} - \epsilon_{z_1 m + t}) \\ &\geq 0, \end{aligned}$$

with equality once more requiring $z_1 = 0$ and so $d_{D_\ell,r} = d_{D_\ell,mr} = \ell$. □

2. Unipotent elements

In this section we shall establish the minimal centralizer dimension of unipotent elements of prescribed order in a simple algebraic group H. These results will be utilized in the following section to prove the main theorem of this paper.

We shall proceed by considering the Jordan structure of unipotent elements. For H classical we take the action of H on its natural module, and use results linking Jordan structure and centralizer structure due to [20] for good characteristic and [7] for types B, C and D in characteristic 2. For H exceptional we take the adjoint action of H, and use results of [8] on Jordan structure and [3, 5, 14, 15, 17, 18] on centralizer structure. (Note that [15] contains some errors involving centralizers in E_8 in characteristic 2, but these concern component groups, not the structure of connected centralizers.)

We use the following notation throughout this section and the next. Let q be a power of the characteristic p of the simple algebraic group H, and set

$$d_q(H) = \min_{u \in H_{[q]}} \dim C_H(u).$$

Given $x \in \mathbb{Z}$ we set $\zeta_x = 1$ or 0 according to whether $x = 0$ or $x \neq 0$, and as above set $\epsilon_x = 1$ or 0 according to whether x is odd or even. For $y \in \mathbb{R}$ we write $\lfloor y \rfloor$ for the greatest integer less than or equal to y, and $\lceil y \rceil$ for the least integer greater than or equal to y.

Lemma 2.1. *Take $m \in \mathbb{N}$.*

(i) *If $m + 1 = \gamma q + \delta$ with $0 \leq \delta < q$, then $d_q(A_m) = \gamma^2 q + (2\gamma + 1)\delta - 1$.*
(ii) *If $m = \gamma q + \delta$ with $0 \leq \delta < q$, then $d_q(A_m) = \gamma^2 q + (2\gamma + 1)(\delta + 1) - 1$.*

Proof. Take H of type A_m; take $u \in H_{[q]}$ and write r_i for the number of Jordan blocks of size i in the action of u on the natural module, so that we have $\sum i r_i =$

$m+1$, and $r_i = 0$ for $i > q$ since $u^q = 1$. By [20],

$$\dim C_H(u) = \sum_i (r_i + r_{i+1} + \cdots + r_q)^2 - 1;$$

among elements of $H_{[q]}$ this is clearly minimized when at most one Jordan block has size less than q.

In (i) we require Jordan structure q^γ, δ, so that $r_q = \gamma$, $r_\delta = 1$ if $\delta > 0$ and $r_i = 0$ otherwise. This gives $\dim C_H(u) = \delta(\gamma+1)^2 + (q-\delta)\gamma^2 - 1 = \gamma^2 q + (2\gamma+1)\delta - 1$ as required. Now consider (ii). If $\delta < q-1$, then we have Jordan structure $q^\gamma, \delta+1$, whence $\dim C_H(u) = (\delta+1)(\gamma+1)^2 + (q-\delta-1)\gamma^2 - 1 = \gamma^2 q + (2\gamma+1)(\delta+1) - 1$ as required; if $\delta = q-1$ the Jordan structure is $q^{\gamma+1}$, giving $\dim C_H(u) = q(\gamma+1)^2 - 1$, again as required. □

Lemma 2.2. *Take $m \in \mathbb{N}$.*

(i) *If $2m = \gamma q + \delta$ with $0 \le \delta < q$, then*

$$d_q(C_m) = \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)\delta + \epsilon_q \lceil \tfrac{\gamma}{2} \rceil.$$

(ii) *If $2m+1 = \gamma q + \delta$ with $0 \le \delta < q$, then*

$$d_q(C_{m+1}) = \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)(\delta+1) + \epsilon_q \lceil \tfrac{\gamma}{2} \rceil.$$

(iii) *If $2m+1 = \gamma q + \delta$ with $0 \le \delta < q$, then*

$$d_q(C_m) = \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)(\delta-1) + \epsilon_q \lceil \tfrac{\gamma}{2} \rceil.$$

Proof. Take H of type C_k; as in the previous result take $u \in H_{[q]}$ and write r_i for the number of Jordan blocks of size i in the action of u on the natural module, so that $\sum i r_i = 2k$ and $r_i = 0$ for $i > q$. We must have r_i even for all odd i. If p is odd, [20] gives $\dim C_H(u) = \frac{1}{2}\sum_i (r_i + r_{i+1} + \cdots + r_q)^2 + \frac{1}{2}\sum_{i \text{ odd}} r_i$; if $p = 2$, then $\dim C_H(u)$ is not determined simply by the Jordan structure, but [7] gives a formula whose minimal value reduces to that for odd characteristic. As in the previous lemma, the optimal Jordan structure will involve as many blocks of size q as possible.

Begin with (i), so that $k = m$. If $p = 2$, then either $q = 1$, in which case $u = 1$ and $\dim C_H(u) = \dim H = 2m^2 + m$; or $q > 1$, when we have Jordan structure q^γ, δ (note that $\delta = 2m - \gamma q$ is even here), giving $\dim C_H(u) = \frac{1}{2}[\delta(\gamma+1)^2 + (q-\delta)\gamma^2] = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta$. If instead p is odd, we take separately the cases where γ (and hence δ) is even and odd: if γ is even the Jordan structure is q^γ, δ, whence $\dim C_H(u) = \frac{1}{2}[\delta(\gamma+1)^2 + (q-\delta)\gamma^2 + \gamma] = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2}\gamma$; if γ is odd the Jordan structure is $q^{\gamma-1}, q-1, \delta+1$ (or $q^{\gamma-1}, (q-1)^2$ if $\delta = q-2$), whence $\dim C_H(u) = \frac{1}{2}[(\delta+1)(\gamma+1)^2 + (q-\delta-2)\gamma^2 + (\gamma-1)^2 + \gamma - 1] = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2}(\gamma+1)$. Thus in all cases the minimal centralizer dimension is $\frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \epsilon_q \lceil \frac{\gamma}{2} \rceil$, as required.

For (ii) and (iii) we proceed by comparing with (i). For (ii) we have $k = m+1$; thus if we write $2k = \gamma' q + \delta'$ with $0 \le \delta' < q$, we have $\gamma' q + \delta' - 1 = \gamma q + \delta$. If $\delta < q-1$ we then have $\gamma' = \gamma$, $\delta' = \delta + 1$ and the minimal centralizer dimension is $\frac{1}{2}{\gamma'}^2 q + \frac{1}{2}(2\gamma'+1)\delta' + \epsilon_q \lceil \frac{\gamma'}{2} \rceil = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)(\delta+1) + \epsilon_q \lceil \frac{\gamma}{2} \rceil$; if $\delta = q-1$ we have $\gamma' = \gamma+1$, $\delta' = 0$ and the minimal centralizer dimension is $\frac{1}{2}(\gamma+1)^2 q + \epsilon_q \lceil \frac{\gamma+1}{2} \rceil = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)q + \epsilon_q \lceil \frac{\gamma}{2} \rceil$ (note that if q is odd here, then $\gamma' = \gamma + 1$ must be even, in which case $\lceil \frac{\gamma+1}{2} \rceil = \lceil \frac{\gamma}{2} \rceil$). In either case we obtain the required formula. For (iii) we have $k = m$; thus if we set $2k = \gamma' q + \delta'$ with $0 \le \delta' < q$ we have

$\gamma' q + \delta' + 1 = \gamma q + \delta$. If $\delta > 0$, then $\gamma' = \gamma$, $\delta' = \delta - 1$ and the minimal centralizer dimension is $\frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)(\delta-1) + \epsilon_q\lceil\frac{\gamma}{2}\rceil$; if $\delta = 0$, then $\gamma' = \gamma - 1$, $\delta' = q - 1$ and the minimal centralizer dimension is $\frac{1}{2}(\gamma-1)^2 q + \frac{1}{2}(2\gamma-1)(q-1) + \epsilon_q\lceil\frac{\gamma-1}{2}\rceil = \frac{1}{2}(\gamma^2 q - 2\gamma + 1) + \epsilon_q\lceil\frac{\gamma-1}{2}\rceil = \frac{1}{2}\gamma^2 q - \frac{1}{2}(2\gamma+1) + \epsilon_q\lceil\frac{\gamma}{2}\rceil$ (note that both γ and q must be odd here, so $1 + \epsilon_q\lceil\frac{\gamma-1}{2}\rceil = \epsilon_q\lceil\frac{\gamma}{2}\rceil$). Again the required formula follows. □

Lemma 2.3. *Take $m \in \mathbb{N}$.*

(i) *If $2m = \gamma q + \delta$ with $0 \leq \delta < q$, then*

$$d_q(B_m) = \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)\delta + \epsilon_q\lceil\tfrac{\gamma}{2}\rceil.$$

(ii) *If $2m+1 = \gamma q + \delta$ with $0 \leq \delta < q$, then*

$$d_q(B_{m+1}) = \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)(\delta+1) + \epsilon_q\lceil\tfrac{\gamma}{2}\rceil.$$

(iii) *If $2m+1 = \gamma q + \delta$ with $0 \leq \delta < q$, then*

$$d_q(B_m) = \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)(\delta-1) + \epsilon_q\lceil\tfrac{\gamma}{2}\rceil.$$

Proof. Take H of type B_k and $u \in H_{[q]}$, and as before write r_i for the number of Jordan blocks of size i in the action of u on the natural module, so that $\sum ir_i = 2k+1$ and $r_i = 0$ for $i > q$. If p is odd, we must have r_i even for all even i; [20] gives $\dim C_H(u) = \frac{1}{2}\sum_i(r_i + r_{i+1} + \cdots + r_q)^2 - \frac{1}{2}\sum_{i \text{ odd}} r_i$. If $p = 2$, we must instead have r_i even for all odd $i > 1$; here $\dim C_H(u)$ is again not determined simply by the Jordan structure, but [7] gives a formula whose minimal value reduces to $\frac{1}{2}\sum_i(r_i + r_{i+1} + \cdots + r_q)^2 - \frac{1}{2}\sum_{i \text{ odd}} r_i - \sum_{i \text{ even}} r_i$. Once more, the optimal Jordan structure will involve as many blocks of size q as possible. Note that it suffices to prove (i), as (ii) and (iii) will then follow by identical calculations to those in the previous result.

Thus let $k = m$. If $p = 2$, then either $q = 1$, in which case $u = 1$ and $\dim C_H(u) = \dim H = 2m^2 + m$; or $q > 1$, when $\delta = 2m - \gamma q$ must be even and we must distinguish the cases $\delta = 0$ and $\delta > 0$. In the case $\delta = 0$ we have Jordan structure $q^\gamma, 1$, giving $\dim C_H(u) = \frac{1}{2}[(\gamma+1)^2 + (q-1)\gamma^2 - 1 - 2\gamma] = \frac{1}{2}\gamma^2 q$; for $\delta > 0$ the Jordan structure is $q^\gamma, \delta, 1$, giving

$$\begin{aligned}\dim C_H(u) &= \tfrac{1}{2}[(\gamma+2)^2 + (\delta-1)(\gamma+1)^2 + (q-\delta)\gamma^2 - 1 - 2\gamma - 2] \\ &= \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)\delta.\end{aligned}$$

If p is odd instead, as in the previous result we take the cases where γ (and hence δ) is even and odd separately. If γ is even the Jordan structure is $q^\gamma, \delta+1$ (or $q^{\gamma+1}$ if $\delta = q-1$), whence

$$\begin{aligned}\dim C_H(u) &= \tfrac{1}{2}[(\delta+1)(\gamma+1)^2 + (q-\delta-1)\gamma^2 - \gamma - 1] \\ &= \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)\delta + \tfrac{1}{2}\gamma.\end{aligned}$$

If γ is odd the Jordan structure is $q^\gamma, \delta, 1$ (or $q^\gamma, 1^2$ if $\delta = 1$), whence

$$\begin{aligned}\dim C_H(u) &= \tfrac{1}{2}[(\gamma+2)^2 + (\delta-1)(\gamma+1)^2 + (q-\delta)\gamma^2 + (\gamma-1)^2 - \gamma - 2] \\ &= \tfrac{1}{2}\gamma^2 q + \tfrac{1}{2}(2\gamma+1)\delta + \tfrac{1}{2}(\gamma+1).\end{aligned}$$

Thus in all cases the minimal centralizer dimension is $\frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \epsilon_q\lceil\frac{\gamma}{2}\rceil$, as required. □

Lemma 2.4. *Take* $m \in \mathbb{N}$.

(i) *If* $2m-1 = \gamma q + \delta$ *with* $0 \leq \delta < q$, *then* $d_q(D_m) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil - \epsilon_\gamma$.

(ii) *If* $2m = \gamma q + \delta$ *with* $0 \leq \delta < q$, *then* $d_q(D_{m+1}) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)(\delta+1) + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil - \epsilon_\gamma$.

(iii) *If* $2m = \gamma q + \delta$ *with* $0 \leq \delta < q$, *then* $d_q(D_m) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)(\delta-1) + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil - \epsilon_\gamma + 2\epsilon_\gamma\zeta_\delta$.

Proof. This is very similar to the previous result, but the details are rather more complicated. Take H of type D_k and $u \in H_{[q]}$, and again write r_i for the number of Jordan blocks of size i in the action of u on the natural module, so that $\sum ir_i = 2k$ and $r_i = 0$ for $i > q$. If p is odd, we must again have r_i even for all even i, and [20] once more gives $\dim C_H(u) = \frac{1}{2}\sum_i (r_i + r_{i+1} + \cdots + r_q)^2 - \frac{1}{2}\sum_{i \text{ odd}} r_i$. If $p = 2$, we must instead have r_i even for all odd i, and $\sum r_i$ must also be even; again, the minimal value taken by the formula in [7] reduces to

$$\frac{1}{2}\sum_i (r_i + r_{i+1} + \cdots + r_q)^2 - \frac{1}{2}\sum_{i \text{ odd}} r_i - \sum_{i \text{ even}} r_i.$$

First consider (i), and let $k = m$. If p is odd, we separate into two cases according to the parity of γ. If γ is odd, then δ is even and the required Jordan structure is $q^\gamma, \delta+1$ (or $q^{\gamma+1}$ if $\delta = q-1$), giving $\dim C_H(u) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2}\gamma$; if γ is even, then δ is odd and the Jordan structure is $q^\gamma, \delta, 1$ (or $q^\gamma, 1^2$ if $\delta = 1$), and we have $\dim C_H(u) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2}(\gamma+1)$. If $p = 2$, then either $q = 1$, in which case $u = 1$ and $\dim C_H(u) = \dim H = 2m^2 - m$; or $q > 1$, when δ must be odd and we again separate into two cases according to the parity of γ. If γ is odd, the Jordan structure is $q^\gamma, \delta+1$ (or $q^{\gamma+1}$ if $\delta = q-1$), and $\dim C_H(u) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta - \frac{1}{2}$. If γ is even, the Jordan structure can take several forms depending on δ and q: if $\delta = q-1$ we have $q^\gamma, q-2, 2$ if $q > 4$, or $4^\gamma, 2^2$ if $q = 4$, or $2^\gamma, 1^2$ if $q = 2$; if $3 < \delta < q-1$ we have $q^\gamma, \delta-1, 2$; if $\delta = 3$ we have $q^\gamma, 2^2$; and if $\delta = 1$ we have $q^\gamma, 1^2$. For each of these possibilities we find $\dim C_H(u) = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2}$. Thus in all cases the minimal centralizer dimension is $\frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)\delta + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil - \epsilon_\gamma$, as required.

For (ii) and (iii) we again proceed by comparing with (i). For (ii) we have $k = m+1$; thus if we write $2k-1 = \gamma' q + \delta'$ with $0 \leq \delta' < q$, we have $\gamma' q + \delta' - 1 = \gamma q + \delta$. If $\delta < q-1$ we then have $\gamma' = \gamma$, $\delta' = \delta+1$ and the minimal centralizer dimension is $\frac{1}{2}\gamma'^2 q + \frac{1}{2}(2\gamma'+1)\delta' + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma'}{2}\rceil - \epsilon_{\gamma'} = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)(\delta+1) + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil - \epsilon_\gamma$; if $\delta = q-1$ we have $\gamma' = \gamma+1$, $\delta' = 0$ and the minimal centralizer dimension is $\frac{1}{2}(\gamma+1)^2 q + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma+1}{2}\rceil - \epsilon_{\gamma+1} = \frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)q + \frac{1}{2} + \frac{1}{2}\gamma$ (note that here q is odd and γ even). In either case we obtain the required formula. For (iii) we have $k = m$; thus if we set $2k-1 = \gamma' q + \delta'$ with $0 \leq \delta' < q$ we have $\gamma' q + \delta' + 1 = \gamma q + \delta$. If $\delta > 0$, then $\gamma' = \gamma$, $\delta' = \delta - 1$ and the minimal centralizer dimension is $\frac{1}{2}\gamma^2 q + \frac{1}{2}(2\gamma+1)(\delta-1) + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil - \epsilon_\gamma$. If on the other hand $\delta = 0$, then $\gamma' = \gamma - 1$, $\delta' = q-1$ and the minimal centralizer dimension is $\frac{1}{2}(\gamma-1)^2 q + \frac{1}{2}(2\gamma-1)(q-1) + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma-1}{2}\rceil - \epsilon_{\gamma-1} = \frac{1}{2}\gamma^2 q - \frac{1}{2}(2\gamma+1) + 1\frac{1}{2} + \epsilon_q\lceil\frac{\gamma-1}{2}\rceil - \epsilon_{\gamma-1} = \frac{1}{2}\gamma^2 q - \frac{1}{2}(2\gamma+1) + \frac{1}{2} + \epsilon_q\lceil\frac{\gamma}{2}\rceil + \epsilon_\gamma$ (note that γq must be even here, so that if q is odd, then γ must be even, whence $\epsilon_q\lceil\frac{\gamma-1}{2}\rceil = \epsilon_q\lceil\frac{\gamma}{2}\rceil$; and $1 - \epsilon_{\gamma-1} = \epsilon_\gamma$). Again the required formula follows. $\square$

We may summarize the above results as follows.

Lemma 2.5. *If $a = cq + d$ with $0 \leq d < q$ and $\zeta \in \{0, 1\}$ we have the following:*

(i) $d_q(A_{a-\zeta}) = c^2q + (2c+1)(a - \zeta - cq + 1) - 1$;

(ii) $d_q(B_{\lceil \frac{a}{2} \rceil - \epsilon_a\zeta}) = \frac{1}{2}c^2q + (2c+1)(\lceil \frac{a}{2} \rceil - \epsilon_a\zeta - \frac{1}{2}cq) + \epsilon_q\lceil \frac{c}{2} \rceil$;

(iii) $d_q(C_{\lceil \frac{a}{2} \rceil - \epsilon_a\zeta}) = \frac{1}{2}c^2q + (2c+1)(\lceil \frac{a}{2} \rceil - \epsilon_a\zeta - \frac{1}{2}cq) + \epsilon_q\lceil \frac{c}{2} \rceil$;

(iv) $d_q(D_{\lceil \frac{a+1}{2} \rceil - \epsilon_{a+1}\zeta}) = \frac{1}{2}c^2q + (2c+1)(\lceil \frac{a+1}{2} \rceil - \epsilon_{a+1}\zeta - \frac{1}{2}cq) + \epsilon_q\lceil \frac{c}{2} \rceil - c - \epsilon_c + 2\epsilon_{c(a+1)}\zeta_{a-cq}\zeta$.

We have also shown the following.

Corollary 2.6. *Take $m \in \mathbb{N}$.*

(i) *If $m + 1 = \gamma q + \delta$ with $0 \leq \delta < q$, then $d_q(A_{m+1}) - d_q(A_m) = 2\gamma + 1$.*

(ii) *If $2m + 1 = \gamma q + \delta$ with $0 \leq \delta < q$, then $d_q(B_{m+1}) - d_q(B_m) = 2\gamma + 1$.*

(iii) *If $2m + 1 = \gamma q + \delta$ with $0 \leq \delta < q$, then $d_q(C_{m+1}) - d_q(C_m) = 2\gamma + 1$.*

(iv) *If $2m = \gamma q + \delta$ with $0 \leq \delta < q$, then $d_q(D_{m+1}) - d_q(D_m) = 2\gamma + 1 - 2\epsilon_\gamma\zeta_\delta$.*

We now turn to the exceptional groups; here we simply obtain the values of $d_q(H)$ by comparing the Jordan structure of unipotent elements on the adjoint module given in [8] with the centralizer structure as given in [3, 5, 14, 15, 17, 18]. We find the following.

Lemma 2.7. *If H is of exceptional type with root system Ψ, then $d_q(H) = d_{\Psi,q}$.*

3. The general case

In this section we shall consider arbitrary elements of a simple algebraic group G defined over an algebraically closed field of characteristic p, and shall prove the various statements involved in Theorem 1. We begin with some notation which will be used throughout this section. Let Φ be the root system of G, taken with respect to some maximal torus T, and let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fundamental system, numbered in accordance with [1, Planches I–IX]; write α_0 for the highest root of Φ with respect to Π, set $m_0 = 1$ and define m_i for $1 \leq i \leq \ell$ by

$$\alpha_0 = \sum_{i=1}^{\ell} m_i\alpha_i.$$

Let $h = \sum_{i=0}^{\ell} m_i$ be the Coxeter number of Φ. Now take $r \in \mathbb{N}$, and let the p'-part and p-part of r be n and q, respectively, so that $r = nq$. Given $g \in G_{[r]}$, let the Jordan decomposition of g be $g = su = us$, where s is semisimple and u unipotent; we then have $s \in G_{[n]}$ and $u \in (C_G(s)^0)_{[q]}$. Extending the notation of the previous section, we write

$$d_r(G) = \min_{g \in G_{[r]}} \dim C_G(g);$$

we then have

$$d_r(G) = \min_{s \in G_{[n]}} d_q(C_G(s)^0) = \min_{Z = C_G(s)^0,\ s \in G_{[n]}} d_q(Z),$$

where we set $d_q(H_1 \ldots H_tT') = d_q(H_1) + \cdots + d_q(H_t)$ for $H_1, \ldots, H_t$ simple and T' a torus. We can immediately provide an important interpretation of the number $d_r(G)$.

Lemma 3.1. *With the notation established,* $\operatorname{codim} G_{[r]} = d_r(G)$.

Proof. Taking $g = su$ as above, by conjugation we may assume $s \in T$; as $|T_{[n]}| = n^\ell$, and $C_G(s)^0$ contains finitely many unipotent classes, it follows that the number of classes in $G_{[r]}$ is finite. Thus

$$\dim G_{[r]} = \max_{g \in G_{[r]}} \dim g^G,$$

and hence

$$\operatorname{codim} G_{[r]} = \dim G - \max_{g \in G_{[r]}} \dim g^G = \min_{g \in G_{[r]}} \dim C_G(g) = d_r(G),$$

as required. □

Our approach to calculating the value of $d_r(G)$ will be to use a result of Hartley and Kuzucuoğlu in [6], given n, to restrict the possibilities for the connected centralizer Z of a semisimple element of order n. Among the possible groups Z we then select one having minimal value of $d_q(Z)$, using the results of the previous section. We shall observe that this minimal value in fact depends only on r and not on the factorization $r = nq$, so that it is independent of the characteristic p.

Unless otherwise stated, we assume from now on that

G is of *adjoint* type.

At the end of this section we shall consider the case of arbitrary isogeny type.

The result from [6] which we need is (part of) Theorem 4.2 there, and follows the approach of Deriziotis in [4]; under our assumption on the isogeny type of G it may be stated as follows. Choose non-negative integers $b_0, b_1, \ldots, b_\ell$ with $\gcd(b_0, b_1, \ldots, b_\ell) = 1$ satisfying $\sum_{i=0}^{\ell} b_i m_i = n$. The roots α_i (or $-\alpha_0$ in the case $i = 0$) for which $b_i = 0$ then form a simple system Ψ whose corresponding subsystem subgroup of G is the centralizer of a semisimple element of order n; moreover, up to conjugacy all centralizers of semisimple elements of order n arise in this way.

Now from this result it follows that if $n \geq h$, then by choosing $b_1 = \cdots = b_\ell = 1$ and $b_0 = n + 1 - h$ we may take $Z = T$, whence $d_r(G) = \ell = d_{\Phi,r}$ as required (note that if $n \geq h$, then certainly $r \geq h$); it therefore suffices to assume in what follows that

$$n < h.$$

The following result shows that in this case we may substantially restrict the non-negative integers b_i which need be considered.

Lemma 3.2. *With the notation established, if $n < h$ the minimal value of $d_q(Z)$ occurs for some $b_0, b_1, \ldots, b_\ell$ with $b_i \in \{0, 1\}$ for all i.*

Proof. Take $b_0, b_1, \ldots, b_\ell$ satisfying $\gcd(b_0, b_1, \ldots, b_\ell) = 1$ and $\sum_{i=0}^{\ell} b_i m_i = n$. Since $n < h = \sum_{i=0}^{\ell} m_i$ we must have $b_i = 0$ for some i; let $v = \min\{m_i : b_i = 0\}$. We shall show that if not all the b_i are 0 or 1, then it is always possible to replace some of them in such a way that the conditions are still satisfied and the value of $d_q(Z)$ is either reduced or unchanged, but the quantity $\Delta = h^2|\Psi| + h\sum_{i=0}^{\ell} b_i + v$ is decreased; thus after a finite number of steps this process must terminate in a sequence $b_0, b_1, \ldots, b_\ell$ in which all terms are 0 or 1 as required.

Thus assume $b_x > 1$ for some x; choose y such that $m_y = v$ and $b_y = 0$. We split into two cases according to the relative sizes of m_x and m_y. First assume $m_x \geq m_y$. If $m_x = 1$ we may replace the pair (b_x, b_y) by $(b_x - 1, 1)$; if $m_x > 1$ and $y = 0$ replace (b_x, b_y) by $(1, (b_x - 1)m_x)$; if $m_x > 1$ and $y \neq 0$ replace the triple

(b_x, b_y, b_0) by $(b_x - 1, 1, b_0 + m_x - m_y)$. In each instance we obtain ${b_0}', {b_1}', \ldots, {b_\ell}'$ satisfying the required conditions with $\Psi' = \Psi \setminus \{\alpha_y\}$, whence $d_q(Z') \leq d_q(Z)$; as $|\Psi'| = |\Psi| - 1$ and $h \sum_{i=0}^{\ell} {b_i}' + v' \leq hn + v' < h(h-1) + h = h^2$ we clearly have $\Delta' < \Delta$ as required.

Now assume that $m_x < m_y$. Here we use an observation relating to the extended Dynkin diagram; we recall that this has nodes $0, 1, \ldots, \ell$ corresponding to roots $-\alpha_0, \alpha_1, \ldots, \alpha_\ell$. Set $m = \max\{m_i : 0 \leq i \leq \ell\}$; then there is a sequence $0 = k_1, k_2, \ldots, k_m$ of nodes satisfying $m_{k_t} = t$ for $t \leq m$ and with k_t joined to $k_{t'}$ if and only if $|t - t'| = 1$. Now the choice of y implies that for any node t with $m_t < m_y$ we must have $b_t > 0$; thus by interchanging the values b_x and $b_{k_{m_x}}$ if necessary we may assume that $k_{m_x} = x$, i.e., the node x occurs in the sequence. Let $k_{m_y} = z'$ and $k_{m_y - 1} = z$, while if $x \neq 0$ let $k_{m_x - 1} = x'$; then $b_z > 0$, again by the choice of y, and if $b_{z'} = 0$ we may assume that y is chosen to be z'. We therefore have $1 = m_0 \leq m_x \leq m_z < m_z + 1 = m_y$ with 0, x and z all appearing in the sequence, and if $x \neq 0$, then x' is also in the sequence with $m_{x'} = m_x - 1$. Now if $m_0 = m_x = m_z$ we may replace the pair (b_x, b_y) by $(b_x - 2, 1)$; if $m_0 < m_x = m_z$ replace the triple (b_0, b_x, b_y) by $(b_0 - 1, b_x - 1, 1)$; if $m_0 = m_x < m_z$ replace (b_x, b_z, b_y) by $(b_x - 1, b_z - 1, 1)$; lastly if $m_0 < m_x < m_z$ replace the quadruple $(b_{x'}, b_x, b_z, b_y)$ by $(b_{x'} + 1, b_x - 1, b_z - 1, 1)$. In every instance $|\Psi'| \leq |\Psi|$; in each but the last we have $\sum_{i=0}^{\ell} {b_i}' < \sum_{i=0}^{\ell} b_i$, while in the last $\sum_{i=0}^{\ell} {b_i}' = \sum_{i=0}^{\ell} b_i$ but if $|\Psi'| = |\Psi|$, then $v' = m_z = v - 1$. Thus the value of Δ decreases as required; and one irreducible component of Ψ (that containing α_y) is reduced in rank by 1, while at most an A_1 component is introduced (containing $-\alpha_0$ in the first two instances and α_z in the second two), so clearly $d_q(Z') \leq d_q(Z)$. This completes the proof. □

We may now work our way through the types of irreducible root system.

Proposition 3.3. *With the notation established,* $d_r(A_\ell) = d_{A_\ell, r}$.

Proof. We have $h = \ell + 1$; and $m_i = 1$ for all $0 \leq i \leq \ell$. By Lemma 3.2 we therefore require coefficients $b_0, b_1, \ldots, b_\ell$ with n of the b_i equal to 1 and the remainder equal to 0. Suppose $b_{i_1} = \cdots = b_{i_n} = 1$ with $i_1 < \cdots < i_n$; we may assume $i_1 = 0$, and set $i_{n+1} = \ell + 1$. We then have Z as a product of n factors $A_{i_{j+1} - i_j - 1}$ (where a factor A_0 is interpreted as being trivial) and an $(n-1)$-dimensional torus T_{n-1}. Now if two of the factors are A_k and $A_{k'}$ with $k - k' > 1$, we may alter the b_i so as to replace these two factors with A_{k-1} and $A_{k'+1}$; it follows from Corollary 2.6(i) that $d_q(A_k A_{k'}) - d_q(A_{k-1} A_{k'+1}) \geq 0$. Thus we may assume that, for some a, all factors are either A_a or A_{a-1}; as rank $Z = \ell$ we must have

$$Z = (A_a)^b (A_{a-1})^{n-b} T_{n-1},$$

where $\ell + 1 = an + b$ with $0 \leq b < n$.

Now write $a = cq + d$ with $0 \leq d < q$, and apply the formulæ from Lemma 2.5(i); we obtain

$$\begin{aligned} d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (n-b)(c^2q + (2c+1)d - 1) + n - 1 \\ &= c^2nq + (2c+1)(dn+b) - 1 \\ &= c^2nq + (2c+1)(\ell + 1 - cqn) - 1 \\ &= (2c+1)(\ell+1) - c(c+1)nq - 1 \\ &= (2c+1)(\ell+1) - c(c+1)r - 1. \end{aligned}$$

Since we have $zr + e = \ell + 1 = an + b = (cq + d)n + b = cr + (dn + b)$ with $dn + b \leq (q-1)n + (n-1) = r - 1$, it follows that $c = z$, so

$$d_r(A_\ell) = (2z+1)(\ell+1) - z(z+1)r - 1;$$

substituting from $\ell + 1 = zr + e$ gives $d_r(A_\ell) = d_{A_\ell,r}$ as required. □

Proposition 3.4. *With the notation established,* $d_r(C_\ell) = d_{C_\ell,r}$.

Proof. We have $h = 2\ell$; and $m_i = 2$ for all $1 \leq i \leq \ell - 1$ while $m_0 = m_\ell = 1$. Here we shall need to argue separately for n odd and n even; first assume n is odd and set $n = 2s + 1$. By Lemma 3.2 we therefore require coefficients $b_0, b_1, \dots, b_\ell$ in which the b_i which are equal to 1 correspond to s nodes i with $1 \leq i \leq \ell - 1$, together with either node 0 or node ℓ; this means that Z has s factors of type A, one factor of type C and an s-dimensional torus. We shall need to determine a configuration which minimizes $d_q(Z)$.

As in the proof of Proposition 3.3, the type A factors may be assumed to take the form $(\tilde{A}_x)^z(\tilde{A}_{x-1})^{s-z}$, for some x and some $0 \leq z < s$; suppose the type C factor is C_y. If we write $x = \gamma_1 q + \delta_1$ and $2y - 1 = \gamma_2 q + \delta_2$ with $0 \leq \delta_1, \delta_2 < q$, then by Lemma 2.5 $d_q(\tilde{A}_{x-1}C_y) - d_q(\tilde{A}_x C_{y-1}) = 2(\gamma_2 - \gamma_1)$; thus if $2y - 1 > x$ we could replace the pair $\tilde{A}_{x-1}C_y$ by $\tilde{A}_x C_{y-1}$ without increasing $d_q(Z)$, so we may assume that $2y \leq x+1$. Likewise if $2y \leq x-2$ we could replace $\tilde{A}_{x-1}C_y$ by $\tilde{A}_{x-2}C_{y+1}$ (and then possibly adjust further the factors of type A), so we may assume $x - 1 \leq 2y$; thus we have $x - 1 \leq 2y \leq x + 1$. If there are A_x factors present (so that $z > 0$ above), then we may argue similarly to deduce that $x \leq 2y \leq x + 2$; putting these conditions together we have $x \leq 2y \leq x + 1$, so that $y = \lceil \frac{x}{2} \rceil$. If there are no A_x factors, then either $y = \lceil \frac{x}{2} \rceil$, or $2y = x - 1$; if the latter holds we may set $x' = x - 1$, $y' = y$, and then we have $(\tilde{A}_{x'})^s C_{y'}$ with $y' = \frac{x'}{2}$ (note that in this case, in which $z = s$ in the form for the type A factors given above, we have x' even). Writing a for x (or x'), we have $\ell = \operatorname{rank} Z = az + (a-1)(s-z) + \lceil \frac{a}{2} \rceil + s = as + \lceil \frac{a}{2} \rceil + z$, so $2\ell = 2as + 2\lceil \frac{a}{2} \rceil + 2z = 2as + a + \epsilon_a + 2z$ with $\epsilon_a + 2z < 2s + 1$ (because if $z = s$, then a is even). Therefore we have

$$Z = (\tilde{A}_a)^{\lfloor \frac{b}{2} \rfloor}(\tilde{A}_{a-1})^{s - \lfloor \frac{b}{2} \rfloor} C_{\lceil \frac{a}{2} \rceil} T_s,$$

where $2\ell = a(2s+1) + b$ with $0 \leq b < 2s + 1$.

Having determined the form of Z, write $a = cq + d$ with $0 \leq d < q$, and apply the formulæ from Lemma 2.5(i,iii); we obtain

$$\begin{aligned}
d_q(Z) &= \lfloor \tfrac{b}{2} \rfloor (c^2 q + (2c+1)(d+1) - 1) + (s - \lfloor \tfrac{b}{2} \rfloor)(c^2 q + (2c+1)d - 1) \\
&\quad + \tfrac{1}{2}c^2 q + (2c+1)(\lceil \tfrac{a}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil + s \\
&= c^2 q(s + \tfrac{1}{2}) + (2c+1)(ds + \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil \\
&= c^2 q(s + \tfrac{1}{2}) + (2c+1)(ds + \tfrac{a+b}{2} - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil \\
&= c^2 q(s + \tfrac{1}{2}) + (2c+1)(\ell - cq(s + \tfrac{1}{2})) + \epsilon_q \lceil \tfrac{c}{2} \rceil \\
&= (2c+1)\ell - c(c+1)q(s + \tfrac{1}{2}) + \epsilon_q \lceil \tfrac{c}{2} \rceil \\
&= (2c+1)\ell - \tfrac{1}{2}c(c+1)r + \epsilon_r \lceil \tfrac{c}{2} \rceil
\end{aligned}$$

(note that the assumption that n is odd means that $\epsilon_q = \epsilon_r$). As in the proof of Proposition 3.3, since we have $zr + e = 2\ell = an + b = (cq + d)n + b = cr + (dn + b)$

with $dn+b \leq (q-1)n+(n-1) = r-1$, it follows that $c = z$, so

$$d_r(C_\ell) = (2z+1)\ell - \tfrac{1}{2}z(z+1)r + \epsilon_r\lceil \tfrac{z}{2}\rceil;$$

substituting from $2\ell = zr+e$ gives $d_r(C_\ell) = d_{C_\ell,r}$ for n odd, as required.

Now assume that n is even and set $n = 2s$. Here there are two types of possibilities for the coefficients b_i, and thus for the centralizer Z, depending on whether $b_0 = b_\ell = 1$ or $b_0 = b_\ell = 0$; we shall need to consider each. First consider the possibility $b_0 = b_\ell = 1$; here there are $s-1$ nodes i with $1 \leq i \leq \ell$ such that $b_i = 1$, and so much as in the previous proposition we must have

$$Z = (\tilde{A}_a)^b(\tilde{A}_{a-1})^{s-b}T_s,$$

where $\ell = as+b$ with $0 \leq b < s$. Again write $a = cq+d$ with $0 \leq d < q$, and apply the formulæ from Lemma 2.5(i); we obtain

$$\begin{aligned} d_q(Z) &= b(c^2q+(2c+1)(d+1)-1)+(s-b)(c^2q+(2c+1)d-1)+s \\ &= c^2qs+(2c+1)(ds+b) \\ &= c^2qs+(2c+1)(\ell-cqs) \\ &= (2c+1)\ell - c(c+1)qs \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)r. \end{aligned}$$

Here we have $zr+e = 2\ell = an+2b = (cq+d)n+2b = cr+dn+2b$ with $dn+2b \leq (q-1)n+2(s-1) = r-2$, so again we have $c = z$ and

$$d_q(Z) = (2z+1)\ell - \tfrac{1}{2}z(z+1)r;$$

substituting from $2\ell = zr+e$ (and noting that $\epsilon_r = 0$ here) gives $d_q(Z) = d_{C_\ell,r}$.

The second possibility here is that $b_0 = b_\ell = 0$; here there are s nodes i with $1 \leq i \leq \ell$ such that $b_i = 1$. Again we must determine an optimal form for Z, which has $s-1$ factors of type A, two factors of type C and an $(s-1)$-dimensional torus. As before we may assume that the type A factors are $(\tilde{A}_x)^z(\tilde{A}_{x-1})^{s-z}$, for some x and some $0 \leq z < s$; the type C factors likewise may be taken to be either $(C_y)^2$ or C_yC_{y-1}. Using the reasoning already given, if the type C factors are of equal rank we must have either $y = \lceil \frac{x}{2}\rceil$, or $z = 0$ and $2y = x-1$; in the latter case we set $x' = x-1$, $y' = y$ and obtain $(\tilde{A}_{x'})^{s-1}(C_{y'})^2$ with $y' = \frac{x'}{2}$ (and $z = s-1$). On the other hand, if we have C_yC_{y-1}, then both y and $y-1$ must satisfy these conditions; the only way this can happen is if we have $2y = x+1$ with $z = 0$, in which case we obtain $(\tilde{A}_{x-1})^{s-1}C_{\lceil \frac{x}{2}\rceil}C_{\lceil \frac{x}{2}\rceil-1}$ with x odd. Putting together these possibilities, writing a for x (or x') and equating $\operatorname{rank} Z$ to ℓ, we find that we have

$$Z = (\tilde{A}_a)^{b-\epsilon_a(1-\zeta_b)}(\tilde{A}_{a-1})^{s-1-b+\epsilon_a(1-\zeta_b)}C_{\lceil \frac{a}{2}\rceil}C_{\lceil \frac{a}{2}\rceil-\epsilon_a\zeta_b}T_{s-1},$$

where $\ell = as+b$ with $0 \leq b < s$.

As usual write $a = cq + d$ with $0 \leq d < q$, and apply the formulæ from Lemma 2.5(i), (iii); noting that $\epsilon_q = 1$ because n is even, we obtain

$$\begin{aligned}
d_q(Z) &= (b - \epsilon_a(1-\zeta_b))(c^2q + (2c+1)(d+1) - 1) \\
&\quad + (s - 1 - b + \epsilon_a(1-\zeta_b))(c^2q + (2c+1)d - 1) \\
&\quad + \tfrac{1}{2}c^2q + (2c+1)(\lceil \tfrac{a}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil \\
&\quad + \tfrac{1}{2}c^2q + (2c+1)(\lceil \tfrac{a}{2} \rceil - \epsilon_a\zeta_b - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil + s - 1 \\
&= c^2qs + (2c+1)(d(s-1) + b + 2\lceil \tfrac{a}{2} \rceil - \epsilon_a - cq) + 2\lceil \tfrac{c}{2} \rceil \\
&= c^2qs + (2c+1)(\ell - cqs) + 2\lceil \tfrac{c}{2} \rceil \\
&= (2c+1)\ell - c(c+1)qs + 2\lceil \tfrac{c}{2} \rceil \\
&= (2c+1)\ell - \tfrac{1}{2}c(c+1)r + 2\lceil \tfrac{c}{2} \rceil
\end{aligned}$$

(observe that $2\lceil \frac{a}{2} \rceil - \epsilon_a = a$). As above we have $c = z$; this time however we have $d_q(Z) - d_{C_\ell,r} = 2\lceil \frac{c}{2} \rceil \geq 0$. Combining the two types of possibilities, we conclude that $d_r(C_\ell) = d_{C_\ell,r}$ for n even, as required. This completes the proof. □

Proposition 3.5. *With the notation established,* $d_r(D_\ell) = d_{D_\ell,r}$.

Proof. We have $h = 2\ell - 2$; and $m_i = 2$ for all $2 \leq i \leq \ell - 2$ while $m_0 = m_1 = m_{\ell-1} = m_\ell = 1$. Again we shall need to argue separately for n odd and n even; first assume n is odd and set $n = 2s + 1$. By Lemma 3.2 there are two types of possibilities for the coefficients b_i, and hence the centralizer Z, according to whether the number of b_i equal to 1 for which $i \in \{0, 1, \ell - 1, \ell\}$ is 1 or 3; we shall consider each. First assume that the b_i which are equal to 1 correspond to $s - 1$ nodes i with $2 \leq i \leq \ell - 2$, together with three of the four nodes 0, 1, $\ell - 1$ and ℓ; this means that Z has s factors of type A and an $(s + 1)$-dimensional torus. In similar fashion to the above, we see that we must have

$$Z = (A_a)^b (A_{a-1})^{s-b} T_{s+1},$$

where $\ell - 1 = as + b$ with $0 \leq b < s$. As usual write $a = cq + d$ with $0 \leq d < q$, and apply the formulæ from Lemma 2.5(i); we obtain

$$\begin{aligned}
d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (s-b)(c^2q + (2c+1)d - 1) + s + 1 \\
&= c^2qs + (2c+1)(ds + b) + 1 \\
&= c^2qs + (2c+1)(\ell - 1 - cqs) + 1 \\
&= (2c+1)\ell - c(c+1)qs - 2c \\
&= (2c+1)\ell - \tfrac{1}{2}c(c+1)q(n-1) - 2c \\
&= (2c+1)\ell - \tfrac{1}{2}c(c+1)(r-q) - 2c.
\end{aligned}$$

Here we have $zr + e = 2\ell - 2 = a(n-1) + 2b = (cq + d)(n-1) + 2b = c(r-q) + d(n-1) + 2b$ with $d(n-1) + 2b \leq (q-1)(n-1) + 2(s-1) = r - q - 2$; thus if we set $f = d(n-1) + 2b$, then we have $zr + e = c(r-q) + f$ with $0 \leq f < r - q$. It follows that $c \geq z$, and we find that

$$\begin{aligned}
d_q(Z) - d_{D_\ell,r} &= (c-z)(f + \tfrac{1}{2}(r-q)(c-z-1)) + \tfrac{1}{2}z((z+1)q - 3) \\
&\quad + (\tfrac{z}{2} - \epsilon_r\lceil \tfrac{z}{2} \rceil + \epsilon_z) \\
&\geq 0
\end{aligned}$$

(note that $\epsilon_r = \epsilon_q$ as n is odd, and thus if $z = q = 1$ the second term is $-\frac{1}{2}$ but the third is $\frac{1}{2}$).

We now consider the other type of possibility for Z. Here we assume that the b_i which are equal to 1 correspond to s nodes i with $2 \leq i \leq \ell - 2$, together with one of the four nodes 0, 1, $\ell - 1$ and ℓ; this means that Z has s factors of type A, one factor of type D and an s-dimensional torus. We shall again need to determine a configuration which minimizes $d_q(Z)$.

As before, the type A factors may be assumed to take the form $(A_x)^z(A_{x-1})^{s-z}$, for some x and some $0 \leq z < s$; suppose the type D factor is D_y. If we write $x = \gamma_1 q + \delta_1$ and $2y - 2 = \gamma_2 q + \delta_2$ with $0 \leq \delta_1, \delta_2 < q$, then by Lemma 2.5 $d_q(A_{x-1}D_y) - d_q(A_x D_{y-1}) = 2(\gamma_2 - \gamma_1 - \epsilon_{\gamma_2}\zeta_{\delta_2})$; thus if $2y - 2 > x$ we could replace the pair $A_{x-1}D_y$ by $A_x D_{y-1}$ without increasing $d_q(Z)$ (note that in this case if $\gamma_2 = \gamma_1$, then $\delta_2 > 0$ and so $\zeta_{\delta_2} = 0$), so we may assume that $2y \leq x + 2$. Likewise if $2y \leq x - 1$ we could replace $A_{x-1}D_y$ by $A_{x-2}D_{y+1}$ (and then possibly adjust further the factors of type A), so we may assume $x \leq 2y$; thus we have $x \leq 2y \leq x + 2$. The remainder of the argument is exactly similar to that in the proof of Proposition 3.4, and we conclude that we have

$$Z = (A_a)^{\lfloor \frac{b}{2} \rfloor}(A_{a-1})^{s-\lfloor \frac{b}{2} \rfloor} D_{\lceil \frac{a+1}{2} \rceil} T_s,$$

where $2\ell - 1 = a(2s + 1) + b$ with $0 \leq b < 2s + 1$.

Having determined the form of Z, write $a = cq + d$ with $0 \leq d < q$, and apply the formulæ from Lemma 2.5(i),(iv); we obtain

$$\begin{aligned} d_q(Z) &= \lfloor \tfrac{b}{2} \rfloor (c^2 q + (2c+1)(d+1) - 1) + (s - \lfloor \tfrac{b}{2} \rfloor)(c^2 q + (2c+1)d - 1) \\ &\quad + \tfrac{1}{2}c^2 q + (2c+1)(\lceil \tfrac{a+1}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c + s \\ &= c^2 q(s + \tfrac{1}{2}) + (2c+1)(ds + \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a+1}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c \\ &= c^2 q(s + \tfrac{1}{2}) + (2c+1)(ds + \tfrac{a+b+1}{2} - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c \\ &= c^2 q(s + \tfrac{1}{2}) + (2c+1)(\ell - cq(s + \tfrac{1}{2})) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c \\ &= (2c+1)\ell - c(c+1)q(s + \tfrac{1}{2}) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)r + \epsilon_r \lceil \tfrac{c}{2} \rceil - c - \epsilon_c \end{aligned}$$

(note once more that the assumption that n is odd means that $\epsilon_q = \epsilon_r$). Here we have $zr + e = 2\ell - 2 = an + b - 1 = (cq + d)n + b - 1 = cr + (dn + b - 1)$ with $dn + b \leq (q-1)n + (n-1) = r - 1$. Thus either $c = z$, or we must have $c = z + 1$ and $2\ell - 1 = (z+1)r$; but if the latter holds then, using the fact that c and r are both odd, we see that $(2c+1)\ell - \frac{1}{2}c(c+1)r + \epsilon_r \lceil \frac{c}{2} \rceil - c - \epsilon_c = (2z+1)\ell - \frac{1}{2}z(z+1)r + \epsilon_r \lceil \frac{z}{2} \rceil - z - \epsilon_z$ anyway. Thus we have

$$d_q(Z) = (2z+1)\ell - \tfrac{1}{2}z(z+1)r + \epsilon_r \lceil \tfrac{z}{2} \rceil - z - \epsilon_z = d_{D_\ell, r};$$

so combining the two possibilities we have $d_r(D_\ell) = d_{D_\ell, r}$ for n odd, as required.

Now assume n is even and set $n = 2s$. Here the number of b_i equal to 1 for which $i \in \{0, 1, \ell - 1, \ell\}$ must be even, and thus can be 0, 2 or 4; this gives a number of possibilities for the type of Z. If $b_i = 1$ for all $i \in \{0, 1, \ell - 1, \ell\}$, there must be $s - 1$ factors of type A and an $(s + 1)$-dimensional torus; we find that

$$Z = (A_a)^b (A_{a-1})^{s-1-b} T_{s+1},$$

where $\ell - 2 = a(s-1) + b$ with $0 \le b < s-1$. Writing $a = cq + d$ with $0 \le d < q$ in the usual fashion, we obtain

$$\begin{aligned} d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (s-1-b)(c^2q + (2c+1)d - 1) + s + 1 \\ &= c^2q(s-1) + (2c+1)(d(s-1)+b) + 2 \\ &= c^2q(s-1) + (2c+1)(\ell - 2 - cq(s-1)) + 2 \\ &= (2c+1)\ell - c(c+1)q(s-1) - 4c \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)q(n-2) - 4c \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)(r-2q) - 4c. \end{aligned}$$

We have $zr + e = 2\ell - 2 = a(n-2) + 2b + 2 = c(r-2q) + d(n-2) + 2b + 2$, and $d(n-2) + 2b + 2 \le (q-1)(n-2) + 2(s-2) + 2 = r - 2q$; thus $c \ge z$, and if we set $f = d(n-2) + 2b$ we find that

$$d_q(Z) - d_{D_\ell, r} = (c-z)(f + \tfrac{1}{2}(r-2q)(c-z-1)) + z((z+1)q - 3) + \epsilon_z \ge 0$$

(note that if $z = q = 1$, then the second term is -1 but the third is 1).

Next assume that the number of b_i equal to 1 for which $i \in \{0, 1, \ell-1, \ell\}$ is 2; here we must subdivide yet further, because the two nodes concerned might be at different ends of the extended Dynkin diagram or at the same end. If they are at different ends, then Z has s factors of type A and an s-dimensional torus, and we obtain

$$Z = (A_a)^b (A_{a-1})^{s-b} T_s,$$

where $\ell = as + b$ with $0 \le b < s$. Setting $a = cq + d$ with $0 \le d < q$ we obtain

$$\begin{aligned} d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (s-b)(c^2q + (2c+1)d - 1) + s \\ &= c^2q(s-1) + (2c+1)(ds+b) \\ &= c^2q(s-1) + (2c+1)(\ell - cqs) \\ &= (2c+1)\ell - c(c+1)qs \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)r. \end{aligned}$$

Here we have $zr + e = 2\ell - 2 = an + 2b - 2 = cr + dn + 2b - 2$, and $dn + 2b - 2 \le (q-1)n + 2(s-1) - 2 = r - 4$. Thus either $c = z$, or we must have $c = z+1$ and $2\ell = (z+1)r$; but if the latter holds then we see that $(2c+1)\ell - \frac{1}{2}c(c+1)r = (2z+1)\ell - \frac{1}{2}z(z+1)r$ anyway. Thus in either case we have

$$\begin{aligned} d_q(Z) - d_{D_\ell, r} &= (2z+1)\ell - \tfrac{1}{2}z(z+1)r - \tfrac{1}{2}z^2r - \tfrac{1}{2}e(2z+1) - \epsilon_r\lceil \tfrac{z}{2} \rceil - z - 1 + \epsilon_z \\ &= z + \epsilon_z \\ &\ge 0. \end{aligned}$$

If however the two nodes are at the same end of the extended Dynkin diagram, then Z has $s-1$ factors of type A, one factor of type D and an s-dimensional torus. Proceeding in a similar fashion to the case above with n odd, we find that

$$Z = (A_a)^{\lfloor \frac{b}{2} \rfloor} (A_{a-1})^{s-1-\lfloor \frac{b}{2} \rfloor} D_{\lceil \frac{a+1}{2} \rceil} T_s,$$

where $2\ell-3=a(2s-1)+b$ with $0\le b<2s-1$. Setting $a=cq+d$ with $0\le d<q$ we obtain

$$\begin{aligned} d_q(Z) &= \lfloor\tfrac{b}{2}\rfloor(c^2q+(2c+1)(d+1)-1)+(s-1-\lfloor\tfrac{b}{2}\rfloor)(c^2q+(2c+1)d-1)\\ &\quad +\tfrac12 c^2q+(2c+1)(\lceil\tfrac{a+1}{2}\rceil-\tfrac12 cq)+\epsilon_q\lceil\tfrac{c}{2}\rceil-c-\epsilon_c+s\\ &= c^2q(s-\tfrac12)+(2c+1)(d(s-1)+\lfloor\tfrac{b}{2}\rfloor+\lceil\tfrac{a+1}{2}\rceil-\tfrac12 cq)+\lceil\tfrac{c}{2}\rceil-c-\epsilon_c+1\\ &= c^2q(s-\tfrac12)+(2c+1)(d(s-1)+\tfrac{a+b+1}{2}-\tfrac12 cq)+\lceil\tfrac{c}{2}\rceil-c-\epsilon_c+1\\ &= c^2q(s-\tfrac12)+(2c+1)(\ell-1-cq(s-\tfrac12))+\lceil\tfrac{c}{2}\rceil-c-\epsilon_c+1\\ &= (2c+1)\ell-c(c+1)q(s-\tfrac12)+\lceil\tfrac{c}{2}\rceil-3c-\epsilon_c\\ &= (2c+1)\ell-\tfrac12 c(c+1)(r-q)+\lceil\tfrac{c}{2}\rceil-3c-\epsilon_c. \end{aligned}$$

Here we have $zr+e=2\ell-2=a(n-1)+b+1=c(r-q)+d(n-1)+b+1$, and $d(n-1)+b+1\le(q-1)(n-1)+2s-2+1=r-q$; thus $c\ge z$, and if we set $f=d(n-1)+b$ we find that

$$\begin{aligned} d_q(Z)-d_{D_\ell,r} &= (c-z)(f+\tfrac12(r-q)(c-z-1))+\tfrac12 z((z+1)q-3)\\ &\quad +(\tfrac12(c-z)+\epsilon_z-\tfrac12\epsilon_c)\\ &\ge 0 \end{aligned}$$

(note again that if $z=q=1$, then the second term is $-\frac12$ but the third is at least $\frac12$).

Finally we must consider the possibility that $b_i=0$ for all $i\in\{0,1,\ell-1,\ell\}$; here Z has $s-1$ factors of type A, two of type D and an $(s-1)$-dimensional torus. As usual the type A factors may be taken to be $(A_x)^z(A_{x-1})^{s-z}$, for some x and some $0\le z<s$, while the type D factors may be taken to be either $(D_y)^2$ or D_yD_{y-1} for some y. Arguing in precisely similar fashion to the analogous case in Proposition 3.4 shows that

$$Z=(A_a)^{b-\epsilon_{a+1}(1-\zeta_b)}(A_{a-1})^{s-1-b+\epsilon_{a+1}(1-\zeta_b)}D_{\lceil\frac{a+1}{2}\rceil}D_{\lceil\frac{a+1}{2}\rceil-\epsilon_{a+1}\zeta_b}T_{s-1},$$

where $\ell-1=as+b$ with $0\le b<s$. As usual, write $a=cq+d$ with $0\le d<q$ and apply the formulæ from Lemma 2.5(i), (iv); noting that $\epsilon_q=1$ because n is even, and that if $d=0$, then $c(a+1)$ is even because $a=cq\equiv c \bmod 2$, we obtain

$$\begin{aligned} d_q(Z) &= (b-\epsilon_{a+1}(1-\zeta_b))(c^2q+(2c+1)(d+1)-1)\\ &\quad +(s-1-b+\epsilon_{a+1}(1-\zeta_b))(c^2q+(2c+1)d-1)\\ &\quad +\tfrac12 c^2q+(2c+1)(\lceil\tfrac{a+1}{2}\rceil-\tfrac12 cq)+\epsilon_q\lceil\tfrac{c}{2}\rceil-c-\epsilon_c\\ &\quad +\tfrac12 c^2q+(2c+1)(\lceil\tfrac{a+1}{2}\rceil-\epsilon_{a+1}\zeta_b-\tfrac12 cq)+\epsilon_q\lceil\tfrac{c}{2}\rceil-c-\epsilon_c+2\epsilon_{c(a+1)}\zeta_d\zeta_b\\ &\quad +s-1\\ &= c^2qs+(2c+1)(d(s-1)+b+2\lceil\tfrac{a+1}{2}\rceil-\epsilon_{a+1}-cq)+2\lceil\tfrac{c}{2}\rceil-2c-2\epsilon_c\\ &= c^2qs+(2c+1)(\ell-cqs)-c-\epsilon_c\\ &= (2c+1)\ell-c(c+1)qs-c-\epsilon_c\\ &= (2c+1)\ell-\tfrac12 c(c+1)r-c-\epsilon_c. \end{aligned}$$

Here $zr+e=2\ell-2=an+2b=cr+dn+2b$ and $dn+2b\le(q-1)n+2(s-1)=r-2$, so $c=z$, and we have

$$d_q(Z)=(2z+1)\ell-\tfrac12 z(z+1)r-z-\epsilon_z=d_{D_\ell,r}$$

(note that $\epsilon_r = 0$ as n is even here). Combining all the possibilities we conclude that $d_r(D_\ell) = d_{D_\ell,r}$ for n even, as required. This completes the proof. □

Proposition 3.6. *With the notation established, $d_r(B_\ell) = d_{B_\ell,r}$.*

Proof. We have $h = 2\ell$; and $m_i = 2$ for all $2 \le i \le \ell$ while $m_0 = m_1 = 1$. Yet again we shall need to argue separately for n odd and n even; first assume n is odd and set $n = 2s + 1$. By Lemma 3.2 and the fact that α_ℓ is short, there are two types of possibilities for the coefficients b_i, and hence Z, according as $b_\ell = 0$ or 1. First assume that $b_\ell = 1$. Then Z has s factors of type A and an s-dimensional torus, and we have

$$Z = (A_a)^b (A_{a-1})^{s-b} T_s,$$

where $\ell = as + b$ with $0 \le b < s$. Setting $a = cq + d$ with $0 \le d < q$ we obtain

$$\begin{aligned} d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (s-b)(c^2q + (2c+1)d - 1) + s \\ &= c^2qs + (2c+1)(ds+b) \\ &= c^2qs + (2c+1)(\ell - cqs) \\ &= (2c+1)\ell - c(c+1)qs \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)(r-q). \end{aligned}$$

Here $zr + e = 2\ell = a(n-1) + 2b = c(r-q) + d(n-1) + 2b$, and $d(n-1) + 2b \le (q-1)(n-1) + 2(s-1) = (r-q) - 2$; thus $c \ge z$, and if we set $f = d(n-1) + 2b$ we find that

$$\begin{aligned} d_q(Z) - d_{B_\ell,r} &= (c-z)(f + \tfrac{1}{2}(r-q)(c-z-1)) + \tfrac{1}{2}z((z+1)q-2) + (z - \epsilon_r\lceil \tfrac{z}{2}\rceil) \\ &\ge 0. \end{aligned}$$

Now assume instead that $b_\ell = 0$. Then Z has s factors of type A, one of type B and an s-dimensional torus. Arguing precisely as in the proof of Proposition 3.4, we see that

$$Z = (A_a)^{\lfloor \frac{b}{2} \rfloor} (A_{a-1})^{s - \lfloor \frac{b}{2} \rfloor} B_{\lceil \frac{a}{2} \rceil} T_s,$$

where $2\ell = a(2s+1) + b$ with $0 \le b < 2s+1$. On setting $a = cq + d$ with $0 \le d < q$, the calculation for $d_q(Z)$ is identical to that given in the proof of Proposition 3.4 for n odd, and gives

$$d_q(Z) = (2c+1)\ell - \tfrac{1}{2}c(c+1)r + \epsilon_r\lceil \tfrac{c}{2}\rceil.$$

Since $zr + e = 2\ell = an + b = cr + dn + b$ with $dn + b \le (q-1)n + (n-1) = r - 1$, we have $c = z$ and thus

$$d_q(Z) = (2z+1)\ell - \tfrac{1}{2}z(z+1)r + \epsilon_r\lceil \tfrac{z}{2}\rceil = d_{B_\ell,r}.$$

Combining the two possibilities we see that $d_r(B_\ell) = d_{B_\ell,r}$ for n odd, as required.

Now assume that n is even, and set $n = 2s$. Here either $b_0 = b_1 = 0$ or $b_0 = b_1 = 1$, and $b_\ell = 0$ or 1; so there are four types of possibility for Z. First take the case where $b_0 = b_1 = b_\ell = 1$. Here Z has $s - 1$ factors of type A and an s-dimensional torus, and we obtain

$$Z = (A_a)^b (A_{a-1})^{s-1-b} T_s,$$

where $\ell - 1 = a(s-1) + b$ with $0 \le b < s-1$. Setting $a = cq + d$ with $0 \le d < q$ gives

$$\begin{aligned} d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (s-1-b)(c^2q + (2c+1)d - 1) + s \\ &= c^2q(s-1) + (2c+1)(d(s-1)+b) + 1 \\ &= c^2q(s-1) + (2c+1)(\ell - 1 - cq(s-1)) + 1 \\ &= (2c+1)\ell - c(c+1)q(s-1) - 2c \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)(r-2q) - 2c. \end{aligned}$$

Here $zr + e = 2\ell = a(n-2) + 2b + 2 = c(r-2q) + d(n-2) + 2b + 2$, and $d(n-2) + 2b + 2 \le (q-1)(n-2) + 2(s-2) + 2 = r - 2q$. Thus $c \ge z$, and if we set $f = d(n-2) + 2b$ we find that

$$d_q(Z) - d_{B_{\ell},r} = (c-z)(f + \tfrac{1}{2}(r-2q)(c-z-1)) + z((z+1)q - 2) \ge 0.$$

Next consider $b_0 = b_1 = 1$, $b_\ell = 0$. Here Z has $s-1$ factors of type A, one factor of type B and an s-dimensional torus, and arguing as before we find

$$Z = (A_a)^{\lfloor \frac{b}{2} \rfloor}(A_{a-1})^{s-1-\lfloor \frac{b}{2} \rfloor} B_{\lceil \frac{a}{2} \rceil} T_s,$$

where $2\ell - 2 = a(2s-1) + b$ with $0 \le b < 2s-1$. Write $a = cq + d$ with $0 \le d < q$. Noting that $\epsilon_q = 1$ here and using the formulæ from Lemma 2.5(i),(ii) we obtain

$$\begin{aligned} d_q(Z) &= \lfloor \tfrac{b}{2} \rfloor (c^2q + (2c+1)(d+1) - 1) + (s - 1 - \lfloor \tfrac{b}{2} \rfloor)(c^2q + (2c+1)d - 1) \\ &\quad + \tfrac{1}{2}c^2q + (2c+1)(\lceil \tfrac{a}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil + s \\ &= c^2q(s - \tfrac{1}{2}) + (2c+1)(d(s-1) + \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a}{2} \rceil - \tfrac{1}{2}cq) + \lceil \tfrac{c}{2} \rceil + 1 \\ &= c^2q(s - \tfrac{1}{2}) + (2c+1)(d(s-1) + \tfrac{a+b}{2} - \tfrac{1}{2}cq) + \lceil \tfrac{c}{2} \rceil + 1 \\ &= c^2q(s - \tfrac{1}{2}) + (2c+1)(\ell - 1 - cq(s - \tfrac{1}{2})) + \lceil \tfrac{c}{2} \rceil + 1 \\ &= (2c+1)\ell - c(c+1)q(s - \tfrac{1}{2}) + \lceil \tfrac{c}{2} \rceil - 2c \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)(r-q) + \lceil \tfrac{c}{2} \rceil - 2c. \end{aligned}$$

Here $zr + e = 2\ell = a(n-1) + b + 2 = c(r-q) + d(n-1) + b + 2$, with $d(n-1) + b + 2 \le (q-1)(n-1) + (2s-2) + 2 = r - q + 1$. Thus $zr + e \le (c+1)(r-q) + 1$. If we were to have $c < z$ this would give $zr + e \le z(r-q) + 1$, or $e \le 1 - zq$, which would force $e = 0$, $z = q = 1$, whence $2\ell = r = n$, contrary to our assumption that $n < h$. Therefore we have $c \ge z$, and if we set $f = d(n-1) + b$ we find that

$$\begin{aligned} d_q(Z) - d_{B_{\ell},r} &= (c-z)(f + \tfrac{1}{2}(r-q)(c-z-1)) + \tfrac{1}{2}z((z+1)q - 3) \\ &\quad + \tfrac{1}{2}((c-z) + \epsilon_c) \\ &\ge 0 \end{aligned}$$

(again note that if $z = q = 1$, then the second term is $-\frac{1}{2}$ but the third is at least $\frac{1}{2}$).

Now consider the case $b_0 = b_1 = 0$, $b_\ell = 1$. Here Z has $s-1$ factors of type A, one factor of type D and an $(s-1)$-dimensional torus, and we find

$$Z = (A_a)^{\lfloor \frac{b}{2} \rfloor}(A_{a-1})^{s-1-\lfloor \frac{b}{2} \rfloor} D_{\lceil \frac{a+1}{2} \rceil} T_{s-1},$$

where $2\ell-1 = a(2s-1)+b$ with $0 \le b < 2s-1$. Setting $a = cq+d$ with $0 \le d < q$ and recalling that $\epsilon_q = 1$ we obtain

$$\begin{aligned} d_q(Z) &= \lfloor \tfrac{b}{2} \rfloor (c^2q + (2c+1)(d+1) - 1) + (s-1-\lfloor \tfrac{b}{2} \rfloor)(c^2q + (2c+1)d - 1) \\ &\quad + \tfrac{1}{2}c^2q + (2c+1)(\lceil \tfrac{a+1}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c + s - 1 \\ &= c^2q(s-\tfrac{1}{2}) + (2c+1)(d(s-1) + \lfloor \tfrac{b}{2} \rfloor + \lceil \tfrac{a+1}{2} \rceil - \tfrac{1}{2}cq) + \lceil \tfrac{c}{2} \rceil - c - \epsilon_c \\ &= c^2q(s-\tfrac{1}{2}) + (2c+1)(d(s-1) + \tfrac{a+b+1}{2} - \tfrac{1}{2}cq) - \lceil \tfrac{c}{2} \rceil \\ &= c^2q(s-\tfrac{1}{2}) + (2c+1)(\ell - cq(s-\tfrac{1}{2})) - \lceil \tfrac{c}{2} \rceil \\ &= (2c+1)\ell - c(c+1)q(s-\tfrac{1}{2}) - \lceil \tfrac{c}{2} \rceil \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)(r-q) - \lceil \tfrac{c}{2} \rceil. \end{aligned}$$

Here we have $zr + e = 2\ell = a(n-1)+b+1 = c(r-q)+d(n-1)+b+1$, and $d(n-1)+b+1 \le (q-1)(n-1)+2s-2+1 = r-q$. Thus $c \ge z$, and if we set $f = d(n-1)+b$ we find that

$$d_q(Z) - d_{B_\ell,r} = (c-z)(f + \tfrac{1}{2}(r-q)(c-z-1)) + \tfrac{1}{2}z((z+1)q-2) + \tfrac{1}{2}(c-\epsilon_c) \ge 0.$$

Finally we must consider the case $b_0 = b_1 = b_\ell = 0$. Here Z has $s-1$ factors of type A, one factor of type B, one factor of type D and an $(s-1)$-dimensional torus. Again we must determine an optimal configuration. As usual we may take the type A factors to be $(A_x)^z(A_{x-1})^{s-1-z}$ for some x and some $0 \le z < s-1$, and easy considerations show that the factors of types B and D may be either B_yD_y or $B_{y-1}D_y$ for some y. Using the reasoning already given for comparing factors of type A with those of type B or D, we see that if we have B_yD_y, then either $2y = x+1$, or $z = 0$ and $2y = x$. If the latter holds we set $x' = x-1$, $y' = y$ and obtain $(A_{x'})^{s-1}B_{y'}D_{y'}$ with $2y' = x'+1$. Likewise if we have $B_{y-1}D_y$, then either $2y = x+2$, or $z = 0$ and $2y = x+1$. Again, if the latter holds we set $x' = x-1$, $y' = y$ and obtain $(A_{x'})^{s-1}B_{y'-1}D_{y'}$ with $2y' = x'+2$. Putting all of these together we see that we have

$$Z = (A_a)^b(A_{a-1})^{s-1-b}B_{\lceil \frac{a}{2} \rceil}D_{\lceil \frac{a+1}{2} \rceil}T_{s-1},$$

where $\ell - 1 = as+b$ with $0 \le b < s$. As usual write $a = cq+d$ with $0 \le d < q$; using the formulæ in Lemma 2.5(i), (ii), (iv) and noting that $\epsilon_q = 1$ here we obtain

$$\begin{aligned} d_q(Z) &= b(c^2q + (2c+1)(d+1) - 1) + (s-1-b)(c^2q + (2c+1)d - 1) \\ &\quad + \tfrac{1}{2}c^2q + (2c+1)(\lceil \tfrac{a}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil \\ &\quad + \tfrac{1}{2}c^2q + (2c+1)(\lceil \tfrac{a+1}{2} \rceil - \tfrac{1}{2}cq) + \epsilon_q \lceil \tfrac{c}{2} \rceil - c - \epsilon_c + s - 1 \\ &= c^2qs + (2c+1)(d(s-1) + b + \lceil \tfrac{a}{2} \rceil + \lceil \tfrac{a+1}{2} \rceil - cq) + 2\lceil \tfrac{c}{2} \rceil - c - \epsilon_c \\ &= c^2qs + (2c+1)(d(s-1) + b + a + 1 - cq) \\ &= c^2qs + (2c+1)(\ell - cqs) \\ &= (2c+1)\ell - c(c+1)qs \\ &= (2c+1)\ell - \tfrac{1}{2}c(c+1)r. \end{aligned}$$

Here $zr + e = 2\ell = an + 2b + 2 = cr + dn + 2b + 2$, with $dn + 2b + 2 \le (q-1)n + 2(s-1) + 2 = r$. Thus either $c = z$, or we must have $c = z-1$ and $2\ell = zr$; but if the latter holds, then we see that $(2c+1)\ell - \frac{1}{2}c(c+1)r = (2z+1)\ell - \frac{1}{2}z(z+1)r$ anyway. Thus in either case we have

$$d_q(Z) = (2z+1)\ell - \tfrac{1}{2}z(z+1)r = d_{B_\ell,r}$$

(note that $\epsilon_r = 0$ as n is even here). Combining all possibilities we see that $d_r(B_\ell) = d_{B_\ell,r}$ for n even, as required. This completes the proof. □

Proposition 3.7. *With the notation established, if G has root system Φ of exceptional type, then $d_r(G) = d_{\Phi,r}$.*

Proof. We begin by considering the cases in which $r < h$. Here we proceed by inspection, using the same general strategy as in the classical cases: given n, Lemma 3.2 is used to restrict the possibilities for Z which must be considered, and then the formulæ of Lemmas 2.5 and 2.7 are used to calculate $d_q(Z)$ for each Z. For convenience, we present the relevant information in tabular form: given Φ, for each $r < h$ we give the value $d_{\Phi,r}$ and various pairs (p, Z), one for each prime p dividing r and then finally one for an arbitrary p coprime to r, such that Z is a centralizer of a semisimple element of order n with minimal value of $d_q(Z)$. (Note that if Φ is of type E_7 or E_8, the standard notation is used to distinguish between isomorphic but non-conjugate subsystems: in E_7 an $({A_1}^3)'$ subsystem is one lying in an A_7 subsystem, while in E_8 an $({A_1}^4)'$ subsystem is one lying in an A_8 subsystem.)

r	$d_{G_2,r}$	(p, Z)
1	14	$(-, G_2)$
2	6	$(2, G_2), (-, A_1\tilde{A}_1)$
3	4	$(3, G_2), (-, \tilde{A}_1T_1)$
4	4	$(2, G_2), (-, A_1T_1)$
5	4	$(5, G_2), (-, A_1T_1)$

r	$d_{F_4,r}$	(p, Z)
1	52	$(-, F_4)$
2	24	$(2, F_4), (-, A_1C_3)$
3	16	$(3, F_4), (-, A_2\tilde{A}_2)$
4	12	$(2, F_4), (-, A_1\tilde{A}_2T_1)$
5	12	$(5, F_4), (-, A_1\tilde{A}_2T_1)$
6	8	$(2, A_2\tilde{A}_2), (3, A_1C_3), (-, A_1\tilde{A}_1T_2)$
7	8	$(7, F_4), (-, A_1\tilde{A}_1T_2)$
8	6	$(2, F_4), (-, \tilde{A}_1T_3)$
9	6	$(3, F_4), (-, \tilde{A}_1T_3)$
10	6	$(2, B_2T_2), (5, A_1C_3), (-, \tilde{A}_1T_3)$
11	6	$(11, F_4), (-, \tilde{A}_1T_3)$

r	$d_{E_6,r}$	(p, Z)
1	78	$(-, E_6)$
2	38	$(2, E_6), (-, A_5A_1)$
3	24	$(3, E_6), (-, {A_2}^3)$
4	20	$(2, E_6), (-, {A_2}^2A_1T_1)$
5	16	$(5, E_6), (-, A_2{A_1}^2T_2)$
6	12	$(2, {A_2}^3), (3, A_5A_1), (-, {A_1}^3T_3)$
7	12	$(7, E_6), (-, {A_1}^3T_3)$
8	10	$(2, E_6), (-, {A_1}^2T_4)$
9	8	$(3, E_6), (-, A_1T_5)$
10	8	$(2, A_2{A_1}^2T_2), (5, A_5A_1), (-, A_1T_5)$
11	8	$(11, E_6), (-, A_1T_5)$

r	$d_{E_7,r}$	(p, Z)
1	133	$(-, E_7)$
2	63	$(2, E_7), (-, A_7)$
3	43	$(3, E_7), (-, A_5A_2)$
4	33	$(2, E_7), (-, A_4A_2T_1)$
5	27	$(5, E_7), (-, A_3A_2A_1T_1)$
6	21	$(2, A_5A_2), (3, A_7), (-, {A_2}^2A_1T_2)$
7	19	$(7, E_7), (-, A_2{A_1}^3T_2)$
8	17	$(2, E_7), (-, A_2{A_1}^2T_3)$
9	15	$(3, E_7), (-, {A_1}^4T_3)$
10	13	$(2, A_3A_2A_1T_1), (5, A_7), (-, ({A_1}^3)'T_4)$
11	13	$(11, E_7), (-, ({A_1}^3)'T_4)$
12	11	$(2, A_5A_2), (3, {A_3}^2A_1), (-, {A_1}^2T_5)$
13	11	$(13, E_7), (-, {A_1}^2T_5)$
14	9	$(2, A_2{A_1}^3T_2), (7, A_7), (-, A_1T_6)$
15	9	$(3, A_3A_2A_1T_1), (5, A_5A_2), (-, A_1T_6)$
16	9	$(2, E_7), (-, A_1T_6)$
17	9	$(17, E_7), (-, A_1T_6)$

r	$d_{E_8,r}$	(p, Z)
1	248	$(-, E_8)$
2	120	$(2, E_8), (-, D_8)$
3	80	$(3, E_8), (-, A_8)$
4	60	$(2, E_8), (-, D_5A_3)$
5	48	$(5, E_8), (-, {A_4}^2)$
6	40	$(2, A_8), (3, D_8), (-, A_4A_3T_1)$
7	36	$(7, E_8), (-, A_4A_2A_1)$
8	30	$(2, E_8), (-, A_3A_2{A_1}^2T_1)$
9	28	$(3, E_8), (-, A_3A_2A_1T_2)$
10	24	$(2, {A_4}^2), (5, D_8), (-, {A_2}^2{A_1}^2T_2)$
11	24	$(11, E_8), (-, {A_2}^2{A_1}^2T_2)$
12	20	$(2, A_8), (3, D_5A_3), (-, A_2{A_1}^3T_3)$
13	20	$(13, E_8), (-, A_2{A_1}^3T_3)$
14	18	$(2, A_4A_2A_1), (7, D_8), (-, A_2{A_1}^2T_4)$
15	16	$(3, {A_4}^2), (5, A_8), (-, ({A_1}^4)'T_4)$
16	16	$(2, E_8), (-, ({A_1}^4)'T_4)$
17	16	$(17, E_8), (-, ({A_1}^4)'T_4)$
18	14	$(2, A_3A_2A_1T_2), (3, D_8), (-, {A_1}^3T_5)$
19	14	$(19, E_8), (-, {A_1}^3T_5)$
20	12	$(2, {A_4}^2), (5, D_5A_3), (-, {A_1}^2T_6)$
21	12	$(3, A_4A_2A_1T_1), (7, A_8), (-, {A_1}^2T_6)$
22	12	$(2, {A_2}^2{A_1}^2T_2), (11, D_8), (-, {A_1}^2T_6)$
23	12	$(23, E_8), (-, {A_1}^2T_6)$
24	10	$(2, A_8), (3, A_3A_2{A_1}^2T_1), (-, A_1T_7)$
25	10	$(5, E_8), (-, A_1T_7)$
26	10	$(2, A_2{A_1}^3T_3), (13, D_8), (-, A_1T_7)$
27	10	$(3, E_8), (-, A_1T_7)$
28	10	$(2, A_4A_2A_1T_1), (7, D_5A_3), (-, A_1T_7)$
29	10	$(29, E_8), (-, A_1T_7)$

In all cases we see that $d_q(Z) = d_{\Phi,r}$, as required. This completes the proof in the cases where $r < h$.

Now assume $r \geq h$; here we have $d_{\Phi,r} = \ell$, so we seek to show that there are regular elements in $G_{[r]}$. As we are assuming $n < h$, we may consider the nth row of the relevant table above, and take Z to be the group appearing in the final entry of the row (so that Z is the centralizer of a semisimple element of order n, provided the characteristic does not divide n). We then wish to show that there are regular unipotent elements in $Z_{[q]}$. To see this we set h_Z to be the maximum of the Coxeter numbers of the simple factors of Z, and observe that in almost all cases we have $h_Z = \lceil \frac{h}{n} \rceil$; the exceptions are listed in the table below:

G	n	Z	h_Z	$\frac{h}{n}$
G_2	2	$A_1\tilde{A}_1$	2	3
F_4	3	$A_2\tilde{A}_2$	3	4
E_6	3	${A_2}^3$	3	4
E_7	2	A_7	8	9
E_8	2	D_8	14	15
E_8	3	A_8	9	10
E_8	5	${A_4}^2$	5	6

In each of these exceptions we have $h_Z = \frac{h}{n} - 1$; and we observe that in each of them h_Z fails to be coprime to n and so cannot be a power of p. Thus in all instances the conditions $p^a \geq h_Z$ and $p^a \geq \frac{h}{n}$ are equivalent. We now employ Testerman's order formula (which is proved in [21] for good characteristic, but may easily be verified in bad characteristic in the case of regular unipotent elements): this states that the order of the regular unipotent elements of Z is $\min\{p^a : p^a \geq h_Z\}$, which we may now write as $\min\{p^a : p^a \geq \frac{h}{n}\}$. Since by assumption $q \geq \frac{h}{n}$, it follows that $Z_{[q]}$ does indeed contain regular unipotent elements as required. This completes the proof in the cases where $r \geq h$. □

Combining Lemma 3.1 with Propositions 3.3, 3.4, 3.5, 3.6 and 3.7, we have proved the following.

Theorem 3.8. *If G is a simple algebraic group of adjoint type with root system Φ, and $r \in \mathbb{N}$, then* $\operatorname{codim} G_{[r]} = d_{\Phi,r}$.

At this point we may observe that taking this result together with Lemma 1.2 gives the following immediate consequence concerning dimensions rather than codimensions.

Corollary 3.9. *If G is a simple algebraic group of adjoint type with root system Φ, and $r \in \mathbb{N}$, set $x = (1 - \frac{1}{r}) \dim G - \dim G_{[r]}$. Then $x \geq 0$ if r is a product of very good primes, and if G is of classical type, the value of x is given by*

$$x = \begin{cases} \frac{1}{r}(e-1)(r-e-1) & \text{if } \Phi = A_\ell, \\ \frac{1}{2r}e(r-e-1) + (\epsilon_r \lceil \frac{z}{2} \rceil - \frac{z}{2}) & \text{if } \Phi = B_\ell \text{ or } C_\ell, \\ \frac{1}{2r}(e+2)(r-e-1) + (\epsilon_r \lceil \frac{z}{2} \rceil - \frac{z}{2} - \epsilon_z) & \text{if } \Phi = D_\ell. \end{cases}$$

We may now prove our result concerning $G_{(r)}$.

Theorem 3.10. *If G is a simple algebraic group of adjoint type with root system Φ, and $G_{(r)} \neq \emptyset$ for some $r \in \mathbb{N}$, then* $\operatorname{codim} G_{(r)} = d_{\Phi,r}$.

Proof. Given G, suppose r is a minimal counterexample to the statement to be proved. Then $G_{[r]}$ contains elements of order r, but the minimal centralizer dimension occurs for elements of order r' for some $r' < r$. Write as usual $r = nq$.

The calculations in Propositions 3.3, 3.4, 3.5, 3.6 and 3.7 provide a centralizer Z of a semisimple element of order n with the required value of $d_q(Z)$, so we must have $r' = nq'$ for some $q' < q$. Since elements of order r' therefore lie in $G_{[r/p]}$, we must have $\operatorname{codim} G_{[r/p]} = d_{\Phi,r}$, i.e., $d_{\Phi,r/p} = d_{\Phi,r}$; by Lemma 1.3 this forces $d_{\Phi,r/p} = \ell$ (and thus $r/p \geq h$). Note that the minimality of r, and the fact that $G_{(r/p)}$ contains the pth powers of the elements in $G_{(r)}$ and thus is non-empty, mean that $\operatorname{codim} G_{(r/p)} = \ell$; so we may assume that $r' = r/p$.

Let

$$H_n = \bigcup_{s \in G_{(n)}} \{u \in G : u \text{ a regular unipotent element of } C_G(s)^0\},$$

and let $q'' = \max_{u \in H_n} o(u)$. Since regular unipotent elements in any algebraic group have the maximal orders among the unipotent elements there, we must have $q'' \geq q$. If we had $q'' = q$ we would have regular elements of order r as required, so we must have $q'' > q$. Thus on the one hand we have regular elements of order rp^x, where $q'' = qp^x$, and on the other we have regular elements of order $r' = r/p$. We shall show that there must then be regular elements of order r, contrary to the choice of r.

Our approach for G of classical type is to begin with a regular element $g = su$ of order r' and successively change the element while maintaining the order of its semisimple part. Thus if Φ is of type A_ℓ, the centralizer Z of s is a product of factors of type A and a torus. If the largest type A factor has rank k, we must have $q'/p \leq k < q'$, because the order of a regular unipotent element in A_k is the smallest power of p which is greater than k. Now if $A_{k'}$ is any other type A factor, replace $A_k A_{k'}$ by $A_{k+1} A_{k'-1}$ and iterate until Z has just one non-trivial type A factor; then iterate replacing $A_{\ell-y} T_y$ by $A_{\ell-y+1} T_{y-1}$ until $A_{\ell-1} T_1$ is reached. (Note that each such change still leaves a centralizer of a semisimple element of order n. In terms of the result from [6], we may take all but one of the non-zero coefficients b_i to be 1, and the last to be determined by the requirement that $\sum b_i = n$.) At each stage the order of regular unipotent elements can change by at most a factor of p. Since the order begins at $q' = q/p$ and finishes at $q'' = qp^x$, after the first increase we must have regular unipotent elements of order q and therefore regular elements of order r as required.

The argument for the other classical groups proceeds in like fashion. If Φ is of type C_ℓ, the centralizer Z has factors of type A, at most two factors of type C and a torus. If there is no type C factor, begin by replacing some A_k by C_k. Iterate replacing $C_k A_{k'}$ by $C_{k+1} A_{k'-1}$ until there are no non-trivial type A factors remaining. If there are two type C factors, iterate replacing $C_k C_{k'}$ (where $k \geq k'$) by $C_{k+1} C_{k'-1}$ until there is only one. Finally iterate replacing $C_{\ell-y} T_y$ by $C_{\ell-y+1} T_{y-1}$ until $C_{\ell-1} T_1$ is reached. Again, at each stage the order of regular unipotent elements can change by at most a factor of p, so the result follows. Types D_ℓ and B_ℓ are precisely similar.

For G of exceptional type we proceed differently, as it is necessary to be more careful about the coefficients b_i which determine the centralizer Z. Here we simply form a list of all possible centralizers Z, and for each record the possible values of n_1 for which it is the centralizer of a semisimple element of order n_1, together with the order q_1 of regular unipotent elements in Z. This therefore lists all possible pairs (n_1, q_1) for which there is a regular element of order $n_1 q_1$. Thus if there are elements of order ${n_1}' {q_1}'$, then there is a pair (n_1, q_1) on the list with ${n_1}' = n_1$

and ${q_1}' \le q_1$. We next delete any lines for which the possible pairs (n_1, q_1) form a subset of those on some other line. (For example, if Φ is of type E_8, then the pairs (n_1, q_1) corresponding to $Z = A_1T_7$ are those with $n_1 \ge 24$ and $q_1 = p$, while those corresponding to $Z = {A_1}^2T_6$ are those with $n_1 \ge 20$ and $q_1 = p$; the first line may be deleted as each of its pairs occurs in the second.) We then form a second list, entering opposite each remaining Z all pairs $(n_1, {q_1}')$ satisfying $n_1{q_1}' \ge h$ for which there is a pair (n_1, q_1) on the first list with ${q_1}' < q_1$. For example, the result for Φ of type E_6 is as follows, in which the values of ${q_1}'$ in the final column correspond to values of n_1 for which $n_1{q_1}' \ge 12$:

Z	n_1	q_1	${q_1}'$
T_6	≥ 12	1	$-$
${A_1}^3T_3$	≥ 6	p	1
$A_3A_1T_2$	≥ 4	$4, 9, p > 3$	$1, 2, 3$
${A_2}^2A_1T_1$	≥ 4	$4, p > 2$	$1, 2$
$A_4A_1T_1$	≥ 3	$8, 9, p > 3$	$1, 2, 3, 4$
A_5T_1	≥ 3	$8, 9, 25, p > 5$	$1, 2, 3, 4, 5$
D_5T_1	≥ 2	$8, 9, 25, 49, p > 7$	$1, 2, 3, 4, 5, 7$
${A_2}^3$	3	$4, p > 3$	$-$
A_5A_1	2	$9, 25, p > 5$	$-$
E_6	1	$16, 27, 25, 49, 121, p > 11$	$-$

By inspection we find that each pair on the second list occurs on the first list opposite some other possibility for Z: those with ${q_1}' = 1$ occur for $Z = T_6$ (for $n \ge 12$); those with ${q_1}' = 2$ for $Z = {A_1}^3T_3$ (for $n \ge 6$); those with ${q_1}' = 3$ for $Z = {A_2}^2A_1T_1$ (for $n \ge 4$); those with ${q_1}' = 4$ for $Z = {A_2}^3$ (for $n = 3$) or $Z = {A_2}^2A_1T_1$ (for $n \ge 4$); those with ${q_1}' = 5$ for $Z = A_4A_1T_1$ (for $n \ge 3$); and those with ${q_1}' = 7$ for $Z = A_5A_1$ (for $n = 2$) or $Z = A_5T_1$ (for $n \ge 3$). This happens for each exceptional group G. Thus if there are elements in G of a given order greater than or equal to h, then there are regular elements of that order. Since by the above it suffices to consider regular elements, this proves the result. $\square$

Finally we consider groups of arbitrary isogeny type.

Theorem 3.11. *If G is a simple algebraic group with root system Φ, and $r \in \mathbb{N}$, then* $\operatorname{codim} G_{[r]} \ge d_{\Phi,r}$.

Proof. Let G_{ad} be the adjoint group of the same type and over the same field as G, and let $\phi : G \to G_{\mathrm{ad}}$ be an isogeny; for $x \in G$ write $\hat{x}$ for $\phi(x)$. Take $g \in G_{[r]}$; then $\hat{g} \in (G_{\mathrm{ad}})_{[r]}$. Given $h \in G$, the set $G_{g,h} = \{x \in G : [g, x] = h\}$ is either empty or a right coset of $C_G(g)$. As $C_{G_{\mathrm{ad}}}(\hat{g}) = \bigcup_{h \in \ker \phi} \phi(G_{g,h})$ and $\ker \phi$ is finite, Theorem 3.8 gives $\dim C_G(g) = \dim C_{G_{\mathrm{ad}}}(\hat{g}) \ge d_{\Phi,r}$. By Lemma 3.1 we then have $\operatorname{codim} G_{[r]} = d_r(G) \ge d_{\Phi,r}$, as required. $\square$

Note that an inequality is the best possible result here, as may be seen by considering groups of type A_1 in odd characteristic with $r = 2$. We have $d_{A_1,2} = 1$: if G is the adjoint group $PGL_2(K)$, the involution which is the image of $\mathrm{diag}(1, -1)$ is a regular semisimple element, giving $\operatorname{codim} G_{[2]} = 1$ as required by Theorem 3.8. However, if G is the simply-connected group $SL_2(K)$, the only involution in G is the central element $\mathrm{diag}(-1, -1)$, so that $\operatorname{codim} G_{[2]} = 3$.

Combining Lemma 1.2, Theorem 3.8 and Theorem 3.11 completes the proof of Theorem 1.

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