

ON THE p -COMPACT GROUPS CORRESPONDING TO THE p -ADIC REFLECTION GROUPS $G(q, r, n)$

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ABSTRACT. There exists an infinite family of p -compact groups whose Weyl groups correspond to the finite p -adic pseudoreflection groups $G(q, r, n)$ of family 2a in the Clark-Ewing list. In this paper we study these p -compact groups. In particular, we construct an analog of the classical Whitney sum map, a family of monomorphisms and a spherical fibration which produces an analog of the classical J -homomorphism. Finally, we also describe a faithful complexification homomorphism from these p -compact groups to the p -completion of unitary compact Lie groups.

INTRODUCTION

The notion of a Lie group was introduced in the last century. The classical examples are matrix groups such as orthogonal, unitary or symplectic groups. The program for the understanding of the homotopy properties of compact Lie groups led to the concept of a p -compact group that was introduced by Dwyer and Wilkerson [9] in 1994. They are p -local versions of finite loop spaces. Namely, a p -compact group is a triple $X = (X, BX, e)$, where BX is a p -complete pointed space, X is a space such that $H^*(X; \mathbb{F}_p)$ is finite, and $e : X \rightarrow \Omega BX$ is a homotopy equivalence.

The first examples of p -compact groups are the p -completions (in the sense of Bousfield-Kan [4]) of compact connected Lie groups and their classifying spaces. Many properties of compact Lie groups can be reinterpreted as homotopy theoretic properties of the classifying spaces in such a way that the concept extends to the category of p -compact groups (see [10]). For example, they admit a concept of maximal torus and Weyl group. The Weyl group is a finite p -adic reflection group, that is, a pair (W, L) , where L is a $\hat{\mathbb{Z}}_p$ -lattice and W is a finite subgroup of $GL(L)$ generated by reflections.

The recently completed classification of the p -compact groups at odd primes [2] states that connected p -compact groups are in one-to-one correspondence with finite $\hat{\mathbb{Z}}_p$ -reflection groups via the Weyl group action on the maximal torus. Since finite $\hat{\mathbb{Z}}_p$ -reflection groups, p odd, were classified by Notbohm [23] (from the classification over \mathbb{C} by Shephard and Todd [25] and over \mathbb{Q}_p^\wedge by Clark and Ewing [8]), this gives a rather complete picture. This classification contains four infinite families, and we

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also use the numbering of the different cases in [8], [23], and [25] (namely, families 1, 2a, 2b, 3).

Those p -compact groups whose Weyl group belongs to the infinite family 2a in the classification of $\hat{\mathbb{Z}}_p$ -pseudoreflection groups generalize classical matrix groups, in the sense that their Weyl group looks like the one of classical Lie groups, but allowing matrix entries to be finite roots of unity in $(\hat{\mathbb{Z}}_p)^*$. Their Weyl groups, as pseudoreflection groups, correspond to family no. 2a in the Clark-Ewing list [8].

Let us describe their Weyl groups $(G(q, r, n), L_n)$ where $L_n = (\hat{\mathbb{Z}}_p)^n$. For any $q \geq 1$, fix an embedding $\mu_q \subset (\hat{\mathbb{Z}}_p)^*$ of the group of q th roots of the unity where $q|p-1$. If $r|q$, $n > 1$, define

$$A(q, r, n) = \{ (z_1, \dots, z_n) \in \mu_q^n \mid z_1 \cdots z_n \in \mu_{q/r} \}.$$

$G(q, r, n)$ is a split extension of Σ_n by $A(q, r, n)$ (Σ_n acts on $A(q, r, n)$ by permuting the factors). $G(q, r, n)$ has a representation in $GL(n, \hat{\mathbb{Z}}_p)$ generated by $A(q, r, n)$ regarded as diagonal matrices, and the group Σ_n of permutation matrices,

$$G(q, r, n) \cong A(q, r, n) \rtimes \Sigma_n.$$

Because of the uniqueness property in [22, Theorem 5.2], and the fact that any classical group in the above sense not in family no. 2a in the Clark-Ewing list [8] (families no. 1, 2b, 3) is either the Weyl group of a Lie group (family no. 1) or nonmodular (families no. 2b, 3), we can speak of the unique p -compact group $X(q, r, n)$ realizing the Weyl group data $(G(q, r, n), L_n)$. The Weyl groups of $SO(2n)$ and $SO(2n+1)$ belong to family no. 2a: by taking $(q, r, n) = (2, 2, n)$ and $(q, r, n) = (2, 1, n)$, respectively.

The purpose of this paper is to study these p -compact groups and certain homogeneous spaces which arise from certain monomorphisms between them. In particular, we construct morphisms similar to the classical Whitney sum map, a family of monomorphisms and a spherical fibration which produces an analog of the classical J -homomorphism.

Theorem A. *Let p be an odd prime. There exists a morphism of p -compact groups*

$$\Phi : X(q_1, r_1, n_1) \times \cdots \times X(q_s, r_s, n_s) \rightarrow X(q, r, n),$$

where $r_i|q_i$, $q_i|q|(p-1)$, $\frac{q_i}{r_i}|r$ and $\sum n_i = n$ such that the induced map on the Weyl groups corresponds to the canonical coordinate-wise inclusion

$$(G(q_1, r_1, n_1), L_{n_1}) \times \cdots \times (G(q_s, r_s, n_s), L_{n_s}) \rightarrow (G(q, r, n), L_n).$$

Moreover Φ is unique up to homotopy satisfying this condition.

When the p -compact groups involved are p -completions of classifying spaces of matrix groups, the morphisms defined in Theorem A are homotopic to the p -completion of the classical Whitney sum maps.

By considering the homotopy fiber of the above Whitney sum map Φ , we obtain p -compact versions of the classical Grassmann manifolds. In a similar way, p -compact versions of classical Stiefel manifolds can also be constructed.

The restriction of the morphisms Φ in Theorem A to each factor gives monomorphisms between the p -compact groups $X(q, r, n)$. The following theorem describes another type of monomorphism between these p -compact groups.

Theorem B. *Let p be an odd prime. Suppose $r_1|q_1|p-1$ and $r|q|p-1$. There exists a monomorphism of p -compact groups Γ ,*

$$\Gamma : X(q_1, r_1, n) \rightarrow X(q, r, n+m),$$

where $q_1|q$ and $m > 0$ such that the induced map on Weyl groups is

$$\iota : (G(q_1, r_1, n), L_n) \rightarrow (G(q, r, n+m), L_{n+m})$$

defined by

$$\iota((a_1, \dots, a_n), \sigma) = ((a_1, \dots, a_n, (a_1 \cdots a_n)^{-1}, 1, \dots, 1), \sigma),$$

where $(a_1, \dots, a_n) \in A(q_1, r_1, n)$ and $\sigma \in \Sigma_n$, and it is an inclusion on the first coordinates on the $\hat{\mathbb{Z}}_p$ -lattices.

The p -completion of the inclusion of $SO(2n+1)$ into $SO(2(n+1))$ is an example of a morphism of the type described in Theorem B.

This large family of p -compact groups is closely related to the classifying space for orientable mod p spherical fibrations. This relation is explicitly stated in Theorems C and D.

Theorem C. *Let p be an odd prime. There exists a mod p spherical fibration η_n ,*

$$(S^{2n+1})_p^\wedge \rightarrow E \rightarrow BX(p-1, p-1, n+1),$$

whose Euler class is $e \in H^*(BX(p-1, p-1, n+1); \mathbb{F}_p) \cong \mathbb{F}_p[y_1, \dots, y_n, e]$, where $\deg(y_i) = 2i(p-1)$ and $\deg(e) = 2(n+1)$.

For $p = 2$, the existence of such fibration is classical. It can be obtained by considering the action of $U(n+1)$ on the unit sphere in $\mathbb{R}^{2(n+1)}$.

The spherical fibrations in Theorem C allow us to describe an analog of the classical J-homomorphism. Let BSG be the classifying space of the orientable spherical fibrations. We consider the morphisms

$$X(p-1, p-1, n) \rightarrow X(p-1, p-1, n+1),$$

which are restrictions of the monomorphisms Φ in Theorem A to the first factor, and we denote by $BX(p-1)$ the corresponding homotopy colimit, $\text{hocolim}_n BX(p-1, p-1, n)$.

Theorem D. *Let p be an odd prime. There exists a map $J : BX(p-1) \rightarrow BSG_p^\wedge$ such that J^* is an F -isomorphism.*

The cohomology of BSG has been computed by Tsuchiya [26] for odd primes, and by Milgram [20] when $p = 2$.

Finally, we answer the question of whether a p -compact group of type $X(q, r, n)$ admits a monomorphism into a unitary group.

Theorem E. *Let p be an odd prime. For $r|q|p-1$, there exists a monomorphism of p -compact groups*

$$c : X(q, r, n) \rightarrow U(nq)_p^\wedge.$$

When $q = 2$, this morphism is the p -completion of the complexification map for classical matrix compact Lie groups.

The proof of Theorems A and B follows by using the results and methods in [22]. The proof of Theorem D relies on the description of the cohomology of BSG and the computation of the characteristic classes for the mod p spherical fibrations in Theorem C.

From now on, p is an odd prime and $H^*(-) := H^*(-; \mathbb{F}_p)$. This paper is organized as follows. The construction of generalized Whitney sum maps (Theorem A) and a family of monomorphisms (Theorem B) is given in Section 2 and the study of the homogeneous spaces in Section 3. Section 4 analyzes the relation between p -compact groups $X(p-1, p-1, n)$ and the classifying space for mod p spherical fibrations (Theorems C and D). The complexification map (Theorem E) is constructed in Section 5.

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2. MAPS BETWEEN GENERALIZED GRASSMANNIANS

This section is devoted to the proof of Theorems A and B. For completeness, we sketch the construction of the p -compact groups $X(q, r, n)$ as a homotopy colimit following [22].

Let \mathcal{H} be a family of subgroups of a given finite group G . The orbit category $\mathcal{O}_{\mathcal{H}}(G)$ associated to \mathcal{H} is the category whose objects are orbits G/H for each $H \in \mathcal{H}$ and morphisms are given by G -maps between orbits, $\text{Hom}_G(G/H, G/K)$.

Definition 2.1 ([22]). Let \mathcal{H} be the family of subgroups $H \leq G(q, r, n)$ conjugate to some $\Sigma(\Pi) := \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$, where $n_1 + \cdots + n_s = n$ and $n_i = p^j$ for some $j \in \mathbb{N} \cup \{0\}$.

Theorem 2.2 ([22]). Fix any odd prime p , any $r|q|(p-1)$ with $q > 1$, and any $n > 1$. Let $G(q, r, n)$ and $\mathcal{O}_{\mathcal{H}}(G(q, r, n))$ be as above. Then there exists a functor Ψ ,

$$\Psi : \mathcal{O}_{\mathcal{H}}(G(q, r, n)) \longrightarrow \text{Top},$$

such that:

- (1) For any partition $\{n_1, \dots, n_s\}$ of n with $n_i = p^j$ for some $j \in \mathbb{N} \cup \{0\}$, $\Psi(G/\Sigma(\Pi)) \simeq BU(\Pi)_p^\wedge$.
- (2) The composite $H^*(-; \hat{\mathbb{Z}}_p) \circ \Psi$ is isomorphic to the fixed point functor

$$G(q, r, n)/H \mapsto \mathbb{Z}_p^\wedge[x_1, \dots, x_n]^H.$$

- (3) If we set $BX(q, r, n) := (\text{hocolim } \Psi)_p^\wedge$,

$$H^*(BX(q, r, n); \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_n]^{G(q, r, n)}.$$

Notation. For simplicity, we will use the following notation:

$$\begin{aligned} X^\times &:= X(q_1, r_1, n_1) \times \cdots \times X(q_s, r_s, n_s), \\ G^\times &:= G(q_1, r_1, n_1) \times \cdots \times G(q_s, r_s, n_s), \end{aligned}$$

where $r_i|q_i|(p-1)$ for any $i = 1, \dots, s$. \mathcal{H}^\times is the family of subgroups of G^\times corresponding to the product $\mathcal{H}_1 \times \cdots \times \mathcal{H}_s$, where \mathcal{H}_i is the family \mathcal{H} defined in Definition 2.1 corresponding to $G(q_i, r_i, n_i)$.

The special feature of subgroup families \mathcal{H} in Definition 2.1 is described in [22, Lemma 3.1]: each p -subgroup of $G(q, r, n)$ is contained in a unique minimal element in \mathcal{H} . This property is strongly used in order to compute higher limits of certain functors ([22, Section 2]). In the next lemma, we check that this property is also satisfied when we consider products of such families of subgroups.

Lemma 2.3. *Let G_1, G_2 be two finite groups. If \mathcal{H}_i is a family of subgroups of G_i for $i = 1, 2$ such that each p -subgroup of G_i is contained in a unique minimal element of \mathcal{H}_i , then the family $\mathcal{H}_1 \times \mathcal{H}_2$ defined in the obvious way ($H \in \mathcal{H}_1 \times \mathcal{H}_2$ if and only if $H = H_1 \times H_2$, where $H_i \in \mathcal{H}_i$ for $i = 1, 2$) also satisfies the same property with respect to $G_1 \times G_2$.*

Proof. For $i = 1, 2$, let $p_i: G_1 \times G_2 \rightarrow G_i$ be the corresponding projections on each component. We need to check that if P is a p -subgroup of $G_1 \times G_2$, then there exists a unique minimal $H_1 \times H_2 \in \mathcal{H}_1 \times \mathcal{H}_2$ such that $P \leq H_1 \times H_2$.

First of all, we observe that if $P_i = p_i(P)$, then $P \leq P_1 \times P_2$, where $P_i \leq G_i$ are p -subgroups for $i = 1, 2$. By [22, Lemma 3.1], there exists a unique minimal subgroup $H_i \in \mathcal{H}_i$ such that $P_i \leq H_i$. Therefore, $P \leq P_1 \times P_2 \leq H_1 \times H_2$.

Next, we show that this subgroup $H_1 \times H_2$ is minimal. Suppose that there exists $Q_1 \times Q_2 \in \mathcal{H}_1 \times \mathcal{H}_2$ such that $P \leq Q_1 \times Q_2$. Since $P_i = p_i(P) \subset p_i(Q_1 \times Q_2) = Q_i$ and H_i are minimal with this property, it follows that $H_i \subset Q_i$. Finally, we obtain that $H_1 \times H_2 \subset Q_1 \times Q_2$. \square

Proof of Theorem A. We denote $G(q, r, n)$ simply by G . Each subgroup in the family \mathcal{H}^\times also belongs to the family \mathcal{H} of subgroups of G described in Definition 2.1. Therefore, there exists a functor between the corresponding orbit categories,

$$\phi : \mathcal{O}_{\mathcal{H}^\times}(G^\times) \longrightarrow \mathcal{O}_{\mathcal{H}}(G),$$

defined on objects by $\phi(G^\times/H) = G \times_{G^\times} G^\times/H \cong G/H$ and, in the obvious way, on morphisms (a G^\times -equivariant map $G^\times/H \rightarrow G^\times/K$ induces a G -equivariant map $G \times_{G^\times} G^\times/H \rightarrow G \times_{G^\times} G^\times/K$).

Let $\Psi : \mathcal{O}_{\mathcal{H}}(G) \rightarrow Top$ be the functor in Theorem 2.2 such that

$$BX(q, r, n) := (\text{hocolim}_{\mathcal{O}_{\mathcal{H}}(G)} \Psi)_p^\wedge$$

and $\Psi(G/\Sigma(\Pi)) \simeq BU(\Pi)_p^\wedge$.

The composite $\Psi \circ \phi$ provides an induced functor on $\mathcal{O}_{\mathcal{H}^\times}(G^\times)$,

$$\phi_{\natural}(\Psi) = \Psi \circ \phi : \mathcal{O}_{\mathcal{H}^\times}(G^\times) \rightarrow \mathcal{O}_{\mathcal{H}}(G) \rightarrow Top,$$

and a map between the p -completions of the corresponding homotopy colimits,

$$(\text{hocolim}_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)} \phi_{\natural}(\Psi))_p^\wedge \rightarrow (\text{hocolim}_{\mathcal{O}_{\mathcal{H}}(G)} \Psi)_p^\wedge.$$

In order to construct the map Φ , it only remains to show that the homotopy colimit on the left-hand side is homotopy equivalent to BX^\times . We proceed by constructing a map

$$i : (\text{hocolim}_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)} \phi_{\natural}(\Psi))_p^\wedge \rightarrow BX^\times,$$

which induces an isomorphism in mod p cohomology. For each $G^\times/\Sigma(\Pi)$, where $\Sigma(\Pi) \in \mathcal{H}^\times$, there is a map

$$i_\Pi : \phi_{\natural}(\Psi)(G^\times/\Sigma(\Pi)) \simeq \prod BU(n_i) \rightarrow BX^\times.$$

All these maps combine to define an element in the inverse limit

$$[\{i_\Pi\}] \in \lim_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)} [\prod BU(n_i), BX^\times].$$

The obstructions to extend $\{i_\Pi\}$ to a map from the homotopy colimit are described by Wojtkowiak in [28]. These obstructions lie in

$$\varprojlim_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)}^{i+1} \pi_i(\text{Map}(BU(n_1) \times \cdots \times BU(n_s), BX^\times)_{i_\Pi}).$$

We show that these higher limits vanish by describing the homotopy type of the mapping spaces involved. There is a homotopy equivalence of spaces ([22, Proposition 4.6])

$$\text{Map}(BU(n_1) \times \cdots \times BU(n_s), BX^\times)_{i_\Pi} \simeq BZ(\prod BU(n_i)).$$

Since $BZ(U(\Pi))_p^\wedge \simeq K((\hat{\mathbb{Z}}_p)^n)^{\Sigma(\Pi)}, 2)$, the homotopy groups

$$\pi_i(\text{Map}(BU(n_1) \times \cdots \times BU(n_s), BX^\times)_{i_\Pi}) = \pi_i(BT)^{\Sigma(\Pi)}$$

are the invariants by a permutation action. By Lemma 2.3, the family \mathcal{H}^\times satisfies the hypothesis of [22, Proposition 2.3]. Therefore, the higher limits vanish,

$$(1) \quad \varprojlim_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)}^i \pi_j(BT)^{\Sigma(\Pi)} = \begin{cases} 0, & i > 0, \\ \pi_j(BT)^{G^\times}, & i = 0. \end{cases}$$

It remains to show that the map just constructed induces an isomorphism on mod p cohomology. In order to compute the cohomology of the homotopy colimit of the functor $\phi_{\mathfrak{H}}$, we shall use the Bousfield-Kan spectral sequence [4]. It is a spectral sequence with E_2 -term

$$E_2^{i,j} \cong \varprojlim_{G^\times/\Sigma(\Pi) \in \mathcal{O}_{\mathcal{H}^\times}(G^\times)}^i H^j(BU(\Pi))$$

converging to $H^*(\text{hocolim}_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)} \phi_{\mathfrak{H}}(\Psi))$.

This spectral sequence collapses at the E_2 term because of the vanishing of the following higher limits ([22, Proposition 2.3]):

$$\begin{aligned} \varprojlim_{G^\times/\Sigma(\Pi) \in \mathcal{O}_{\mathcal{H}^\times}(G^\times)}^i H^j(BU(\Pi)) &\cong \varprojlim_{G^\times/\Sigma(\Pi) \in \mathcal{O}_{\mathcal{H}^\times}(G^\times)}^i \mathbb{F}_p[x_1, \dots, x_n]^{\Sigma(\Pi)} \\ &\cong \begin{cases} 0, & i > 0, \\ \mathbb{F}_p[x_1, \dots, x_n]^{G^\times}, & i = 0. \end{cases} \end{aligned}$$

Therefore, $H^*(BX^\times) \cong \mathbb{F}_p[x_1, \dots, x_n]^{G^\times}$, and, by construction, the map i induces an isomorphism in mod p cohomology.

Summarizing, we have constructed a morphism of p -compact groups

$$BX^\times \simeq (\text{hocolim}_{\mathcal{O}_{\mathcal{H}^\times}(G^\times)} \phi_{\mathfrak{H}}(\Psi))_p^\wedge \rightarrow BX$$

which induces the coordinate-wise inclusion of Weyl groups

$$(G(q_1, r_1, n_1), L_{n_1}) \times \cdots \times (G(q_s, r_s, n_s), L_{n_s}) \xrightarrow{\iota} (G(q, r, n), L_n).$$

Moreover, Φ is a regular (also called p -toric) map (see [21, Proposition 2.4]). Therefore, the induced morphism between the Weyl groups is unique.

Assume that $f : X^\times \rightarrow X$ is another morphism of p -compact groups such that the induced map in the Weyl groups is ι . Since $H^*(BX; \mathbb{Q}_p^\wedge) \cong H^*(BT; \mathbb{Q}_p^\wedge)^{W^\times}$ ([9, Theorem 9.7]), Bf and $B\Phi$ induce the same morphism in rational cohomology. The restrictions of Bf and $B\Phi$ to each $BU(\Pi)_p^\wedge$ also induce the same homomorphism in $H^*(-; \hat{\mathbb{Z}}_p) \otimes \mathbb{Q}$. Applying [22, Proposition 3.3] to the situation in which $X =$

$X(q, r, n)$, $G = U(n)$ and $H = U(\Pi)$, we obtain that, in fact, these restrictions are homotopic. That means that $B\Phi$ and Bf define the same element in the inverse limit

$$\lim_{\mathcal{O}_{\mathcal{H}(\Pi)}(G^\times)} [BU(\Pi)_p^\wedge, BX].$$

The obstructions for the two maps to be homotopic lie in the following higher limits (see [28]):

$$\lim_{\leftarrow}^i_{G/\Sigma(\Pi) \in \mathcal{O}_{\mathcal{H}^\times}(G^\times)} \pi_i(\text{Map}(BU(\Pi)_p^\wedge, BX)_{i_\Pi})$$

for $i \geq 1$. These higher limits have been computed in (1), and they all vanish. Therefore, the obstructions for uniqueness vanish, and $Bf \simeq B\Phi$. \square

Lemma 2.4. *Let $i: B\mathbb{Z}/p\mathbb{Z} \rightarrow BU(n) \times (BS^1)^m \rightarrow BX(q, r, m+n)$, where the first map is given by the inclusion of $\mathbb{Z}/p\mathbb{Z}$ in the centre of $U(n)$ and the last map is the corresponding inclusion of*

$$\Psi(G(q, r, n+m)/(\Sigma_n \times 1 \times \cdots \times 1)) \simeq BU(n)_p^\wedge \times ((BS^1)_p^\wedge)^m$$

into the homotopy colimit $BX(q, r, m+n)$. Then,

$$\text{Map}(B\mathbb{Z}/p, BX(q, r, n+m))_i \simeq BU(n) \times BX(q, r, m),$$

up to p -completion.

Proof. The cohomology of the mapping space can be computed using Lannes T -functor machinery [16]. The components of the T -functor applied to algebras of invariants can be computed by means of the isotropy subgroups of each component (see [11, Proof of Theorem 1.1]). Since the isotropy subgroup of i^* is $\Sigma_n \times G(q, r, m)$, then

$$\begin{aligned} T_{i^*}(H^*(BX(q, r, n+m))) &\cong T_{i^*}(\mathbb{F}_p[x_1, \dots, x_{n+m}]^{G(q, r, m+n)}) \\ &\cong \mathbb{F}_p[x_1, \dots, x_{n+m}]^{\Sigma_n \times G(q, r, m)} \\ &\cong H^*(BU(n) \times BX(q, r, m)). \end{aligned}$$

Next, we consider the composite

$$B\mathbb{Z}/p\mathbb{Z} \times BU(n)_p^\wedge \times BX(q, r, m) \xrightarrow{\mu \times \text{id}} BU(n)_p^\wedge \times BX(q, r, m) \rightarrow BX(q, r, n+m),$$

where the first map is given by multiplication with the central subgroup $\mathbb{Z}/p\mathbb{Z}$ in $U(n)$ and the last one is a morphism described in Theorem A. From the previous considerations on the cohomology of that mapping space, it follows that the adjoint of the above composite of morphisms

$$BU(n)_p^\wedge \times BX(q, r, m) \rightarrow \text{Map}(B\mathbb{Z}/p\mathbb{Z}, BX(q, r, n+m))_i$$

induces an isomorphism in mod p cohomology. \square

Proof of Theorem B. We consider the composite

$$g_\Pi: BU(\Pi) \xrightarrow{i_1} BU(\Pi) \times (BS^1)^m \rightarrow BX(q, r, n+m),$$

where the first map is the inclusion in the first factor and the last map is the corresponding inclusion of

$$\Psi(G(q_1, r_1, n)/(\Sigma(\Pi) \times 1 \times \cdots \times 1)) \simeq BU(\Pi)_p^\wedge \times ((BS^1)_p^\wedge)^m$$

into the homotopy colimit $BX(q, r, m+n)$.

Next, we check that these maps commute up to homotopy with maps induced by morphism in the orbit category $\mathcal{O}_{\mathcal{H}}(G(q_1, r_1, n))$. Let f be such a morphism in the orbit category inducing the following diagram:

$$\begin{array}{ccc}
 BU(\Pi)_p^\wedge & \xrightarrow{i_1} & BU(\Pi)_p^\wedge \times ((BS^1)_p^\wedge)^m & \longrightarrow & BX(q, r, n + m) \\
 \downarrow f & & \downarrow g & \nearrow & \\
 BU(\Pi')_p^\wedge & \xrightarrow{i_1} & BU(\Pi')_p^\wedge \times ((BS^1)_p^\wedge)^m & &
 \end{array}$$

The maps induced by morphisms in the orbit category are of three types. We consider them separately. If f is an inclusion map or a permutation of the factors, then we take $g = f \times \text{id}$. If f is a product of Adams operations $\Psi^{a_1} \times \dots \times \Psi^{a_s}$, then we let $g = \Psi^{a_1} \times \dots \times \Psi^{a_s} \times \Psi^{(a_1 \dots a_s)^{-1}} \times \text{id}$. With this choice of g , the left square is homotopy commutative. The right triangle is homotopy commutative because it commutes in $H^*(-; \hat{\mathbb{Z}}_p) \otimes \mathbb{Q}$ ([22, Proposition 3.3]).

Summarizing, we have obtained an element in the following inverse limit:

$$\{[g_\Pi]\} \in \varprojlim_{\mathcal{O}_{\mathcal{H}}(G(q_1, r_1, n))} [BU(\Pi)_p^\wedge, BX(q, r, n + m)].$$

The obstructions to extend these compatible maps to a morphism from $BX(q_1, r_1, n)$ lie in the following higher limits ([28]):

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G(q_1, r_1, n))}^{i+1} \pi_i(\text{Map}(BU(\Pi)_p^\wedge, BX(q, r, n + m))_{g_\Pi}).$$

First, we analyze the homotopy type of these mapping spaces,

$$\begin{aligned}
 & \text{Map}(BU(\Pi)_p^\wedge, BX(q, r, n + m))_{g_\Pi} \\
 & \simeq \text{Map}(B\mathbb{Z}/p \times BU(\Pi)_p^\wedge, BX(q, r, n + m))_{g_\Pi \circ Bmult} \\
 & \simeq \text{Map}(BU(\Pi)_p^\wedge, \text{Map}(B\mathbb{Z}/p, BX(q, r, n + m))_{g_\Pi|_{B\mathbb{Z}/p}})_{ad},
 \end{aligned}$$

where $\mathbb{Z}/p\mathbb{Z}$ is a central subgroup of $U(\Pi)$ which is also central in $U(n)$. The first homotopy equivalence follows from [9, Lemma 7.5]. Moreover, by Lemma 2.4,

$$\text{Map}(B\mathbb{Z}/p, BX(q, r, n + m))_{g_\Pi|_{B\mathbb{Z}/p}} \simeq BU(n) \times BX(q, r, m),$$

up to p -completion. Therefore, by [22, Proposition 4.6],

$$\begin{aligned}
 \text{Map}(BU(\Pi)_p^\wedge, BX(q, r, n + m))_{g_\Pi} & \simeq \text{Map}(BU(\Pi)_p^\wedge, BU(n)_p^\wedge \times BX(q, r, m))_{incl \times *} \\
 & \simeq BZ(U(\Pi))_p^\wedge \times BX(q, r, m).
 \end{aligned}$$

Applying [22, Proposition 2.3], we obtain the desired vanishing result

$$\begin{aligned}
 & \varprojlim^i \pi_j(\text{Map}(BU(\Pi), BX(q, r, n + m))_{g_\Pi}) \\
 & \cong \varprojlim^i \pi_j(BT^n)^{\Sigma(\Pi)} \times \pi_j(BX(q, r, m)) = 0,
 \end{aligned}$$

if $i > 0$.

Finally, $B\Gamma$ is a monomorphism since the cohomology of its homotopy fiber is \mathbb{F}_p -finite (Definition 3.1 and Proposition 3.2). □

The uniqueness properties of the maps constructed in Theorems A and B can be used to check that the following diagrams are homotopy commutative:

$$\begin{array}{ccc}
 BX(q_1, r_1, n_1) \times BX(q_2, r_2, n_2) \times BX(q_3, r_3, n_3) & \xrightarrow{B\Phi_{n_1, n_2} \times \text{id}} & BX(q, r, n_1 + n_2) \times BX(q_3, r_3, n_3) \\
 \text{id} \times B\Phi_{n_2, n_3} \downarrow & & B\Phi_{n_1 + n_2, n_3} \downarrow \\
 BX(q_1, r_1, n_1) \times BX(q, r, n_2 + n_3) & \xrightarrow{B\Phi_{n_1, n_2 + n_3}} & BX(q', r', n_1 + n_2 + n_3) \\
 & & \\
 BX^\times & \xrightarrow{\tau} & BX^\times \\
 B\Phi \downarrow & & B\Phi \downarrow \\
 BX(q, r, n) & \xrightarrow{=} & BX(q, r, n)
 \end{array}$$

where τ is any permutation of the factors. These diagrams can be interpreted as the homotopy associative and commutative property of the morphisms described in Theorem A. Moreover, we see next that $B\Phi$ commutes with certain self maps.

The classification of self maps of the p -compact groups $X(q, r, n)$ is described in [22, Theorem 7.2]. Among them there are unstable Adams' operations ψ^a for each $a \in (\hat{\mathbb{Z}}_p)^*$,

$$\psi^a : BX(q, r, n) \rightarrow BX(q, r, n).$$

The induced map on maximal tori $BT_p^\wedge \rightarrow BT_p^\wedge$ is multiplication by a . For each $a = (a_1, \dots, a_n) \in ((\hat{\mathbb{Z}}_p)^*)^n$, we can consider a self map of type

$$\psi^a := \psi^{a_1} \times \dots \times \psi^{a_n} : BX^\times \rightarrow BX^\times.$$

There is another type of self maps φ^a of $BX(q, r, n)$ which we consider. They are given by elements $a = (a_1, {}^{n_1}a_1, \dots, a_s, {}^{n_s}a_s) \in \mu_q^n \subset ((\hat{\mathbb{Z}}_p)^*)^n$. Its restriction to the maximal torus is $(\psi^{a_1})^{n_1} \times \dots \times (\psi^{a_s})^{n_s}$. Again, by the uniqueness property of the morphisms Φ constructed in Theorem A, the following square is homotopy commutative:

$$\begin{array}{ccc}
 BX^\times & \xrightarrow{\psi^{a_1} \times \dots \times \psi^{a_s}} & BX^\times \\
 B\Phi \downarrow & & \downarrow B\Phi \\
 BX & \xrightarrow{\varphi^a} & BX
 \end{array}$$

Proposition 2.5. *The following diagram is homotopy commutative:*

$$\begin{array}{ccc}
 BX(q, r, n) \times BX(q, r_1, s) & \xrightarrow{B\Gamma \times \text{id}} & BX(q, r_1, n + m) \times BX(q, r_1, s) \\
 B\Phi \downarrow & & B\Phi \downarrow \\
 BX(q, r, m + n) & \xrightarrow{B\Gamma} & B(q, r_1, m + n + s)
 \end{array}$$

where $\frac{q}{r_1} \mid \frac{q}{r}$.

Proof. First, we check that the diagram commutes when restricted to subgroups $(BU(\Pi) \times BU(\Pi'))_p^\wedge$ in $BX(q, r, n) \times BX(q, r_1, s)$. This fact follows from the factorization of this restriction map through the corresponding diagram of unitary

subgroups

$$\begin{array}{ccc}
 BU(\Pi)_p^\wedge \times BU(\Pi')_p^\wedge & \xrightarrow{i \times \text{id}} & (BU(\Pi)_p^\wedge \times ((BS^1)_p^\wedge)^m) \times BU(\Pi') \\
 \text{id} \downarrow & & \downarrow \tau \\
 BU(\Pi)_p^\wedge \times BU(\Pi') & \xrightarrow{\text{id} \times 1} & BU(\Pi)_p^\wedge \times BU(\Pi') \times ((BS^1)_p^\wedge)^m \\
 & & \downarrow \\
 & & BX(q, r_1, m + n + s)
 \end{array}$$

Therefore $B\Phi \circ (B\Gamma \times \text{id})$ and $B\Gamma \circ B\Phi$ define the same element in the inverse limit

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G(q,r,n) \times G(q,r_1,m))} [(BU(\Pi) \times BU(\Pi'))_p^\wedge, BX(q, r_1, n + m + s)].$$

The obstructions to uniqueness of the extension lie in the following higher limits ([28]):

$$\varprojlim^i \pi_i(\text{Map}((BU(\Pi) \times BU(\Pi'))_p^\wedge, BX(q, r_1, m + n + s))_{g_{\Pi}}).$$

If we show that these higher limits vanish, then the proof is complete. We first describe the homotopy type of these mapping spaces.

$$\begin{aligned}
 & \text{Map}((BU(\Pi) \times BU(\Pi'))_p^\wedge, BX(q, r_1, m + n + s))_{g_{\Pi}} \\
 & \simeq \text{Map}(B\mathbb{Z}/p \times BU(\Pi)_p^\wedge \times BU(\Pi')_p^\wedge, BX(q, r_1, m + n + s))_{g_{\Pi \times \Pi'} \circ B\text{mult}} \\
 & \simeq \text{Map}(BU(\Pi)_p^\wedge \times BU(\Pi')_p^\wedge, \text{Map}(B\mathbb{Z}/p, BX(q, r_1, n + m + s)))_{g_{\Pi|B\mathbb{Z}/p} \text{ad}},
 \end{aligned}$$

where $\mathbb{Z}/p\mathbb{Z}$ is a central subgroup of $U(\Pi) \times U(\Pi')$ which is also central in $U(n + s)$. The first homotopy equivalence follows from Lemma [9, Lemma 7.5] and, by Lemma 2.4,

$$\text{Map}(B\mathbb{Z}/p, BX(q, r_1, m + n + s))_{g_{\Pi \times \Pi'}|B\mathbb{Z}/p} \simeq BU(n + s) \times BX(q, r_1, m),$$

up to p -completion.

Therefore,

$$\begin{aligned}
 & \text{Map}(BU(\Pi)_p^\wedge \times BU(\Pi')_p^\wedge, BX(q, r, n + 1))_{g_{\Pi}} \\
 & \simeq \text{Map}(BU(\Pi) \times BU(\Pi'), BU(m + n) \times BX(r_1, r_1, 1))_{\text{incl} \times *} \\
 & \simeq BZ(U(\Pi) \times U(\Pi'))_p^\wedge \times BX(r_1, r_1, 1),
 \end{aligned}$$

where the last equivalence follows from [22, Proposition 4.6] and [12, Theorem 9.3].

Finally, [22, Proposition 2:3] implies that, if $i > 0$,

$$\begin{aligned}
 & \varprojlim^i \pi_j(\text{Map}(BU(\Pi), BX(q, r_1, m + n + s))_{g_{\Pi}}) \\
 & \cong \varprojlim^i \pi_j(BT(n + s))^{\Sigma(\Pi) \times \Sigma(\Pi')} \times \pi_j(((BS^1)_p^\wedge)^m) = 0.
 \end{aligned}$$

□

Proposition 2.6. *Let φ^a be the automorphism of $BX(q, r_1, m)$ defined by $a = (1, \dots, 1, a') \in \mu_q^m$. Then $\varphi^a \circ B\Gamma \simeq B\Gamma$.*

Proof. We note that $\varphi^a \circ B\Gamma$ and $B\Gamma$ are homotopic when restricted to any subgroup $BU(\Pi)$ by construction. Hence, by uniqueness of extension, we obtain that $\varphi^a \circ B\Gamma \simeq B\Gamma$. □

The monomorphisms of p -compact groups

$$i_n : X(q, r, n) \rightarrow X(q, r, n) \times X(q, r, 1) \rightarrow X(q, r, n + 1),$$

where the second map is a morphism described in Theorem A, allow us to define the following homotopy colimit:

$$BX(q, r) := \operatorname{hocolim}_{n \in \mathbb{N}} BX(q, r, n).$$

We can also consider the infinite disjoint union of p -compact groups,

$$W(q, r) := \coprod_{n \geq 0} BX(q, r, n).$$

The map $\coprod B\Phi_{n,m}$ provides a structure of homotopy associative and commutative topological monoid with unit given by $*$ $=$ $BX(q, r, 0)$ in $W(q, r)$. Both spaces are related by the group completion theorem [19]. The group completion theorem says that if K is a homotopy commutative topological monoid and if $Gr(K) := \Omega BK$ is its topological group completion, then $\pi_0(Gr(K)) \cong Gr(\pi_0(K))$, and there is a homology equivalence

$$\operatorname{hocolim}_{x \in \pi_0(K)} K_x \rightarrow Gr(K)_1,$$

where $Gr(K)_1$ is the component of the identity.

Corollary 2.7. *There is a homology equivalence*

$$\operatorname{hocolim}_{\mathbb{N}} BX(q, r, n) \rightarrow (\Omega BW(q, r))_1$$

and $\pi_0(\Omega BW(q, r)) \cong \mathbb{Z}$.

Moreover, the homotopy type of $BX(q, r)$ does not depend on r .

Proposition 2.8.

$$BX(q, r) \simeq_p BX(q, q)$$

for any $r|q|p - 1$.

Proof. There is a map $Bi : BX(q, r) \rightarrow BX(q, q)$ induced by the morphisms of p -compact groups $B\Phi : BX(q, r, n) \rightarrow BX(q, q, n)$ constructed in Theorem A. It is easily seen that the cohomology of these spaces does not depend on r ,

$$H^*(BX(q, r)) \cong \mathbb{F}_p[y_1, y_2, \dots],$$

where $deg(y_i) = 2qi$. Since Bi induces an isomorphism in mod p cohomology, they are homotopy equivalent, after p -completion. □

3. HOMOGENEOUS SPACES

This section is devoted to the study of several homogeneous spaces arising from the existence of the morphisms Φ in Theorem A. These homogeneous spaces are the homotopy fibres of several morphisms between p -compact groups defined in Section 2. In particular, we show that they are \mathbb{F}_p -finite. The main tool used is the Eilenberg-Moore spectral sequence developed in [14] and applied to the study of homogeneous spaces in [3] and [18].

Definition 3.1. Let $G_{n,m}(q_1, r_1, q, r)$, $V_{n,m}(q_1, r_1, q, r)$ and $W_{n,m}(q_1, r_1, q, r)$ be the following homogeneous spaces.

- (1) $G_{n,m}(q_1, r_1, q, r)$ is the homotopy fibre of the map

$$B\Phi: BX(q_1, r_1, m) \times BX(q_1, r_1, n) \rightarrow BX(q, r, m + n)$$

in Theorem A.

- (2) $V_{n,m}(q_1, r_1, q, r)$ is the homotopy fibre of the inclusion map

$$\iota: BX(q_1, r_1, n) \rightarrow BX(q_1, r_1, n) \times ((BS^1_p)^\wedge)^m \rightarrow BX(q, r, n + m).$$

- (3) $W_{n,m}(q_1, r_1, q, r)$ is the homotopy fibre of the map

$$B\Gamma: BX(q_1, r_1, n) \rightarrow BX(q, r, n + m)$$

in Theorem B.

Inspired by the situation with matrix groups, we obtain the following diagrams of fibrations up to homotopy:

$$\begin{array}{ccccc} X(q, r, m) & \xlongequal{\quad\quad\quad} & X(q, r, m) & & \\ \downarrow & & \downarrow & & \\ V_{n,m}(q_1, r_1, q, r) & \longrightarrow & BX(q_1, r_1, n) & \longrightarrow & BX(q, r, m + n) \\ \downarrow & & \downarrow & & \parallel \\ G_{n,m}(q_1, r_1, q, r) & \longrightarrow & BX(q_1, r_1, n) \times BX(q_1, r_1, m) & \longrightarrow & BX(q, r, n + m) \end{array}$$

The next proposition describes the cohomology of several homogeneous spaces defined in Definition 3.1. In particular, it shows that they are \mathbb{F}_p -finite and therefore that the maps in Definition 3.1 are monomorphisms.

Proposition 3.2.

- (1) If $m > 0$,

$$H^*(V_{n,m}(q, r_1, q, r); \mathbb{F}_p) \cong \Lambda(a_{n+1}, \dots, a_{m+n-1}, b) \otimes \mathbb{F}_p[h]/(h^{r_1}),$$

where $\deg(a_i) = 2iq - 1$, $\deg(b) = 2(m + n)\frac{q}{r} - 1$, $\deg(h) = 2n\frac{q}{r_1}$.

$$H^*(V_{n,0}(q, r_1, q, r); \mathbb{F}_p) \cong \mathbb{F}_p[h]/(h^{r/r_1}),$$

where $\deg(h) = 2n\frac{q}{r_1}$.

- (2)

$$H^*(W_{n,m}(q, r_1, q, r); \mathbb{F}_p) \cong \Lambda(a_{n+1}, \dots, a_{m+n-1}, f) \otimes \mathbb{F}_p[e]/(e^{r_1}),$$

where $\deg(a_i) = 2iq - 1$, $\deg(e) = 2n\frac{q}{r_1}$ and $\deg(f) = 2(n + m)\frac{q}{r} - 1$.

Proof. If (E, p, B) is a fibration over the simply connected space B , the Eilenberg-Moore spectral sequence is a spectral sequence of commutative algebras $\{E_r, d_r\}$ converging to $H^*(F; k)$ where F is the fiber, and $E_2 = \text{Tor}_{H^*(B; k)}(H^*(E; k), k)$.

The convergence of the spectral sequence when the spaces E and B are finite loop spaces has been discussed in [18]. The results in [18] imply the convergence of the spectral sequence in the E_2 -term and $H^*(F) \cong \text{Tor}_{H^*(B)}(H^*(E), \mathbb{F}_p)$, where $H^*(E)$ is a $H^*(B)$ -module via p^* .

When both algebras $H^*(E)$ and $H^*(B)$ are polynomial, $\text{Tor}_{H^*(B)}(H^*(E), \mathbb{F}_p)$ is computed in [3, Lemma 4.11]. It is an exterior algebra on the desuspension of the

kernel of p^* tensor the cokernel of p^* . The mod p cohomology of the p -compact groups $X(q, r, n)$ is polynomial. In fact, it is given by

$$H^*(BX(q, r_1, n)) \cong \mathbb{F}_p[y_1, \dots, y_{n-1}, e]$$

and

$$H^*(BX(q, r, m+n)) \cong \mathbb{F}_p[z_1, \dots, z_{m+n-1}, f],$$

where $\deg(y_i) = \deg(z_i) = 2qi$, $\deg(e) = 2n\frac{q}{r_1}$ and $\deg(f) = 2(m+n)\frac{q}{r}$ (see [22, Proposition 1.4]).

(1) The induced morphism $B\iota^*$ in mod p cohomology is

$$B\iota^*(z_i) = \begin{cases} y_i, & i = 1, \dots, n-1, \\ e^{r_1}, & i = n, \\ 0, & i = n+1, \dots, m+n-1, \end{cases}$$

and $B\iota^*(f) = 0$ if $m > 0$ or $B\iota^*(f) = e^{r_1/r}$ if $m = 0$. Therefore, by [3, Lemma 4.11] we obtain the following description of the E_2 -term of the spectral sequence:

$$\begin{aligned} \text{Tor}_{H^*(BX(q,r,m+n))}(H^*(BX(q, r_1, n)); \mathbb{F}_p) \\ \cong \Lambda(a_{n+1}, \dots, a_{m+n-1}, b_m) \otimes \mathbb{F}_p[h]/(h^{r_1}), \end{aligned}$$

where $\deg(a_i) = 2qi - 1$, $\deg(b_m) = 2(m+n)\frac{q}{r} - 1$ and $\deg(f) = 2n\frac{q}{r_1}$.

If $m = 0$, then

$$\text{Tor}_{H^*(BX(q,r,n))}(H^*(BX(q, r_1, n)), \mathbb{F}_p) \cong \mathbb{F}_p[h]/(h^{r/r_1}),$$

where $\deg(h) = 2n\frac{q}{r_1}$.

(2) The morphism Γ induces the following algebra morphism in mod p cohomology:

$$B\Gamma^*(z_i) = \begin{cases} y_i, & i = 1, \dots, n-1, \\ e^{r_1}, & i = n, \\ 0, & i = n+1, \dots, m+n-1, \end{cases}$$

and $B\Gamma^*(f) = 0$. Therefore, the E_2 -term of the spectral sequence is of the form

$$\begin{aligned} \text{Tor}_{H^*(BX(q,r,m+n))}(H^*(BX(q, r_1, n)), \mathbb{F}_p) \\ \cong \Lambda(a_{n+1}, \dots, a_{m+n-1}, b) \otimes \mathbb{F}_p[h]/(h^{r_1}), \end{aligned}$$

where $\deg(a_i) = 2iq - 1$, $\deg(h) = 2n\frac{q}{r_1}$ and $\deg(b) = 2(n+m)\frac{q}{r} - 1$. □

Corollary 3.3.

(1) $H^*(V_{n,0}(q, r_1, q, r)) \cong \mathbb{F}_p[h]/(h^{r_1/r})$, $\deg(f) = 2n\frac{q}{r_1}$. In particular,

$$V_{n,0}(q, 2r, q, r) \simeq (S^{(nq)/r})_p^\wedge.$$

(2) $H^*(V_{n,1}(q, r, q, r); \mathbb{F}_p) \cong \Lambda(b) \otimes \mathbb{F}_p[h]/(h^r)$, $\deg(b) = 2(n+1)\frac{q}{r} - 1$ and $\deg(h) = 2n\frac{q}{r_1}$. In particular, $V_{n,1}(q, 1, q, 1) \simeq (S^{2(n+1)q-1})_p^\wedge$.

(3) $W_{n,1}(p-1, 1, p-1, p-1) \simeq (S^{2n+1})_p^\wedge$.

The proof of the following proposition follows by induction, and it is analogous to that of the classical case for compact Lie groups (see, for example, [27, Chapter IV, 10.II]).

Proposition 3.4. *The monomorphisms between p -compact groups in Theorem A induce the morphisms between homotopy groups with the following properties:*

(1)

$$\pi_i(BX(q, q, m-1)) \rightarrow \pi_i(BX(q, q, m))$$

is an isomorphism for $i < 2mq - 1$ and an epimorphism for $i = 2mq - 1$.

(2)

$$\pi_i(BX(q, r, m)) \rightarrow \pi_i(BX(q, 2r, m))$$

is an isomorphism for $i < 2mr$ and an epimorphism for $i = 2mr$.

(3)

$$\pi_i(V_{n-1, m-1}(q, q)) \rightarrow \pi_i(V_{n, m}(q, q))$$

is an isomorphism for $i < 2mq - 2$ and an epimorphism for $i = 2mq - 2$.

(4)

$$\pi_i(V_{n, m}(q, q)) = 0 \quad \text{for } i < 2(m - n + 1)q - 1.$$

Corollary 3.5.

$$\pi_i(V_{n, \infty}(q, q)) = 0 \quad \forall i.$$

Corollary 3.5 is relevant when applied to the following fibration:

$$X(q, q, n) \rightarrow V_{n, \infty}(q, q) \rightarrow G_{n, \infty}(q, q).$$

Since $V_{n, \infty}(q, q)$ is weakly contractible, it follows that $\Omega G_{n, \infty}(q, q)$ is weakly equivalent to $X(q, q, n)$.

Hopf fibrations arise as particular cases of the fibrations described. When we consider unitary groups, we obtain the Hopf fibration

$$S^1 \rightarrow S^3 \rightarrow S^2$$

as a special case of a fibration of a Stiefel variety over a Grassmann manifold. When we are dealing with p -compact groups $X(q, r, n)$, we obtain mod p fibrations of spheres by spheres.

Proposition 3.6. *There exist fibrations of spheres by spheres mod p ,*

$$(S^{2q-1})_p^\wedge \rightarrow (S^{2(n+1)q-1})_p^\wedge \rightarrow G_{n,1}(q, 1, q, 1).$$

Proof. Consider the fibration

$$X(q, 1, 1) \rightarrow V_{n,1}(q, 1, q, 1) \rightarrow G_{n,1}(q, 1, q, 1).$$

The p -compact group $X(q, 1, 1) \simeq (S^{2q-1})_p^\wedge$ is a Sullivan sphere, and from Corollary 3.3, $V_{n,1}(q, 1, q, 1) \simeq (S^{2(n+1)q-1})_p^\wedge$. \square

These fibrations correspond to fibrations of type I in the paper by Aguadé [1]. $G_{n,1}(q, 1, q, 1)$ is called a S^{2q-1} -projective $(n+1)$ -space whose cohomology is

$$H^*(G_{n,1}(q, 1, q, 1); \mathbb{F}_p) \cong \mathbb{F}_p[f]/(f^{n+1}),$$

where $\deg(f) = 2q$.

Remark 3.7. When $n = 1$ we obtain a fibration of mod p spheres

$$(S^{2q-1})_p^\wedge \rightarrow (S^{4q-1})_p^\wedge \rightarrow (S^{2q})_p^\wedge$$

analogous to the Hopf fibration.

4. MOD p SPHERICAL FIBRATIONS

A mod p spherical fibration is an orientable Hurewicz fibration $\pi : E \rightarrow B$ whose fibre has the homotopy type of a p -complete sphere. The associated Thom space is the homotopy cofiber of π , and its reduced mod p cohomology is also an example of an $H^*(B)$ -Thom module.

In Corollary 3.3, we described several homogeneous spaces arising from p -compact groups $X(q, r, n)$ which are p -completed spheres.

In particular, we consider the fibration whose fibre is the homogeneous space $W_{n,1}(p-1, 1, p-1, p-1) \simeq (S^{2n+1})_p^\wedge$.

Proposition 4.1. *The Euler class of the spherical fibration η_{n+1} ,*

$$(S^{2n+1})_p^\wedge \rightarrow BX(p-1, 1, n) \rightarrow BX(p-1, p-1, n+1),$$

is $e \in H^*(BX(p-1, p-1, n+1)) \cong \mathbb{F}_p[y_1, \dots, y_n, e]$, $\text{deg}(y_i) = 2i(p-1)$ and $\text{deg}(e) = 2(n+1)$.

Proof. [5] Note that $B\Gamma^*$ is onto (see the proof of Proposition 3.2) and, by Corollary 3.3, $W(p-1, 1, p-1, p-1) \simeq (S^{2n+1})_p^\wedge$. Then, the Gysin long exact sequence associated to the fibration

$$(S^{2n+1})_p^\wedge \rightarrow BX(p-1, 1, n) \xrightarrow{B\Gamma} BX(p-1, p-1, n+1)$$

gives rise to a short exact sequence

$$0 \rightarrow \mathbb{F}_p[y_1, \dots, y_n, e] \xrightarrow{\chi} \mathbb{F}_p[y_1, \dots, y_n, e] \rightarrow \mathbb{F}_p[y_1, \dots, y_n] \rightarrow 0.$$

It is easy to check that the mod p Euler class is $\chi = e$. □

Remark 4.2. Let R be an unstable algebra over the Steenrod algebra \mathcal{A} . A Thom module M over R is a free $R \circ \mathcal{A}$ -module of rank 1 as an R -module, where $R \circ \mathcal{A}$ denotes the semi-tensor product. A Thom module is unstable if it is unstable as a \mathcal{A} -module. The notion of a Thom module appears for the first time in Handel [15]. The classification of Thom modules was addressed in [6] via the universal Thom modules $T(n)$ over $\mathbb{F}_p[q_1, \dots, q_n] = \mathbb{F}_p[t_1, \dots, t_n]^{G(p-1, 1, n)}$ with Thom class $E = t_1 \cdots t_n$. The cohomology of the Thom space associated to η_{n+1} is $T(n)$. This fact can be used to realize Thom modules as the cohomology of a Thom space.

Let $F \rightarrow E \xrightarrow{p_1} B$ and $F' \rightarrow E' \xrightarrow{p_2} B'$ be two fibrations. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 F & \longleftarrow & F \times F' & \longrightarrow & F' \\
 \downarrow & & \downarrow & & \downarrow \\
 E \times B' & \xleftarrow{1 \times p_2} & E \times E' & \xrightarrow{p_1 \times 1} & B \times E' \\
 & \searrow p_1 & \downarrow & \swarrow p_2 & \\
 & & B \times B' & &
 \end{array}$$

By Puppe's theorem [24], taking homotopy colimits in the fibre and total spaces produces another fibration whose fibre is $F \star F'$,

$$F \star F' \rightarrow \bar{E} \rightarrow B \times B',$$

where \bar{E} is the homotopy colimit of the diagram in the middle row. This construction is the fibrewise join fibration.

Proposition 4.3. *Let η_n be the mod p spherical fibration defined in Proposition 4.1. Let $B\Phi^*(\eta_{n+m})$ be the pullback of η_n along $B\Phi$, where Φ is a morphism described in Theorem A,*

$$B\Phi : BX(p-1, p-1, n) \times BX(p-1, p-1, m) \rightarrow BX(p-1, p-1, n+m).$$

Then, the mod p spherical fibration $B\Phi^(\eta_{n+m})$ is the fibrewise p -completion of the join fibration $\eta_n \star \eta_m$.*

Proof. We construct a map of fibrations $\eta_n \star \eta_m \rightarrow B\Phi^*(\eta_{n+m})$. The total space of $\eta_n \star \eta_m$ is the homotopy colimit of

$$\begin{array}{ccc} BX(p-1, 1, n-1) \times BX(p-1, p-1, m) & \xleftarrow{\text{id} \times B\Gamma} & BX(p-1, 1, n-1) \times BX(p-1, 1, m-1) \\ & & \downarrow B\Gamma \times \text{id} \\ & & BX(p-1, p-1, n) \times BX(p-1, 1, m-1) \end{array}$$

and the total space of $B\Phi^*(\eta_{n+m})$ is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & BX(p-1, 1, n+m-1) \\ & & \downarrow B\Gamma \\ BX(p-1, p-1, n) \times BX(p-1, p-1, m) & \xrightarrow{B\Phi} & BX(p-1, p-1, n+m) \end{array}$$

In order to construct the map between the total spaces, we use the maps $B\Phi$ described in Theorem A. Because of the above description of both total spaces, we only need to check the homotopy commutativity of the following two diagrams:

$$\begin{array}{ccc} BX(p-1, 1, n-1) \times BX(p-1, 1, m-1) & \xrightarrow{\text{id} \times B\Gamma} & BX(p-1, 1, n-1) \times BX(p-1, p-1, m) \\ \downarrow B\Gamma \times \text{id} & & \downarrow B\Phi \\ BX(p-1, p-1, n) \times BX(p-1, 1, m-1) & \xrightarrow{B\Phi} & BX(p-1, 1, n+m-1) \\ \\ BX(p-1, 1, n-1) \times BX(p-1, 1, m-1) & \xrightarrow{\text{id} \times B\Gamma} & BX(p-1, 1, n-1) \times BX(p-1, p-1, m) \\ \downarrow B\Gamma \times \text{id} & & \downarrow B\Gamma \times \text{id} \\ BX(p-1, p-1, n) \times BX(p-1, 1, m-1) & \xrightarrow{\text{id} \times B\Gamma} & BX(p-1, p-1, n) \times BX(p-1, p-1, m) \end{array}$$

The commutativity of the first diagram is easy to check. The commutativity of the second one follows from Proposition 2.5. Therefore, we obtain a map between the total spaces. This map also commutes with the projections: the corresponding diagrams commute up to homotopy because of the compatibility of Γ with the morphisms in Theorem A described in Proposition 2.5.

Finally, we use the Serre spectral sequence in order to prove that the map between the fibres induces an isomorphism in mod p cohomology. Since the transgression morphism in the Serre spectral sequence is natural, the mod p Euler class of η_{n+m} , $e_{n+m} \in H^*(BX(p-1, p-1, n+m))$, maps to the mod p Euler class of the pullback fibration

$$B\Phi^*(\eta_{n+m}), B\Phi^*(e_{n+m}) = e_n e_m \in H^*(BX(p-1, p-1, n)) \otimes H^*(BX(p-1, p-1, m)),$$

which is nontrivial. Hence, the induced morphism in the fibres is an isomorphism in mod p cohomology and, therefore, it is a homotopy equivalence after p -completion. \square

Corollary 4.4. *Let $Bi_n : BX(p-1, p-1, n) \rightarrow BX(p-1, p-1, n+1)$. The pullback of the mod p spherical fibration η_{n+1} along Bi_n satisfies $Bi_n^*(\eta_{n+1}) \cong \eta_n \star S^1$.*

Proof. The monomorphism i_n factors through the morphism

$$B\Phi : BX(p-1, p-1, n) \times BX(p-1, p-1, 1) \rightarrow BX(p-1, p-1, n+1)$$

in Theorem A. By Proposition 4.3, the pullback is the join fibration of the restriction to both factors. Therefore, when restricted further to the first component $BX(p-1, p-1, n)$, we obtain $Bi_n^*(\eta_{n+1}) \cong \eta_n \star S^1$. \square

Let BSG be the classifying space of the orientable spherical fibrations. The mod p cohomology (p odd) of BSG is

$$H^*(BSG; \mathbb{F}_p) \cong \mathbb{F}_p[q_1, q_2, \dots] \otimes \Lambda(\beta q_1, \beta q_2, \dots) \otimes C,$$

where q_i are the Wu classes in dimensions $2i(p-1)$ and C is the tensor product of an exterior algebra and a divided power algebra (see [26]).

To show the relation between $BX(p-1)$ and BSG we will use the techniques of Nil-localization (see [7] for more details).

Proof of Theorem D. Each spherical fibration η_n is classified by a map

$$J_n : BX(p-1, p-1, n) \rightarrow BSG_p^\wedge.$$

Since the restriction $\eta_n|_{BX(p-1, 1, n-1)} \cong \eta_{n-1} * S^1$ (Corollary 4.4), these maps are compatible with respect to the monomorphisms $BX(p-1, p-1, n-1) \rightarrow BX(p-1, p-1, n)$. Therefore, they induce a map in the homotopy colimit $J : BX(p-1) \rightarrow BSG_p^\wedge$.

The mod p Euler class of η_{n+1} is e (Proposition 4.1), and the spherical characteristic classes are given by applying Steenrod operations to the Thom class. Recall that

$$e = x_1 \cdots x_{n+1} \in \mathbb{F}_p[x_1, \dots, x_{n+1}]^{G(p-1, p-1, n+1)},$$

therefore,

$$\mathcal{P}(e) = \left(\prod_{i=1}^{n+1} (1 + x_i^{p-1}) \right) e = (1 + q_1 + \cdots + q_n + e^{p-1})e.$$

The morphism J^* is an isomorphism onto the polynomial part $\mathbb{F}_p[q_1, q_2, \dots]$.

The mod p cohomology of BSG is computed in [26],

$$H^*(BSG) \cong \mathbb{F}_p[q_1, q_2, \dots] \otimes C,$$

where C is the tensor product of an exterior algebra and a divided power algebra mod p . In particular, C is nilpotent. Since S^* is an epimorphism and the kernel is nilpotent, J^* is an F -isomorphism. \square

Corollary 4.5. *Let V be an elementary abelian p -group. Then the orientable spherical fibrations mod p over BV are classified by $BX(p-1)$. That is, there is a bijection*

$$[BV, BX(p-1)] \cong [BV, BSG_p^\wedge].$$

Proof. One consequence of Theorem D is that J^* induces an isomorphism

$$Hom_{\mathcal{K}}(H^*(BSG), H^*(BV)) \cong Hom_{\mathcal{K}}(H^*(BX(p-1)), H^*(BV)).$$

It follows from the fact that $H^*(BV)$ is an injective object in the category of unstable modules over the Steenrod algebra, and that it is also a reduced module (see [17]).

Since $BX(p - 1)$ and BSG are spaces of finite type, by Lannes theory [16] we obtain the following bijections:

$$\begin{aligned} [BV, BX(p - 1)] &\cong \text{Hom}_{\mathcal{K}}(H^*(BX(p - 1)), H^*(B\mathbb{Z}/p\mathbb{Z})), \\ [BV, BSG] &\cong \text{Hom}_{\mathcal{K}}(H^*(BSG), H^*(B\mathbb{Z}/p\mathbb{Z})), \end{aligned}$$

which complete the proof. □

5. A COMPLEXIFICATION MAP

This section contains the proof of Theorem E. First, we describe the injective morphism between the corresponding Weyl groups $(G(q, r, n), L_n) \rightarrow (\Sigma_{qn}, L_{qn})$ which will be topologically realized by a morphism between p -compact groups $X(q, r, n) \rightarrow U(nq)_p^\wedge$.

There is a natural action of $G(q, r, n)$ on \mathbb{C}^n by linear transformations. We consider the basis $\{e_1, \dots, e_n\}$ of the \mathbb{C} -vector space where $e_i = (0, \dots, 1^{(i)}, \dots, 0)$. It is easy to check that the orbit of e_1 by the action of $G(q, r, n)$ is

$$\{e_1, \dots, e_n, \xi e_1, \dots, \xi e_n, \xi^2 e_1, \dots, \xi^2 e_n, \dots, \xi^{q-1} e_1, \dots, \xi^{q-1} e_n\},$$

where ξ is a primitive complex q th root of the unity. $G(q, r, n)$ acts transitively on the orbit as a group of permutations, and this correspondence gives a group morphism $\phi : G \rightarrow \Sigma_{nq}$.

Note that ϕ is injective: if $g \in G(q, r, n)$ satisfies $\phi(g) = id$, in particular, $ge_i = e_i$, this means that $g = id$.

Proof of Theorem E. We consider the group of q th roots of the unity $\mu_q \subset \mu_{p-1}$, and we use them to define a map

$$Bc := \psi^0 \times \psi^1 \times \dots \times \psi^{q-1} : BU(n)_p^\wedge \rightarrow BU(n)_p^\wedge \times \dots \times BU(n)_p^\wedge \leq BU(nq)_p^\wedge,$$

where ψ^α are unstable Adams' operations on $BU(n)_p^\wedge$ of order ξ^α .

The restriction of Bc to each $BU(\Pi)$

$$Bc|_{BU(\Pi)_p^\wedge} : BU(\Pi)_p^\wedge \rightarrow BU(\Pi)_p^\wedge \times \dots \times BU(\Pi)_p^\wedge \leq BU(nq)_p^\wedge$$

defines a family of morphisms. In fact we check that they all fit to produce an element in the inverse limit

$$\{[Bc|_{BU(\Pi)_p^\wedge}]\} \underset{\mathcal{O}_{\mathcal{H}}(G(q, r, n))}{\varprojlim}^0 [BU(\Pi), BU(nq)_p^\wedge].$$

We have to prove that the following diagrams are homotopy commutative, where f is a morphism in $\mathcal{O}_{\mathcal{H}}(G(q, r, n))$:

$$\begin{array}{ccccc} BU(\Pi)_p^\wedge & \longrightarrow & BU(\Pi)_p^\wedge \times \dots \times BU(\Pi)_p^\wedge & \longrightarrow & BU(nq)_p^\wedge \\ \downarrow f & & \downarrow g & \nearrow & \\ BU(\Pi)_p^\wedge & \longrightarrow & BU(\Pi)_p^\wedge \times \dots \times BU(\Pi)_p^\wedge & & \end{array}$$

Recall that there are three kinds of maps induced by morphisms in the orbit category. On the one hand, there are inclusions and permutations of the factors f ; in this case just take $g = f \times \dots \times f$. On the other hand, there are unstable Adams'

operations ψ^α ; recall that $\psi^r \circ \psi^s \simeq \psi^{rs}$, thus we have to take on the right-hand side the permutation of the factors that corresponds to $g = \phi(f) \in \Sigma_{nq}$.

The triangle on the right is homotopy commutative because Whitney sum morphisms are homotopy commutative and associative (g is always given by a permutation of the factors).

The obstructions to extend $\{[Bc|_{BU(\Pi)_p^\wedge}]\}$ (see [28]) lie in

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G(q,r,n))}^{i+1} \pi_i(\text{Map}(BU(\Pi), BU(nq)_p^\wedge)_{Bc|_{BU(\Pi)}}).$$

In order to compute the above higher limits, we first deal with the homotopy type of the mapping spaces. Consider the fibration

$$B\mathbb{Z}/p \rightarrow BU(\Pi) \times B\mathbb{Z}/p \rightarrow BU(\Pi)$$

induced by the exact sequence $\mathbb{Z}/p\mathbb{Z} \rightarrow U(\Pi) \times \mathbb{Z}/p\mathbb{Z} \rightarrow U(\Pi)$, where the last morphism is given by group multiplication by a central element $\mathbb{Z}/p\mathbb{Z}$.

We have the following homotopy equivalences, up to p -completion:

$$\begin{aligned} \text{Map}(BU(\Pi), BU(nq))_{Bc} &\simeq \text{Map}(BU(\Pi) \times B\mathbb{Z}/p, BU(nq))_{\Psi \circ Bmult} \\ &\simeq \text{Map}(BU(\Pi), \text{Map}(B\mathbb{Z}/p, BU(nq))_{Bc|_{B\mathbb{Z}/p}})_{ad(Bc)} \\ &\simeq \text{Map}(BU(\Pi), BU(n)^q)_{Bc} \\ &\simeq \prod_{i=1}^q \text{Map}(BU(\Pi), BU(n))_{\psi^i} \simeq (BZ(U(\Pi))_p^\wedge)^q. \end{aligned}$$

The first homotopy equivalence follows from [9, Lemma 7.5], and the third one by [13]. Therefore,

$$\pi_i(\text{Map}(BU(\Pi), BU(nq)_p^\wedge)_{Bc|_{BU(\Pi)}}) \cong (\pi_j(BT^n)^{\Sigma(\Pi)})^q.$$

By [22, Proposition 2.3],

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G(q,r,n))}^i \pi_j(\text{Map}(BU(\Pi), BU(nq)_p^\wedge)_{Bc}) = 0,$$

if $i > 0$ and $j \neq 2$, and

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G(q,r,n))}^i \pi_j(\text{Map}(BU(\Pi), BU(nq)_p^\wedge)_{Bc}) = (\pi_j((BT^n)_p^\wedge)^{G(q,r,n)})^q,$$

if $i = 0$ and $j = 2$.

It remains to show that the morphism $Bc : BX(q, r, n) \rightarrow BU(nq)_p^\wedge$ is a monomorphism. We shall compute the mod p cohomology of the homotopy fibre of Bc . The method is exactly the same as the one used in the proof of Proposition 3.2.

$$\begin{aligned} Bc^*(1 + c_1 + \dots + c_{nq}) &= Bc^*\left(\prod_{i=1}^{qn} (1 + x_i)\right) = \prod_{i=1}^n \prod_{j=1}^q (1 + a^j t_i) \\ &= \prod_{i=1}^n (1 + (-1)^{q+1} t_i^q) = 1 + (-1)^{q+1} y_1 + (-1)^{2(q+1)} y_2 + \dots + (-1)^{n(q+1)} y_n, \end{aligned}$$

$$Bc^*(c_j) = \begin{cases} (-1)^{\frac{j}{q}(q+1)} q_{\frac{j}{q}}, & j \equiv 0(q), j \neq qn, \\ (-1)^{n(q+1)} e^r, & j = qn, \\ 0, & \text{otherwise,} \end{cases}$$

where y_i are the elementary symmetric polynomials in t_1^q, \dots, t_n^q , $e = t_1^r \cdots t_n^r$, $y_n = e^{q/r}$. Therefore, the cohomology of the homotopy fibre of Bc is an exterior algebra on a finite number of generators tensor a truncated polynomial on one generator $\Lambda(a_j : j = 1, \dots, nq, j \neq 0(q)) \otimes \mathbb{F}_p[b]/b^r$, where $\deg(a_i) = 2iq - 1$ and $\deg(b) = 2n\frac{q}{r}$, and hence \mathbb{F}_p -finite.

Note that Ψ is a regular map (also called p -toric [21]),

$$\text{Map}(BT^n, BU(nq)_p^\wedge)_\Psi \simeq (BT^n)^q,$$

therefore, the induced morphism in the Weyl groups ϕ is unique ([21, Proposition 2.4]). □

Remark 5.1. Since $Bc^*(c_1) = 0$, Ψ factors through $SU(nq)_p^\wedge$.

We finish this section by studying the compatibility of the complexification map with unstable Adams' operations on $BU(nq)_p^\wedge$.

Proposition 5.2. *The complexification map is equivariant up to homotopy with respect to the action of unstable Adams' operations of order q on $BU(nq)_p^\wedge$.*

Proof. Let ξ be a primitive q th root of unity and ψ^ξ an unstable Adams' operation of order q on $BU(nq)_p^\wedge$. The restriction to $BU(\Pi)$ of the composite $\psi^\xi \circ B\Psi$ fits in the following homotopy commutative diagram:

$$\begin{CD} BU(\Pi) @>{\psi^0 \times \dots \times \psi^{q-1}}>> BU(\Pi)_p^\wedge \times \dots \times BU(\Pi)_p^\wedge @>{\oplus}>> BU(nq)_p^\wedge \\ @V{\psi^\xi}VV @. @V{\psi^\xi}VV \\ BU(\Pi) @>{\Psi^0 \times \dots \times \Psi^{q-1}}>> BU(\Pi)_p^\wedge \times \dots \times BU(\Pi)_p^\wedge @>{\oplus}>> BU(nq)_p^\wedge \end{CD}$$

In particular, this means that the complexification map fits in the following homotopy commutative diagram:

$$\begin{CD} BX(q, r, n) @>{Bc}>> BU(nq)_p^\wedge \\ @V{\simeq \text{id}}VV @V{\psi^\xi}VV \\ BX(q, r, n) @>{Bc}>> BU(nq)_p^\wedge \end{CD}$$

□

REFERENCES

1. J. Aguadé, *Fiberings of spheres by spheres mod p*, Quart. J. Math. Oxford (2) **31** (1980), 129–137. MR0576332 (81j:55017)
2. K. Andersen, J. Grodal, J. Møller, A. Viruel, *The classification of p-compact groups, p odd*, to appear in Ann. of Math.
3. B.F. Baum, *On the cohomology of homogeneous spaces*, Topology **7** (1968), 15–38. MR0219085 (36:2168)
4. A. Bousfield, D. Kan, *Homotopy limits, Completion and Localisations*, SLNM vol. 304, Springer, Berlin, 1972. MR0365573 (51:1825)
5. C. Broto, *Thom modules and mod p spherical fibrations*, Proc. Amer. Math. Soc. **114** No.4 (1992), 1131–1137. MR1065084 (92g:55018)
6. C. Broto, L. Smith, R. Stong, *Thom modules and pseudoreflexion groups*, J. Pure Appl. Algebra **60** (1989), 1–20. MR1014604 (91c:55021)
7. C. Broto, S. Zarati, *Nil-localization of unstable algebras over the Steenrod algebra*, Math. Z. **199** (1988), 525–537. MR0968318 (89i:55015)

8. A. Clark, J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. of Math. **5** (1974), 425–434. MR0367979 (51:4221)
9. W. G. Dwyer, C. W. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, Ann. of Math. **139** (1994), 395–442. MR1274096 (95e:55019)
10. W. G. Dwyer, C. W. Wilkerson, *The elementary geometric structure of compact Lie groups*, Bull. London Math. Soc. **30** (1998), 337–364. MR1620888 (99g:22007)
11. W. G. Dwyer, C. W. Wilkerson, *Kähler differentials, the T -functor, and a theorem of Steinberg*, Trans. Amer. Math. Soc. **350** (1998), no. 12, 4919–4930. MR1621741 (99e:55027)
12. W. G. Dwyer, C. W. Wilkerson, *The center of a p -compact group*, The Cech centennial (Boston, MA, 1993), Contemp. Math. **181** (1995), 119–157. MR1320990 (96a:55024)
13. W. G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, Algebraic Topology, Barcelona, 1986, Lecture Notes in Math. **1298**, 106–119, Springer. MR0928826 (89b:55018)
14. S. Eilenberg and J. C. Moore, *Homology and fibrations I*, Comment. Math. Helv. **40** (1966), 199–236. MR0203730 (34:3579)
15. D. Handel, *Thom modules*, J. Pure Appl. Algebra **36** (1985), 237–252. MR0790616 (86k:55003)
16. J. Lannes, *Sur les espaces fonctionnelles dont le source est le classifiant d'un p -groupe abélien élémentaire*, Ins. Hautes Etudes Sci. Publ. Math. **75** (1992), 135–224. MR1179079 (93j:55019)
17. J. Lannes and S. Zarati, *Sur les U -injectifs*, Ann. Scient. Ec. Norm. Sup. **19** (1986), 1–31. MR0868302 (89a:55032)
18. P. May and F. Neumann, *On the cohomology of generalized homogeneous spaces*, Proc. Amer. Math. Soc. **130** (2002), no. 1, 267–270. MR1855645 (2002m:57049)
19. D. McDuff and G. Segal, *Homology fibrations and the group-completion theorem*, Invent. Math. **31** (1975/76), no. 3, 279–284. MR0402733 (53:6547)
20. R. J. Milgram, *The mod 2 spherical characteristic classes*, Ann. of Math. **92** (1970), no. 2, 238–261. MR0263100 (41:7705)
21. J. Møller, *Toric morphisms of p -compact groups*, Cohomological Methods in Homotopy Theory (Bellaterra, 1998), J. Aguadé et al. (eds.), Prog. Math. **196**, Birkhäuser, 2001, 271–306. MR1851259 (2002i:55010)
22. D. Notbohm, *Topological realization of a family of pseudoreflection groups*, Fund. Math. **155** (1998), 1–31. MR1487985 (99a:55005)
23. D. Notbohm, *p -adic lattices of pseudo reflection groups*, Proceedings of the Barcelona Conference 1994, Algebraic Topology: New trends in localization and periodicity, ed.: C. Broto, C. Casacuberta and G. Mislin, 337–352. MR1397742 (97e:20005)
24. V. Puppe, *A remark on homotopy fibrations*, Manuscr. Math. **12** (1974), 113–120. MR0365556 (51:1808)
25. G. C. Sheppard, J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954) 274–304. MR0059914 (15:600b)
26. A. Tsuchiya, *Characteristic classes for spherical fiber spaces*, Nagoya Math. J. **43** (1971), 1–39. MR0298687 (45:7736)
27. G. W. Whitehead, *Elements of homotopy theory*, Graduate Text in Mathematics **61**, Springer-Verlag, New York, 1978. MR0516508 (80b:55001)
28. Z. Wojtkowiak, *On maps from $holim F$ to Z* , Algebraic Topology (Barcelona 1986), Lecture Notes in Math. **1298**, Springer, 1987, 227–236. MR0928836 (89a:55034)