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THE PARAMETERIZED STEINER PROBLEM AND THE SINGULAR PLATEAU PROBLEM VIA ENERGY

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Abstract. The Steiner problem is the problem of finding the shortest network connecting a given set of points. By the singular Plateau Problem, we will mean the problem of finding an area-minimizing surface (or a set of surfaces adjoined so that it is homeomorphic to a 2-complex) spanning a graph. In this paper, we study the parametric versions of the Steiner problem and the singular Plateau problem by a variational method using a modified energy functional for maps. The main results are that the solutions of our one- and two-dimensional variational problems yield length and area minimizing maps respectively, i.e. we provide new methods to solve the Steiner and singular Plateau problems by the use of energy functionals. Furthermore, we show that these solutions satisfy a natural balancing condition along its singular sets. The key issue involved in the two-dimensional problem is the understanding of the moduli space of conformal structures on a 2-complex.

1. Introduction

Given a domain Ω and a boundary condition for a class of maps $\mathcal{F}$ from Ω (i.e. a map g from $\partial\Omega$ so that $f|_{\partial\Omega} = g$ for $f \in \mathcal{F}$), the problem of finding a map which minimizes certain integrals among other competitors in $\mathcal{F}$ arises naturally in geometry as well as in physics. In particular, the energy functional, the L^2 norm of the gradient of $f \in \mathcal{F}$, has been widely studied, and its minimizer is called a harmonic map. In this paper, we consider a modification of the standard energy functional to consider the parametric version of the Steiner problem of finding the shortest network connecting given points in space and the problem of finding an area minimizing surface with soap film-like singularities considered previously in geometric measure theory.

We point out two applications of the harmonic map theory important to this paper. The first is the problem of finding the shortest curve connecting two points p and q in space. Here, we consider the class of maps f from the unit interval $[0,1]$ so that $f(0) = p$ and $f(1) = q$. By finding the map f which minimizes the energy,

$$\int_0^1 \left|\frac{\partial f}{\partial t}\right|^2 dt,$$

we have found a parameterized curve whose image is a curve from p to q and which minimizes the length among all other such curves.

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The second is the problem of finding a disc-type surface which minimizes the area among all surfaces spanning a given simple closed curve. This is the classical Plateau Problem (for a reference, see [La]), and the main ingredient in the solution proposed by Douglas is the minimization of the energy functional,

$$\int_{\triangle} \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 dxdy,$$

of a map f from a disc $\triangle$. The conformal class of the unit disc Δ is represented by a monotone map $\psi : \partial\Delta \to \partial\Delta$ with $\psi(z_i) = z_i$ for three distinct fixed points $z_1, z_2, z_3 \in \partial\Delta$. Finding a minimal surface involves the following procedure: if φ is a monotone map from $\partial\Delta$ to Γ, $u_\psi : \Delta \to \mathbf{R}^n$ is the energy minimizing map with $u_\psi|_{\partial\Delta} = \varphi \circ \psi$, and $E(\psi)$ is the energy of u_ψ, then we can obtain a weakly conformal harmonic map as a limit of the sequence $\{u_{\psi_i}\}$, where $\{\psi_i\}$ is the minimizing sequence of $E(\cdot)$.

We consider two problems analogous to the examples above. In the first problem, we seek a minimal network in the spirit of Steiner, where one hopes to find a one-dimensional network of least length which connects a prescribed set Γ of points $p_1, \ldots, p_k$ in $\mathbf{R}^n$. By a one-dimensional network, we mean a subset of $\mathbf{R}^n$ which is a homeomorphic image of a graph and Γ is a subset of the image of the vertex set. In some cases, a network which has additional vertices besides $p_1, \ldots, p_n$ is shorter than the shortest network consisting of only those vertices; that is, we can shorten some networks by introducing *hubs* or *junctions* where several edges would meet. To determine the optimal locations of the hubs analytically, take a graph G with k vertices in ∂G (∂G consists of degree 1 vertices) and endow it with a metric so that each edge is isometric to a unit interval.

Now consider smooth maps α compatible with a boundary map Φ which associates each boundary point of ∂G to a unique point of Γ. Instead of directly finding the map which minimizes the image length, we consider a minimization problem involving the c-weighted energy of maps α. Here, a weight c is a non-negative valued function $e_i \mapsto c_i$ from the set of edges $\{e_i\}$ of G so that $\sum c_i = 1$. The c-weighted energy of α is sum over the edges e_i of the energy of α restricted to e_i multiplied by $1/c_i$. The minimizer of the weighted energy among (i) the space of maps α and (ii) the space of the weights c is a parameterized length minimizing network with a prescribed topological type (modulo possible collapsing of some edges).

Given a graph G and a weight c, we show the existence of a c-energy minimizer. Moreover, we show the existence of a weight c whose energy minimizer is the absolute energy minimizer, i.e. the solution to the above inf-inf problem. We then find that the absolute minimizer has the properties that, at each hub where several edges meet, the edges are balanced. In particular, when the degree of the non-boundary vertex is three, the meeting angles of the three edges are $\pi/3$, as expected of the solutions of the Steiner minimal network problem [IT]. Note that the solution to the problem of minimizing the energy is dependent on the choice of the graph G, and apriori one does not know which choice of G would produce the Steiner network. On the other hand, for the very same reason, this approach is useful when the topology of the network is part of the prescribed data, which then produces a solution previously unavailable in non-parameterized approaches (see for example [IT]).

In the second problem, we attempt to reproduce soap films, possessing singularities as investigated by Taylor [T], by constructing a map α from a two-dimensional simplicial complex X into $\mathbf{R}^n$ with a given boundary condition. For simplicity, we will restrict ourselves to a complex X which is topologically a union of three half-discs, with the three straight edges of the half-discs glued together, and call the image of this domain under α a singular surface. Let Γ be an embedded graph in $\mathbf{R}^n$ consisting of three arcs A_i sharing common endpoints q_1 and q_2 ($A_i \cap A_j = \{q_1, q_2\}$ for $i \neq j$). The boundary condition is the requirement that the map α sends the boundary of X to Γ. The image under α of the straight edges of the half-discs will be referred to as the free boundary. The approach we take here is analogous to the one-dimensional problem above; we seek a map which minimizes a certain weighted energy. The additional complication here is that there is a *Teichmüller space* of X, which consists of diffeomorphisms of the three discs which in turn defines a space of gluing functions along the three straight edges. When the domain is a single disc, the energy is minimized among maps with differing boundary parameterizations which is equivalent to varying the conformal structures of the open disc while fixing the boundary parameterization. In this paper, we introduce a weighted energy functional whose minimizer among (i) the space of the maps, (ii) the space of weights, and (iii) the space of the conformal structures of X (an inf inf inf approach) is a parameterized area minimizing (singular) surface. The minimizer of this variational problem is "balanced" along the free boundary where three surfaces which make up the singular surface meet, a result first proven by Taylor [T] in the non-parametric setting of geometric measure theory; i.e. a minimizing map finds a position of equilibrium. For a certain boundary set Γ which we call non-degenerate, we solve the variational problem by proving existence of an area minimizing map whose boundary map parameterizes Γ. As in the one-dimensional case, this yields a solution with a prescribed singular set. If Γ represents a wire frame, the soap film spanning this wire frame is modelled by this solution, and the meeting curve of the three surfaces is modelled by the singular set of this solution.

We end this introduction by mentioning some work which studies harmonic maps between spaces more general than Riemannian manifolds. Motivated by rigidity problems of discrete groups, harmonic map theory into singular spaces was initiated by the work of M. Gromov and R. Schoen [GS]. The study of harmonic maps into singular targets was further developed by N. Korevaar and R. Schoen [KS1], [KS2], [KS3] and also by J. Jost [J]. Jost also considered singular domains, defined the energy functional from measure spaces and proved existence results for energy minimizing maps. J. Chen [Ch], J. Eells-B. Fuglede [EF], B. Fuglede [F] and C. Mese [M] have studied the regularity of energy minimizing maps whose domain is a simplicial complex. In [DM], G. Daskalopoulos and C. Mese develop the theory of harmonic maps from a 2-complex and apply it to the study group actions on trees. The subject of this paper can be seen as yet another context in which harmonic maps from singular spaces play a role.

2. The one-dimensional case

Given two points p and q, let α be a C^1 map from $[0,1]$ so that $\alpha(0) = p$ and $\alpha(q) = 1$. Let $L(\alpha)$ be the length of the image curve. We may try to obtain the shortest curve by taking a minimizing sequence $\{\alpha_i\}$ of the length functional. The

problem with this approach is that the sequence $\{\alpha_i\}$ may not have any subsequences which converge in any geometric sense. So instead, we take the minimizing sequence of the energy functional

$$\int_0^1 \left|\frac{\partial \alpha}{\partial t}\right|^2 dt$$

which gives control over the parameterizations of the minimizing sequence of curves. In this section, we give an analogous approach to the Steiner problem of finding the minimal network connecting several points in $\mathbf{R}^n$.

2.1. **The variational problem.** Let

$$\mathcal{C} = \{c = (c_1, c_2, c_3) \in \mathbf{R}^3 : c_1 + c_2 + c_3 = 1,\ c_i \geq 0\ (i = 1, 2, 3)\}.$$

An element of $\mathcal{C}$ will be called a weight. Let $I = [0, 1]$, fix three points $p_1, p_2, p_3 \in \mathbf{R}^n$ and let

$$\begin{aligned}\mathcal{A} \quad = \quad & \{\alpha = (\alpha_1, \alpha_2, \alpha_3) : \alpha_i : I \to \mathbf{R}^n \in C^\infty \text{ so that } \alpha_i(0) = p_i \text{ for } i = 1, 2, 3 \\ & \text{and } \alpha_1(1) = \alpha_2(1) = \alpha_3(1)\}.\end{aligned}$$

An element of $\mathcal{A}$ will be called a map; in fact, α can be seen as a map from a tripod (the tree with three edges incident to the same vertex).

Let $c = (c_1, c_2, c_3) \in \mathcal{C}$. We say that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{A}$ is compatible with c if $\alpha_i(t)$ is constant whenever $c_i = 0$. Otherwise, we say α is incompatible with c. Furthermore, we define the c-energy of $\alpha \in \mathcal{A}$ as

$$E_c(\alpha) = \begin{cases} \displaystyle\sum^c \frac{1}{c_i} \int_0^1 |\frac{d\alpha_i}{dt}|^2 dt & \text{if } \alpha \text{ is compatible with } c, \\ \infty & \text{if } \alpha \text{ is incompatible with } c. \end{cases}$$

Here, $\sum^c$ denotes the sum over i with $c_i \neq 0$.

We consider the variational problem contained in

$$\inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{A}} E_c(\alpha) \tag{1}$$

and show that its minimizing element is the solution to the Steiner problem.

2.2. **The properties of a minimizer.** The length $L(\alpha)$ of α is the sum of the lengths of the curves defined by $\alpha_1, \alpha_2, \alpha_3$; in other words,

$$L(\alpha) = \sum \int_0^1 \left|\frac{d\alpha_i}{dt}\right| dt.$$

Lemma 1. *For every $c \in \mathcal{C}$ and $\alpha \in \mathcal{A}$, $L(\alpha) \leq (E_c(\alpha))^{1/2}$. The equality $L(\alpha) = (E_c(\alpha))^{1/2}$ is achieved if and only if $\left|\frac{\partial \alpha_i}{\partial t}\right| = l_i$ and $c_i = \frac{l_i}{\sum_{i=1}^3 l_i}$ $(i = 1, 2, 3)$.*

Proof. By the Cauchy-Schwartz inequality, as well as the fact that the energy of a parameterized curve bounds from above the square of the length, gives

$$\begin{aligned} L(\alpha) = \sum l_i = \sum^c \sqrt{c_i} \frac{l_i}{\sqrt{c_i}} &\leq \left(\sum c_i\right)^{1/2} \left(\sum^c \frac{l_i^2}{c_i}\right)^{1/2} \\ &\leq 1 \cdot \left(\sum^c \frac{1}{c_i} \int_0^1 \left|\frac{\partial \alpha_i}{\partial t}\right|^2\right)^{1/2} = (E_c(\alpha))^{1/2}. \end{aligned}$$

We have equality in the first inequality above if and only if $c_i = \frac{l_i}{\sum_{i=1}^3 l_i}$ and in the second inequality if and only if $\left|\frac{\partial \alpha_i}{\partial t}\right| = l_i$. □

We are now ready show that the minimizing element of our variational problem is in fact a Steiner solution.

Theorem 2. *If*

$$E_{c^*}(\alpha^*) = \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}} E_c(\alpha)$$

for $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{A}$, then $L(\alpha^*) \le L(\alpha)$ for all $\alpha \in \mathcal{A}$.

Proof. First, note that if $c \in \mathcal{C}$ and $\alpha^c \in \mathcal{A}$ satisfies $E_c(\alpha^c) = \inf_{\alpha\in\mathcal{A}} E_c(\alpha)$, then $\alpha_i^c : I \to \mathbf{R}^n$ $(i = 1, 2, 3)$ must be energy minimizing, i.e.

$$\int_0^1 \left|\frac{d\alpha_i^c}{dt}\right|^2 \le \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt$$

for all $\gamma : I \to \mathbf{R}^n \in C^\infty$ with $\gamma(0) = \alpha_i^c(0)$ and $\gamma(1) = \alpha_i^c(1)$. (Otherwise, we can replace α_i^c by γ to lower the c-weighted energy.)

In particular, this implies that α_i^* is a one-dimensional harmonic map, which in turn implies it is linear and thus $\left|\frac{d\alpha_i^*}{dt}\right|$ is a constant, say λ_i. In particular, this shows that the image of α^* consists of three line segments meeting at a point $\alpha_1^*(1) = \alpha_2^*(1) = \alpha_3^*(1)$. Now note that if $\Lambda = \sum_{i=1}^3 \lambda_i$, then $\lambda = (\frac{\lambda_1}{\Lambda}, \frac{\lambda_2}{\Lambda}, \frac{\lambda_3}{\Lambda}) \in \mathcal{C}$ and

$$E_{c^*}(\alpha^*) \le E_\lambda(\alpha^*) = \sum^{\lambda} \frac{\lambda_i^2}{\left(\frac{\lambda_i}{\Lambda}\right)} = \Lambda\left(\sum \lambda_i\right) = L(\alpha^*)^2 \tag{2}$$

by the minimality of c^*. Furthermore, $L(\alpha^*)^2 \le E_{c^*}(\alpha^*)$ by Lemma 1, and this shows $L(\alpha^*)^2 = E_{c^*}(\alpha^*)$.

For an arbitrary choice of $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{A}$, we wish to show $L(\alpha^*) \le L(\alpha)$. Since reparameterizing a curve does not change the length of the image, without the loss of generality, we may assume that α_i $(i = 1, 2, 3)$ is parameterized proportional to arclength and let l_i equal the length of the image of α_i. By the minimizing property of c^* and α^*, $E_{c^*}(\alpha^*) \le E_c(\alpha)$ for every $c \in \mathcal{C}$ and every $\alpha \in \mathcal{A}$. Thus, if we set $c = (c_1, c_2, c_3) \in \mathcal{C}$, where $c_i = \frac{l_i}{\sum_{j=1}^3 l_j}$, we obtain

$$E_c(\alpha) = \sum^{c} \frac{1}{c_i} l_i^2 = \sum^{c} \frac{1}{\left(\frac{l_i}{\sum_{j=1}^3 l_j}\right)} l_i^2 = \left(\sum_{j=1}^{3} l_j\right)\left(\sum^{c} l_i\right) = L(\alpha)^2,$$

and this implies

$$L(\alpha^*)^2 \le E_{c^*}(\alpha^*) \le E_c(\alpha) = L(\alpha)^2.$$

Therefore, $L(\alpha^*) \le L(\alpha)$. □

Lemma 3. *If*

$$E_{c^*}(\alpha^*) = \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}} E_c(\alpha)$$

for $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{A}$, we have $c_i^* = \frac{\lambda_i}{\sum_{j=1}^3 \lambda_j}$, where $\left|\frac{d\alpha_i^*}{dt}\right| = \lambda_i$.

Proof. Immediate from (2) and Lemma 1. □

Theorem 4. *Let $c^0 = (c_1^0, c_2^0, c_3^0) \in \mathcal{C}$ and $\alpha^0 = (\alpha_1^0, \alpha_2^0, \alpha_3^0) \in \mathcal{A}$ be c^0-energy minimizing. If $c_i^0 \neq 0$ $(i = 1, 2, 3)$, then $\sum \frac{1}{c_i^0} \frac{\partial \alpha_i^0}{\partial t}(1) = 0$.*

Proof. Let V be an arbitrary vector field on $\bigcup_{i=1}^3 \alpha_i^0(I)$, the image of α^0. In particular, this means that $V(\alpha_1^0(1)) = V(\alpha_2^0(1)) = V(\alpha_3^0(1))$. Let $\alpha^s = (\alpha_1^s, \alpha_2^s, \alpha_3^s) \in \mathcal{A}$ be defined by setting $\alpha_i^s(t) = \alpha_i^0 + sV(\alpha_i^0(t))$ $(i = 1, 2, 3)$. By the minimizing property of α^0,

$$\begin{aligned}
0 &= \frac{d}{ds} E_{c^0}(\alpha^s)|_{s=0} \\
&= \frac{d}{ds}\left(\sum \frac{1}{c_i^0} \int_0^1 \left\langle \frac{\partial \alpha^s}{\partial t}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt\right)|_{s=0} \\
&= \sum \frac{1}{c_i^0}\left(\frac{\partial}{\partial s} \int_0^1 \left\langle \frac{\partial \alpha^s}{\partial t}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt\right)|_{s=0} \\
&= \sum \frac{1}{c_i^0}\left(\int_0^1 \frac{\partial}{\partial s} \left\langle \frac{\partial \alpha^s}{\partial t}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt\right)|_{s=0} \\
&= \sum \frac{1}{c_i^0}\left(\int_0^1 2\left\langle \frac{\partial^2 \alpha^s}{\partial t \partial s}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt\right)|_{s=0} \\
&= 2\sum \frac{1}{c_i^0}\left(\int_0^1 \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha^s}{\partial s}, \frac{\partial \alpha^s}{\partial t} \right\rangle dt\right)|_{s=0} \\
&= 2\sum \frac{1}{c_i^0} \cdot \left\langle V(\alpha_i^0(1)), \frac{\partial \alpha^0}{\partial t}(1) \right\rangle,
\end{aligned}$$

where $\frac{\partial \alpha^0}{\partial t}(1)$ is defined by continuity. Note that we have used the fact that $\frac{\partial \alpha^0}{\partial s}(0) = 0$, which is implied by the boundary condition. Since $V(\alpha_i^0(1))$ is arbitrary, we have shown $\sum \frac{1}{c_i^0} \frac{\partial \alpha_i^0}{\partial t}(1) = 0$. □

Recall the well-known fact that the solution to the Steiner problem has vertices at which three edges meet at 120° angles. (These are called Steiner points.) This fact can now be seen as a special case of the general phenomena for c-energy minimizer described in Theorem 4.

Corollary 5. *Let α^* and c^* be as in Theorem 2. If $c_i^* \neq 0$ $(i = 1, 2, 3)$, then the three line segments which comprise the image of α^* meet at 120° angles.*

Proof. Let $\lambda_i = \left|\frac{d\alpha_i^*}{dt}\right|$ and $\Lambda = \sum_{i=1}^3 \lambda_i$. From Lemma 3 we obtain $c_i^* = \frac{\lambda_i}{\Lambda}$, and thus Theorem 4 implies

$$\sum_{i=1}^3 \frac{1}{\lambda_i} \frac{\partial \alpha_i^0}{\partial t}(1) = \Lambda \sum_{i=1}^3 \frac{1}{c_i^*} \frac{\partial \alpha_i^*}{\partial t}(1) = 0.$$

Since $\frac{1}{\lambda_i} \frac{\partial \alpha_i^*}{\partial t}(1)$ is a unit vector which indicates the outward direction of the image of α_i^*, this shows that the three line segments that comprise the image of α^* must meet at 120° angles. □

2.3. **The existence problem.** We now consider the general existence question. Let G be a connected finite graph. Denote by ∂G the set of vertices of G incident with only one edge. Suppose there are m edges in G and n vertices in ∂G. Label the edges of G by $e_1, ..., e_m$ so that $e_1, ..., e_n$ corresponds to the n edges incident to

a vertex in ∂G. Now let the vertices incident to the edge e_i be labelled $e_{i,0}$ and $e_{i,1}$. Here, the labelling for an edge e_i with $i = 1, ..., n$ is chosen so that $e_{i,0} \in \partial G$. Let $e_{i,j} \sim e_{i',j'}$ $(i, i' = 1, ..., m,\ j, j' = 0, 1)$ if $e_{i,j}$ and $e_{i',j'}$ represent the same vertex in G.

Let

$$\mathcal{C}_G = \{c = (c_1, ..., c_m) : c_1 + ... + c_m = 1,\ \ c_i \geq 0\ (i = 1, ..., m)\}.$$

Fix $p_1, ..., p_n \in \mathbf{R}^n$ and define

$$\begin{aligned}\mathcal{A}_G \quad = \quad & \{\alpha = (\alpha_1, ..., \alpha_m) : \alpha_i : I \to \mathbf{R}^n \in C^\infty \text{ so that } \alpha_i(0) = p_i\ (i = 1, ..., n) \\ & \text{and } \alpha_i(j) = \alpha_{i'}(j')\ (i = 1, ..., m, j = 0, 1) \text{ if } e_{i,j} \sim e_{i',j'}\}.\end{aligned}$$

Note that $\alpha \in \mathcal{A}_G$ can be seen as a map from G to $\mathbf{R}^n$ satisfying $\alpha(\partial G) = \{p_1, ..., p_n\}$. For $c \in \mathcal{C}_G$ and $\alpha \in \mathcal{A}_G$, define $E_c(\alpha)$ and $L(\alpha)$ analogously to the case when G is a tripod.

Proposition 6. *For each $c \in \mathcal{C}$, there exists a c-energy minimizer $\alpha^c \in \mathcal{A}$. In other words, there exists $\alpha^c \in \mathcal{A}$ so that*

$$E_c(\alpha^c) = \inf_{\alpha \in \mathcal{A}} E_c(\alpha).$$

Proof. Fix $c = (c_1, ..., c_m) \in \mathcal{C}_G$ and let $\{\alpha^j = (\alpha_1^j, ..., \alpha_m^j)\} \subset \mathcal{A}_G$ be a minimizing sequence, i.e.

$$E_c(\alpha^j) \to \inf_{\alpha \in \mathcal{A}} E_c(\alpha).$$

If we reparameterize α_i^j with respect to arclength and call it $\bar{\alpha}_i^j$, then

$$E_c(\bar{\alpha}^j) = L(\bar{\alpha}^j)^2 = L(\alpha^j)^2 \leq E_c(\alpha^j).$$

Therefore, we may assume that α_i^j is arclength parameterized with speed l_i^j.

Assume $E_c(\alpha^j) \leq M$. Thus, $(l_i^j)^2 \leq c_i E_c(\alpha_i^j) \leq c_i M \leq M$. This in turn implies that α^j is an equicontinuous family of maps. By the Arzela-Ascoli Theorem, there exists a subsequence of α^j (which we still denote by α^j by abuse of notation) which converges uniformly to $\alpha^c \in \mathcal{A}$. In particular, $\lim_{j\to\infty} l_i^j = l_i$ where l_i is the arclength of $\alpha^c([0, 1])$. Hence

$$\begin{aligned}\inf_{\alpha \in \mathcal{A}} E_c(\alpha) \quad &\leq \quad E_c(\alpha^c) \\ &= \quad \sum^c \frac{1}{c_i} l_i^2 \\ &= \quad \sum^c \frac{1}{c_i} \lim_{j\to\infty} (l_i^j)^2 \\ &= \quad \lim_{j\to\infty} E_c(\alpha^j) \\ &= \quad \inf_{\alpha \in \mathcal{A}} E_c(\alpha),\end{aligned}$$

and this shows the existence of a c-energy minimizer α^c. □

Theorem 7. *The parameterized Steiner problem can be solved. In other words, there exists $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{A}$ so that*

$$E_{c^*}(\alpha^*) = \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{A}} E_c(\alpha).$$

Proof. Let $c^j = (c_1^j, ..., c_m^j) \in \mathcal{C}$ be a minimizing sequence. In other words, if we let $\alpha^j \in \mathcal{A}$ be a c^j-energy minimizer whose existence is guaranteed by Proposition 6, then

$$\lim_{j\to\infty} E_{c^j}(\alpha^j) = \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}} E_c(\alpha).$$

Since α_i^j is energy minimizing (see the proof of Theorem 2), it is parameterized by arclength.

Since $\mathcal{C} \subset \mathbf{R}^n$ is compact, there exists a subsequence of c^j (still denoted by c^j) which converges to $c^* \in \mathcal{C}$. Without the loss of generality, we may assume $E_{c^j}(\alpha^j) \le M$ and $c_i^j \ne 0$ for all $j = 1, 2, ...$ and $i = 1, 2, ..., m$. If l_i^j is the arclength of $\alpha_i^j([0,1])$, then $(l_i^j)^2 \le c_i^j E_{c^j}(\alpha^j) \le c_i^j M \le M$. Thus, α_i^j $(j = 1, 2, ...)$ is an equicontinuous family of maps, and there exists a subsequence of α_i^j (still denoted by α_i^j) which converges uniformly to α_i^*. Let l_i^* is the arclength of $\alpha^*([0,1])$.

If $c_i^* \ne 0$, then

$$\begin{aligned}
\frac{1}{c_i^*}(l_i^*)^2 &= \lim_{j\to\infty} \frac{1}{c_i^*}(l_i^j)^2 \\
&= \lim_{j\to\infty} \frac{1}{c_i^j}(l_i^j)^2 + \lim_{j\to\infty} \left(\frac{1}{c_i^*} - \frac{1}{c_i^j}\right)(l_i^j)^2.
\end{aligned}$$

Since $(l_i^j)^2 \le M$, the last term on the right hand side equals 0. Therefore,

$$\begin{aligned}
\inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}} E_c(\alpha) &\le E_{c^*}(\alpha^*) \\
&= \sum^{c^*} \frac{1}{c_i^*}(l_i^*)^2 \\
&= \lim_{j\to\infty} \sum^{c^*} \frac{1}{c_i^j}(l_i^j)^2 \\
&\le \lim_{j\to\infty} E_{c^j}(\alpha^j) \\
&= \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}} E_c(\alpha),
\end{aligned}$$

and thus $E_{c^*}(\alpha^*) = \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{A}} E_c(\alpha)$. □

Remark. The absolute minimizer as above can have various degenerations of edges. In particular, when the domain graph G has a non-trivial topology, there are various ways the topology of the image network $\alpha(G)$ result from degenerations of edges. Consequently we do not expect uniqueness in this length minimizing solution. For some examples where the uniqueness fails, see [IT].

3. The two-dimensional case

In this section, we study the two-dimensional analog of the Steiner problem, the problem of finding a minimal surface that is topologically a finite 2-complex. We introduce the appropriate weighted energy and propose a variational problem which, when solved, produces a minimal surface with the prescribed singularity.

3.1. **The variational problem.**

3.1.1. *The classical Plateau problem.* The classical Plateau problem is formulated as follows. Let $\Gamma \subset \mathbf{R}^n$ be a Jordan curve, i.e. a subset homeomorphic to the circle, and let $\tilde{\triangle}$ be the unit disc. We denote the closure of $\tilde{\triangle}$ by $\tilde{\triangle}'$. Let

$$\mathcal{F} = \{\alpha : \tilde{\triangle} \to \mathbf{R}^n : \alpha \in W^{1,2}(\tilde{\triangle}) \cap C^0(\tilde{\triangle}') \\ \text{and } \alpha|_{\partial\tilde{\triangle}} \text{ is a homeomorphism of } \Gamma\}.$$

Define the area functional $A : \mathcal{F} \to \mathbf{R}^+ \cup \{\infty\}$ by

$$A(\alpha) : \int_{\tilde{\triangle}} \sqrt{\left|\frac{\partial\alpha}{\partial x}\right|^2 + \left|\frac{\partial\alpha}{\partial y}\right|^2 - 2\frac{\partial\alpha}{\partial x}\cdot\frac{\partial\alpha}{\partial y}}\, dxdy.$$

The Plateau problem. Find $\alpha^* \in \mathcal{F}$ so that $A(\alpha^*) \leq A(\alpha)$ for all $\alpha \in \mathcal{F}$.

Similar to the one-dimensional case of finding the shortest curve, the difficulty of dealing with the area functional is the lack of control over the parameterizations of the same surface. Therefore Douglas [D] proposed minimizing the Dirichlet energy,

$$E(\alpha) = \int_{\tilde{\triangle}} \left|\frac{\partial\alpha}{\partial x}\right|^2 + \left|\frac{\partial\alpha}{\partial x}\right|^2 \, dxdy,$$

instead. More precisely, let $\psi : \partial\tilde{\triangle} \to \Gamma \subset \mathbf{R}^n$ be a monotone map and define

$$\mathcal{F}_\psi = \{\alpha \in \mathcal{F} : \alpha|_{\partial\tilde{\triangle}} = \psi\}.$$

For each ψ so that $E_\psi := \inf\{E(\alpha) : \alpha \in \mathcal{F}_\psi\} < \infty$, the Dirichlet principle implies that there exists a unique map $\alpha_\psi \in \mathcal{F}_\psi$ so that $E(\alpha_\psi) = E_\psi$. If $\{\psi_i\}$ is a sequence so that $E_{\psi_i} \to \inf_\psi E_\psi$, we can prove (a subsequence of) α_{ψ_i} converges uniformly to a weakly conformal harmonic map $\alpha^* \in \mathcal{F}_\psi$ and that $E(\alpha^*) = \inf_\psi \inf\{E(\alpha) : \alpha \in \mathcal{F}_\psi\} = \inf_{\alpha\in\mathcal{F}} A(\alpha)$.

The Douglas solution α^* can also be interpreted as follows. Let $\triangle$ be the unit disc, let $\psi : \partial\triangle \to \Gamma$ be a constant speed parameterization of Γ and let $\mathcal{P}$ be a set of diffeomorphism $\phi : \triangle \to \tilde{\triangle}$ that is homeomorphic up to the boundary, satisfying the so-called three-point condition to disregard the conformal transformations of the disc. For each $\phi \in \mathcal{P}$, let

$$\mathcal{F}(\phi) = \{\alpha \in \mathcal{F} : \alpha \circ \phi\Big|_{\partial\triangle} = \psi\}.$$

Then $\alpha^* \in \mathcal{F}$ is the map which satisfies

$$E(\alpha^*) = \inf_{\phi\in\mathcal{P}} \inf_{\alpha\in\mathcal{F}(\phi)} E(\alpha). \tag{3}$$

We can identify ϕ to an element of $\mathcal{M}(\triangle)$, the moduli space of conformal structures on $\triangle$, via

$$\phi \mapsto \phi^*(g_0)$$

and interpret the first inf of equation (3) as a variational problem over $\mathcal{M}(\triangle)$. Conversely, the important feature of the Douglas solution of the Plateau problem is that it gives us a way to solve this variational problem over $\mathcal{M}(\triangle)$; namely by considering a family of parameterization of Γ.

3.1.2. *The generalized Plateau problem with prescribed singularity.* We now consider the case when the domain is topologically a union of three discs with the lower-semicircles identified. Let $\triangle_i, \tilde{\triangle}_i$ $(i = 1, 2, 3)$ be unit discs and define

$$\mathcal{P} = \{\Phi = \{\phi_1, \phi_2, \phi_3\} |\ \phi_i : \triangle_i \to \tilde{\triangle}_i \text{ with the three-point condition}\},$$

where ϕ_i $(i = 1, 2, 3)$ is a Lipschitz diffeomorphism in the interior of the disc $\triangle_i$ and homeomorphism up to the boundary and the three point condition dictates that $\phi_i(1, 0) = (1, 0), \phi_i(0, 1) = (0, 1)$ and $\phi_i(-1, 0) = (-1, 0)$. Denote by A_i (resp. $\tilde{A}_i$) the lower semicircle bounded by $(1, 0)$ and $(-1, 0)$ of the unit circle $\partial\triangle_i$ (resp. $\partial\tilde{\triangle}_i$). Let A be the lower semi-circle obtained by mutually identifying A_i by the identity map $\mathrm{Id} : \triangle_i \to \triangle_j$ and let X_{Id} be the union of the three discs $\triangle_i$ $(i = 1, 2, 3)$ with the identification on A_i. Now consider an identification of the $\tilde{A}_i$'s defined by

$$t \sim s \text{ if and only if } \phi_i^{-1}(t) = \phi_j^{-1}(s) \text{ on } A$$

for $t \in \tilde{A}_i$ and $s \in \tilde{A}_j$ and let X_Φ be a union of the three discs $\tilde{\triangle}_i$ $(i = 1, 2, 3)$ along with this identification.

With this, the moduli space needed for our variational problem can be described in two different ways. One way is as the space of conformal structures on X_{Id} defined by the conformal structures $\phi_i^*(g_0)$ on $\triangle_i$ $(i = 1, 2, 3)$. The second way is as the collection of $\{X_\Phi : \Phi \in \mathcal{P}\}$ with Φ defining the gluing maps of A_i's and with each disc $\tilde{\triangle}_i$ equipped with the standard Euclidean conformal structure.

Let Γ be a graph embedded in $\mathbf{R}^n$ consisting of three arcs Γ_i $(i = 1, 2, 3)$ sharing common end points q_1, q_2. Let ψ_i $(i = 1, 2, 3)$ be a constant speed parameterization with speed $\leq L$ (for a large enough L) which sends $\partial\triangle_i \backslash A \subset \partial X_{\mathrm{Id}}$ to Γ_i and so that $\psi_i(1, 0) = q_1$ and $\psi_i(-1, 0) = q_2$. Set $\Psi = (\psi_1, \psi_2, \psi_3)$.

Define

$$\begin{aligned} \mathcal{F}(\Phi) \quad = \quad & \{\alpha = (\alpha_1, \alpha_2, \alpha_3) |\ \alpha_i : \tilde{\triangle}_i \to \mathbf{R}^n \in W^{1,2}(\tilde{\triangle}_i) \cap C^0(\tilde{\triangle}_i') \\ & \text{satisfies the trace conditions } *\}, \end{aligned}$$

where the trace conditions $*$ are as follows:

$$\alpha_i \circ \phi_i \Big|_A = \alpha_j \circ \phi_j \Big|_A \quad \text{(matching condition)}$$

and

$$\alpha_i \circ \phi_i \Big|_{\partial\triangle_i \backslash A} = \psi_i \Big|_{\partial\triangle_i \backslash A_i} \quad \text{(boundary condition)}.$$

The set Γ will be called the fixed boundary, and $e_\alpha := \alpha_i(\tilde{A}_i)$ will be called the free boundary (associated with α) for $\alpha \in \mathcal{F}(\Phi)$.

As before, let

$$\mathcal{C} = \{(c_1, c_2, c_3) \in \mathbf{R}^n |\ c_1 + c_2 + c_3 = 1,\ \ c_i \geq 0\ (i = 1, 2, 3)\}.$$

For $c = (c_1, c_2, c_3) \in \mathcal{C}$, we say that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}(\Phi)$ is compatible with c if $E(\alpha_i) = 0$ whenever $c_i = 0$. Otherwise, we say α is incompatible with c. We define the area of $\alpha \in \mathcal{F}(\Phi)$ as $A(\alpha) = \sum A(\alpha_i)$ and the c-weighted energy as

$$E_c(\alpha) = \begin{cases} \Big(\sum^c \frac{1}{c_i}(E(\alpha_i))^2\Big)^{1/2} & \text{if } \alpha \text{ is compatible with } c\ , \\ \infty & \text{if } \alpha \text{ is incompatible with } c, \end{cases}$$

where $\sum^c$ denotes the sum over i with $c_i \neq 0$.

We consider the variational problem contained in the following:

$$\inf_{\Phi\in\mathcal{P}}\inf_{c\in\mathcal{C}}\inf_{\alpha\in\mathcal{F}(\Phi)} E_c(\alpha) \tag{4}$$

and show that its minimizing element is a parameterized soap-film.

3.2. **The properties of a minimizer.**

Lemma 8. *For every $\Phi \in \mathcal{P}, c \in \mathcal{C}$ and $\alpha \in \mathcal{F}(\Phi)$, the area of α is bounded from above by the c-energy of α, i.e $A(\alpha) \le E_c(\alpha)$. The equality $A(\alpha) = E_c(\alpha)$ is achieved if and only if α_i is weakly conformal and $c_i = E(\alpha_i)/\Big(\sum^c E(\alpha_i)\Big)$.*

Proof. An application of the Cauchy-Schwarz inequality together with the fact that Dirichlet energy always dominates the area of a map gives the following inequality for any $\alpha \in \bigcup_{\Phi\in\mathcal{P}} \mathcal{F}(\Phi)$ and $c \in \mathcal{C}$:

$$A(\alpha) = \sum_i A(\alpha_i) = \sum^c \sqrt{c_i}\frac{A(\alpha_i)}{\sqrt{c_i}}$$
$$\le \Big(\sum^c c_i\Big)^{1/2}\Big(\sum^c \frac{A(\alpha_i)^2}{c_i}\Big)^{1/2} \le 1\cdot\Big(\sum^c \frac{E(\alpha_i)^2}{c_i}\Big)^{1/2} = E_c(\alpha).$$

We have equality in the first inequality above if and only if $c_i = A(\alpha_i)/\Big(\sum^c A(\alpha_j)\Big)$ and in the second inequality if and only if α_i is a weakly conformal map, in which case $c_i = E(\alpha_i)/\Big(\sum^c E(\alpha_j)\Big)$ $(i = 1, 2, 3)$. □

We are now ready to show that the solution to our variational problem is in fact an area minimizing map (i.e. a parameterized soap film).

Theorem 9. *Let $\Phi^* \in \mathcal{P}$, $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{F}(\Phi^*)$ be such that*

$$E_{c^*}(\alpha^*) = \inf_{\Phi\in\mathcal{P}}\inf_{c\in\mathcal{C}}\inf_{\alpha\in\mathcal{A}} E_c(\alpha).$$

Then $A(\alpha^*) \le A(\alpha)$ for all $\alpha \in \bigcup_{\Phi\in\mathcal{P}} \mathcal{F}(\Phi)$.

Proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_2) \in \mathcal{F}(\Phi)$. By the definition of $\mathcal{F}(\Phi)$, $\alpha_i|_{\partial\tilde{\triangle}_i}$ is continuous. Define $\alpha_{i,\epsilon} : \tilde{\triangle}_i \to \mathbf{R}^{n+2}$ $(i = 1, 2, 3)$ by setting

$$\alpha_{i,\epsilon}(x, y) = (\alpha^1_{i,\epsilon}(x, y), \dots, \alpha^n_{i,\epsilon}(x, y), \epsilon x, \epsilon y).$$

Then $\alpha_{i,\epsilon}$ is a homeomorphism of $\partial\tilde{\triangle}_i$ and $\alpha_{i,\epsilon}(\partial\tilde{\triangle}_i)$, and thus $\alpha_{i,\epsilon}(\partial\tilde{\triangle}_i)$ is a Jordan curve. Let $\beta_i : \tilde{\triangle}_i \to \mathbf{R}^{n+2}$ $(i = 1, 2, 3)$ be a Plateau solution with $\beta_i(\partial\tilde{\triangle}_i) = \alpha_{i,\epsilon}(\partial\tilde{\triangle}_i)$ and $\beta_i(1, 0) = \alpha_{i,\epsilon}(1, 0)$, $\beta_i(0, 1) = \alpha_{i,\epsilon}(0, 1)$, $\beta_i(-1, 0) = \alpha_{i,\epsilon}(-1, 0)$. Since $\beta_i|_{\partial\tilde{\triangle}}$ is a homeomorphism of $\partial\tilde{\triangle}_i$ and $\alpha_{i,\epsilon}(\partial\tilde{\triangle}_i)$ (cf. [La]), the following definition of $\varphi_i : \partial\triangle_i \to \partial\tilde{\triangle}_i$ $(i = 1, 2, 3)$ makes sense:

$$\varphi_i(t) = \begin{cases} \beta_i^{-1}\circ\psi_i(t) & \text{if } t \in \partial\triangle_i\backslash A, \\ \beta_i^{-1}\circ\beta_1\circ \mathrm{Id}_A(t) & \text{if } t \in A \end{cases}$$

for $i = 1, 2, 3$, where $\mathrm{Id}_A : A \to \tilde{A}_i$ is the identity map. Note that φ_i is a homeomorphism. Furthermore, for $t \in A \subset \partial\triangle_i$, we have that

$$\beta_i\circ\varphi_i(t) = \beta_i\circ\beta_i^{-1}\circ\beta_1(t)\circ\mathrm{Id}_A = \beta_1\circ\mathrm{Id}_A(t) = \beta_1\circ\varphi_1(t), \tag{5}$$

and for $t \in \partial\triangle_i \backslash A$, we have that

$$\beta_i \circ \varphi_i(t) = \beta_i \circ \beta_i^{-1} \circ \psi_i(t) = \psi_i(t). \tag{6}$$

We extend $\varphi_i : \partial\triangle_i \to \partial\tilde{\triangle}_i$ $(i = 1, 2, 3)$ to a diffeomorphism $\phi_i : \triangle_i \to \tilde{\triangle}_i$. For example, ϕ_i can be defined to be the harmonic extension of φ_i which is known to be a diffeomorphism by a result of Rado (cf. [SY], Part I).

Let Π be the projection map of $\mathbf{R}^{n+2}$ to the first n components and $\bar{\beta}_i = \Pi \circ \beta_i$. Then

$$\bar{\beta}_i \circ \varphi_i(t) = \bar{\beta}_1 \circ \varphi_1(t)$$

for $t \in A \subset \partial\triangle_i$ by equation (5) and

$$\bar{\beta}_i \circ \varphi_i(t) = \psi_i(t)$$

for $t \in \partial\triangle_i - A$ by equation (6). Therefore, $\bar{\beta} := (\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)$ satisfies the trace conditions $*$ defined by $\Phi = (\phi_1, \phi_2, \phi_3)$ and thus $\bar{\beta} \in \mathcal{F}(\Phi)$.

Since β_i is conformal, we have $A(\beta) = E_c(\beta)$ for $c = (c_1, c_2, c_3)$ with $c_i = E(\beta_i)/\Big(\overset{c}{\sum} E(\beta_j)\Big)$. Therefore,

$$A(\alpha^*) \le E_{c^*}(\alpha^*) \le E_c(\bar{\beta}) \le E_c(\beta) + O(\epsilon) = A(\beta) + O(\epsilon),$$

where $O(\epsilon) \to 0$ as $\epsilon \to 0$. Since β_i is the Plateau solution with same boundary data as $\alpha_{i,\epsilon}$, we see that

$$A(\beta) \le A(\alpha_\epsilon) \le A(\alpha) + O(\epsilon),$$

where $\beta := (\beta_1, \beta_2, \beta_3)$ and $\alpha_\epsilon := (\alpha_{1,\epsilon}, \alpha_{2,\epsilon}, \alpha_{3,\epsilon})$, which in turn implies

$$A(\alpha^*) \le A(\beta) + O(\epsilon) \le A(\alpha) + O(\epsilon).$$

Since ϵ is arbitrary, $A(\alpha^*) \le A(\alpha)$. □

In analogy to the situation of the Plateau problem of disc-type minimal surface, we have the following observation for the area minimizer.

Lemma 10. *Let $\Phi^* \in \mathcal{P}, c^* \in \mathcal{C}$ and $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*) \in \mathcal{F}(\Phi^*)$ be as in Theorem 9. If $\alpha_i^* : \tilde{\triangle}_i \to \mathbf{R}^n$ is continuous up to the boundary for $i = 1, 2, 3$, then $c_i^* = \frac{E(\alpha_i^*)}{\sum^{c^*} E(\alpha_i^*)}$ and α_i^* is a weakly conformal map.*

Proof. If β is the Plateau solution as defined in the proof of Theorem 9 with $\alpha = \alpha^*$ and if $c_i = \frac{E(\beta_i)}{\overset{c}{\sum} E(\beta_j)}$, then

$$A(\alpha^*) \le E_{c^*}(\alpha^*) \le E_c(\beta) + O(\epsilon) = A(\beta) + O(\epsilon) \le A(\alpha^*) + O(\epsilon).$$

Hence $A(\alpha^*) = E_{c^*}(\alpha^*)$. Lemma 8 implies that $A(\alpha^*) = E_{c^*}(\alpha^*)$ only when α_i^* is weakly conformal and $c_i^* = \frac{E(\alpha_i^*)}{\overset{c}{\sum} E(\alpha_j^*)}$ for $i = 1, 2, 3$. □

Theorem 11. *For a fixed $\Phi \in \mathcal{P}$ and $c \in \mathcal{C}$ with $c_i \neq 0$ for each i, let $\alpha^0 = (\alpha_1^0, \alpha_2^0, \alpha_3^0) \in \mathcal{F}(\Phi)$ be so that*

$$E_c(\alpha^0) = \inf_{\alpha \in \mathcal{F}(\Phi)} E_c(\alpha).$$

Let η be the outward pointing unit normal to $\partial\tilde{\triangle}_i$ and let s be the arclength parameter of $\partial\tilde{\triangle}_i$. Assume α_i^0 is C^1 up to $\tilde{A}_i \subset \partial\tilde{\triangle}_i$. Then along $\tilde{A}_i$ we have

$$\sum_{i=1,2,3} \frac{E(\alpha_i^0)}{c_i} \left|\frac{\partial \alpha_i^0}{\partial s}\right|^{-1} \frac{\partial \alpha_i^0}{\partial \eta} = \vec{0},$$

where $\vec{0} = (0, ..., 0) \in \mathbf{R}^n$.

Proof. Let V be a vector field defined in a neighborhood of the image of α^0 and compactly supported away from $\alpha^0(\partial X_\Phi) = \alpha^0(\bigcup_{i=1}^3 \partial\tilde{\triangle}_i \backslash \tilde{A}_i)$. Let $\alpha^s = (\alpha_1^s, \alpha_2^s, \alpha_3^s) \in \mathcal{F}(\Phi)$ be defined by setting $\alpha_i^s(z) = \alpha_i^0(z) + sV(\alpha_i^0(z))$. With $\langle \cdot, \cdot \rangle_{\mathbf{R}^n}$ denoting the usual inner product in $\mathbf{R}^n$, the minimizing property of α^0 implies

$$\begin{aligned}
0 &= \frac{d}{ds}(E_c(\alpha^s))\Big|_{s=0} \\
&= \frac{d}{ds} \sum_{i=1,2,3} \frac{1}{c_i}(E(\alpha_i^s))^2\Big|_{s=0} \\
&= 2\sum_{i=1}^3 \frac{1}{c_i}(E(\alpha_i^s))\frac{d}{ds}E(\alpha_i^s)\Big|_{s=0} \\
&= 2\sum_{i=1}^3 \frac{1}{c_i}(E(\alpha_i^s))\frac{d}{ds}\int_{\tilde{\triangle}_i} \left\langle \frac{\partial \alpha_i^s}{\partial x}, \frac{\partial \alpha_i^s}{\partial x}\right\rangle_{\mathbf{R}^n} + \left\langle \frac{\partial \alpha_i^s}{\partial y}, \frac{\partial \alpha_i^s}{\partial y}\right\rangle_{\mathbf{R}^n} dxdy\Big|_{s=0} \\
&= 4\sum_{i=1}^3 \frac{1}{c_i}(E(\alpha_i^s)) \int_{\tilde{\triangle}_i} \left\langle \frac{\partial}{\partial x}(\frac{\partial \alpha_i^s}{\partial s}), \frac{\partial \alpha_i^s}{\partial x}\right\rangle_{\mathbf{R}^n} + \left\langle \frac{\partial}{\partial y}(\frac{\partial \alpha_i^s}{\partial s}), \frac{\partial \alpha_i^s}{\partial y}\right\rangle_{\mathbf{R}^n} dxdy\Big|_{s=0}.
\end{aligned}$$

But if we write α_i^s in coordinates as $(\alpha_{i1}^s, ..., \alpha_{in}^s)$, then for each $j = 1, ..., n$, we can use the fact that $\triangle\alpha_{ij} = 0$ to see

$$\begin{aligned}
&\int_{\tilde{\triangle}_i} \left\langle \frac{\partial}{\partial x}(\frac{\partial \alpha_{ij}^s}{\partial s}), \frac{\partial \alpha_{ij}^s}{\partial x}\right\rangle_{\mathbf{R}^n} + \left\langle \frac{\partial}{\partial y}(\frac{\partial \alpha_{ij}^s}{\partial s}), \frac{\partial \alpha_{ij}^s}{\partial y}\right\rangle_{\mathbf{R}^n} dxdy \\
&\quad = \int_{\tilde{\triangle}} \mathrm{div}_{\mathbf{R}^2}(\frac{\partial \alpha_{ij}^s}{\partial s}\nabla\alpha_{ij}^s)dxdy \\
&\quad = \int_{\partial\tilde{\triangle}_i} \frac{\partial \alpha_{ij}^s}{\partial s}\nabla\alpha_{ij} \cdot \eta ds \\
&\quad = \int_{\partial\tilde{\triangle}_i} \frac{\partial \alpha_{ij}^s}{\partial s}\frac{\partial \alpha_{ij}}{\partial \eta} ds
\end{aligned}$$

by the divergence theorem. Therefore

$$\begin{aligned}
0 &= 4\sum_{i=1}^3 \frac{1}{c_i}(E(\alpha_i^s)) \int_{\partial\tilde{\triangle}_i} \left\langle \frac{\partial \alpha_i^s}{\partial s}, \frac{\partial \alpha_i^s}{\partial \eta}\right\rangle_{\mathbf{R}^n} ds\Big|_{s=0} \\
&= 4\sum_{i=1}^3 \int_{A_i} \left\langle V(\alpha_i^0(z)), \frac{E(\alpha_i^0)}{c_i}\frac{\partial \alpha_i^0}{\partial \eta}\right\rangle_{\mathbf{R}^n} ds. \qquad (7)
\end{aligned}$$

Let σ be the arclength parameter of the free boundary $e_\alpha = \alpha_i(A_i)$. Then

$$d\sigma = \frac{d\sigma}{ds}ds = \left|\frac{\partial \alpha_i}{\partial s}\right| ds.$$

Therefore, we can rewrite equality (7) as

$$\begin{aligned} 0 &= 4\sum_{i=1}^{3}\int_{A_i}\left\langle V, \frac{E(\alpha_i^0)}{c_i}\frac{\partial\alpha_i^0}{\partial\eta}\right\rangle_{\mathbf{R}^n}\left|\frac{\partial\alpha_i}{ds}\right|^{-1} d\sigma \\ &= 4\int_{e_{\alpha^0}}\left\langle V, \sum_{i=1}^{3}\frac{E(\alpha_i^0)}{c_i}\left|\frac{\partial\alpha_i^0}{ds}\right|^{-1}\frac{\partial\alpha_i^0}{\partial\eta}\right\rangle_{\mathbf{R}^n} d\sigma. \end{aligned}$$

Since V is arbitrary,

$$\sum_{i=1}^{3}\frac{E(\alpha_i)}{c_i}\left|\frac{\partial\alpha_i^0}{ds}\right|^{-1}\frac{\partial\alpha_i^0}{\partial\eta} = 0$$

for each point of the free boundary e_{α^0}. □

We now quote the following regularity theorem of soap-bubble/film clusters in $\mathbf{R}^3$ by J. Taylor [T].

Theorem. *A soap bubble/film cluster $\mathcal{C}$ in $\mathbf{R}^3$ (an $(M,0,\delta)$-minimal set) consists of real analytic constant mean curvature surfaces meeting smoothly in threes at $120°$ angles along smooth curves, which are in turn meeting in fours at angles of $\cos^{-1}(-1/3) \approx 109°$.*

The singular curves were proved to be $C^{1,\alpha}$ by Taylor [T], and real-analytic by Kinderlerer-Nirenberg-Spruck [KNS].

The balancing of the three meeting surfaces can now be seen as a particular case of the balancing phenomena for c-energy minimizer as described in Theorem 11. To be more precise, we have

Corollary 12. *Let $\Phi^* \in \mathcal{P}$, $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{F}(\Phi^*)$ be as in Theorem 9. Assume α_i^* is C^1 up to $\tilde{A}_i \subset \partial\tilde{\triangle}_i$. Then the three minimal surfaces meet along the free boundary $e_{\alpha^*} := \alpha_i^*(\tilde{A}_i)$ at $120°$ angles.*

Proof. By Lemma 10, $c_i^* = \frac{E(\alpha_i^*)}{\sum^{c^*} E(\alpha_j^*)}$ and α_j^* is weakly conformal. Thus $\frac{E(\alpha_i^*)}{c_i^*} = \sum_{j=1}^{3} E(\alpha_j^*)$ and $\left|\frac{\partial\alpha_i^*}{\partial s}\right| = \left|\frac{\partial\alpha_i^*}{\partial\eta}\right|$. Theorem 11 implies

$$\sum_{i=1}^{3}\left|\frac{\partial\alpha_i^*}{\partial\eta}\right|^{-1}\frac{\partial\alpha_i^*}{\partial\eta} = \left(\sum_{j=1}^{3}E(\alpha_j^*)\right)^{-1}\sum_{i=1}^{3}\frac{E(\alpha_i^*)}{c_i^*}\left|\frac{\partial\alpha_i^*}{\partial s}\right|^{-1}\frac{\partial\alpha_i^*}{\partial\eta} = \vec{0}.$$

Therefore the three unit vectors $\left|\frac{\partial\alpha_i^*}{\partial\eta}\right|^{-1}\frac{\partial\alpha_i}{\partial\eta}$ balance at each point e_{α^*}, which implies that they must meet at $120°$ angles. □

3.3. **The existence problem.** We see from Section 3.2 that finding the solution to the variational problem contained in (4) is equivalent to proving the existence of an area-minimizing singular surface with fixed boundary Γ.

Let $\Phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{P}$. At a point $z \in \triangle$, the differential $d\phi_i$ is a linear mapping and takes a unit circle about the origin to an ellipse. The dilatation $D_{\phi_i}(z)$ of ϕ_i at z is the ratio of the major axis and the minor axis of this ellipse. The map ϕ_i is called D-quasiconformal if $D_{\phi_i}(z) \le D$ for all $z \in \triangle$.

We define a subset $\mathcal{P}(D)$ of $\mathcal{P}$ by setting

$$\mathcal{P}(D) = \{\Phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{P} |\ \phi_i \text{ is } D\text{-quasiconformal}\}.$$

Definition 13. We say that an embedded graph $\Gamma \subset \mathbf{R}^n$, consisting of three arcs Γ_i $(i = 1, 2, 3)$ sharing common end points q_1, q_2, is a non-degenerate boundary if there exists $1 \leq D < \infty$ so that

$$\inf_{\Phi \in \mathcal{P}} \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi)} E_c(\alpha) = \inf_{\Phi \in \mathcal{P}(D)} \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi)} E_c(\alpha).$$

We will show an existence theorem for the generalized Plateau Problem under the assumption that Γ is a non-degenerate boundary.

Remark. It should be noted that we unfortunately expect *degenerate* boundary Γ to exist, in the sense that for those Γ there is no finite upper bound D as appears in the definition above. Lawlor and Morgan [LM] conjecture ten types of smooth boundary singularities of soap films. According to their descriptions, one expects the induced metric on some parts of the surfaces to have degenerate conformal structures. In particular, the situation where the free boundary e_{α^*} touches the fixed boundary Γ would be such an example.

We start by showing the existence of the c-energy minimizing map in $\mathcal{F}(\Phi)$ for $\Phi \in \mathcal{P}(D)$.

Theorem 14. *For a given $\Phi \in \mathcal{P}(D)$ and $c \in \mathcal{C}$, there exists a c-energy minimizer $\alpha \in \mathcal{F}(\Phi)$, i.e.*

$$E_c(\alpha) = \inf_{\alpha' \in \mathcal{F}(\Phi)} E_c(\alpha').$$

Moreover, α is the unique c-energy minimizer in $\mathcal{F}(\Phi)$.

Proof. Let $\{\alpha^k\}_{k=1}^{\infty} \subset \mathcal{F}(\Phi)$ be a c-energy minimizing sequence, i.e.

$$\lim_{k \to \infty} E_c(\alpha^k) = \inf_{\alpha' \in \mathcal{F}(\Phi)} E_c(\alpha').$$

We may assume that $E_c(\alpha^k) \leq M < \infty$ for some M.

For each i with $c_i \neq 0$,

$$(E(\alpha_i^k))^2 \leq c_i (E_c(\alpha^k))^2 \leq c_i M^2 \leq M^2.$$

Consequently,

$$\|\alpha_i^k\|_{H_1} \leq M'$$

for some $M' < \infty$. The Rellich compactness theorem then says that there exists a subsequence of α^k (which we will again denote by α^k by an abuse of notation) such that α_i^k converges in L^2 and converges weakly in $W^{1,2}$ to some $\alpha_i \in W^{1,2}$ $(i = 1, 2, 3)$.

We claim that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}(\Phi)$ and that

$$E_c(\alpha) = \inf_{\alpha' \in \mathcal{F}(\Phi)} E_c(\alpha').$$

The latter claim is true since

$$\begin{aligned}(E_c(\alpha))^2 &= \sum \frac{1}{c_i} (E(\alpha_i))^2 \leq \liminf_{k \to \infty} \sum \frac{1}{c_i} (E(\alpha_i^k))^2 \\ &= \liminf_{k \to \infty} (E_c(\alpha^k))^2 = \Big(\inf_{\alpha' \in \mathcal{F}(\Phi)} E_c(\alpha') \Big)^2,\end{aligned}$$

where the inequality above is due to the lower semicontinuity of the Dirichlet energy.

We now show that $\alpha \in \mathcal{F}(\Phi)$. We already know that $\alpha_i \in W^{1,2}(\tilde{\triangle}_i)$. It remains to show that $\alpha_i \in C^0(\tilde{\triangle}_i')$ and that α_i satisfies the trace condition $*$. Since $\alpha^k \in \mathcal{F}(\Phi)$, the matching condition says

$$\alpha_i^k \circ \phi_i\Big|_A = \alpha_j^k \circ \phi_j\Big|_A$$

for $i, j = 1, 2, 3$ and $k = 1, 2, \ldots$. Since ϕ_i is Lipschitz, the composition maps $\{\alpha_i^k \circ \phi_i\}$ form a sequence in $W^{1,2}$, and hence there is a subsequence (which we still index by k) of $\{\alpha_i^k \circ \phi_i\}$ converging strongly in L^2 and weakly in $W^{1,2}$. By letting $k \to \infty$, we obtain

$$\operatorname{tr}(\alpha_i \circ \phi_i)\Big|_A =^{L^2} \operatorname{tr}(\alpha_j \circ \phi_j)\Big|_A \tag{8}$$

by the $W^{1,2}$-trace theory.

The boundary condition implies

$$\alpha_i^k \circ \phi_i\Big|_{\partial\triangle_i \setminus A} = \psi_i\Big|_{\partial\triangle_i \setminus A}.$$

Note that the right-hand side is independent of k. Again, letting $k \to \infty$, we obtain

$$\operatorname{tr}(\alpha_i \circ \phi_i)\Big|_{\partial\triangle_i \setminus A} =^{L^2} \psi_i\Big|_{\partial\triangle_i \setminus A}. \tag{9}$$

We now show that α_i is continuous up to the boundary of $\tilde{\triangle}_i$, which in turn will imply that the L^2 equalities (8) and (9) are actually pointwise equalities. It is sufficient to prove:

Claim 15. *The map $\beta_i := \varphi_i \circ \alpha_i : \triangle_i \to \mathbf{R}^n$ is continuous up to the boundary $\partial\triangle_i$.*

Proof of the claim. First, we claim that for $z \in \triangle_i'$ and $\delta < 1$, there exists $r \in (\delta, \sqrt{\delta})$ so that

$$|\beta_i(z_1) - \beta_i(z_2)| \le (4\pi D M')^{\frac{1}{2}} \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} \tag{10}$$

for any $z_1, z_2 \in \partial B_r(z) \cap \triangle_i$ where $B_r(z)$ is a ball of radius r centered at z in $\triangle_i$ (with the radius measured with respect to the Euclidean distance of $\triangle_i$). This is the so-called Courant-Lebesgue Lemma which we prove as follows. Let (r, θ) be the (Euclidean) polar coordinates on $\triangle_i$ about z. With $g := \phi_i^*(g_0)$, write $g = (g_{jk})$ by setting

$$\begin{aligned} g_{11} &= g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right), \\ g_{12} = g_{21} &= g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right), \\ g_{22} &= g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right), \end{aligned}$$

and let $g = g_{11}g_{22} - g_{12}^2$. With (g^{jk}) the inverse matrix of (g_{jk}), we have

$$\begin{aligned} g^{22}\sqrt{g} &= \frac{g_{11}}{g}\sqrt{g} = \frac{g_{11}}{\sqrt{g}} \\ &= \frac{\left|\frac{\partial\phi_i}{\partial r}\right|^2}{\sqrt{\left|\frac{\partial\phi_i}{\partial r}\right|^2\left|\frac{\partial\phi_i}{\partial\theta}\right|^2 - \left(\frac{\partial\phi_i}{\partial r}\frac{\partial\phi_i}{\partial\theta}\right)^2}} \geq \frac{\left|\frac{\partial\phi_i}{\partial r}\right|^2}{\sqrt{\left|\frac{\partial\phi_i}{\partial r}\right|^2\left|\frac{\partial\phi_i}{\partial\theta}\right|^2}} \\ &= \frac{\left|\frac{\partial\phi_i}{\partial r}\right|}{\left|\frac{\partial\phi_i}{\partial\theta}\right|}. \end{aligned}$$

The D-quasiconformality of ϕ_i implies that

$$\frac{\left|\frac{\partial\phi_i}{\partial r}\right|}{\left|\frac{1}{r}\frac{\partial\phi_i}{\partial\theta}\right|} \geq \frac{1}{D},$$

which in turn implies $g^{22}\sqrt{g} \geq \frac{1}{rD}$. Therefore, denoting by the energy of β_i with respect to (the conformal structure defined by) the metric $g = \phi_i^*(g_0)$ by ${}^gE(\beta_i)$, we get

$$\begin{aligned} M' &\geq E(\alpha_i) \\ &= {}^gE(\beta_i) \\ &\geq \int_{B_{\sqrt{\delta}}(z)\cap\triangle} \sum_{j,k} g^{jk}\frac{\partial\beta_i}{\partial x^j}\frac{\partial\beta_i}{\partial x^k}\sqrt{g}dx \\ &\geq \int_{B_{\sqrt{\delta}}(z)\cap\triangle} g^{22}\left|\frac{\partial\beta_i}{\partial\theta}\right|^2\sqrt{g}dx \\ &\geq \frac{1}{D}\int_{B_{\sqrt{\delta}}(z)\cap\triangle}\left|\frac{\partial\beta_i}{\partial\theta}\right|^2\frac{1}{r}drd\theta. \end{aligned}$$

This means that there exists $r \in (\delta, \sqrt{\delta})$ so that

$$\int_{\partial B_r(z)\cap\triangle}\left|\frac{\partial\beta_i}{\partial\theta}\right|^2 d\theta \leq \frac{DM'}{\int_\delta^{\sqrt{\delta}}\frac{1}{r}dr} = \frac{2DM'}{\log\delta^{-1}}.$$

Thus, for $z_1, z_2 \in \partial B_r(z)\cap\triangle$, we have

$$\begin{aligned} |\beta_1(z_1) - \beta_2(z_2)| &\leq \int_{\partial B_r(z)\cap\triangle}\left|\frac{\partial\beta_i}{\partial\theta}\right|d\theta \\ &\leq (2\pi)^{\frac{1}{2}}\left(\int_{\partial B_r(z)\cap\triangle}\left|\frac{\partial\beta_i}{\partial\theta}\right|^2 d\theta\right)^{\frac{1}{2}} \\ &= (4\pi DM')^{\frac{1}{2}}\left(\log\frac{1}{\delta}\right)^{-\frac{1}{2}}, \end{aligned}$$

which proves inequality (10).

Using inequality (10), we prove that β_i is continuous at $t \in \partial\triangle$. We consider the following three cases:

Case 1. $t \in \partial\triangle_i \backslash (A \cup \{(1,0),(-1,0)\})$.

Because α_i is energy minimizing, β_i is also energy minimizing (with respect to the conformal structure defined by the metric $\phi_i^*(g_0)$). Since $\mathrm{tr}(\beta_i)\Big|_{\partial\triangle_i\backslash A} = \psi$ and ψ is Lipschitz (since it is a constant speed parameterization of a Jordan curve), this implies that β_i is continuous at t by the standard regularity theory for harmonic maps.

Case 2. $t \in A$.

Let $\delta > 0$ be given. Inequality (10) implies that there exists $r \in (\delta, \sqrt{\delta})$ so that

$$\mathrm{diam}(\bigcup_{i=1}^{3} \beta_i(\partial B_r(t) \cap \triangle_i)) \leq 3(4\pi DM')^{\frac{1}{2}} \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}}.$$

Assume δ (and hence r) is sufficiently small so that $\partial B_r(t) \cap (\partial\triangle_i \backslash A) = \emptyset$.

We claim $\beta_i(\bigcup_{i=1}^{3} B_r(t) \cap \triangle_i)$ is contained in $\mathrm{Cvx}(\bigcup_{i=1}^{3} \beta_i(\partial B_r(t) \cap \triangle_i))$, the convex hull of $\bigcup_{i=1}^{3} \beta_i(\partial B_r(t) \cap \triangle_i)$. Indeed, let

$$\Pi : \mathbf{R}^n \to \mathrm{Cvx}\left(\bigcup_{i=1}^{3} \beta_i(\partial B_r(t) \cap \triangle_i)\right) \subset \mathbf{R}^n$$

be the orthogonal projection map. Clearly, the map $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$ defined by

$$\tilde{\alpha}_i = \begin{cases} \alpha_i, & z \in \tilde{\triangle}_i \backslash \phi_i(B_r(t) \cap \triangle_i), \\ \Pi \circ \alpha_i, & z \in \phi_i(B_r(t) \cap \triangle_i), \end{cases}$$

for $i = 1,2,3$ is in $\mathcal{F}(\Phi)$. Furthermore, Π is distance decreasing which implies $E(\tilde{\alpha_i}) \leq E(\alpha_i)$. The uniqueness statement in Theorem 14 shows that $\Pi \circ \alpha_i(z) = \alpha_i(z)$ for each $z \in \phi_i(B_r(t) \cap \triangle_i)$, which in turn implies our claim. Using this convex hull property, we have that

$$\mathrm{diam}\left(\bigcup_{i=1}^{3} \beta_i(B_r(t) \cap \triangle_i)\right) \leq 3(4\pi DM')^{\frac{1}{2}} \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}},$$

and this shows that β_i is continuous at t.

Case 3. $t = (1,0)$ or $t = (-1,0)$.

The argument in this case is similar to Case 2, except we need to take into account that $\beta_i = \psi_i$ on $\partial\triangle\backslash A$. Let $\delta > 0$ be given. By the assumption that ψ_i is a constant speed parameterization of Γ_i and that the speed is less than L, we have

$$\mathrm{diam}(\psi_i((\partial\triangle_i \backslash A) \cap B_r(t))) < Lr.$$

Thus, using inequality (10) in Case 2 and noting that $\beta_i\Big|_{\partial\triangle_i\backslash A} = \psi_i$,

$$\mathrm{diam}\left(\bigcup_{i=1}^{3} \beta_i((\partial\triangle_i \backslash A \cap B_r(t)) \cup (\partial B_r(t) \cap \triangle_i))\right) < 3(4\pi DM')^{\frac{1}{2}} \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} + 3Lr$$

for $r \in (\delta, \sqrt{\delta})$. Using the fact that $\bigcup_{i=1}^{3} \beta_i(B_r(t) \cap \triangle_i)$ is contained in the convex hull of $\beta_i((\partial\triangle_i \backslash A \cap B_r(t)) \cup (\partial B_r(t) \cap \triangle_i))$, we have

$$\operatorname{diam}\left(\bigcup_{i=1}^{3} \beta_i(B_r(t) \cap \triangle_i)\right) < 3(4\pi DM')^{\frac{1}{2}} \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} + 3Lr,$$

and this shows that β_i is continuous at t. $\square$(*claim*)

To show uniqueness, let $\alpha^{(0)} := \alpha$, suppose that $\alpha^{(1)} \in \mathcal{F}(\Phi)$ is another c-energy minimizer and let $\alpha_i^{(t)} := (1-t)\alpha_i^{(0)} + t\alpha_i^{(1)}$. Then clearly $\alpha^{(t)} = (\alpha_1^{(t)}, \alpha_2^{(t)}, \alpha_3^{(t)}) \in \mathcal{F}(\Phi)$. By the usual convexity of energy statement (cf. [H], also see [SY], Chapter X (2.6ii)),

$$E(\alpha_i^{(t)}) \leq (1-t)E(\alpha_i^{(0)}) + tE(\alpha_i^{(1)}) - t(1-t)\int_{\tilde{\triangle}_i} |\alpha_i^{(0)} - \alpha_i^{(1)}|^2 dx,$$

which implies that $\alpha_i^{(0)} \equiv \alpha_i^{(1)}$. $\square$

Now we are ready to prove our main existence theorem.

Theorem 16. *If Γ is a non-degenerate boundary, the generalized Plateau problem can be solved. In other words, for some $1 \leq D < \infty$, there exists $\Phi^* \in \mathcal{P}(D)$, $c^* \in \mathcal{C}$ and $\alpha^* \in \mathcal{F}(\Phi^*)$ so that*

$$E_{c^*}(\alpha^*) = \inf_{\Phi \in \mathcal{P}} \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi)} E_c(\alpha).$$

Proof. The choice of $D = D(\Gamma)$ is made by Definition 13. Let $\{\Phi^k = (\phi_1^k, \phi_2^k, \phi_3^k)\}_{k=1}^{\infty} \subset \mathcal{F}(D)$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty} \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi^k)} E_c(\alpha) = \inf_{\Phi \in \mathcal{P}(D)} \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi)} E_c(\alpha).$$

By the compactness of $\mathcal{C}$ and by Theorem 14, there exists $c^k = (c_1^k, c_2^k, c_3^k) \in \mathcal{C}$ so that

$$E_{c^k}(\alpha^k) = \inf_{c \in \mathcal{C}} \inf_{\alpha \in \mathcal{F}(\Phi^k)} E_c(\alpha),$$

where $\alpha^k = (\alpha_1^k, \alpha_2^k, \alpha_3^k) \in \mathcal{F}(\Phi^k)$ is a c^k-energy minimizing map in $\mathcal{F}(\Phi^k)$. We may assume that $E_{c^k}(\alpha^k) < M$ for all $k = 1, 2, ...$, which implies $E(\alpha_i^k) < M'$ for some $M' < \infty$ for all $i = 1, 2, 3$ and $k = 1, 2, ...$ (cf. proof of Theorem 14).

Let $\beta_i^k := \alpha_i^k \circ \phi_i^k$. From the proof of Claim 15, we see that the modulus of continuity of $\beta_i^k\big|_{\partial\triangle_i}$ is independent of k (and dependent only on M', D and L), and hence $\{\beta_i^k\big|_{\partial\triangle_i}\}_{k=1}^{\infty}$ is an equicontinuous family of functions. Furthermore, we claim that $\beta_i^k\big|_{\partial\triangle_i}$ is also uniformly continuous. To see this, let $\epsilon > 0$ be given. From the assumption that ψ_i is a constant speed parameterization of Γ with speed $\leq L$, we get $|\psi_i(t) - \psi_i(s)| < \frac{\epsilon}{2}$ whenever $|t-s| < \frac{\epsilon}{2L}$ and $t, s \in \partial\triangle_i \backslash A$. Since $\beta_i^k\big|_{\partial\triangle_i \backslash A} = \psi_i\big|_{\partial\triangle_i \backslash A}$, we then have

$$|\beta_i^k(t) - \beta_i^k(s)| < \frac{\epsilon}{2} \text{ whenever } |t-s| < \frac{\epsilon}{2L} \text{ and } t, s \in \partial\triangle_i \backslash A.$$

Additionally, choose $\delta > 0$ be sufficiently small so that

$$3(4\pi DM')^{\frac{1}{2}} \left(\log \frac{1}{\delta}\right)^{-\frac{1}{2}} + 3L\sqrt{\delta} < \frac{\epsilon}{2}.$$

Then from the proof of the Claim 15,

$$|\beta_i^k(t) - \beta_i^j(s)| \le \frac{\epsilon}{2} \text{ whenever } |t-s| < \delta \text{ and } t, s \in A \cup \{(1,0), (-1,0)\}.$$

Therefore, if $\eta = \min\{\frac{\epsilon}{2L}, \delta\}$, then for $t, s \in \partial\triangle_i$

$$|\beta_i^k(t) - \beta_i^k(s)| < \epsilon \text{ whenever } |t-s| < \eta. \tag{11}$$

Here, the inequality (11) is independent of $k = 1, 2, ...$ and $i = 1, 2, 3$.

Since ϕ_i^k, and hence $(\phi_i^k)^{-1}$ are D-quasiconformal and fix $(1,0)$, $(-1,0)$ and $(0,1)$, $\{\phi_i^k\}_{i=1}^\infty$ and $\{(\phi_i^k)^{-1}\}_{i=1}^\infty$ respectively are equicontinuous families of maps (cf. [Le], Theorem 2.1). Consequently, $(\phi_i^k)^{-1}\Big|_{\partial\triangle_i}$ is λ-quasisymmetric, where λ is dependent only on D. In particular, this means that $\{(\phi_i^k)^{-1}\Big|_{\partial\tilde{\triangle}_i}\}_{k=1}^\infty$ is an equicontinuous family of functions (cf. [Le], Lemma 5.2). Therefore $\{\alpha_i^k\Big|_{\partial\tilde{\triangle}_i} = \beta_i^k \circ (\phi_i^k)^{-1}\Big|_{\partial\tilde{\triangle}_i}\}_{k=1}^\infty$ is an equicontinuous family of functions. Indeed, given $\tau \in \partial\tilde{\triangle}_i$ and $\epsilon > 0$, there exists $\eta > 0$ so that for $\sigma \in \partial\tilde{\triangle}_i$,

$$|\beta_i^k \circ (\phi_i^k)^{-1}(\tau) - \beta_i^k \circ (\phi_i^k)^{-1}(\sigma)| < \epsilon \text{ whenever } |(\phi_i^k)^{-1}(\tau) - (\phi_j^k)^{-1}(\sigma)| < \eta$$

by inequality (11) and there exists $\theta > 0$ so that

$$|(\phi_i^k)^{-1}(\tau) - (\phi_i^k)^{-1}(\sigma)| < \eta \text{ whenever } |\tau - \sigma| < \theta$$

by the equicontinuity of $\{(\phi_i^k)^{-1}\Big|_{\partial\tilde{\triangle}_i}\}$. Here, θ is chosen independently of k which proves the equicontinuity of $\{\alpha_i^k\Big|_{\partial\tilde{\triangle}_i}\}$. Thus, we can extract subsequences (still indexed by k) so that

$$\phi_i^k, \quad (\phi_i^k)^{-1}, \quad \alpha_i^k\Big|_{\partial\tilde{\triangle}_i}$$

converges uniformly as $k \to \infty$ to

$$\phi_i^* : \triangle_i \to \tilde{\triangle}_i, \quad (\phi_i^*)^{-1} : \tilde{\triangle}_i \to \triangle_i, \quad \gamma_i^* : \partial\tilde{\triangle} \to \mathbf{R}^n$$

respectively for each i. The map ϕ_i^* is D-quasiconformal (cf. [Le], Theorem 2.3). Let α_i^* be the Dirichlet solution with boundary value γ_i^*. The maximum principle implies that α_i^k converges uniformly to α_i^* on the closure $\tilde{\triangle}_i'$ of $\tilde{\triangle}_i$, and thus the lower semicontinuity of energy implies

$$E_{c^*}(\alpha^*) \le \liminf_{k\to\infty} E_{c^k}(\alpha^k) = \inf_{\Phi\in\mathcal{P}(D)} \inf_{c\in\mathcal{C}} \inf_{\alpha\in\mathcal{F}(\Phi)} E_c(\alpha).$$

Moreover,

$$\begin{aligned}
\alpha_i^* \circ \phi_i^*\Big|_{\partial\triangle_i}(t) &= \gamma_i^* \circ \phi_i^*\Big|_{\partial\triangle_i}(t) \\
&= \lim_{k\to\infty} \alpha_i^k \circ \phi_i^k(t) \\
&= \begin{cases} \lim\limits_{k\to\infty} \alpha_j^k \circ \phi_j^k\Big|_{\partial\triangle_i}(t) & \text{for } t \in A, \\ \psi_i(t) & \text{for } t \in \partial\triangle_i \backslash A, \end{cases} \\
&= \begin{cases} \alpha_j^* \circ \phi_j^*\Big|_{\partial\triangle_i}(t) & \text{for } t \in A, \\ \psi_i(t) & \text{for } t \in \partial\triangle_i \backslash A. \end{cases}
\end{aligned}$$

Therefore, $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*) \in \mathcal{F}(\Phi^*)$, where $\Phi^* = (\phi_1^*, \phi_2^*, \phi_3^*)$. □

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