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THE DEGREE OF THE VARIETY OF RATIONAL RULED SURFACES AND GROMOV-WITTEN INVARIANTS

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Abstract. We compute the degree of the variety parametrizing rational ruled surfaces of degree d in $\mathbb{P}^3$ by relating the problem to Gromov-Witten invariants and Quantum cohomology.

1. Introduction

In 1986, D.F. Coray and I. Vainsencher computed the degree of certain strata of the variety parametrizing ruled cubic surfaces [CV]. Here we generalize their result and compute the degree of the variety parametrizing rational ruled surfaces of degree d in $\mathbb{P}^3$ in the projective space parametrizing all surfaces of degree d. We approach this enumerative problem by fixing a suitable parameter space for the objects that we want to count, and expressing the locus of objects satisfying given conditions as a zero-cycle on this parameter space. We then need to evaluate the degree of this zero-dimensional cycle class. This is possible in principle whenever the Chow group of cycles up to numerical equivalence of the parameter space is known in terms of generators and relations.

We use $\mathrm{Mor}(\mathbb{P}^1, G(2,4))$, the variety of morphisms from $\mathbb{P}^1$ to $G(2,4)$, as a parameter space for rational ruled surfaces of degree d. Here $G(2,4)$ denotes the Grassmannian of lines in $\mathbb{P}^3$. By the universal property of the Grassmannian, we can identify a rational ruled surface in $\mathbb{P}^3$ with a rational curve in $G(2,4)$. More precisely, the points of this scheme are parametrized ruled surfaces. The objects we intend to count are the images of these morphisms, which are rational curves of fixed degree d in the Grassmannian $G(2,4)$. To eliminate the data of the parametrization due to the action of the group $\mathrm{PGL}(1)$ on $\mathbb{P}^1$, we impose three point conditions.

The variety of morphisms of a fixed degree from $\mathbb{P}^1$ to the Grassmannian $G(2,4)$ is not a compact parameter space. The Grothendieck Quot scheme parametrizing rank 2 and degree d quotients of a trivial vector bundle $\mathcal{O}^4_{\mathbb{P}^1}$ is a compactification of this variety as shown in [Str]. This Quot scheme has very good properties, it is a fine moduli space equipped with a universal element, and it is a smooth, irreducible scheme. Unfortunately, some of the divisors we want to intersect have a common component contained in the boundary of the Quot scheme. Therefore, our intersection problem does not have enumerative meaning on the Quot scheme.

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We instead use the coarse moduli space $\overline{M}_{0,n}(G(2,4),d)$ of Kontsevich stable maps from $n-$pointed genus 0 curves to the Grassmannian $G(2,4)$ representing d times the positive generator of the homology group $H_2(G(2,4),\mathbb{Z})$. In the Kontsevich space our classes intersect properly and have enumerative meaning. Since the Grassmannian $G(2,4)$ is a homogeneous variety, transversality arguments imply a relationship between Gromov-Witten invariants and enumerative geometry. We proceed in some sense in an opposite direction to Bertram in [Ber1], where he defines the Gromov-Witten invariants and then he reinterprets them as intersections of generalized Schubert cohomology classes in a Grothendieck Quot scheme.

The quantum cohomology ring of a variety is defined in terms of intersection data (the Gromov-Witten invariants) on the spaces of holomorphic maps from pointed curves of genus zero to the variety. The quantum cohomology of $G(2,4)$, $QH^*(G(2,4))$, is an algebra over a polynomial ring in one variable. It has been described in [FI], where the authors present some of the line and conic quantum numbers. The solutions to our enumerative problem occur as coefficients in the multiplication table of $QH^*(G(2,4))$.

The associativity of the quantum cohomology ring gives many relations among the Gromov-Witten invariants and allows us to determine the invariants from a few basic ones. In this work, we have applied Farsta, a computer program due to Andrew Kresch, which computes quantum numbers using associativity relations, to find the invariants we are interested in.

Notation. A_dX and A^dX can be taken to be the Chow homology and cohomology groups for homogeneous varieties. For $\beta \in A_kX$, $\int_\beta c$ is the degree of the zero cycle obtained by evaluating c_k on β, $c \in A^*X$. We use the cup product $\cup$ for the product in A^*X. A closed subvariety T of $G(2,4)$ of pure codimension c determines classes in $A_{n-c}(G(2,4))$ and $A^c(G(2,4))$ via the duality isomorphism. Both of these classes are denoted by $[T]$. We denote the automorphism group of $\mathbb{P}^n$ by $\mathrm{PGL}(n)$.

Throughout this paper, we work over the field of complex numbers $\mathbb{C}$.

2. Rational ruled surfaces

2.1. **Some basic preliminaries.** A *rational ruled surface* is the projectivization of a locally free sheaf of rank 2 on $\mathbb{P}^1$, $Y = \mathbb{P}(E)$, together with the projection morphism $\pi : \mathbb{P}(E) \to \mathbb{P}^1$ (see V.2 of [Har]). Classically the term referred to a birational image of Y in $\mathbb{P}^3$, where the fibers of π are mapped to lines. We will describe how such an f can be obtained.

If E is isomorphic to the locally free sheaf $\mathcal{O}^2_{\mathbb{P}^1}$ in the open set $U \subset \mathbb{P}^1$, then

$$\pi^{-1}U = \mathbb{P}(E)|_U \cong \mathbb{P}(\mathcal{O}^2_{\mathbb{P}^1})\,|_U \cong \mathbb{P}^1 \times U.$$

This means that $\pi : \mathbb{P}(E) \to \mathbb{P}^1$ is a locally trivial fibration. By a theorem of Grothendieck every locally free sheaf over $\mathbb{P}^1$ decomposes as a direct sum of linear sheaves.

The degree of E is the degree of the invertible sheaf $\bigwedge^2 E$. If $E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, then

$$\bigwedge^2 E \cong \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(b) \cong \mathcal{O}_{\mathbb{P}^1}(a+b), \quad d := a+b.$$

We can suppose $a, b \geq 0$ (see V.2.2 of [Har]). In particular this implies E is generated by global sections, or equivalently E is given by a quotient of $\mathcal{O}_{\mathbb{P}^1}^{n+1}$:

$$0 \to \mathcal{N} \to \mathcal{O}_{\mathbb{P}^1}^{n+1} \to E \to 0. \tag{1}$$

Moreover, we will suppose $a, b > 0$, therefore we are excluding the cone of degree d, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$.

The sequence (1) induces the morphism $\mathbb{P}(E) \xrightarrow{i} \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{n+1})$. But $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{n+1}) \cong \mathbb{P}^n \times \mathbb{P}^1$ and by composing this morphism with the projection to the first factor, we obtain a morphism

$$f_n : \mathbb{P}(E) \to \mathbb{P}^n. \tag{2}$$

This is a projective morphism by definition, but in general it is not a birational map. We denote by X the image of f_n.

It is easy to see that the morphism f_n is associated to the linear sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ and maps fibers of π into lines. Moreover, the two projections

$$\pi_a : \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \to \mathcal{O}_{\mathbb{P}^1}(a), \tag{3}$$

$$\pi_b : \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \to \mathcal{O}_{\mathbb{P}^1}(b) \tag{4}$$

correspond to sections (II.7.12 of [Har]) which are mapped to curves C^a and C^b of degrees a and b, respectively. X can be obtained from C^a and C^b: for each point $p \in C^a$ there is a unique point $q \in C^b$ such that $\pi(p) = \pi(q)$. Then X is the union over all $p \in C^a$ of the lines $\overline{pq}$ (see §5.6 of [Fri]).

For $a = b = 1$, there are two distinct families of lines and $f_n(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)))$ is the quadric Q in $\mathbb{P}^n$. In this case, there are two different morphisms $Q \to \mathbb{P}^1$ corresponding to the two projections.

Lemma 2.1. *The morphism* (2) *is a finite morphism.*

Proof. Let us consider the intersection number

$$f^*\mathcal{O}_{\mathbb{P}^n}(1) \cdot f^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}(E)}(1)^2.$$

Let ξ denote the divisor class corresponding to $\mathcal{O}_{\mathbb{P}(E)}(1)$ in the Chow ring $A(\mathbb{P}(E))$. By the definition of the first Chern class $c_1(E) \in A^1(\mathbb{P}(E))$, we have

$$\pi^* c_0(E)\xi^2 - \pi^* c_1(E)\xi = 0,$$

and $c_0(E) = 1$. Hence,

$$\xi^2 = \pi^* c_1(E)\, \xi = deg\, E = d > 0.$$

Now from the fact that $\dim X = \dim \mathbb{P}(E) = 2$, it follows that f_n is generically finite by the fiber-dimension theorem (I. 6.3.7, [Sha]). For each $y \in \mathbb{P}^n$, the preimage $\pi_2^{-1}(y)$ is a section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{n+1}) \to \mathbb{P}^1$, and therefore a quotient $\mathcal{O}_{\mathbb{P}^1}^{n+1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}$. $\pi_2^{-1}(y)$ is either a 0-dimensional fiber or a section of $\mathbb{P}(E)$. If it were a section, we would have

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^1}^{n+1} \arrow[r, two heads] \arrow[dr] & \mathcal{O}_{\mathbb{P}^1} \\ & E \arrow[u] \end{tikzcd}$$

Since $Hom\,(E, \mathcal{O}) = H^0(E^\vee) = 0$ and $a, b > 0$, $\pi_2^{-1}(y) \subset \mathbb{P}(E)$ cannot be a section. Therefore, $\pi_2^{-1}(y)$ is a finite set of points. This shows that the morphism

is quasi-finite. Actually, it is finite (see exercise 3.11.2 of [Har]). We also conclude the following formula connecting the degree of the morphism with the degree of X:

$$deg(X) \cdot deg(f) = f^*(\mathcal{O}_{\mathbb{P}^n}(1))^2, \tag{5}$$

where $deg(f)$, is the degree of the field extension $K(f(X)) \subset K(\mathbb{P}(E))$. □

Remark 2.2. When $n = h^\circ(\mathbb{P}^1, E) = d + 2$, the surface is said to be "linearly normal". The surfaces obtained as the images of the above morphisms are all projections of the linearly normal surface. In this particular case, $deg\, X = \mathcal{O}_{\mathbb{P}(E)}(1)^2 = d$ (see §1 of [EH]). Therefore formula (5) implies that $deg(f) = 1$. Hence, $f_{d+1} : \mathbb{P}(E) \to \mathbb{P}^{d+1}$ is generically injective, that is, birational onto its image.

2.2. **A parameter space for rational ruled surfaces of degree d in $\mathbb{P}^3$.** In order to solve a specific counting problem, one of the most successful approaches is to apply the methods of intersection theory to parameter spaces. We will put the family of rational ruled surfaces in one-to-one correspondence with the points of an algebraic variety, the parameter space or moduli space.

Lemma 2.3. *There exists a bijective correspondence between the sets of isomorphism classes of rank* 2 *and degree* d *quotients on* $\mathbb{P}^1$ *of* $\mathcal{O}_{\mathbb{P}^1}^{n+1}$ *and the set of morphisms of degree* d *of* $\mathbb{P}^1$ *in the Grassmannian of* 2*-dimensional subspaces of a* $\mathbb{C}-$*vector space of dimension* $n+2$, $\mathrm{Mor}(\mathbb{P}^1, G(2, n+1))$.

Proof. Let E be a bundle of rank 2 on $\mathbb{P}^1$, with $\mathcal{O}_{\mathbb{P}^1}^{n+1} \to E \to 0$, and let us consider the universal exact sequence on the Grassmannian,

$$0 \to \mathcal{S} \to \mathcal{O}_G^{n+1} \to \mathcal{Q} \to 0. \tag{6}$$

Then by the universal property of the Grassmannian, there exists a unique morphism $\varphi : \mathbb{P}^1 \to G(2, n)$, so that

$$(\varphi^* \mathcal{O}_G^{n+1} \to \varphi^* \mathcal{Q} \to 0) = \mathcal{O}_{\mathbb{P}^1}^{n+1} \to E_{\mathbb{P}^1} \to 0.$$

The degree of φ is defined as

$$deg(\varphi) := deg(\varphi^* \mathcal{Q}).$$

The pull-back under φ of the universal quotient on the Grassmannian $\varphi^* \mathcal{Q}$ is a bundle of rank 2 on $\mathbb{P}^1$ locally isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$, where $(0 \le a \le [\frac{d}{2}])$. On the other hand $c_1(\mathcal{Q}) = \mathcal{O}_G(1)$ and $\varphi^*(c_1(\mathcal{Q})) = c_1(\varphi^*(\mathcal{Q})) = c_1(E)$, its first Chern class, therefore $deg(\varphi) = deg(E) = d$. □

Now we concentrate our attention on the case $n = 3$, i.e. quotients

$$\mathcal{O}_{\mathbb{P}^1}^4 \to E \to 0. \tag{7}$$

In this case the morphism f_3 given in Section 2.1 maps the rational ruled surface $\mathbb{P}(E)$ into $\mathbb{P}^3$ and, by the previous identification, we can see it as a curve of degree d in $G(2, 4)$.

If we fix the degree, d, and the rank, 2, of a locally free sheaf E on $\mathbb{P}^1$, we are fixing its Hilbert polynomial,

$$P(t) = \chi(E(t)) = 2t + d + 2.$$

The moduli $\mathrm{Quot}(\mathbb{P}^1, P(t))$ of quotients with fixed Hilbert polynomial $P(t)$ is a fine moduli space by a theorem of Grothendieck [Gro]. We will denote it as R_d. We observe that the quotient $\mathcal{O}_{\mathbb{P}^1}^4 \to E \to 0$ determines a point $q \in \mathrm{Quot}(\mathbb{P}^1, P(t))$ and

a morphism $f_q : \mathbb{P}^1 \to G(2,4)$ by the universal property of the Grassmannian. By definition, there is a universal quotient,

$$\mathcal{O}^4_{\text{Quot}\times\mathbb{P}^1} \to \mathcal{E}_{\text{Quot}\times\mathbb{P}^1}, \tag{8}$$

with the property that for all $k-$schemes S, the set of morphisms $f : S \to \text{Quot}$ is in one-to-one correspondence with the set of isomorphism classes of short exact sequences over $S \times \mathbb{P}^1$,

$$\mathcal{O}^4_{S\times\mathbb{P}^1} \to \mathcal{E}_{S\times\mathbb{P}^1} \to 0,$$

where $\mathcal{E}_{S\times\mathbb{P}^1}$ is flat over S with Hilbert polynomial

$$\chi(\mathcal{E}_{\{s\}}) = 2t + d + 2 \text{ on the fibers of } \pi_{\mathrm{s}} : \mathrm{S} \times \mathbb{P}^1 \to \mathrm{S}.$$

Equivalently, $\mathcal{E}_{\{s\}\times\mathbb{P}^1}$ has rank 2 and degree d for every $s \in S$. There exists an open neighborhood U of $\{q\}$ on the Quot scheme so that $\mathcal{E}_{U\times\mathbb{P}^1}$ is locally free of rank 2 and degree d.

Let R^0_d be the maximal open set where $\mathcal{E}$ is locally free and of constant rank 2; then $R^0_d \cong \mathrm{Mor}_d(\mathbb{P}^1, G(2,4))$. R_d is a natural compactification of R^0_d as shown in [Str]. In [Str] it is proved that R^0_d is a quasi-projective, smooth, rational and irreducible variety of dimension $4d+4$. This means that quotients (7) are points of the scheme R^0_d. That is, it parametrizes ruled surfaces $X \subset \mathbb{P}^3$ together with a projection map $\pi : X \to \mathbb{P}^1$ whose fibers are isomorphic to $\mathbb{P}^1$.

Lemma 2.4. *The space R^0_d is a fine moduli space for degree d maps from $\mathbb{P}^1$ to $G(2,4)$.*

Proof. Let us consider the evaluation map from $R^0_d \times \mathbb{P}^1$ to the Grassmannian $G(2,4)$, and the projection map to the first component:

$$\begin{array}{ccc} R^0_d \times \mathbb{P}^1 & \xrightarrow{\ e\ } & G(2,4) \\ {\scriptstyle \pi_1}\downarrow & & \\ R^0_d & & \end{array} \tag{9}$$

This family is a universal family. By the universal property of the Grassmannian, the pull back under e of the universal exact sequence on $G(2,4)$ (6) gives us a universal exact sequence on R^0_d, which is the restriction of the universal one on $R_d \times \mathbb{P}^1$. □

We will denote by R^{00}_d the Zariski open set in R^0_d corresponding to a morphism birational onto its image. This open set is precisely the set of automorphism-free maps.

Lemma 2.5. *If $d \geq 2$, the complement of R^{00}_d in R^0_d, the locus parametrizing multiple covers, has codimension at least 2.*

Proof. A birational morphism on R^0_d corresponds to a birational map to $\mathbb{P}^3$ (see Section 2.1 and Lemma 2.3), and consequently to a rational ruled surface of degree d in $\mathbb{P}^3$. For $d \geq 2$, formula (5) implies that the maps in the complement of R^{00}_d are the multiple covers. For $k|d$, every $k-$multiple cover factorizes as

$$\mathbb{P}^1 \xrightarrow{\rho} \mathbb{P}^1 \xrightarrow{\xi} G(2,4),$$

where ρ is a $k-$sheeted cover of $\mathbb{P}^1$ and $\xi \in \mathrm{Mor}_{d/k}(\mathbb{P}^1, G(2,4))$. Then there is a natural morphism

$$\begin{array}{ccccc} \mathrm{Mor}_k(\mathbb{P}^1,\mathbb{P}^1) & \times & \mathrm{Mor}_{d/k}(\mathbb{P}^1, G(2,4)) & \to & \mathrm{Mor}_d(\mathbb{P}^1, G(2,4)) \\ \rho & & \xi & & \rho\circ\xi. \end{array}$$

The dimension of the product is $(2k+1)+(4\frac{d}{k}+4)+3$. The fibers of this morphism, PGL(1), are of dimension 3, therefore the image has codimension

$$4d+4+3-(2k+1)-(4\frac{d}{k}+4) = 4d-4\frac{d}{k}+2-2k.$$

The study of this function for $2 \le k \le d$ shows that the maximum value must be attained at the end points $k = 2, d$. If $k = 2$,

$$4d-2d-4 = 2\,(d-1) \ge 2.$$

If $k = d$ since $d \ge 2$,

$$2d-2 = 2\,(d-1) \ge 2.$$

This shows that the complement $(R_d^{00})^c$ is of codimension at least 2. □

Remark 2.6. In addition to the problem of multiple covers, reparametrizations of the same curve in $G(2,4)$ are considered distinct objects in R_d^0. This means that the actual space for rational ruled surfaces ought to be $R_d^{00}/\mathrm{PGL}(1)$, and this is in fact a quotient in the sense of Mumford.

Proposition 2.7. *There exists a morphism from the variety R_d^{00} to the Hilbert scheme $\mathbb{P}^{\binom{d+3}{3}-1}$ of surfaces in $\mathbb{P}^3$ of degree d.*

Proof. From the universal quotient (8) we obtain a morphism

$$\mathbb{P}(\mathcal{E}_{\mathbb{P}^1\times R_d^0}) \to \mathbb{P}(\mathcal{O}^4_{\mathbb{P}^1\times R_d^0}) \cong \mathbb{P}^1\times R_d^0\times\mathbb{P}^3.$$

Projecting to the last two components, we obtain a morphism

$$\mathbb{P}(\mathcal{E}_{R_d^0\times\mathbb{P}^1}) \to R_d^0\times\mathbb{P}^3. \tag{10}$$

For all $q \in R_d^0$, we have that $\mathbb{P}(E_q) \to \{q\}\times\mathbb{P}^3$ with $E_q := \mathcal{E}_{q\times\mathbb{P}^1}$ is the rational surface corresponding to that point by the morphism constructed in Section 2.1. Although we have seen that this morphism is not always birational, the map (10) restricted to the open set $R_d^{00} \subset R_d^0$ is a birational map. Now composing with the projection morphism to the first component, we have a morphism

$$\mathbb{P}(\mathcal{E}_{R_d^{00}\times\mathbb{P}^1}) \xrightarrow{\varphi} R_d^{00}. \tag{11}$$

We now consider the image of this morphism,

$$\varphi(\mathbb{P}(\mathcal{E}_{R_d^{00}\times\mathbb{P}^1})) \overset{i}{\hookrightarrow} R_d^{00}. \tag{12}$$

We observe that for each $r \in R_d^{00}$, $i^{-1}(r) \subset \mathbb{P}^3$ is a rational ruled surface with constant Hilbert polynomial (see ex. V.1.2 of [Har]), therefore (12) is a universal family. By the universal property of the Hilbert scheme parametrizing all surfaces of degree d in $\mathbb{P}^3$, $\mathbb{P}^{\binom{d+3}{3}-1}$, there exists a unique morphism,

$$R_d^{00} \xrightarrow{\phi} \mathbb{P}^{\binom{d+3}{3}-1}, \tag{13}$$

which sends a point $q \in R_d^{00}$ to the corresponding surface $f_3(\mathbb{P}(E_q))$, where f_3 is the morphism defined in Section 2.1. □

3. The degree of the variety of rational ruled surfaces

We want to compute the **degree** of the **variety** parametrizing the family of **rational ruled surfaces** of degree d in $\mathbb{P}^3$ in the projective space of surfaces in $\mathbb{P}^3$ of degree d, $\mathbb{P}^{\binom{d+3}{3}-1}$.

We will express the locus of objects satisfying given geometric conditions as a zero-cycle on the parameter space we have fixed for rational ruled surfaces. In other words, we want to compute *the degree* N_d of $\mathrm{Im}(\phi)$ for $d \geq 3$, where ϕ is the morphism

$$R_d^{00} \xrightarrow{\phi} \mathbb{P}^{\binom{d+3}{3}-1}, \tag{1}$$

defined in Proposition 2.7.

In order to compute this degree, the divisors we have to intersect are described geometrically as sets of rational curves verifying certain incidence conditions with some Schubert varieties on the Grassmannian. For this reason, we first describe the cohomology ring of the Grassmannian. We again consider the universal sequence on the Grassmannian $G(2,4)$:

$$0 \to \mathcal{S} \to \mathcal{O}_G^4 \to \mathcal{Q} \to 0. \tag{2}$$

We will represent the special Schubert cycles on $G(2,4)$ as

$$\begin{aligned} T_1 &= c_1(\mathcal{Q}) \text{ lines meeting a given line,} \\ T_b &= c_2(\mathcal{S}) \text{ lines contained in a given plane,} \\ T_a &= c_2(\mathcal{Q}) \text{ lines containing a given point.} \end{aligned}$$

Also, $T_3 \in H_2(G(2,4),\mathbb{Z})$ will stand for the class of a line and T_4 for the class of a point.

We now give a description of the two Weil divisors on R_d^0 we are interested in. These are constructed by means of the evaluation map in (9):

(A) The locus of morphisms whose image meets an $a-$plane in the Grassmannian associated to a point $P_i \in \mathbb{P}^3$. We denote it by D_i:

$$D_i := \{\varphi \in R_d^0 |\ \ e(t,\varphi) \cap T_{a_i} \neq \emptyset\}.$$

(B) The set of morphisms Y_i sending a fixed point $t_i \in \mathbb{P}^1$ to a hyperplane T_1 on the Grassmannian:

$$Y_i := \{\varphi \in R_d^0 |\ e(t_i,\varphi) \in T_1 \text{ for a fixed } \ t_i \in \mathbb{P}^1\}.$$

Theorem 3.1. *For generic choices of $a-$planes and hyperplanes on the Grassmannian $G(2,4)$, the degree N_d coincides with the intersection of the divisors:*

$$W := Y_1 \cap Y_2 \cap Y_3 \cap D_1 \cap \ldots \cap D_{4d+1} \tag{3}$$

or equivalently, the product in the Chow ring $A(R_d^0)$ of the one cycles:

$$[Y_1][Y_2][Y_3][D_1]\ldots[D_{4d+1}]. \tag{4}$$

Although the divisors we intersect are in R_d^0, we will see in the following lemma that the intersection is in fact in the open set R_d^{00}, and it is a finite number of reduced points which coincides with the degree N_d of $\mathrm{Im}(\phi)$.

The Grassmannian of lines $G(2,4)$ is a homogeneous variety under the action of the group PGL(3).

Lemma 3.2. *The intersection W consists of a finite number of reduced points supported in the locus R_d^{00} of maps without automorphisms.*

Proof. Let $G(2,4)^{4d+4} = G(2,4)\times \overset{4d+4}{\cdots} \times G(2,4)$ be the product of $4d+4$ factors equal to $G(2,4)$. We fix three points to be $0, 1, \infty$ in $\mathbb{P}^1$, and we consider the multiple evaluation map,

$$R_d^0 \times \{0\} \times \{1\} \times \{\infty\} \times \mathbb{P}^1 \times \overset{4d+1}{\cdots} \times \mathbb{P}^1 \times \xrightarrow{\overline{e}} G(2,4)^{4d+4}. \tag{5}$$

Let Υ be the cycle on $G(2,4)^{4d+4}$ given by the product of the irreducible varieties on $G(2,4)$ associated to the divisors in (3), i.e. $W = \overline{e}^{\,-1}(\Upsilon)$.

We consider the action of the product of $4d+4$ copies of PGL(3) on $G(2,4)^{4d+4}$. Let $D_{i,j}$ be the diagonal on $G(2,4)^{4d+4}$ determined by the factors i and j. Restricting to the complement of $\bigcup_{i,j} D_{i,j}$, we get a transitive action. We apply Kleiman's Theorem [Kle] to the complement $(R_d^{00})^c$. We have seen in Lemma 2.5 that this is a closed subvariety of codimension at least 2 in R_d^0. Kleiman's Theorem applied to

$$\begin{array}{ccc} & & (R_d^{00})^c \\ & & \downarrow \\ \Upsilon & \hookrightarrow & G(2,4)^{4d+4} \backslash \{\bigcup_{i,j} D_{i,j}\} \end{array} \tag{6}$$

tells us that the intersection is supported in R_d^{00}.

The intersection W is transverse, smooth and the corresponding class in the Chow ring $A(R_d^0)$ is a top degree class. This is a consequence of the usual Bertini Lemma (see II.8.18 of [Har]). □

Proof of Theorem 3.1. For a general hyperplane H in $\mathbb{P}^{\binom{d+3}{3}-1}$, the intersection $\mathrm{Im}(\phi)\cdot H$, corresponds to the pullback $\phi^* H$.

Geometrically, if P is a general point in $\mathbb{P}^3$, the surfaces containing the point constitute a hyperplane of $\mathbb{P}^{\binom{d+3}{3}-1}$. Let T_{a_P} be the class of the codimension 2 cycle in the Grassmannian representing the set of lines containing the point, and let $[H_P]$ be the class of the hyperplane in $\mathbb{P}^{\binom{d+3}{3}-1}$ of surfaces of degree d containing P. It is clear by the way the morphism ϕ has been constructed that $\phi^* H_P = D_P$, i.e. the pull back of H_P is the divisor of rational curves meeting the a_P-plane, or equivalently the locus of parametrized rational ruled surfaces through P.

We have seen that a morphism from $\mathbb{P}^1$ to $G(2,4)$ depends on $4d+4$ parameters. If we quotient by the action of PGL(1) on $\mathbb{P}^1$, it depends on $4d+1$ parameters. Therefore we need to intersect $4d+1$ divisors of the first kind to fix a rational curve on the Grassmannian, and 3 divisors of the second kind to fix a parametrization. Now the transversality of the intersection follows from Lemma 3.2.

$$N_d = \int_{R_d^0} [D]^{4d+1} \cdot [Y]^3 = \int_{R_d^0} \phi^*[H]^3 \cdot \phi^*[H_P]^{4d+1} = d^3 \int_{\mathbb{P}^{\binom{d+3}{3}-1}} [H_P]^{4d+1} \phi_*(1),$$

where 1 is the class of the total space in the Chow ring $A(R_d^0)$. The last equality is a consequence of the projection formula. □

Corollary 3.3. *The Severi degree of the variety of rational ruled surfaces in $\mathbb{P}^3$ of degree d, for $d \geq 3$, in the projective space of surfaces of degree d, $\mathbb{P}^{\binom{d+3}{3}-1}$, coincides with the number of rational ruled surfaces through $4d+1$ points.*

In order to apply the methods of intersection theory, a compact parameter space is required together with some knowledge of its intersection ring. The Quot scheme

has been used many times as a smooth compactification of the space of morphisms of a fixed degree from $\mathbb{P}^1$ to a Grassmannian, [Ber1]. Strømme in [Str] computes the Betti numbers of R_d and gives a description with generators and relations of its cohomology ring. In particular, he gives a basis for its Picard group $A^1(R_d)$.

It must be checked that the solutions, i.e. the objects satisfying all the conditions (3), are in the dense open set R_d^0. For this purpose the next step will be to intersect the closures of the divisors with the boundary of R_d. The Weil divisors defined in R_d^0 extend to divisors in R_d. We shall denote by $\overline{D}_i$ the closures of the divisors D_i in R_d.

Proposition 3.4. *The intersection of the divisors $\overline{D}_i$ have a common excess component contained in the boundary of the Quot scheme R_d.*

Proof. Let us consider the universal exact sequence on $R_d \times \mathbb{P}^1$,

$$0 \to \mathcal{K} \xrightarrow{\varphi} \mathcal{O}^4 \to \mathcal{E} \to 0 \quad \text{on} \quad R_d \times \mathbb{P}^1.$$

$\mathcal{K}$ is a locally free sheaf of rank 2, therefore $\varphi : \mathcal{K} \to \mathcal{O}^4$ is a morphism of locally free sheaves. The intersection component is given by the points $p \in R_d$, $0 \to N \xrightarrow{\varphi_p} O^4_{\mathbb{P}^1} \xrightarrow{\bullet} E \to 0$ such that there exists $t \in \mathbb{P}^1$ where the morphism $O^4_{\mathbb{P}^1}|_t \xrightarrow{\bullet} E_t$ is an isomorphism, i.e. the points where the rank of the map $\varphi_{r,t} : \mathcal{K}_{r,t} \to \mathcal{O}^4_{r,t}$ is 0. In that case $\mathcal{E}_{\{r\}\times\mathbb{P}^1}$, is not a locally free sheaf. Rather it has a non-zero torsion of degree 2 in $t \in \mathbb{P}^1$, and this condition is satisfied trivially by all the divisors $\overline{D}_i$. These points are in the boundary of R_d (see also [Ber2]), and they are parametrized by the image of the projection $\pi : R_d \times \mathbb{P}^1 \to R_d$ of the determinantal variety,

$$\{(r,t) \in R_d \times \mathbb{P}^1 |\ rk(\varphi_{r,t,0}\}.$$

□

Remark 3.5. The intersection (3) does not have enumerative meaning on the variety R_d. We should separate the divisors $\overline{D}_i$ outside R_d^0. This could be accomplished directly by blowing up their intersection in R_d. There is a formula for the intersection numbers in the blow ups in terms of the intersection rings of the base and the center, and the normal bundle of the center (see Corollary 4.2.2 and Proposition 4.1.(a), [Ful]). The formula minimizes the amount of information needed to perform a single product in the intersection ring of a blow up. Paolo Aluffi computes in [Alu] some characteristic numbers for smooth plane curves by using this method.

The divisors Y_i are already considered by Bertram in [Ber1], where he proves a moving lemma stating that these varieties can be made to intersect transversely. If we denote by $[Y]$ the associated class in the Chow ring $A(R_d)$, then we consider the self-intersection,

$$\int_{[R_d]} [Y]^{4d+4} \cap [R_d]. \tag{7}$$

This intersection number is computed in [RRW], where it is called the degree of the generalized Plücker embedding. It is a Gromov-Witten invariant and can also be obtained by computing in the Quot scheme by means of the formulas of Vafa and Intriligator, [Ber1].

4. Interpretation of N_d as a Gromov-Witten invariant

The Kontsevich moduli space $\overline{M}_{0,n}(G(2,4),d)$ parametrizes isomorphism classes of stable maps f from nodal, $n-$pointed trees of $\mathbb{P}^1$'s $(C,p_1,\ldots,p_n)$ into $G(2,4)$ of Plücker degree d, where $f_*([C]) = dT_3$. The stability condition implies that any $\mathbb{P}^1$ contracted by f has at least three nodes or marked points. This space has dimension $4d+n+1$ and provides a compactification of the space of morphisms from $\mathbb{P}^1$ to $G(2,4)$.

The numbers N_d occur as intersection numbers on the space $\overline{M}_{0,4d+4}(G(2,4),d)$. For each marked point $1 \leq i \leq 4d+4$, there is a canonical evaluation map,

$$\pi_i : \overline{M}_{0,4d+4}(G(2,4),d) \to G(2,4), \quad \pi_i([C,p_1,\ldots,p_n,f]) = f(p_i). \tag{1}$$

A product in the ring $A^*(\overline{M}_{0,n}(G(2,4),d))$ is given by

$$\pi_1^*(\gamma_1) \cup \ldots \cup \pi_n^*(\gamma_n) \in A^*(\overline{M}_{0,n}(G(2,4),d)), \tag{2}$$

where γ_i are cycles on $G(2,4)$.

If $\sum \operatorname{codim}(\gamma_i) = \dim(\overline{M}_{0,n}(G(2,4),d))$, then *the Gromov-Witten invariant* $I_{0,n,d}(\gamma_1,\ldots,\gamma_n)$ is defined as the top degree class

$$I_{0,n,d}(\gamma_1,\ldots,\gamma_n) = \int_{\overline{M}_{0,n}(G(2,4),d)} \pi_1^*(\gamma_1) \cup \ldots \cup \pi_n^*(\gamma_n). \tag{3}$$

If $\gamma_i, \beta_j \in H^*(G(2,4),\mathbb{Z})$ are cohomology classes satisfying

$$\sum_{i=0}^{k} deg\, \gamma_i + \sum_{j=0}^{l} deg\, \beta_j = 4d+4+l, \tag{4}$$

then *the mixed invariant* $\varphi_{0,4d+4,d}\,(\gamma_1,\ldots,\gamma_k|\beta_1,\ldots,\beta_l)$ is defined as the number of maps $\mu : \mathbb{P}^1 \to G(2,4)$ in $\overline{M}_{0,4d+4}(G(2,4),d)$ such that $\mu_*[\mathbb{P}^1] = dT_3$, $\mu(p_i)$ lies in a cycle dual to γ_i and $\mu(\mathbb{P}^1)$ meets a cycle dual to β_j.

As in the previous section we fix the action of the group PGL(3) on $G(2,4)$ and we choose general a_i-planes for $i = 1,\ldots,4d+1$ and 3 general hyperplanes $j = 1,2,3$ in $G(2,4)$ with $[T_{a_i}]$ and $[T_{3_j}]$ as their cohomology classes respectively in the Chow ring $A(G(2,4))$.

Theorem 4.1. *The Gromov-Witten invariant,*

$$I_{0,4d+4,d}(T_3,T_3,T_3,T_{a_1},\ldots,T_{a_{4d+1}}),$$

coincides with the intersection number N_d $(d \geq 3)$, that is, with the number of rational curves in $G(2,4)$ of degree d which have non-empty intersection with all a_i-planes $(i = 1,\ldots,4d+1)$ and $j-$hyperplanes $(j = 1,2,3)$.

Proof. Let Y be the subscheme in $\overline{M}_{0,4d+4}(G(2,4),d)$ defined as the intersection product,

$$Y = \pi_{4d+2}^{-1}(T_3) \cap \pi_{4d+3}^{-1}(T_3) \cap \pi_{4d+4}^{-1}(T_3) \cap \pi_1^{-1}(T_{a_1}) \cap \ldots \cap \pi_{4d+1}^{-1}(T_{a_{4d+1}}). \tag{5}$$

The evaluation maps are flat by generic flatness [Alt], therefore the inverse image of a hyperplane is of codimension 1, and the inverse image of an a-plane is of codimension 2. We have seen that $\dim(\overline{M}_{0,4d+4}(G(2,4),d)) = 8d+5$ and that the codimension of the total intersection by the previous observation is $2\cdot(4d+1)+3 = 8d+5$. Y is the locus of maps $(C;P_1,\ldots,P_{4d+4};\mu)$ such that $\mu(P_i) \in T_{a_i}$ for $i = 1,\ldots,4d+1$ and $\mu(P_{4d+1+j}) \in T_{3_j}$ for $j = 1,2,3$. Since $G(2,4)$ is a homogeneous variety, the number of these maps equals $I_{0,4d+4,d}(T_3,T_3,T_3,T_{a_1},\ldots,T_{a_{4d+1}})$ (see

Lemma 7.14 of [FP]). Since the number of marked points n is ≥ 3, we can work in the variety of morphisms R_d^0 (see Lemma 2.4). If we fix three of the marked points to be $0, 1, \infty \in \mathbb{P}^1$ and let $V = \mathbb{P}^1 \times \overset{4d+1}{\cdots} \times \mathbb{P}^1 \backslash \{\triangle_{i,j}, L_{0,i}, L_{1,i}, L_{\infty,i}\}$ (where $\triangle_{i,j}$ is the large diagonal determined by factors i and j, $L_{0,i}$ is the locus where the ith factor is $0 \in \mathbb{P}^1$ and $L_{1,i}, L_{\infty,i}$ are defined similarly), there is a universal family of Kontsevich stable degree d maps of $4d+4-$pointed automorphism free curves with smooth and irreducible source,

$$\mathbb{P}^1 \times V \times R_d^{00} \to G(2,4). \tag{6}$$

By the universal property there is an injection

$$V \times R_d^{00} \hookrightarrow \overline{M}_{0,4d+4}(G(2,4), d).$$

Now arguing as in the proof of Lemma 3.2, it follows by Kleiman's Theorem that the intersection (5) is a finite set of reduced points and is supported in the open set $V \times R_d^{00}$.

Next we prove that counting stable maps is the same as counting rational curves. First we check if there is repetition, that is, the solution curves intersect any cycle in just one point. In the case of T_{3_j}, this type of repetition is unavoidable. By Bezout's theorem, a curve of degree d will always meet a codimension one space. This is just the divisor-property of the Gromov-Witten invariants. This gives us a contribution factor d^3,

$$I_{0,4d+4,d}(T_3, T_3, T_3, T_{a_1}, \ldots, T_{a_{4d+1}}) = d^3 I_{0,4d+1,d}(T_{a_1}, \ldots, T_{a_{4d+1}}). \tag{7}$$

For the cycles of codimension 2, another transversality argument (with respect to the PGL(3)-action) implies that each solution map intersects the cycle in only one point. This part is similar to the analogous result for plane curves (see Lemma 3.5.5 of [KoVa]).

The other behavior we want to exclude is when the same curve passes twice through the same point. In order to prove this, we have to see that the locus of maps μ for which $\mu^{-1}(\mu(P_i))$ contains at least one point distinct from P_i, or equivalently, the locus of maps with nodes, is of positive codimension. Let us consider the diagram

$$\begin{array}{ccc} T & \to & \triangle \\ \downarrow & & \downarrow \\ X := R_d^0 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{e} & G(2,4) \times G(2,4) \end{array} \tag{8}$$

where $\triangle$ is the diagonal in $G(2,4) \times G(2,4)$, and T is the fiber product $X \times_{G \times G} \triangle$ which is the locus of morphisms with at least one node. We observe that the action of the group PGL(3)×PGL(3) on $G(2,4) \times G(2,4)$ is not transitive, in fact there are two orbits, a dense orbit and the diagonal orbit. A slightly more general result on the transversality concerning the orbits of an action [Spe] apply to see that T has positive codimension. □

We define Q_d to be the Gromov-Witten invariant $I_{0,4d+1,d}(T_{a_1}, \ldots, T_{a_{4d+1}})$.

Corollary 4.2. *Q_d counts the number of rational ruled surfaces in $\mathbb{P}^3$ through $4d+1$ points for each degree $d \geq 3$.*

Remark 4.3. We are counting maps of degree d, $f : \mathbb{P}^1 \to G(2,4)$ verifying $f(\mathbb{P}^1) \cap T_{a_i} \neq \emptyset$ $(i = 1, \ldots, 4d+1)$ and $f(P_j) \in T_{3_j}$ $(j = 1, 2, 3)$, and the cycles verify the

assumption (4) on the dimension, so N_d is also the mixed invariant

$$\varphi_{0,4d+4,d}(T_3, T_3, T_3 | T_{a_1}, \ldots, T_{a_{4d+1}}).$$

5. Efective computation of degree and quantum cohomology

We now describe the quantum cohomology ring of $G(2,4)$, $QH^*(G(2,4))$. The basic reference for this section is [FI]. We introduce the new variable T_0 for the class of the total G(2,4).

The classical ring with unit T_0 is given by the relations

$$T_1^2 = T_a + T_b,$$
$$T_1 T_a = T_1 T_b = T_3,$$
$$T_1 T_3 = T_a^2 = T_b^2 = T_4.$$

The Gromov-Witten invariants count the numbers

$$N(\alpha, \beta, \gamma, \delta; d) = I_{0,4d+1,d}(T_a^\alpha, T_b^\beta, T_3^\gamma, T_4^\delta), \tag{9}$$

which are non-zero when $\alpha + \beta + 2\,\gamma + 3\,\delta = dim\,\overline{M}_{0,0}(G(2,4), d) = 4d + 1$, with $\alpha, \beta, \gamma,\ \delta \geq 0$.

We define the numbers $g_{ij} = \int_G T_i \cup T_j, \quad i, j \in \{1, a, b, 3, 4\}$, and (g^{ij}) for the inverse matrix. The only non-vanishing elements of the intersection form are $(g_{04} = g_{aa} = g_{bb} = g_{13} = 1)$.

$$T_i \cup T_j = \sum_{e,f} \int_G (T_i \cup T_j \cup T_e)\, g^{ef}\, T_f = \sum_{e,f} I_0(T_i, T_j, T_e)\, g^{ef} T_f. \tag{10}$$

A "quantum deformation" of the cup multiplication is defined by allowing non-zero classes.

The classical potential function is given by

$$f_{cl}(T_0, T_1, T_a, T_b, T_3, T_4) = \frac{1}{2}\, T_0\, (T_4 T_0 + T_a^2 + T_b^2) + \frac{1}{2} T_1^2 (T_a + T_b) + T_0 T_1 T_3. \tag{11}$$

The classical potential encodes the numerical values of all triple products. Now we introduce the **deformation parameters** y_0, y_1, y_a, y_b, y_3 and y_4. The genus 0 free energy F splits into $F = f_{cl} + f$ where

$$f = \sum_{\alpha,\beta,\gamma,\delta > 0\ \ \alpha+\beta+2\gamma+3\delta = 4d+1} N(\alpha, \beta, \gamma, \delta | d) \frac{y_a^\alpha}{\alpha!} \frac{y_b^\beta}{\beta!} \frac{y_3^\gamma}{\gamma!} \frac{y_4^\delta}{\delta!}\, e^{d y_1} \tag{12}$$

Quantum corrections are due to non-trivial maps $\mathbb{P}^1 \to G(2,4)$ with the necessary markings to rigidify them. The splitting of the free energy into F_{cl} and F is according to maps for which the image of $\mathbb{P}^1$ is a point or an irreducible curve.

$$F_{ijk} := \frac{\partial^3 F}{\partial\, y_i \partial\, y_j \partial\, y_k}, \quad 0 \leq i, j, k \leq m. \tag{13}$$

The **quantum product** is defined as

$$T_i * T_j = \sum_{e,f} F_{ijk} g^{\,ef} T_f, \text{ where } T_f \text{ is Poincaré dual to } T_e. \tag{14}$$

This product is extended $Q[[y]]$-linearly to the $Q[[y]]$-module $A^*G \otimes_{\mathbb{Z}} Q[[y]]$, thus making it a $Q[[y]]$-algebra. The product is commutative, since the partial derivatives are symmetric in the subscripts. It is easy to see that $t_0 = 1$ is a unit. The essential

point is the associativity, [FP], which allows us to obtain interesting relations among the quantum numbers $N(\alpha, \beta, \gamma, \delta; d)$.

There are 55 relations expressing the associativity of the deformed ring of the Grassmannian and the obvious symmetry relations $N(\alpha, \beta, \gamma, \delta|d) = N(\beta, \alpha, \gamma, \delta|d)$. Via the identification (9), the numbers Q_d are of the form $N(4d+1, 0, 0, 0|d)$. To compute these numbers we have applied Farsta, a program due to Andrew Kresch (http://www.maths.warwick.ac.uk/ kresch/).

The following table lists the number of rational ruled surfaces in $\mathbb{P}^3$ through $4d+1$ points for each degree $3 \leq d \leq 9$. In the case $d = 2$, Q_2 is twice the number of quadrics through 9 points, that is 1, since as we saw in Section 2.1 a quadric has 2 different rulings.

d	Q_d	$4d+1$
1	0	5
2	2	9
3	504	13
4	1044120	17
5	5335687360	21
6	67992124121040	25
7	1743784747544391896	29
8	82475300124495938244352	33
9	6608238869716397977928547520	37

The case $d = 3$ has been studied by Vainsencher and Coray in [CV], where he showed there are 504 rational ruled cubic surfaces passing through 13 general points or, equivalently, 504 rational curves in the Grassmannian $G(2, 4)$ meeting 13 general $a-$planes.

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References

[Alt] A. B. Altman and S. L. Kleiman. *Introduction to Grothendieck duality theory.* **146** in Lecture Notes in Mathematics. Springer-Verlag, 1970. MR0274461 (43:224)

[Alu] Paolo Aluffi, *Two characteristic numbers for smooth plane curves of any degree*, Trans. Amer. Math. Soc. **329** (1992), 73-96. MR1041042 (92e:14049)

[Ber1] A. Bertram, *Quantum Schubert Calculus*, Adv. Math. **128** (1997), no.2, 289-305. MR1454400 (98j:14067)

[Ber2] A. Bertram, *Towards a Schubert Calculus for Maps from a Riemann Surface to a Grassmannian*, Internat. J. Math. **5** (1994), no. 6, 811-825. MR1298995 (96h:14070)

[CV] D. F. Coray and I. Vainsencher, *Enumerative formulae for ruled cubic surfaces and rational quintic curves*, Comment. Math. Helvetici **61**, (1986) 501-518. MR0860137 (87m:14061)

[EH] D. Eisenbud and J. Harris, *On varieties of minimal degree*, Proc. Sympos. Pure Math. **46**, Part 1. Amer. Math. Soc., Providence, R1, 1987. MR0927946 (89f:14042)

[FI] P. di Francesco and C. Itzykson, *Quantum Intersection Rings,* in *The Moduli Space of Curves,* R. Dijkgraaf, C. Faber, and G. van der Geer, eds., Birkhäuser, 1995, 335-368. MR1363054 (96k:14041a)

[Fri] R. Friedman, *Algebraic Surfaces and holomorphic vector bundles*, Springer Verlag, 1998. MR1600388 (99c:14056)

[FP] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology-Santa Cruz*, Proc. Sympos. Pure Math. **62** (1995), 45-96. MR1492534 (98m:14025)

[Ful] W. Fulton, *Intersection theory*, Springer-Verlag, 1984. MR0732620 (85k:14004)

[Gro] A. Grothendieck *Techniques de construction et théorèmes d'existence en géometrie algebrique IV: Les schémas de Hilbert*, Séminaire Bourbaki 221 (1960/1961). MR1611822

[Har] R. Harshorne. *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977. MR0463157 (57:3116)

[Kle] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287-297. MR0360616 (50:13063)

[KoVa] J. Kock and I. Vainsencher, *Kontsevich's Formula for Rational Plane Curves*, 22nd Brazilian Mathematics Colloquium, Recife, 1999. MR1722053 (2000m:14054)

[KM] M. Kontsevich and Y. Manin, *Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry*, Commun. Math. Phys. **164** (1994) 525-562. MR1291244 (95i:14049)

[Mar] C. Martínez, *On a stratification of the moduli* $\overline{M}_{0,n}(G(2,4),d)$ *and enumerative geometry*, in preparation.

[RRW] M.S. Ravi, J. Rosenthal and X.Wang, *Degree of the generalized Plücker embedding of a quot scheme and quantum cohomology*, Math. Ann. **311** (1998), no. 1, 11-26. MR1624330 (2000e:14095)

[Spe] R. Speiser, *Transversality theorems for families of maps*, Algebraic geometry (Sundance, UT, 1986), Lecture Notes in Math., Springer, Berlin, **1311**, 1988, 235-252. MR0951649 (89k:14005)

[Sha] J.R. Shafarevich, *Basic Algebraic Geometry*, New York, 1977. MR0447223 (56:5538)

[Str] S. A. Strømme. *On parametrized rational curves in Grassmann Varieties*, Lecture Notes in Math., 1266, Springer, Berlin, 1987, 251-272. MR0908717 (88i:14020)

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