

SIGNATURE OF RELATIONS IN MAPPING CLASS GROUPS AND NON-HOLOMORPHIC LEFSCHETZ FIBRATIONS

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Dedicated to Professor Yukio Matsumoto for his 60th birthday

ABSTRACT. We introduce the notion of signature for relations in mapping class groups and show that the signature of a Lefschetz fibration over the 2-sphere is the sum of the signatures for basic relations contained in its monodromy. Combining explicit calculations of the signature cocycle with a technique of substituting positive relations, we give some new examples of non-holomorphic Lefschetz fibrations of genus 3, 4 and 5 which violate slope bounds for non-hyperelliptic fibrations on algebraic surfaces of general type.

1. INTRODUCTION

The study of Lefschetz fibrations has turned out to be interesting and important by virtue of the remarkable works of Donaldson [10] and Gompf [17] which demonstrate a close relationship between symplectic 4-manifolds and Lefschetz fibrations.

The geography problem of Lefschetz fibrations is one of the most interesting topics to be investigated. Originally, the geography problem for complex surfaces was to find minimal surfaces of general type with a prescribed pair of Chern numbers. Using the pair of Euler characteristic and signature instead of the pair of Chern numbers, we can consider also the geography problem of Lefschetz fibrations. The Euler characteristic of a given Lefschetz fibration is easily computed, while the signature is not. If the Lefschetz fibration is over the 2-sphere, Ozbagci [33] and Smith [36] gave signature formulae: the former's is suited for explicit computation and the latter's for showing some qualitative properties of Lefschetz fibrations. If the Lefschetz fibration is hyperelliptic, a local signature, which is a generalization of the σ -number [28] and the fractional signature [29] due to Matsumoto, was defined by the first author [12] and a method of computing signatures was established.

It is also interesting to find various kinds of examples of non-holomorphic Lefschetz fibrations, which imply the difference between the geography of algebraic surfaces fibered over curves and that of Lefschetz fibrations. Ozbagci and Stipsicz [34] constructed examples of non-holomorphic genus-2 Lefschetz fibrations whose total space never appear in the table of the Enriques-Kodaira classification of complex surfaces. Korkmaz [25] generalized their examples to higher genera. Fintushel

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and Stern [14] produced minimal symplectic Lefschetz fibrations which do not satisfy the Noether inequality.

In the present paper we would like to propose a useful method of computing signatures of Lefschetz fibrations and give some new examples of non-holomorphic Lefschetz fibrations. We first introduce the notion of signature for relations in mapping class groups. The signature of a Lefschetz fibration over the 2-sphere is equal to the sum of the signatures for basic relations appearing in its monodromy. Combining explicit computations of signatures for basic relations with a technique of substituting positive relations, which is a generalization of a method of Fuller and Smith [37], we construct new examples of non-holomorphic Lefschetz fibrations of genus 3, 4 and 5 which violate lower bounds of the slope of non-hyperelliptic fibrations in algebraic geometry.

This paper is organized as follows. In Section 2 we recall a well-known theorem of Hopf and some facts on 2-cocycles of groups. In Section 3 we define the signature of a relator of mapping class groups and carry out explicit computations of signatures for basic relators. In Section 4 we construct examples of non-holomorphic Lefschetz fibrations using the results of Section 3 and some results on slope bounds for algebraic surfaces.

The idea of “the signature of a relation” occurred to the first author when he worked on signature of surface bundles at the University of Munich in 2000 with the co-authors of [13]. He is grateful to the co-authors of [13], especially D. Kotschick for helpful discussions and comments. The authors are grateful to K. Konno for a detailed explanation of his works on the lower bound of the slope of non-hyperelliptic fibrations and to T. Morifuji for useful comments on his formula of Meyer’s function in [30].

2. HOPF’S THEOREM AND 2-COCYCLE

In this section we review some facts about cohomology of groups (cf. Brown [6] and Meyer [27]).

Let G be a group. We define the homology $H_*(G)$ of G as the homology of the co-invariants of the standard resolution of \mathbb{Z} over $\mathbb{Z}G$. The standard chain complex of G is denoted by $C_*(G)$. We describe the second homology group $H_2(G)$ in terms of a given presentation of G .

Theorem 2.1 (Hopf [19]). *Let G be a group, F a free group and $\pi : F \rightarrow G$ an epimorphism. Then the following isomorphism holds:*

$$H_2(G) \cong R \cap [F, F] / [R, F],$$

where $R = \text{Ker } \pi$ is the kernel of π .

An explicit homomorphism which induces the isomorphism in Hopf’s theorem is given as follows.

Lemma 2.2. *Let G be a group, $F = F(S)$ the free group generated by a set S and $\pi : F \rightarrow G$ an epimorphism. The homomorphism*

$$R \rightarrow C_2(G) : r \mapsto \sum_{s \in S} \left[\frac{\partial r}{\partial s} \middle| \bar{s} \right]$$

induces the isomorphism

$$R \cap [F, F] / [R, F] \rightarrow H_2(G) = H_2(C_*(G)),$$

where $\partial/\partial s : \mathbb{Z}F \rightarrow \mathbb{Z}F$ is the Fox free derivative with respect to s , \bar{f} is the image of $f \in F$ under π and the symbol $[\cdot | \cdot]$ stands for the bar notation extended linearly over $\mathbb{Z}G$.

Proof. See Brown [6], §2.5, Exercise 4. \square

Using the lemma above, we have the following description of the evaluation map $H_2(G) \rightarrow \mathbb{Z}$ for the cohomology class of a 2-cocycle of G .

Proposition 2.3 (cf. Meyer [27], §3). *Let G be a group, $F = F(S)$ the free group generated by a set S and $\pi : F \rightarrow G$ an epimorphism. If a 2-cocycle $z : G \times G \rightarrow \mathbb{Z}$ of G satisfies the condition*

$$(*) \quad z(g, 1) = z(1, g) = z(g, g^{-1}) = 0 \quad \text{for all } g \in G,$$

then the homomorphism $H_2(C_(G)) \rightarrow \mathbb{Z}$ defined by the cohomology class of z together with the Kronecker product is induced by the homomorphism*

$$c : R \longrightarrow \mathbb{Z} : r \mapsto \sum_{j=1}^m z(\overline{s_1 \cdots s_{j-1}}, \overline{s_j})$$

$$(r = s_1 \cdots s_m, s_1, \dots, s_m \in S \cup S^{-1})$$

under the isomorphism in Lemma 2.2.

Proof. Let r be an element of R . We express r as a word in s_1, \dots, s_m : $r = s_1^{\epsilon_1} \cdots s_m^{\epsilon_m}$, where $\epsilon_1, \dots, \epsilon_m \in \{\pm 1\}$. By virtue of Lemma 2.2, the image of r under the composition $c : R \rightarrow C_2(G) \rightarrow \mathbb{Z}$ is equal to

$$\sum_{s \in S} z\left(\overline{\frac{\partial r}{\partial s}}, \overline{s}\right) = \sum_{j=1}^m z\left(\overline{\frac{\partial r}{\partial s_j}}, \overline{s_j}\right) = \sum_{j=1}^m z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}} \frac{\partial s_j^{\epsilon_j}}{\partial s_j}}, \overline{s_j}\right).$$

If $\epsilon_j = +1$, we have

$$\begin{aligned} z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}} \frac{\partial s_j^{\epsilon_j}}{\partial s_j}}, \overline{s_j}\right) &= z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}} \frac{\partial s_j}{\partial s_j}}, \overline{s_j}\right) = z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}}}, \overline{s_j}\right) \\ &= z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}}}, \overline{s_j^{\epsilon_j}}\right) \end{aligned}$$

and if $\epsilon_j = -1$, we have

$$\begin{aligned} z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}} \frac{\partial s_j^{\epsilon_j}}{\partial s_j}}, \overline{s_j}\right) &= z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}} \frac{\partial s_j^{-1}}{\partial s_j}}, \overline{s_j}\right) = z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}} \cdot -s_j^{-1}}, \overline{s_j}\right) \\ &= z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}}}, \overline{s_j^{-1}}\right) - z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}}}, \overline{1}\right) - z\left(\overline{s_j^{-1}}, \overline{s_j}\right) \\ &= z\left(\overline{s_1^{\epsilon_1} \cdots s_{j-1}^{\epsilon_{j-1}}}, \overline{s_j^{\epsilon_j}}\right) \end{aligned}$$

from $(*)$ and the cocycle condition. \square

Remark 2.4. The homomorphism $c : R \rightarrow \mathbb{Z}$ naturally extend to a map $c : F \rightarrow \mathbb{Z}$ by virtue of the definition above. It satisfies $c(xy) = c(x) + c(y) + z(\overline{x}, \overline{y})$ for $x, y \in F$ (i.e. $\delta c = -\pi^* z$) (see the formula (25) of Meyer [27]).

3. SIGNATURE OF RELATIONS IN MAPPING CLASS GROUPS

In this section we introduce the notion of a signature for relations in mapping class groups of surfaces and calculate it for the relations in presentations due to Gervais [16] and Luo [26].

Let Σ_g be a closed oriented surface of genus g and \mathcal{M}_g the mapping class group of Σ_g , namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g . We follow the functional notation: for $\varphi, \psi \in \mathcal{M}_g$, the symbol $\varphi\psi$ means that we apply ψ first and then φ . We denote by \mathcal{F} the free group generated by all isotopy classes \mathcal{S} of simple closed curves on Σ_g : $\mathcal{F} = F(\mathcal{S})$. There is a natural homomorphism $\varpi : \mathcal{F} \rightarrow \mathcal{M}_g$ which sends (the isotopy class of) a simple closed curve a on Σ_g to the right-handed Dehn twist t_a along a . We often denote the image $\varpi(W)$ of a word W in generators \mathcal{S} by \overline{W} . In particular, $\varpi(a) = t_a = \overline{a}$ for a simple closed curve $a \in \mathcal{S}$. It is well known as a theorem of Dehn [9] that this homomorphism ϖ is surjective. We set $\mathcal{R} := \text{Ker } \varpi$ and call each element of \mathcal{R} a *relator* in the generators \mathcal{S} of \mathcal{M}_g .

Gervais [16] gave an infinite presentation of \mathcal{M}_g with generators \mathcal{S} .

Theorem 3.1 (Gervais [16]). *The kernel \mathcal{R} of ϖ is normally generated by the following relators:*

- (I) *all the braid relators T_0, T_1 and T_{2_0} ;*
- (II) *all the chain relators C_3 of length 3;*
- (III) *all the lantern relators L .*

If $g \geq 1$, (II) and (III) can be replaced by

- (IV) *all the star relators E .*

Luo [26] improved Gervais' infinite presentation to show that all the relators were essentially discovered by Dehn.

Theorem 3.2 (Luo [26]). *The kernel \mathcal{R} of ϖ is normally generated by the following relators:*

- (I) *all the braid relators T_0 and T_1 ;*
- (II) *all the chain relators C_2 of length 2;*
- (III) *all the lantern relators L .*

We will give definitions of the relators exhibited in the two theorems above together with their signature computations in the latter half of this section.

We now define the signature of a relator of \mathcal{M}_g .

Definition 3.3. We set $G = \mathcal{M}_g$, $F = \mathcal{F}$, $R = \mathcal{R}$, $\pi = \varpi$ and $z = \tau_g$ in Proposition 2.3, where $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z}$ is the signature cocycle due to Meyer [27] (see Appendix A). Then we get an explicit homomorphism $c_g : \mathcal{R} \rightarrow \mathbb{Z}$ inducing the evaluation $H_2(G) \rightarrow \mathbb{Z}$. For a relator $\varrho \in \mathcal{R}$, we define $I_g(\varrho) := -c_g(\varrho) - s(\varrho)$ and call it the *signature* of ϱ , where $s(\varrho)$ is the total exponent of separating simple closed curves contained in the word ϱ . For example, the signature $I_g(A)$ of the *identity relator* $A := a \in \mathcal{R}$, where a is a null-homotopic simple closed curve on Σ_g , is equal to -1 .

A compact connected oriented surface of genus g with n boundary components is denoted by $\Sigma_{g,n}$. For (isotopy classes of) simple closed curves a, b , we denote their geometric intersection number by $i(a, b)$.

Let $\varrho, \varrho' \in \mathcal{R}$ be relators. We express ϱ as a word in generators \mathcal{S} : $\varrho = c_1^{\epsilon_1} \cdots c_m^{\epsilon_m}$, where c_1, \dots, c_m are simple closed curves on Σ_g and $\epsilon_1, \dots, \epsilon_m = \pm 1$. If there

exists an orientation-preserving homeomorphism $h : \Sigma_g \rightarrow \Sigma_g$ such that $\varrho' = h(c_1)^{\epsilon_1} \cdots h(c_m)^{\epsilon_m}$, we call that ϱ and ϱ' are *topologically equivalent* or have the same *topological type*. We call ϱ a *positive* relator if $\epsilon_1 = \cdots = \epsilon_m = +1$.

Definition 3.4 (Smith [37]). Let $\varrho \in \mathcal{R}$ be a relator and $\varsigma \in \mathcal{R}$ a (positive) relator. Suppose that there exist (positive) words $U, V, W_1, W_2 \in \mathcal{F}$ such that $\varrho = W_1^{-1}W_2$ and $\varsigma = UW_1V$. Then we can construct a new (positive) relator $\varsigma' = \varsigma V^{-1}\varrho V = UW_2V$ in \mathcal{R} . This operation is called a ϱ -*substitution* to ς . When ς' is obtained by applying a sequence of $\varrho^{\pm 1}$ -substitutions to ς , we denote it by $\varsigma \equiv \varsigma' \pmod{\varrho}$.

Lemma 3.5. *The signature I_g for relators of \mathcal{M}_g has the following properties:*

- (1) $I_g(\varrho^{-1}) = -I_g(\varrho)$ for $\varrho \in \mathcal{R}$;
- (2) $I_g(W\varrho W^{-1}) = I_g(\varrho)$ for $\varrho \in \mathcal{R}$ and $W \in \mathcal{F}$;
- (3) $I_g(\varrho_1\varrho_2) = I_g(\varrho_1) + I_g(\varrho_2)$ for $\varrho_1, \varrho_2 \in \mathcal{R}$;
- (4) $I_g(\varsigma') = I_g(\varsigma) + I_g(\varrho)$ if ς' is obtained by applying a ϱ -substitution to ς , where $\varrho, \varsigma, \varsigma' \in \mathcal{R}$;
- (5) $I_g(\varrho) = I_g(\varrho')$ if $\varrho \in \mathcal{R}$ and $\varrho' \in \mathcal{R}$ are topologically equivalent.

Proof. (1)–(3): It is clear that $s(\varrho^{-1}) = -s(\varrho)$, $s(W\varrho W^{-1}) = s(\varrho)$ and $s(\varrho_1\varrho_2) = s(\varrho_1) + s(\varrho_2)$ for $\varrho, \varrho_1, \varrho_2 \in \mathcal{R}$ and $W \in \mathcal{F}$. It is trivial that $c_g(\varrho^{-1}) = -c_g(\varrho)$ and $c_g(\varrho_1\varrho_2) = c_g(\varrho_1) + c_g(\varrho_2)$ for $\varrho, \varrho_1, \varrho_2 \in \mathcal{R}$ because c_g is a homomorphism on \mathcal{R} . A proof of $c_g(W\varrho W^{-1}) = c_g(\varrho)$ for $\varrho \in \mathcal{R}$ and $W \in \mathcal{F}$ are found in Meyer [27], p. 253. (4): We use the notation of Definition 3.4 above. Then we have

$$I_g(\varsigma') = I_g(\varsigma) + I_g(V^{-1}\varrho V) = I_g(\varsigma) + I_g(\varrho)$$

from (2) and (3). We postpone the proof of (5) until the next subsection. \square

3.1. Braid relation. Let a, b be simple closed curves on Σ_g and put $c = t_b(a)$. The relation

$$t_c = t_b t_a t_b^{-1}$$

in \mathcal{M}_g is called the *braid relation*. If $i(a, b) = n$, we put $T_n = T(a, b) := bab^{-1}c^{-1} \in \mathcal{R}$. Especially, we denote T_2 by T_{2_0} if the algebraic intersection number of a and b is equal to 0.

Proposition 3.6. *The signature $I_g(T_n)$ of a braid relator T_n ($n = 0, 1, 2, \dots$) is equal to 0.*

Proof. We first note that $c_g : \mathcal{R} \rightarrow \mathbb{Z}$ can naturally be extended to $c_g : \mathcal{F} \rightarrow \mathbb{Z}$ and satisfies $c_g(uv) = c_g(u) + c_g(v) + \tau_g(\overline{u}, \overline{v})$ for $u, v \in \mathcal{F}$ (see Remark 2.4). Then we have

$$\begin{aligned} c_g(T_n) &= c_g(bab^{-1}c^{-1}) = c_g(bab^{-1}) + c_g(c^{-1}) + \tau_g(\overline{bab^{-1}}, \overline{c^{-1}}) \\ &= c_g(a) - c_g(c) + \tau_g(\overline{c}, \overline{c^{-1}}) \\ &= 0 - 0 + 0 = 0 \end{aligned}$$

(cf. Meyer [27], p. 253). Hence we have $I_g(T_n) = -c_g(T_n) - s(T_n) = -0 - 0 = 0$. \square

Proof of Lemma 3.5 (5). We express ϱ as a word in generators \mathcal{S} : $\varrho = c_1^{\epsilon_1} \cdots c_m^{\epsilon_m}$, where c_1, \dots, c_m are simple closed curves on Σ_g and $\epsilon_1, \dots, \epsilon_m = \pm 1$. There exists an orientation-preserving homeomorphism $h : \Sigma_g \rightarrow \Sigma_g$ such that $\varrho' =$

$h(c_1)^{\epsilon_1} \cdots h(c_m)^{\epsilon_m}$. Choose a word $W \in \mathcal{F}$ such that $\varpi(W) = h$. Applying T_n^{-1} -substitutions to each $h(c_i)$, we have $h(c_i) \equiv Wc_iW^{-1} \pmod{T_n}$. Hence we get

$$I_g(\varrho') = I_g(h(c_1)^{\epsilon_1} \cdots h(c_m)^{\epsilon_m}) = I_g(Wc_1^{\epsilon_1} \cdots c_m^{\epsilon_m} W^{-1}) = I_g(\varrho)$$

from (2), (4) and Proposition 3.6. \square

Remark 3.7. (1) For each $n = 0, 1, 2, \dots$, the topological type of T_n need not be unique. T_1 has a unique topological type because both a and b must be non-separating.

(2) If $i(a, b) = 1$, we also have the braid relation $t_b = t_a t_c t_a^{-1}$. This relation together with the original relation $t_c = t_b t_a t_b^{-1}$ yields Artin's relation $t_b = t_a t_b t_a t_b^{-1} t_a^{-1}$. If we put $T_A := abab^{-1}a^{-1}b^{-1}$, then we have $I_g(T_A) = -c_g(T_A) = -2c_g(T_1) = 0$.

3.2. Chain relation.

Definition 3.8 (cf. Johnson [21], Wajnryb [44]). An ordered n -tuple (c_1, \dots, c_n) of simple closed curves on Σ_g is called a *chain* of length n if it satisfies the next conditions:

- (i) c_i and c_{i+1} intersect transversely at one point ($i = 1, \dots, n-1$);
- (ii) $c_i \cap c_j = \emptyset$ if $|i - j| > 1$.

When the length n is even, a regular neighborhood of a chain (c_1, \dots, c_n) is a subsurface of Σ_g which is of genus $h = n/2$ and has one boundary component. We denote a simple closed curve parallel to the boundary by d . The relation

$$t_d = (t_{c_1} \cdots t_{c_{2h}})^{4h+2}$$

is called the *chain relation* of length $2h$, or the *even chain relation* (see Wajnryb [44]). We put $C_{2h} := C(c_1, \dots, c_{2h}) = (c_1 \cdots c_{2h})^{4h+2} d^{-1} \in \mathcal{R}$. The topological type of C_{2h} is unique.

Proposition 3.9. *The signature $I_g(C_{2h})$ of a chain relator C_{2h} of length $2h$ is equal to $-4h(h+1) + 1$.*

Proof. From direct computations of the signature cocycle, we have

$$\begin{aligned} c_g(C_{2h}) &= c_g((c_1 \cdots c_{2h})^{4h+2}) - c(d) + \tau_g(\bar{d}, \overline{d^{-1}}) \\ &= (2h+1)c_g((c_1 \cdots c_{2h})^2) + \sum_{i=1}^{2h-1} \tau_g(\overline{(c_1 \cdots c_{2h})^2}, \overline{(c_1 \cdots c_{2h})^{2i}}) \\ &= (2h+1)(2c_g(c_1 \cdots c_{2h}) + \tau_g(\overline{c_1 \cdots c_{2h}}, \overline{c_1 \cdots c_{2h}})) + 2h \\ &= (2h+1)(0+2h) + 2h \\ &= 4h(h+1). \end{aligned}$$

See also [12] Lemma 3.5 for similar computations. This completes the proof because $s(C_{2h}) = -1$. \square

When the length n is odd, a regular neighborhood of a chain (c_1, \dots, c_n) is a subsurface of Σ_g which is of genus $h = (n-1)/2$ and has two boundary components. We denote simple closed curves parallel to two boundary components by d_1 and d_2 . The relation

$$t_{d_1} t_{d_2} = (t_{c_1} \cdots t_{c_{2h+1}})^{2h+2}$$

is called the chain relation of length $2h+1$, or the *odd chain relation* (see Wajuryb [44]). We put $C_{2h+1} := C(c_1, \dots, c_{2h+1}) = (c_1 \cdots c_{2h+1})^{2h+2} d_1^{-1} d_2^{-1} \in \mathcal{R}$. The topological type of C_{2h+1} is not unique. We denote C_{2h+1} also by $C_{2h+1, \text{I}}$ if both d_1 and d_2 are non-separating, while we denote it by $C_{2h+1, \text{II}}$ if both d_1 and d_2 are separating.

Proposition 3.10. *The signature $I_g(C_{2h+1})$ of a chain relator C_{2h+1} of length $2h+1$ is equal to $-2h(h+2)$. In particular, it does not depend on the topological type of C_{2h+1} .*

Proof. If d_1 and d_2 are separating, then we have

$$\begin{aligned} I_g(C_{2h+1, \text{II}}) &= -c_g(C_{2h+1, \text{II}}) - s(C_{2h+1, \text{II}}) = -2(h+1)^2 - (-2) \\ &= -2h(h+2) \end{aligned}$$

from the formula (30) of Meyer [27].

If d_1 and d_2 are non-separating, we can compute the value with the help of a work by Arakawa and Ashikaga [2] or by Morifuji [30] as follows. Choose a hyperelliptic involution $\iota : \Sigma_g \rightarrow \Sigma_g$ so that the chain (c_1, \dots, c_{2h+1}) is invariant under the action of ι . Then the map $t_{d_1} t_{d_2} = (t_{c_1} \cdots t_{c_{2h+1}})^{2h+2}$ is the monodromy of a degeneration of hyperelliptic curves, whose local signature is equal to $(2h(g-h-1)-1)/(2g+1)-1$ (see [2]). According to a theorem of Terasoma [42], the value $\phi_g(t_{d_1} t_{d_2})$ of Meyer's function on $t_{d_1} t_{d_2}$ must be equal to $-(2h(g-h-1)-1)/(2g+1)$. On the other hand, we have

$$\begin{aligned} \phi_g(t_{d_1} t_{d_2}) &= \phi_g((t_{c_1} \cdots t_{c_{2h+1}})^{2h+2}) \\ &= (2h+2)\phi_g(t_{c_1} \cdots t_{c_{2h+1}}) - \sum_{i=1}^{2h+1} \tau_g(\overline{c_1 \cdots c_{2h+1}}, \overline{(c_1 \cdots c_{2h+1})^i}) \\ &= \frac{g+1}{2g+1} \cdot (2h+1)(2h+2) - \sum_{i=1}^{2h+1} \tau_g(\overline{c_1 \cdots c_{2h+1}}, \overline{(c_1 \cdots c_{2h+1})^i}) \end{aligned}$$

from direct computations. Hence we conclude that

$$\sum_{i=1}^{2h+1} \tau_g(\overline{c_1 \cdots c_{2h+1}}, \overline{(c_1 \cdots c_{2h+1})^i}) = 2h^2 + 4h + 1.$$

As a result, we obtain

$$\begin{aligned} c_g(C_{2h+1, \text{I}}) &= c_g((c_1 \cdots c_{2h+1})^{2h+2} d_1^{-1} d_2^{-1}) \\ &= c_g((c_1 \cdots c_{2h+1})^{2h+2}) - c_g(d_1) - c_g(d_2) - \tau_g(\overline{d_1}, \overline{d_2}) \\ &= (2h+2)c_g(c_1 \cdots c_{2h+1}) \\ &\quad + \sum_{i=1}^{2h+1} \tau_g(\overline{c_1 \cdots c_{2h+1}}, \overline{(c_1 \cdots c_{2h+1})^i}) - 0 - 0 - 1 \\ &= (2h+2) \cdot 0 + (2h^2 + 4h + 1) - 1 \\ &= 2h(h+2) \end{aligned}$$

and then

$$I_g(C_{2h+1, \text{I}}) = -c_g(C_{2h+1, \text{I}}) - s(C_{2h+1, \text{I}}) = -2h(h+2) - 0 = -2h(h+2).$$

We have thus proved the proposition. \square

Remark 3.11. T. Morifuji kindly told the authors that the value $\phi_g(t_{d_1}t_{d_2})$ can directly be computed by using a method similar to one developed in his paper (see [30], Remark 2.5 and Example 2.6).

3.3. Lantern relation. Let a and b be simple closed curves on Σ_g with $i(a, b) = 2$ and algebraic intersection number 0. We orient a and b locally on a neighborhood of each intersection point $p \in a \cap b$ such that the intersection number $(a \cdot b)_p$ at p is $+1$. Resolving all intersection points according to the local orientations, we obtain a new simple closed curve c . A regular neighborhood of $a \cup b$, which can be chosen to include c , is a genus-0 subsurface Σ of Σ_g with 4 boundary components. We denote simple closed curves parallel to 4 boundary components of Σ by d_1, d_2, d_3 and d_4 . The relation

$$t_a t_b t_c = t_{d_1} t_{d_2} t_{d_3} t_{d_4}$$

is called the *lantern relation*, which was discovered by Dehn and rediscovered by Johnson [21]. We put

$$L := L(a, b) = abcd_1^{-1}d_2^{-1}d_3^{-1}d_4^{-1} \in \mathcal{R}.$$

The topological type of L is not unique. We denote L also by the following symbols according to the place where L sits in Σ_g :

- L_I if all of d_1, d_2, d_3 and d_4 are non-separating and $\Sigma_g - \Sigma$ is connected;
- L_{II} if one of d_1, d_2, d_3 and d_4 is separating and the rest are non-separating;
- L_{III} if all of d_1, d_2, d_3 and d_4 are non-separating and $\Sigma_g - \Sigma$ is not connected;
- L_{IV} if two of d_1, d_2, d_3 and d_4 are separating and the rest are non-separating;
- L_V if all of d_1, d_2, d_3 and d_4 are separating.

Proposition 3.12. *The signature $I_g(L)$ of a lantern relator L is equal to $+1$. In particular, it does not depend on the topological type of L .*

Proof. From the definition of the map c_g , we have

$$\begin{aligned} c_g(L) &= \tau_g(\overline{a}, \overline{b}) + \tau_g(\overline{ab}, \overline{c}) + \tau_g(\overline{abc}, \overline{d_1^{-1}}) + \tau_g(\overline{abcd_1^{-1}}, \overline{d_2^{-1}}) \\ &\quad + \tau_g(\overline{abcd_1^{-1}d_2^{-1}}, \overline{d_3^{-1}}) + \tau_g(\overline{d_4}, \overline{d_4^{-1}}), \end{aligned}$$

where the last term $\tau_g(\overline{d_4}, \overline{d_4^{-1}})$ is obviously equal to 0. Carrying out explicit computation of τ_g , we obtain the following table:

type	$\tau_g(\overline{a}, \overline{b})$	$\tau_g(\overline{ab}, \overline{c})$	$\tau_g(\overline{abc}, \overline{d_1^{-1}})$	$\tau_g(\overline{abcd_1^{-1}}, \overline{d_2^{-1}})$	$\tau_g(\overline{abcd_1^{-1}d_2^{-1}}, \overline{d_3^{-1}})$
L_I	0	0	-1	0	0
L_{II}	0	+1	-1	0	0
L_{III}	0	0	-1	0	-1
L_{IV}	0	+1	-1	0	0
L_V	0	0	0	0	0

Thus we have $c_g(L_I) = -1$, $c_g(L_{III}) = -2$ and $c_g(L_{II}) = c_g(L_{IV}) = c_g(L_V) = 0$. It is easy to see that $s(L_I) = 0$, $s(L_{III}) = +1$ and $c_g(L_{II}) = c_g(L_{IV}) = c_g(L_V) = -1$. Therefore we have $I_g(L) = +1$ in each case $L = L_I, L_{II}, L_{III}, L_{IV}, L_V$. \square

3.4. Star relation. Let a_1, a_2, a_3 and b be simple closed curves on Σ_g with $i(a_i, a_j) = 0$ ($i \neq j$) and $i(a_i, b) = 1$ ($i = 1, 2, 3$). A regular neighborhood of $a_1 \cup a_2 \cup a_3 \cup b$ is a genus-1 subsurface of Σ_g with 3 boundary components. We denote simple closed curves parallel to 3 boundary components by d_1, d_2 and d_3 . The relation

$$t_{d_1} t_{d_2} t_{d_3} = (t_{a_1} t_{a_2} t_{a_3} t_b)^3$$

is called the *star relation*, which was discovered by Gervais [16]. We put $E := (a_1 a_2 a_3 b)^3 d_1^{-1} d_2^{-1} d_3^{-1} \in \mathcal{R}$. The topological type of E is not unique. We denote E also by the following symbols according to the place where E sits in Σ_g :

E_I if all of d_1, d_2 and d_3 are non-separating;

E_{II} if one of d_1, d_2 and d_3 is separating and the rest are non-separating;

E_{III} if all of d_1, d_2 and d_3 are separating.

Proposition 3.13. *The signature $I_g(E)$ of a star relator E is equal to $+5$. In particular, it does not depend on the topological type of E .*

Proof. It follows from the proof of Theorem 1.6 of Gervais [16] that

$$E_I \equiv C_{3,I}^{-1} L_I^{-1}, \quad E_{II} \equiv C_{3,II}^{-1} L_{III}^{-1}, \quad E_{III} \equiv C_{3,II}^{-1} L_{IV}^{-1} \pmod{T_0, T_1, T_{2_0}}.$$

We have values of c_g on E_I, E_{II} and E_{III} .

$$\begin{aligned} c_g(E_I) &= -c_g(C_{3,I}) - c_g(L_I) = -6 - (-1) = -5, \\ c_g(E_{II}) &= -c_g(C_{3,II}) - c_g(L_{III}) = -8 - (-2) = -6, \\ c_g(E_{III}) &= -c_g(C_{3,II}) - c_g(L_{IV}) = -8 - 0 = -8. \end{aligned}$$

It is easy to see that $s(E_I) = 0$, $s(E_{II}) = +1$ and $s(E_{III}) = +3$. Therefore we have $I_g(E) = +5$ in each case $E = E_I, E_{II}, E_{III}$. \square

3.5. Hyperelliptic relation. The longest chain on Σ_g is a chain (c_1, \dots, c_{2g+1}) of length $2g + 1$. The relation

$$(t_{c_1} \cdots t_{c_{2g}} t_{c_{2g+1}}^2 t_{c_{2g}} \cdots t_{c_1})^2 = 1$$

is called the *hyperelliptic relation*, which appeared in Birman-Hilden's paper [5]. We put $H_g = H(c_1, \dots, c_{2g+1}) := (c_1 \cdots c_{2g} c_{2g+1}^2 c_{2g} \cdots c_1)^2 \in \mathcal{R}$. The topological type of H_g is unique.

Proposition 3.14. *The signature $I_g(H_g)$ of a hyperelliptic relator H_g is equal to $-4(g + 1)$.*

Proof. We have

$$I_g(H_g) = -c_g(H_g) - s(H_g) = -4(g + 1) - 0 = -4(g + 1)$$

from the formula (31) of Meyer [27]. \square

The signature $I_g(\varrho)$ of a relator $\varrho \in \mathcal{R}$ is equal to that of another relator with the same topological type from Lemma 3.5. It seems to the authors that $I_g(\varrho)$ does not depend even on the topological type of ϱ for arbitrary relator $\varrho \in \mathcal{R}$ because of Theorems 3.1 and 3.2 and Propositions 3.6, 3.9, 3.10, 3.12 and 3.13.

4. NON-HOLOMORPHIC LEFSCHETZ FIBRATIONS

In this section we exhibit some known examples of Lefschetz fibrations and computations of their signatures using the function $I_g : \mathcal{R} \rightarrow \mathbb{Z}$. We also give some new examples of non-holomorphic Lefschetz fibrations over S^2 .

We first review the definition and basic properties of Lefschetz fibrations. More details can be found in Matsumoto [29] and Gompf and Stipsicz [18].

Definition 4.1. Let M and B be compact oriented smooth manifolds of dimension 4 and 2, respectively. A smooth map $f : M \rightarrow B$ is called a *Lefschetz fibration* of genus g if it satisfies the following conditions:

- (i) $\partial M = f^{-1}(\partial B)$;
- (ii) f has finitely many critical values $b_1, \dots, b_n \in \text{int} B$ and f is a smooth fiber bundle with fiber Σ_g over $B - \{b_1, \dots, b_n\}$;
- (iii) for each i ($1 \leq i \leq n$), there exists a unique critical point p_i in the *singular fiber* $F_i := f^{-1}(b_i)$ such that f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around p_i and b_i which are compatible with orientations of M and B ;
- (iv) no fibers contain a (-1) -sphere.

We take a small disk $D_i \subset B$ centered at each b_i and set $B_0 := B - (\text{int} D_1 \cup \dots \cup \text{int} D_n)$ and $M_0 := f^{-1}(B_0)$. Since $f_0 := f|_{B_0} : M_0 \rightarrow B_0$ is a smooth fiber bundle with fiber Σ_g , we consider the homomorphism

$$\chi : \pi_1(B_0) \rightarrow \pi_1(\text{BDiff}_+ \Sigma_g) \cong \pi_0(\text{Diff}_+ \Sigma_g) = \mathcal{M}_g$$

induced by the classifying map $B_0 \rightarrow \text{BDiff}_+ \Sigma_g$ of f_0 . χ is called the *holonomy homomorphism* of f_0 (cf. Morita [31]). If $g \geq 2$, there exists a one-to-one correspondence between the isomorphism classes of the bundle f_0 and the conjugacy classes of the homomorphism χ by virtue of a result of Earle and Eells [11]. Moreover, Matsumoto [29] proved that if $g \geq 2$, there exists a one-to-one correspondence between the isomorphism classes of the Lefschetz fibration $f : M \rightarrow B$ and the conjugacy classes of the homomorphism χ which sends each boundary curve to a right-handed Dehn twist along an essential simple closed curve on Σ_g .

Suppose that the base B is the 2-sphere S^2 . Let γ_i ($i = 1, \dots, n$) denote the loop consisting of ∂D_i oriented clockwise and a path connecting a point on ∂D_i to the base point $b_0 \in \text{int} B$. We choose these loops $\gamma_1, \dots, \gamma_n$ so that the composition $\gamma_1 \cdots \gamma_n$ is null-homotopic on B_0 and any two of them intersect only at b_0 . Thus we obtain a presentation

$$\pi_1(B_0) = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle$$

of $\pi_1(B_0)$. For each i , $\chi(\gamma_i)$ is known to be a right-handed Dehn twist t_{c_i} along some essential simple closed curve c_i on Σ_g . Hence we have a *positive* relation

$$t_{c_1} \cdots t_{c_n} = \chi(\gamma_1 \cdots \gamma_n) = 1 \in \mathcal{M}_g$$

or a *positive* relator $c_1 \cdots c_n \in \mathcal{R}$ associated to the Lefschetz fibration $f : M \rightarrow S^2$.

Theorem 4.2. Let $f : M \rightarrow S^2$ be a Lefschetz fibration of genus g over S^2 and $c_1 \cdots c_n \in \mathcal{R}$ a positive relator associated to f . Then the signature $\text{Sign}(M)$ of the total space M is equal to $I_g(c_1 \cdots c_n)$: namely

$$\text{Sign}(M) = I_g(c_1 \cdots c_n).$$

Proof. It is an immediate consequence of Meyer's signature formula that $\text{Sign}(M_0) = -c_g(c_1 \cdots c_n)$ because $f_0 : M_0 \rightarrow B_0$ is a fiber bundle over B_0 with fiber Σ_g (see Meyer [27], Satz 1). The signature $\text{Sign}(f^{-1}(D_i))$ of a fibered neighborhood of the singular fiber F_i is equal to 0 (resp. -1) if c_i is non-separating (resp. separating). Hence we obtain

$$\text{Sign}(M) = -c_g(c_1 \cdots c_n) - s(c_1 \cdots c_n) = I_g(c_1 \cdots c_n)$$

from the Novikov additivity. \square

Although the positive relator $c_1 \cdots c_n \in \mathcal{R}$ actually depends on a choice of a loop system $(\gamma_1, \dots, \gamma_n)$ on B_0 , its equivalence class modulo conjugations of all elements c_1, \dots, c_n by a fixed element of \mathcal{F} and *elementary transformations*

$$\begin{aligned} c_1 \cdots c_i c_{i+1} \cdots c_n &\sim c_1 \cdots c_{i+1} (c_i)_{c_{i+1}}^{-1} \cdots c_n, \\ c_1 \cdots c_i c_{i+1} \cdots c_n &\sim c_1 \cdots (c_{i+1})_{c_i} c_i \cdots c_n, \end{aligned}$$

where $(a)_b = bab^{-1}$ and $i = 1, \dots, n-1$, is uniquely determined by the isomorphism class of the Lefschetz fibration $f : M \rightarrow S^2$. Conversely, any positive relator $\varrho \in \mathcal{R}$ can be realized as a relator associated to some Lefschetz fibration over S^2 . It is worth noting that $(a)_b = bab^{-1}$ is not in \mathcal{S} but in \mathcal{F} for $a, b \in \mathcal{S}$. We think of $(a)_b$ as the element $t_b(a)$ of \mathcal{S} because of the braid relation $t_{t_b(a)} = t_b t_a t_b^{-1}$.

Precisely speaking, the elementary transformation

$$c_1 \cdots c_i c_{i+1} \cdots c_n \sim c_1 \cdots (c_{i+1})_{c_i} c_i \cdots c_n = c_1 \cdots t_{c_i}(c_{i+1}) c_i \cdots c_n$$

is the composition of inserting a *trivial relator* $c_i^{-1} c_i \in \mathcal{R}$ and applying $T(c_{i+1}, c_i)^{-1}$ -substitution.

Theorem 4.3. *Let $f : M \rightarrow S^2$, $f' : M' \rightarrow S^2$ be Lefschetz fibrations of genus g over S^2 and $\varsigma, \varsigma' \in \mathcal{R}$ positive relators associated to f, f' , respectively. Suppose that a ϱ -substitution to ς yields ς' for some relator $\varrho \in \mathcal{R}$. Then we have*

$$\text{Sign}(M') = \text{Sign}(M) + I_g(\varrho).$$

Proof. Straightforward from Theorem 4.2 and Lemma 3.5. \square

The two theorems above are generalized to those for Lefschetz fibrations over closed surfaces of arbitrary genus. As a consequence, we conclude that signatures of Lefschetz fibrations corresponding to relators obtained in §3 of [13] do not depend on topological types of relators.

Notation 4.4. We denote the signature and the Euler characteristic of a compact oriented smooth 4-manifold M by $\sigma = \text{Sign}(M)$ and e , respectively. For an almost complex closed 4-manifold M , we set $\chi_h := (\sigma + e)/4$ (the *holomorphic Euler characteristic*) and $K^2 := 3\sigma + 2e$. We define relative numerical invariants $\chi_f := \chi_h + g - 1$, $K_f^2 := K^2 + 8(g-1)$ and the *slope* $\lambda_f := K_f^2/\chi_f$ for a Lefschetz fibration $f : M \rightarrow B$ of genus g over a closed surface B . It is easily seen that $e = -4(g-1) + n$ for a Lefschetz fibration $f : M \rightarrow S^2$ of genus g with n singular fibers.

4.1. Three hyperelliptic Lefschetz fibrations. We shall begin signature computations with the following well-known Lefschetz fibrations.

Let (c_1, \dots, c_{2g+1}) be a chain of length $2g+1$ on Σ_g . Since a hyperelliptic relator $H_g = (c_1 \cdots c_{2g} c_{2g+1}^2 c_{2g} \cdots c_1)^2$ of genus g is a positive relator, we obtain

a Lefschetz fibration $M_1 \rightarrow S^2$ of genus g corresponding to H_g . A chain relator $C_{2g+1} = (c_1 \cdots c_{2g+1})^{2g+2} d_1^{-1} d_2^{-1}$ of length $2g+1$ combined with two identity relators $A(d_1) = d_1, A(d_2) = d_2$ is a positive relator $C_{2g+1} A(d_2) A(d_1) = (c_1 \cdots c_{2g+1})^{2g+2}$. A chain relator $C_{2g} = (c_1 \cdots c_{2g})^{4g+2} d^{-1}$ of length $2g$ combined with an identity relator $A(d) = d$ is a positive relator $C_{2g} A(d) = (c_1 \cdots c_{2g})^{4g+2}$. We have Lefschetz fibrations $M_2 \rightarrow S^2$ and $M_3 \rightarrow S^2$ corresponding to $C_{2g+1} A(d_2) A(d_1)$ and $C_{2g} A(d)$, respectively. By virtue of Theorem 4.2 and 4.3, signatures of M_1, M_2 and M_3 are equal to $I_g(H_g), I_g(C_{2g+1}) + 2I_g(A)$ and $I_g(C_{2g}) + I_g(A)$, respectively. Invariants of M_1, M_2 and M_3 are calculated as in the following table:

4-manifold	σ	e	χ_h	K^2
M_1	$-4(g+1)$	$4(g+2)$	1	$-4(g-1)$
M_2	$-2(g+1)^2$	$2(2g^2+g+3)$	$g(g-1)/2+1$	$2(g-1)(g-3)$
M_3	$-4g(g+1)$	$4(2g^2+1)$	g^2-g+1	$4(g-1)(g-2)$

Each of M_1, M_2 and M_3 admits a (-1) -section, then cannot be decomposed to any non-trivial fiber sum and is a simply connected non-spin 4-manifold (cf. Stipsicz [39], Theorem 1.3, Siebert and Tian [35], Proposition 3.2, and Stipsicz [40], Theorem 1.3). If $g = 1$, all of them are the 4-manifold $E(1)$, which is $CP^2 \# 9\overline{CP}^2$ equipped with an elliptic fibration. If $g = 2$, M_1, M_2 and M_3 are $CP^2 \# 13\overline{CP}^2, E(2) \# 2\overline{CP}^2$ and $H'(1)$, respectively (see Gompf and Stipsicz [18] for notation) and are studied minutely by Matsumoto [29] as Examples A, C and D. According to Ito [20], M_1 can be obtained by splitting singular fibers of a certain holomorphic fibration $f_g : M_g \rightarrow S^2$ of genus g and is diffeomorphic to $CP^2 \# (4g+5)\overline{CP}^2$ for arbitrary genus g .

A fiber sum of copies of M_1, M_2 and M_3 are isomorphic to one of $\#_F n M_1, M_2 \#_F (n-1) M_1$ and $\#_F n M_3$, where $\#_F$ stands for the fiber sum. $\#_F n M_1$ and $M_2 \#_F (n-1) M_1$ are spin if and only if g is odd and n is even. $\#_F n M_3$ is spin if and only if n is even (see Nagami [32] and Stipsicz [40]).

The slope λ_f of these three fibrations and their fiber sums are equal to $4 - 4/g$, which is known as the lower bound of the slope of relatively minimal holomorphic fibrations of curves of genus g on non-singular algebraic surfaces (see Xiao [45]).

Remark 4.5. The three Lefschetz fibrations above are all hyperelliptic in the sense of [12]. Their signature can also be computed by using the local signature for hyperelliptic fibrations (see [2], [3], [12], [35] and [36]).

4.2. Cadavid-Korkmaz's examples. Matsumoto's genus-2 Lefschetz fibration (Example B of [29]) were generalized to that of arbitrary genus g independently by Cadavid [7] and Korkmaz [25].

We have constructed a positive relator $C_{2g+1} A(d_2) A(d_1) = (c_1 \cdots c_{2g+1})^{2g+2}$ in §4.1. If we put

$$\Gamma := (c_1 c_2 \cdots c_g) \cdots (c_1 c_2 c_3) (c_1 c_2) c_1, \quad \Delta := (c_1 c_2 \cdots c_{2g+1}) \Gamma,$$

then we have $\Gamma^{-1} C_{2g+1} A(d_2) A(d_1) \Gamma \equiv \Delta^2 \pmod{T_0, T_1}$. We can take certain simple closed curves B_0, B_1, \dots, B_g to verify

$$\Delta \equiv B_0 B_1 \cdots B_g (c_g \cdots c_2 c_1)^{2g+2} \pmod{T_0, T_1}$$

(see Korkmaz [25], Figure 2 and Theorem 2.4). Consider the chain relator $C_g = C(c_1, \dots, c_g)$ of length g and let c (resp. a, b) be simple closed curve(s) parallel to

the boundary of a regular neighborhood of C_g in Σ_g when g is even (resp. odd). Applying C_g^{-1} -substitutions to $B_0B_1 \cdots B_g(c_g \cdots c_2c_1)^{2g+2}$, we obtain

$$\begin{aligned}\Delta C_g^{-1} &\equiv B_0B_1 \cdots B_g c \pmod{T_0, T_1} \quad (g : \text{even}), \\ \Delta C_g^{-2} &\equiv B_0B_1 \cdots B_g a^2 b^2 \pmod{T_0, T_1} \quad (g : \text{odd}).\end{aligned}$$

Therefore we have a positive relator

$$CK_g := \begin{cases} (B_0B_1 \cdots B_g c)^2 & (g : \text{even}) \\ (B_0B_1 \cdots B_g a^2 b^2)^2 & (g : \text{odd}) \end{cases} \in \mathcal{R}$$

and the corresponding Lefschetz fibration $M^{CK} \rightarrow S^2$ of genus g .

The signature of M^{CK} is easily computed from signatures of chain relators by virtue of Theorem 4.2 and 4.3:

$$\begin{aligned}\text{Sign}(M^{CK}) &= \begin{cases} I_g(C_{2g+1}) + 2I_g(A) - 2I_g(C_g) & (g : \text{even}), \\ I_g(C_{2g+1}) + 2I_g(A) - 4I_g(C_g) & (g : \text{odd}), \end{cases} \\ &= \begin{cases} -4 & (g : \text{even}), \\ -8 & (g : \text{odd}). \end{cases}\end{aligned}$$

Invariants of M^{CK} are calculated as in the following table:

4-manifold	σ	e	χ_h	K^2	λ_f
$M^{CK} (g : \text{even})$	-4	$8 - 2g$	$1 - g/2$	$4 - 4g$	$8(g - 1)/g$
$M^{CK} (g : \text{odd})$	-8	$14 - 2g$	$(3 - g)/2$	$4 - 4g$	$8(g - 1)/(g + 1)$

The Lefschetz fibration M^{CK} admits a (-1) -section and cannot be decomposed to a non-trivial fiber sum. M^{CK} is known to be diffeomorphic to $\Sigma_{g/2} \times S^2 \# 4\overline{CP}^2$ (resp. $\Sigma_{(g-1)/2} \times S^2 \# 8\overline{CP}^2$) if g is even (resp. odd) (see Matsumoto [29], Stipsicz [40] and Korkmaz [25]). $M^{CK} \rightarrow S^2$ is hyperelliptic if g is even while it neither is it if g is odd.

4.3. A generalization of Fuller's construction. Fuller gave an example of non-hyperelliptic Lefschetz fibration of genus 3 (see Ozbagci [33] and Smith [37]). We generalize his example to that of arbitrary genus g and show that some of them are not isomorphic to any holomorphic Lefschetz fibration.

We construct an “odd subchain part” in a longer chain relator using braid relations and apply C -substitutions to it in order to destroy hyperellipticity.

We first need to show the following lemma.

Lemma 4.6. *Let (c_1, \dots, c_n) be a chain of length n on Σ_g . Then we have the following equivalence:*

$$\begin{aligned}(c_1c_2 \cdots c_n)^{k+1} &\equiv (c_1c_2 \cdots c_k)^{k+1} (c_{k+1}c_k \cdots c_2c_1)(c_{k+2}c_{k+1} \cdots c_3c_2) \\ &\quad \cdots (c_nc_{n-1} \cdots c_{n-k+1}c_{n-k}) \pmod{T_0, T_1}\end{aligned}$$

for $k = 1, \dots, n - 1$.

Proof. Fix an integer k such that $1 \leq k \leq n - 1$. We prove the next equivalence by induction on i ,

$$\begin{aligned}(c_1c_2 \cdots c_n)^{k+1} &\equiv (c_1c_2 \cdots c_k)^i (c_{k+1}c_k \cdots c_{k-i+2})(c_{k+2}c_{k+1} \cdots c_{k-i+3}) \\ &\quad \cdots (c_nc_{n-1} \cdots c_{n-i+1})(c_1c_2 \cdots c_n)^{k-i+1} \pmod{T_0, T_1}\end{aligned}$$

for $i = 1, \dots, k+1$. It is trivial for $i = 1$. Suppose that it is verified for i . We use

$$\begin{aligned}
 & (c_{k+1}c_k \cdots c_{k-i+1})(c_{k+2}c_{k+1} \cdots c_{k-i+2}) \cdots (c_n c_{n-1} \cdots c_{n-i}) \cdot c_j \\
 (*) \quad & \equiv c_{j+k-n} \cdot (c_{k+1}c_k \cdots c_{k-i+1})(c_{k+2}c_{k+1} \cdots c_{k-i+2}) \cdots (c_n c_{n-1} \cdots c_{n-i}) \\
 & \pmod{T_0, T_1}
 \end{aligned}$$

for $j = n - i + 1, \dots, n$ (see Korkmaz [25], Lemma 2.1) to show

$$\begin{aligned}
 & (c_{k+1}c_k \cdots c_{k-i+2})(c_{k+2}c_{k+1} \cdots c_{k-i+3}) \cdots (c_n c_{n-1} \cdots c_{n-i+1})c_1c_2 \cdots c_n \\
 & \equiv c_1c_2 \cdots c_{k-i}(c_{k+1}c_k \cdots c_{k-i+2})(c_{k+2}c_{k+1} \cdots c_{k-i+3}) \cdots (c_n c_{n-1} \cdots c_{n-i+1}) \\
 & \quad \cdot c_{k-i+1}c_{k-i+2} \cdots c_n \\
 & \equiv c_1c_2 \cdots c_{k-i}(c_{k+1}c_k \cdots c_{k-i+1})(c_{k+2}c_{k+1} \cdots c_{k-i+2}) \cdots (c_n c_{n-1} \cdots c_{n-i}) \\
 & \quad \cdot c_{n-i+1}c_{n-i+2} \cdots c_n \\
 & \equiv c_1c_2 \cdots c_{k-i+1}(c_{k+1}c_k \cdots c_{k-i+1})(c_{k+2}c_{k+1} \cdots c_{k-i+2}) \cdots (c_n c_{n-1} \cdots c_{n-i}) \\
 (*) \quad & \quad \cdot c_{n-i+2}c_{n-i+3} \cdots c_n \\
 & \equiv \cdots \\
 (*) \quad & \equiv c_1c_2 \cdots c_k(c_{k+1}c_k \cdots c_{k-i+1})(c_{k+2}c_{k+1} \cdots c_{k-i+2}) \cdots (c_n c_{n-1} \cdots c_{n-i}) \\
 (*) \quad & \pmod{T_0, T_1}.
 \end{aligned}$$

Then we obtain

$$\begin{aligned}
 (c_1c_2 \cdots c_n)^{k+1} & \equiv (c_1c_2 \cdots c_k)^{i+1}(c_{k+1}c_k \cdots c_{k-i+1})(c_{k+2}c_{k+1} \cdots c_{k-i+2}) \\
 & \quad \cdots (c_n c_{n-1} \cdots c_{n-i})(c_1c_2 \cdots c_n)^{k-i} \pmod{T_0, T_1}.
 \end{aligned}$$

Thus we have proven the given equivalence. \square

We have obtained positive relators $C_{2g+1}A(d_2)A(d_1) = (c_1 \cdots c_{2g+1})^{2g+2}$ and $C_{2g}A(d) = (c_1 \cdots c_{2g})^{4g+2}$ in §4.1.

For $h = 0, \dots, g-1$, we set $n = 2g+1, 2g, k = 2h+1$ in Lemma 4.6. Then we have

$$\begin{aligned}
 C_{2g+1}A(d_2)A(d_1) & \equiv (c_1c_2 \cdots c_{2h+1})^{2h+2}(c_{2h+2}c_{2h+1} \cdots c_2c_1)(c_{2h+3}c_{2h+2} \cdots c_3c_2) \\
 & \quad \cdots (c_{2g+1}c_{2g} \cdots c_{2g-2h+1}c_{2g-2h})(c_1c_2 \cdots c_{2g+1})^{2g-2h}, \\
 C_{2g}A(d) & \equiv (c_1c_2 \cdots c_{2h+1})^{2h+2}(c_{2h+2}c_{2h+1} \cdots c_2c_1)(c_{2h+3}c_{2h+2} \cdots c_3c_2) \\
 & \quad \cdots (c_{2g}c_{2g-1} \cdots c_{2g-2h}c_{2g-2h-1})(c_1c_2 \cdots c_{2g})^{4g-2h} \\
 & \pmod{T_0, T_1}.
 \end{aligned}$$

Applying C_{2h+1}^{-1} -substitutions to words in right-hand sides, we get positive relators

$$\begin{aligned}
 F_h^{\text{odd}} & := d_{h,1}d_{h,2}(c_{2h+2}c_{2h+1} \cdots c_2c_1)(c_{2h+3}c_{2h+2} \cdots c_3c_2) \\
 & \quad \cdots (c_{2g+1}c_{2g} \cdots c_{2g-2h+1}c_{2g-2h})(c_1c_2 \cdots c_{2g+1})^{2g-2h}, \\
 F_h^{\text{even}} & := d_{h,1}d_{h,2}(c_{2h+2}c_{2h+1} \cdots c_2c_1)(c_{2h+3}c_{2h+2} \cdots c_3c_2) \\
 & \quad \cdots (c_{2g}c_{2g-1} \cdots c_{2g-2h}c_{2g-2h-1})(c_1c_2 \cdots c_{2g})^{4g-2h},
 \end{aligned}$$

where $d_{h,1}$ and $d_{h,2}$ are simple closed curves parallel to the boundary of a regular neighborhood of the chain (c_1, \dots, c_{2h+1}) .

Let M_h^{odd} and M_h^{even} be Lefschetz fibrations of genus g over S^2 corresponding to positive relators F_h^{odd} and F_h^{even} , respectively. Signatures of M_h^{odd} and M_h^{even} are easily computed from signatures of chain relators by virtue of Theorem 4.2 and 4.3:

$$\begin{aligned}\text{Sign}(M_h^{\text{odd}}) &= I_g(C_{2g+1}) + 2I_g(A) - I_g(C_{2h+1}) \\ &= -2(g+1)^2 + 2h(h+2), \\ \text{Sign}(M_h^{\text{even}}) &= I_g(C_{2g}) + I_g(A) - I_g(C_{2h+1}) \\ &= -4g(g+1) + 2h(h+2).\end{aligned}$$

M_h^{odd} and M_h^{even} are simply connected non-spin 4-manifolds. Other invariants of them are calculated as in the following table:

4-manifold	e	λ_f
M_h^{odd}	$2(2g^2 + g - 2h^2 - 3h + 3)$	$4(g^2 - h^2 - 1)/(g(g+1) - h(h+1))$
M_h^{even}	$2(4g^2 - 2h^2 - 3h + 2)$	$4(2g^2 - 2g - h^2)/(2g^2 - h(h+1))$

Proposition 4.7. *Lefschetz fibrations M_1^{odd} and M_1^{even} of genus 3, Lefschetz fibrations $M_1^{\text{odd}}, M_2^{\text{odd}}, M_1^{\text{even}}$ and M_2^{even} of genus 4 and Lefschetz fibrations $M_1^{\text{odd}}, M_2^{\text{odd}}, M_3^{\text{odd}}, M_1^{\text{even}}, M_2^{\text{even}}$ and M_3^{even} of genus 5 are not isotopic to any holomorphic Lefschetz fibration over \mathbb{CP}^1 .*

Proof. Lefschetz fibrations exhibited above are not hyperelliptic in the sense of [12] because they should have fractional signature from the local signature formula if they were hyperelliptic. If a non-hyperelliptic Lefschetz fibration is isotopic to a holomorphic fibration, the slope λ_f must be greater than or equal to 3 (resp. $24/7, 40/11$) when $g = 3$ (resp. $g = 4, 5$) by virtue of theorems of Konno [23], [24] and Chen [8]. On the other hand, the slope λ_f is equal to $14/5$ and $11/4$ for M_1^{odd} and M_1^{even} of genus 3, to $28/9, 22/7, 46/15$ and $40/13$ for $M_1^{\text{odd}}, M_2^{\text{odd}}, M_1^{\text{even}}$ and M_2^{even} of genus 4 and $23/7, 10/3, 10/3, 13/4, 36/11$ and $62/19$ for $M_1^{\text{odd}}, M_2^{\text{odd}}, M_3^{\text{odd}}, M_1^{\text{even}}, M_2^{\text{even}}$ and M_3^{even} of genus 5, respectively. This completes the proof. \square

For every $g \geq 3$ and $h = 0, \dots, g-1$, M_h^{odd} and M_h^{even} satisfies inequalities $4 - 4/g \leq \lambda_f \leq 4$. The equality $\lambda_f = 4 - 4/g$ holds if and only if $h = g-1$, which is the only case that M_h^{odd} and M_h^{even} are hyperelliptic (see Xiao [45] and Konno [24]).

Remark 4.8. The Lefschetz fibration $M_1^{\text{even}} \rightarrow S^2$ of genus 3 is nothing but Fuller's original example. Smith [37] deduced an inequality equivalent to $\lambda_f \geq 3$ from study of divisors in moduli space and proved non-holomorphicity for this fibration.

4.4. Lefschetz fibrations coming from lantern relations. The lantern relation is essentially non-hyperelliptic. We introduce new type examples of non-hyperelliptic Lefschetz fibrations coming from lantern relations.

We first construct an example peculiar to genus 3. The idea of this example is to construct a “lantern part” in a chain of length 7. We recall the positive relator $C_7A(d_2)A(d_1) = (c_1 \cdots c_7)^8$ on the genus-3 surface Σ_3 obtained in §4.1.

Lemma 4.9. *For a chain (c_1, \dots, c_7) of length 7 on Σ_3 , we have the following equivalence:*

$$\begin{aligned} & (c_1 c_2 c_3 c_4 c_5 c_6 c_7)^4 \\ & \equiv c_4 \cdot c_1 c_2 c_3 c_4 c_5 \cdot c_1 c_2 c_3 c_4 c_5 c_6 \cdot c_1 c_3 c_5 c_7 \cdot c_2 c_3 c_4 c_5 c_6 c_7 \cdot c_3 c_4 c_5 c_6 c_7 \cdot c_4 \\ & \quad (\text{mod } T_0, T_1). \end{aligned}$$

Proof. We apply T_0 - and T_1 -substitutions to the left-hand side repeatedly:

$$\begin{aligned} & (c_1 c_2 c_3 c_4 c_5 c_6 c_7)^4 \\ & = c_1 c_2 c_3 c_4 c_5 c_6 c_7 \cdot c_1 c_2 c_3 c_4 c_5 c_6 c_7 \cdot c_1 c_2 c_3 c_4 c_5 c_6 c_7 \cdot c_1 c_2 c_3 c_4 c_5 c_6 c_7 \\ & \equiv c_1 c_2 c_3 c_4 c_5 c_6 \cdot c_1 c_2 c_3 c_4 c_5 \cdot c_7 c_6 c_7 \cdot c_1 c_2 c_1 \cdot c_3 c_4 c_5 c_6 c_7 \cdot c_2 c_3 c_4 c_5 c_6 c_7 \\ & \equiv c_1 c_2 c_3 c_4 c_5 \cdot c_1 c_2 c_3 c_4 \cdot c_6 c_5 c_6 \cdot c_7 c_6 \cdot c_2 c_1 \cdot c_2 c_3 c_2 \cdot c_4 c_5 c_6 c_7 \cdot c_3 c_4 c_5 c_6 c_7 \\ & \equiv c_1 c_2 c_3 c_4 c_5 \cdot c_1 \cdot c_2 c_3 c_2 \cdot c_4 c_5 c_6 \cdot c_5 c_7 \cdot c_1 c_3 \cdot c_2 c_3 c_4 \cdot c_6 c_5 c_6 \cdot c_7 \cdot c_3 c_4 c_5 c_6 c_7 \\ & \equiv c_1 c_2 \cdot c_4 c_3 c_4 \cdot c_5 \cdot c_1 c_2 c_3 c_4 c_5 c_6 \cdot c_1 c_3 c_5 c_7 \cdot c_2 c_3 c_4 c_5 c_6 c_7 \cdot c_3 \cdot c_5 c_4 c_5 \cdot c_6 c_7 \\ & \equiv c_4 \cdot c_1 c_2 c_3 c_4 c_5 \cdot c_1 c_2 c_3 c_4 c_5 c_6 \cdot c_1 c_3 c_5 c_7 \cdot c_2 c_3 c_4 c_5 c_6 c_7 \cdot c_3 c_4 c_5 c_6 c_7 \cdot c_4 \\ & \quad (\text{mod } T_0, T_1). \end{aligned}$$

Thus the lemma has been proven. \square

The complement of an open regular neighborhood of the curves c_1, c_3, c_5, c_7 is homeomorphic to two copies of $\Sigma_{0,4}$. Take one of two components and draw simple closed curves a, b and c on it as in §3.2 so that d_1, d_2, d_3 and d_4 coincide with c_1, c_3, c_5 and c_7 . Put $L := abcc_1^{-1}c_3^{-1}c_5^{-1}c_7^{-1} \in \mathcal{R}$. Applying L -substitution to the right-hand side of the equivalence in Lemma 4.9, we obtain a positive relator

$$U := c_4 \cdot c_1 \cdots c_5 \cdot c_1 \cdots c_6 \cdot abc \cdot c_2 \cdots c_7 \cdot c_3 \cdots c_7 \cdot c_4 (c_1 \cdots c_7)^4$$

from the positive relator $C_7 A(d_2) A(d_1) = (c_1 \cdots c_7)^8$ and the corresponding Lefschetz fibration $M^U \rightarrow S^2$ of genus 3.

We next treat both chain and lantern relations to construct another example. The idea of this construction is to connect two chains using a lantern as a connector.

Lemma 4.10. *Let (c_1, \dots, c_n) be a chain of length n on Σ_g . Then we have the following equivalence:*

$$\begin{aligned} (c_1 c_2 \cdots c_n)^{k+1} & \equiv (c_{k+1} c_k \cdots c_1) (c_{k+2} c_{k+1} \cdots c_2) \cdots (c_n c_{n-1} \cdots c_{n-k}) \\ & \quad \cdot (c_{n-k+1} c_{n-k+2} \cdots c_n)^{k+1} \quad (\text{mod } T_0, T_1) \end{aligned}$$

for $k = 0, \dots, n-1$.

Proof. Similar to the proof of Lemma 4.6. \square

Let (c_1, \dots, c_{2g+1}) be a chain of length $2g+1$ on Σ_g and $C_{2g} A := (c_1 c_2 \cdots c_{2g})^{4g+2}$ and $\bar{C}_{2g} A := (c_2 c_3 \cdots c_{2g+1})^{4g+2}$ positive relators defined as in §4.1. For $h = 2, \dots, g-2$, we have constructed the positive relator

$$\begin{aligned} F_{h-1}^{\text{even}} & := d_{h-1,1} d_{h-1,2} (c_{2h} c_{2h-1} \cdots c_{2h-2} c_{2h-1}) (c_{2h+1} c_{2h} \cdots c_{2h-1} c_{2h}) \\ & \quad \cdots (c_{2g} c_{2g-1} \cdots c_{2g-2h+2} c_{2g-2h+1}) (c_1 c_2 \cdots c_{2g})^{4g-2h+2} \end{aligned}$$

from $C_{2g}A$ using C_{2h-1}^{-1} -substitution in §4.3. We apply Lemma 4.10 to $\bar{C}_{2g}A$ and obtain the following equivalence:

$$\begin{aligned}\bar{C}_{2g}A &\equiv (c_2c_3 \cdots c_{2g+1})^{2g+2h+2}(c_{2g-2h+1} \cdots c_3c_2)(c_{2g-2h+2} \cdots c_4c_3) \\ &\quad \cdots (c_{2g+1} \cdots c_{2h+3}c_{2h+2})(c_{2h+3}c_{2h+4} \cdots c_{2g+1})^{2g-2h} \\ &\quad (\text{mod } T_0, T_1).\end{aligned}$$

Let $d_{h,1}, d_{h,2}$ be simple closed curves parallel to the boundary of a regular neighborhood of the chain $(c_{2h+3}c_{2h+4} \cdots c_{2g+1})$ of length $2g - 2h - 1$. Put

$$\bar{C}_{2g-2h-1} := (c_{2h+3}c_{2h+4} \cdots c_{2g+1})^{2g-2h}d_{h,1}^{-1}d_{h,2}^{-1} \in \mathcal{R}.$$

We apply $\bar{C}_{2g-2h-1}^{-1}$ -substitution to the right-hand side of the equivalence above and get a positive relator

$$\begin{aligned}\bar{F}_{g-h-1}^{\text{even}} &:= (c_2c_3 \cdots c_{2g+1})^{2g+2h+2}(c_{2g-2h+1} \cdots c_3c_2)(c_{2g-2h+2} \cdots c_4c_3) \\ &\quad \cdots (c_{2g+1} \cdots c_{2h+3}c_{2h+2})d_{h,1}d_{h,2}.\end{aligned}$$

Since $d_{h-1,1}, d_{h-1,2}, d_{h,1}$ and $d_{h,2}$ bound a subsurface of Σ_g homeomorphic to $\Sigma_{0,4}$, we can draw simple closed curves \bar{a}_h, \bar{b}_h and \bar{c}_h on it as in §3.2 so that

$$L_h := \bar{a}_h\bar{b}_h\bar{c}_hd_{h-1,1}^{-1}d_{h-1,2}^{-1}d_{h,1}^{-1}d_{h,2}^{-1}$$

is a lantern relator. Then we apply L_h -substitution to $\bar{F}_{g-h-1}^{\text{even}}F_{h-1}^{\text{even}}$ and obtain a positive relator

$$\begin{aligned}V_h &:= (c_2c_3 \cdots c_{2g+1})^{2g+2h+2}(c_{2g-2h+1} \cdots c_3c_2)(c_{2g-2h+2} \cdots c_4c_3) \\ &\quad \cdots (c_{2g+1} \cdots c_{2h+3}c_{2h+2}) \cdot \bar{a}_h\bar{b}_h\bar{c}_h \\ &\quad \cdot (c_{2h}c_{2h-1} \cdots c_{2g+1})(c_{2h+1}c_{2h} \cdots c_3c_2) \\ &\quad \cdots (c_{2g}c_{2g-1} \cdots c_{2g-2h+2}c_{2g-2h+1})(c_1c_2 \cdots c_{2g})^{4g-2h+2}\end{aligned}$$

and the corresponding Lefschetz fibration $M_h^V \rightarrow S^2$ of genus g .

Signatures of M^U and M_h^V are computed from signatures of chain and lantern relators by virtue of Theorems 4.2 and 4.3:

$$\begin{aligned}\text{Sign}(M^U) &= I_g(C_7) + 2I_g(A) + I_g(L) \\ &= -31, \\ \text{Sign}(M_h^V) &= 2I_g(C_{2g}) + 2I_g(A) - I_g(C_{2h-1}) - I_g(C_{2g-2h-1}) + I_g(L) \\ &= -6g^2 - 8g - 4gh + 4h^2 - 3.\end{aligned}$$

The Euler characteristic and the slope of M^U are 47 and $17/6$, those of M_2^V of $g = 4$ are 255 and $31/10$, and those of M_2^V of $g = 5$ are 393 and $151/46$, respectively. Thus we have the next proposition.

Proposition 4.11. *The Lefschetz fibration $M^U \rightarrow S^2$ of genus 3 and the Lefschetz fibration $M_2^V \rightarrow S^2$ of genus 4 and 5 are not isotopic to any holomorphic Lefschetz fibration over \mathbb{CP}^1 .*

4.5. Concluding remarks. We can generalize methods in §4.3 and §4.4. For example, it is possible to construct two or more “subchain parts” in a chain and apply substitutions twice or more. It is also possible to carry out the construction in §4.4 starting with two chains of length $2g + 1$ instead of two chains of length $2g$, or with $n + 1$ chains and n lanterns instead of two chains and a lantern. As a result, we have more non-holomorphic Lefschetz fibrations of genus 3, 4 and 5.

Many of the known examples of non-holomorphic Lefschetz fibrations are “non-complex” because they do not satisfy the Noether inequality though minimal or they cannot be found in the table of the Enriques-Kodaira classification of compact complex surfaces. For example, Fintushel and Stern [14] gave an example of minimal symplectic Lefschetz fibrations of genus g with $K^2 = g - 2$ and $\chi_h = g + 1$. A computation of the Seiberg-Witten invariants shows the minimality of their examples.

All examples exhibited in §4 satisfy the slope inequality $\lambda_f \geq 4 - 4/g$ due to Xiao [45], but some of them violate lower bounds of the slope for non-hyperelliptic holomorphic fibrations of genus 3, 4 and 5. We began with a typical hyperelliptic Lefschetz fibration, which is on the slope bound $4 - 4/g$, and then destroyed its hyperellipticity not to increase the slope so much. Thus we obtained examples of non-holomorphic Lefschetz fibration without proving their minimality. However, we could not achieve decreasing the slope of fibrations M_1, M_2 and M_3 by the substitution technique. All examples known to the authors have the slope greater than or equal to $4 - 4/g$. For example, the above example due to Fintushel and Stern has the slope $9/2 - 5/g$. We shall restate the following conjecture in terms of the slope.

Conjecture 4.12 (Hain, cf. [1], Question 5.10). *The slope inequality $\lambda_f \geq 4 - 4/g$ holds for every smooth Lefschetz fibration $f : M \rightarrow S^2$ of genus $g \geq 2$ over the 2-sphere.*

We do not know if our examples having slope greater than lower bounds given by Konno [23], [24] and Chen [8] are holomorphic.

APPENDIX A. TWO DEFINITIONS OF THE SIGNATURE COCYCLE

We review two definitions of the signature cocycle due to Meyer [27] and Turaev [43] and prove that they coincide.

Meyer [27] discovered the signature cocycle and gave an explicit description as follows.

Definition A.1 (Meyer [27]). Let $A, B \in Sp(2g, \mathbb{Z})$ be symplectic matrices of rank $2g$. We consider the subspace

$$V_{A,B} := \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\}$$

of the real vector space $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$, where I_{2g} is the identity matrix. The bilinear form $\langle \cdot, \cdot \rangle_{A,B} : (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \times (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \rightarrow \mathbb{R}$ is defined by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} := (x_1 + y_1) \cdot J(I_{2g} - B)y_2,$$

where \cdot is the inner product of \mathbb{R}^{2g} and J is the matrix representing the multiplication by $\sqrt{-1}$ on $\mathbb{R}^{2g} = \mathbb{C}^g$. It is easy to see that the restriction of $\langle \cdot, \cdot \rangle_{A,B}$ on $V_{A,B}$ is symmetric. Then we define

$$\tau_g(A, B) := \text{sign}(V_{A,B}, \langle \cdot, \cdot \rangle_{A,B}).$$

τ_g turns out to be a 2-cocycle of $Sp(2g, \mathbb{Z})$ and called the *signature cocycle* of genus g .

Turaev [43] rediscovered (independently) the cocycle above.

Definition A.2 (Turaev [43]). Let $A, B \in Sp(2g, \mathbb{Z})$ be symplectic matrices of rank $2g$. We consider the subspace

$$W_{A,B} := (A - I_{2g})\mathbb{R}^{2g} \cap (B - I_{2g})\mathbb{R}^{2g}$$

of the real vector space \mathbb{R}^{2g} . The bilinear form $(\ , \)_{A,B} : W_{A,B} \times W_{A,B} \rightarrow \mathbb{R}$ is defined by

$$(\xi_1, \xi_2)_{A,B} := (u + v + \xi_1) \cdot J\xi_2,$$

where $\xi_1, \xi_2 \in W_{A,B}$ and $u, v \in \mathbb{R}^{2g}$ such that $\xi_1 = Au - u = Bv - v$. It is easy to see that $(\ , \)_{A,B}$ is well defined and symmetric. We define

$$\varphi_g(A, B) := \text{sign}(W_{A,B}, (\ , \)_{A,B}).$$

φ_g turns out to be a 2-cocycle of $Sp(2g, \mathbb{Z})$.

For $A, B \in Sp(2g, \mathbb{Z})$, we define the linear map $f_{A,B} : V_{A,B} \rightarrow W_{A,B}$ by

$$f_{A,B}(x, y) := (B - I)y \quad ((x, y) \in V_{A,B}).$$

It is easy to check that this map is well defined.

Proposition A.3. For $A, B \in Sp(2g, \mathbb{Z})$, the linear map $f_{A,B} : V_{A,B} \rightarrow W_{A,B}$ is surjective and satisfies

$$(f_{A,B}(x, y), f_{A,B}(x', y'))_{A,B} = -\langle (x, y), (x', y') \rangle_{A,B},$$

where $(x, y), (x', y') \in V_{A,B}$. In particular, $\varphi_g(A, B) = -\tau_g(A, B)$.

Proof. Let ξ be an element of $W_{A,B}$. We take $u, v \in \mathbb{R}^{2g}$ so that $\xi = Au - u = Bv - v$. This equality means

$$(A^{-1} - I)(Au) + (B - I)v = 0$$

and implies $(Au, u) \in V_{A,B}$. So we have $f_{A,B}(Au, u) = (B - I)v = \xi$.

For $(x, y), (x', y') \in V_{A,B}$, we have

$$\begin{aligned} (f_{A,B}(x, y), f_{A,B}(x', y'))_{A,B} &= ((B - I)y, (B - I)y')_{A,B} \\ &= ((A - I)^{-1}((B - I)y) + (B - I)^{-1}((B - I)y) + (B - I)y) \cdot J(B - I)y' \\ &= ((A - I)^{-1}((A - I)(A^{-1}x)) + y + By - y) \cdot J(B - I)y' \\ &= (A^{-1}x + By) \cdot J(B - I)y' = -(A^{-1}x + By) \cdot (I - B)y' \\ &= -(x + y) \cdot (I - B)y' \\ &= -\langle (x, y), (x', y') \rangle_{A,B}. \end{aligned}$$

Because the kernel of $f_{A,B}$ is included in the annihilator of $\langle \ , \ \rangle_{A,B}$, $f_{A,B}$ induces a linear isomorphism

$$\bar{f}_{A,B} : (V_{A,B}/\text{Ker } f_{A,B}, -\langle \ , \ \rangle_{A,B}) \rightarrow (W_{A,B}, (\ , \)_{A,B})$$

which preserves bilinear forms. \square

The mapping class group \mathcal{M}_g of genus g acts on $H_1(\Sigma_g; \mathbb{Z})$ preserving the intersection form. If we fix a symplectic basis on $H_1(\Sigma_g; \mathbb{Z})$, we obtain a representation $\mathcal{M}_g \rightarrow Sp(2g, \mathbb{Z})$. We call the pull-back of the signature cocycle by this representation also the signature cocycle.

Remark A.4. The signature cocycle satisfies the condition $(*)$ in Proposition 2.3 (see Meyer [27] formula (9)).

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