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THE ARONSSON EQUATION FOR ABSOLUTE MINIMIZERS OF L^∞-FUNCTIONALS ASSOCIATED WITH VECTOR FIELDS SATISFYING HÖRMANDER'S CONDITION

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Abstract. Given a Carnot-Carathéodory metric space (R^n, d_{X}) generated by vector fields $\{X_i\}_{i=1}^m$ satisfying Hörmander's condition, we prove in Theorem A that any absolute minimizer $u \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ to $F(v,\Omega) = \operatorname{ess\,sup}_{x\in\Omega} f(x, Xv(x))$ is a viscosity solution to the Aronsson equation

$$-\sum_{i=1}^m X_i(f(x, Xu(x)))f_{p_i}(x, Xu(x)) = 0, \quad \text{in} \quad \Omega,$$

under suitable conditions on f. In particular, any AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation

$$\Delta_\infty^{(X)} u := -\sum_{i,j=1}^m X_i u X_j u X_i X_j u = 0, \quad \text{in} \quad \Omega.$$

If the Carnot-Carathéodory space is a Carnot group $\mathbf{G}$ and f is independent of the x-variable, we establish in Theorem C the uniqueness of viscosity solutions to the Aronsson equation

$$A(Xu, (D^2u)^*) := -\sum_{i,j=1}^m f_{p_i}(Xu) f_{p_j}(Xu) X_i X_j u = 0, \quad \text{in} \quad \Omega,$$
$$u = \phi, \quad \text{on } \partial\Omega,$$

under suitable conditions on f. As a consequence, the uniqueness of both AMLE and viscosity solutions to the subelliptic ∞-Laplacian equation is established on any Carnot group $\mathbf{G}$.

§1. Introduction

Variational problems in L^∞ are very important because of both their analytic difficulties and their frequent appearance in applications. (See the survey article by Barron [B]). The study began with Aronsson [A1, A2]. The simplest model is to consider minimal Lipschitz extensions (or MLE): for a bounded, Lipschitz domain $\Omega \subset R^n$ and $g \in \mathrm{Lip}(\Omega)$, find $u \in W^{1,\infty}(\Omega)$, with $u|_{\partial\Omega} = g$, such that

$$\|Du\|_{L^\infty(\Omega)} \le \|Dw\|_{L^\infty(\Omega)}, \ \forall w \in W_0^{1,\infty}(\Omega) \text{ with } w|_{\partial\Omega} = g. \tag{1.1}$$

Since MLE may be neither unique nor smooth, Aronsson [A1] introduced the notation of absolutely minimizing Lipschitz extensions (or AMLE for short), and proved

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that any C^2 AMLE solves the ∞-Laplacian equation

$$\Delta_\infty u := -\sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad \text{in } \Omega. \tag{1.2}$$

However, (1.2) is a fully nonlinear, degenerate PDE and may not have C^2 solutions. The issue was finally settled by Jensen [J2], who not only established the equivalence between the AMLE and the viscosity solution to (1.2) (a notion of weak solutions first introduced by Crandall-Lions [CL]; see also Crandall-Ishii-Lions [CIL]), but also proved the uniqueness of viscosity solutions to (1.2) with Dirichlet boundary data. The remarkable analysis of [J2] involves approximation by p-Laplacian and Jensen's maximum principle for semiconvex functions [J1]. The reader can consult with Evans [E] and Lindqvist-Manfredi [LM] for qualitative estimates on ∞-harmonic functions. Crandall-Evans-Gariepy [CEG] developed the comparison principle for viscosity solutions of (1.2) with *cones*, which are of the forms $a+b|x-x_0|$, and gave an alternative proof of the equivalence between AMLE and viscosity solution to (1.2). Furthermore, Crandall-Evans [CE] have utilized this property in their study on the regularity issue of ∞-harmonic functions. Recently, Barron-Jensen-Wang [BJW] considered general L^∞-functionals

$$F(u,\Omega) := \operatorname{ess\,sup}_{x\in\Omega} f(x,u(x),Du(x)),\ \forall u \in W^{1,\infty}(\Omega),$$

and proved, under suitable conditions on f, that any absolute minimizer of $F(\cdot,\Omega)$ is a viscosity solution to the Aronsson equation

$$-\sum_{i=1}^n f_{p_i}(x,u(x),Du(x))\frac{\partial}{\partial x_i}(f(x,u(x),Du(x))) = 0, \quad \text{in} \quad \Omega.$$

Shortly after [BJW], Crandall [C] gave an elegant proof of an improved version of [BJW]. Through [BJW] and [C], it becomes clear that the classical solution to the Hamilton-Jacobi equation $f(x,\phi(x),D\phi(x)) - c = 0$ plays an important role in the analysis. We would like to mention that there is a very interesting survey paper available on this subject by Aronsson-Crandall-Juntinen [ACJ].

Since the notion of AMLE can easily be formulated in any metric space, it is a very natural and interesting problem to study AMLE in Carnot-Carathéodory metric spaces, which include Riemannian manifolds and Subriemannian manifolds arising from Hörmander's vector fields (e.g. Heisenberg groups, Carnot groups). There have been several works in this direction. For example, Juutinen [J] extended the main theorems of [J2] into Riemannian manifolds. Bieske [B1, B2] proved the uniqueness of viscosity solutions to the subelliptic ∞-Laplacian equation on the Heisenberg group $\mathbf{H}^n$ and the Grushin-type space. Inspired by [C], Bieske-Capogna [BC] proved that any AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation for any Carnot group $\mathbf{G}$. In general, it was proved by [BC] that any horizontally C^1 AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation associated with *free* systems of Hörmander's vector fields. Bieske [B3] recently extended [BC] to the Grushin-type space.

In this paper, we are mainly interested in the Aronsson equation of AMLE's and its uniqueness issue for any Carnot-Carathédory metric space generated by vector fields satisfying Hörmander's condition. In this direction, we prove that any AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation. Moreover, if the vector fields are horizontal vector fields of a Carnot group $\mathbf{G}$, then both AMLE

and viscosity solution to the Aronsson equation are unique. In fact some more general results are provided by Theorems A and C below.

In order to state our results, we first recall some preliminary facts.

For a bounded domain $\Omega \subset R^n$ let $X_i \in C^{0,1}(\Omega, R^n)$, $1 \le i \le m$, be given Lipschitz continuous vector fields on Ω. We say that an absolutely continuous curve $r : [0,\delta] \to \Omega$ is *admissible* if there are measurable functions $a_j(t) : [0,\delta] \to R$, $1 \le j \le m$, such that

$$\sum_{j=1}^{m} a_j^2(t) \le 1, \quad r'(t) = \sum_{j=1}^{m} a_j(t) X_j(r(t)), \quad \text{for a.e. } t \in [0,\delta].$$

For any $x, y \in \Omega$, we define the *Carnot-Carathéodory distance* by

(1.3)
$$d_{\mathrm{X}}(x,y) = \inf\{T | \exists \text{ an admissible curve } r : [0,T] \to \Omega, \text{ with } r(0) = x, r(T) = y\}.$$

Definition 1.1 (Carnot-Carathédory space)**.** For a bounded domain $\Omega \subset R^n$ and a family of Lipschitz continuous vector fields, $\{X_i\}_{i=1}^m$, on Ω, if the above-defined $d_{\mathrm{X}}(\cdot,\cdot)$ satisfies the three properties for the distance function, then we call (Ω, d_{X}) a *Carnot-Carathéodory space*, i.e. there exists at least one admissible curve joining any two points in Ω. Furthermore, we assume throughout this paper that d_{X} is uniformly continuous with respect to the euclidean norm $\|\cdot\|$, namely

$$\omega(\delta) = \sup\{d_{\mathrm{X}}(x,y) : x, y \in \Omega, \text{with } \|x-y\| \le \delta\} \to 0, \text{ as } \delta \to 0. \tag{1.4}$$

We will focus on two classes of Carnot-Carathéodory spaces: (i) Hörmander's vector fields, and (ii) Carnot groups.

Example 1.2 (Hörmander's vector fields)**.** Suppose that $\{X_i\}_{i=1}^m \subset C^\infty(\Omega, R^n)$ satisfy Hörmander's finite rank condition, i.e. there exists a step $r \ge 1$ such that $\{X_i(x)\}_{i=1}^m$ and their commutators up to at most order r span R^n for any $x \in \Omega$. Then it follows from Nagel-Stein-Wainger [NSW] that d_{X}, defined by (1.4), is a distance. Moreover, for each compact set $K \subset\subset \Omega$ there exists a $C_K > 0$ so that

$$C_K^{-1}\|x-y\| \le d_{\mathrm{X}}(x,y) \le C_K \|x-y\|^{\frac{1}{r}}, \ \forall x, y \in K, \tag{1.5}$$

where $\|\cdot\|$ is the Euclidean norm on R^n.

The description of Carnot groups will be given later in this section. We now recall the definition of horizontal Sobolev spaces and Lipschitz spaces.

For $u : \Omega \to R$, denote $Xu := (X_1 u, \cdots, X_m u)$ as the horizontal gradient of u. For $1 \le p \le \infty$, the horizontal Sobolev space is defined by

$$W_{\mathrm{X}}^{1,p}(\Omega) \equiv \{u : \Omega \to R \mid \ \|u\|_{W_{\mathrm{X}}^{1,p}(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)} < \infty\}.$$

The Lipschitz space is defined by

$$\mathrm{Lip}_{\mathrm{X}}(\Omega) := \{u : \Omega \to R \mid \|u\|_{\mathrm{Lip}_{\mathrm{X}}(\Omega)} \equiv \sup_{x,y\in\Omega, x\ne y} \frac{|u(x)-u(y)|}{d_{\mathrm{X}}(x,y)} < \infty\}.$$

It was proved by Garofalo-Nieu [GN] and Franchi-Serapioni-Serra [FSS] that for any Carnot-Carathédory space (Ω, d_{X}), $u \in \mathrm{Lip}_{\mathrm{X}}(\Omega)$ iff $u \in W_{\mathrm{X}}^{1,\infty}(\Omega)$.

We now give the definition of absolute minimizers of L^∞-functionals over $W_{\mathrm{X}}^{1,\infty}(\Omega)$.

Definition 1.3 (absolute minimizers)**.** For any nonnegative, continuous function $f \in C(\Omega \times R^m)$, let

$$F(v, \Omega) = \operatorname{ess\,sup}_{x\in\Omega} f(x, Xv(x)),\ \forall v \in W^{1,\infty}_{\mathrm{X}}(\Omega).$$

A function $u \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ is an *absolute minimizer* of $F(\cdot, \Omega)$, if for any open subset $\tilde{\Omega} \subset \Omega$ and $w \in W^{1,\infty}_{\mathrm{X}}(\tilde{\Omega})$, with $w = u$ on $\partial\tilde{\Omega}$, we have

$$F(u, \tilde{\Omega}) \le F(w, \tilde{\Omega}). \tag{1.6}$$

$u \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ is called an *absolutely minimizing Lipschitz extension* (or AMLE) if u is an absolute minimizer of $F(\cdot, \Omega)$ when $f(x, p) = |p|^2$, for $(x, p) \in \Omega \times R^m$.

Formal calculations yield that an absolute minimizer $u \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ to $F(\cdot, \Omega)$ satisfies the Aronsson equation

$$-\sum_{i=1}^{m} X_i(f(x, Xu(x)))f_{p_i}(x, Xu(x)) = 0, \quad \text{in} \quad \Omega. \tag{1.7}$$

In particular, the Aronsson equation of an AMLE is the (subelliptic) ∞-Laplacian equation

$$\Delta^{(X)}_{\infty} u := -\sum_{i,j=1}^{m} X_i u X_j u X_i X_j u = 0, \quad \text{in} \quad \Omega. \tag{1.8}$$

In order to show that an absolute minimizer is a solution to (1.7), we need to recall the notion of viscosity solutions to (1.7), which is a natural extension of that developed by Crandall-Lions [CL] (see also [CIL]) of second order (degenerate) elliptic PDEs.

Let $\mathcal{S}^m$ denote the set of symmetric $m \times m$ matrices, equipped with the usual order. A function $A \in C(R^n \times R^m \times \mathcal{S}^m)$ is called *horizontally elliptic*, if, for any $(x, p) \in R^n \times R^m$,

$$A(x, p, M) \le A(x, p, N),\ \forall M, N \in \mathcal{S}^m, \quad \text{with } N \le M. \tag{1.9}$$

Let $(D^2u)^* \in \mathcal{S}^m$ denote the *horizontal* hessian of u, defined by

$$(D^2u)^*_{ij} = \frac{1}{2}(X_iX_j + X_jX_i)u,\ \forall 1 \le i,\ j \le m.$$

We now have

Definition 1.4. Consider a horizontally elliptic equation:

$$A(x, Xu(x), (D^2u)^*(x)) = 0, \quad \text{in} \quad \Omega. \tag{1.10}$$

A function $u \in C(\Omega)$ is called a viscosity subsolution to (1.10), if, for any pair $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that x_0 is a local maximum point of $(u - \phi)$, then we have

$$A(x_0, X\phi(x_0), (D^2\phi)^*(x_0)) \le 0. \tag{1.11}$$

A function $u \in C(\Omega)$ is called a viscosity supersolution to (1.10) if $-u$ is a viscosity subsolution to (1.10). Finally, a function $u \in C(\Omega)$ is a viscosity solution to (1.10) if it is both a viscosity subsolution and a viscosity supersolution to (1.10).

Remark 1.5. If $\{X_i\}_{i=1}^m \in C^2(\Omega, R^n)$, then we can define the horizontal C^2 space, $\Gamma^2(\Omega)$, by

$$\Gamma^2(\Omega) = \{u : \Omega \to R \mid X_iu, X_iX_ju \in C(\Omega), 1 \le i, j \le m\}.$$

In [B1, B2], [BC], Manfredi [FM], the authors introduced a slightly different notion of viscosity solution to (1.10) by requiring the test functions $\phi \in \Gamma^2(\Omega)$ in Definition 1.4. Since $C^2(\Omega) \subset \Gamma^2(\Omega)$, the notion of viscosity solution introduced here is weaker than that of [B1, B2], [BC], and [FM]. However, when the underlying Carnot-Carathéodory space is induced by a Carnot group, then these two notions are equivalent (see [B1]).

It is easy to check that both (1.7) and (1.8) are horizontally elliptic equations. Now we are ready to state our first theorem.

Theorem A. *Assume that $\{X_i\}_{i=1}^m \subset C^2(\Omega, R^n)$ and (Ω, d_{X}) is a Carnot-Carathéodory space. Suppose that $u \in W_{\mathrm{X}}^{1,\infty}(\Omega)$ is an absolute minimizer of*

$$F(v, \Omega) = \operatorname{ess\,sup}_{x\in\Omega} f(x, Xv(x)), v \in W_{\mathrm{X}}^{1,\infty}(\Omega),$$

where $f \in C^2(\Omega \times R^m, R_+)$ satisfies

(f1) *for any $x \in \Omega$, $f(x, \cdot)$ is quasiconvex:*

$$f(x, tp_1 + (1-t)p_2) \le \max\{f(x, p_1), f(x, p_2)\},\ \forall p_1\ p_2 \in R^m,\ 0 \le t \le 1. \tag{1.12}$$

(f2) *f is homogeneous of degree $\alpha \ge 1$ and $f_p(0, 0) = 0$.*

Then u is a viscosity solution to the Aronsson equation (1.7).

The ideas to prove Theorem A are based on: (1) the observation of rewriting (1.7) into a euclidean form which enables us to adopt Crandall's construction [C] of solutions to the Hamilton-Jacobi equation as test functions (see also [BJW]), and (2) the comparison principle of viscosity solutions to Hamilton-Jacobi equations without u-dependence (see [CIL] or [BJW]).

Since $f(x, p) \equiv |p|^2 : \Omega \times R^m \to R_+$ satisfies both (f1) and (f2), we have, as a consequence of Theorem A,

Corollary B. *Assume that $\{X_i\}_{i=1}^m \subset C^2(\Omega, R^n)$ and (Ω, d_{X}) is a Carnot-Carathéodory space. Suppose that $u \in W_{\mathrm{X}}^{1,\infty}(\Omega)$ is an AMLE. Then u is a viscosity solution to the (subelliptic) ∞-Laplacian equation* (1.8).

Now we turn to the discussion on the uniqueness problem of absolute minimizers of $F(\cdot, \Omega)$ and viscosity solutions to (1.7). For this purpose, we restrict our attention to the case where the vector fields generating the Carnot-Carathédory metric are the horizontal vector fields associated with a Carnot group $\mathbf{G}$.

Recall that a Carnot group of step $r \ge 1$ is a simply connected Lie group $\mathbf{G}$ whose Lie algebra g admits a vector space decomposition in r layers $g = V^1 + V^2 + \cdots + V^r$ having two properties: (i) g is stratified, i.e., $[V^1, V^j] = V^{j+1}, j = 1, \cdots, r-1$; (ii) g is r-nilpotent, i.e. $[V^j, V^r] = 0, j = 1, \cdots, r$. V^1 is called the *horizontal* layer and $V^j, j = 2, \cdots, r$, are *vertical* layers. It is well known (cf. Folland-Stein [FS]) that the exponential map, $\exp : g \to \mathbf{G}$, is a global differmorphism so that we can identify $\mathbf{G}$ with $g \equiv R^n$ via exp and $\mathbf{G}$ has an exponential coordinate system; here $n = \dim(\mathbf{G})$ is the topological dimension of $\mathbf{G}$. More precisely, let $X_{i,j}$, $1 \le i \le m_j = \dim(V^j)$, be a basis of V^j, for $1 \le j \le r$, which is orthonormal with respect to an arbitrarily chosen euclidean norm $\|\cdot\|$ on g, with respect to which the V^j's are mutually orthogonal. Then $p \in \mathbf{G}$ has coordinate $(p_{ij})_{1\le i\le m_j, 1\le j\le r}$ if $p = \exp(\sum_{j=1}^r \sum_{i=1}^{m_j} (p_{ij} X_{i,j})$. Let $\cdot$ denote the group multiplication on $\mathbf{G}$. Then it is known ([FS]) that the group law $(x, y) \to x{\cdot}y$ is a polynomial map with respect to the exponential map. From now on, we set $m = m_1 = \dim(V^1)$ and denote $X_i = X_{i,1}$ for $1 \le i \le m$. Two bi-Lipschitz equivalent metrics, on $\mathbf{G}$, we need are:

(1) the Carnot-Carathédory metric d_{X} on $\mathbf{G}$ generated by $\{X_i\}_{i=1}^m$; (2) the gauge metric d on $\mathbf{G}$ given as follows. For $p = (p_{ij})_{1\le i\le m_j, 1\le j\le r}$,

$$\|p\|_{\mathbf{G}}^{2r!} = \sum_{j=1}^{r}(\sum_{i=1}^{m_j}|p_{ij}|^2)^{\frac{r!}{j}},$$

with the induced gauge distance

$$d(x,y) = \|x^{-1}y\|_{\mathbf{G}}, \quad \forall x, y \in \mathbf{G},$$

satisfying the invariant property:

$$d(z\cdot x, z\cdot y) = d(x,y), \quad \forall x, y, z \in G.$$

Now we mention the Heisenberg group $\mathbf{H}^n$, the simplest Carnot group of step two. $\mathbf{H}^n \equiv \mathbf{C}^n \times R$ endowed with the group law: for $(z_1, \cdots, z_n, t), (z_1', \cdots, z_n', t') \in \mathbf{C}^n \times R$

$$(z_1, \cdots, z_n, t)\cdot(z_1', \cdots, z_n', t') = (z_1 + z_1', \cdots, z_n + z_n', t + t' + 2\mathrm{Im}(\sum_{i=1}^{n} z_i\bar{z_i'})),$$

whose Lie algebra $h = V_1 + V_2$ with $V_1 = \mathrm{span}\{X_i, Y_i\}_{1\le i\le n}$ and $V_2 = \mathrm{span}\{T\}$, where

$$T = 4\frac{\partial}{\partial t}, \ X_i = \frac{\partial}{\partial x_i} - 2y_i\frac{\partial}{\partial t}, \ \ Y_i = \frac{\partial}{\partial y_i} + 2x_i\frac{\partial}{\partial t}, \ \ 1 \le i \le n.$$

Recall that a function $f \in C^2(R^m)$ is strictly convex if there is a $C_0 > 0$ such that $D^2 f(x) \ge C_0$ for any $x \in R^m$. Now we are ready to state the uniqueness theorem.

Theorem C. *Let $\mathbf{G}$ be a Carnot group and let $\Omega \subset G$ be a bounded domain. Assume that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \ge 1$, and $f(p) > 0$ for $p \ne 0$. Then, for any $\phi \in W_{\mathrm{X}}^{1,\infty}(\Omega)$, the Dirichlet problem*

$$\begin{aligned} A(Xu, (D^2u)^*) := -\sum_{i,j=1}^{m} f_{p_i}(Xu)f_{p_j}(Xu)X_iX_ju = 0, \quad &\text{in } \ \Omega, \\ u = \phi, \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.13}$$

has at most one viscosity solution in $C(\bar{\Omega})$.

Although the operator A is horizontally elliptic, one can check that the operator $\bar{A}(x, Du, D^2u) \equiv A(Xu, (D^2u)^*)$ has x-dependence and is not elliptic in the euclidean sense (see [CIL] for its definition). Therefore, the uniqueness theorems, by [J1], Ishii [I], or Jensen-Lions-Souganidis [JLS], on viscosity solutions to second order elliptic PDEs, are not directly applicable. Our ideas are: (i) We observe that (1.13) is invariant under group multiplications: for any $a \in \mathbf{G}$, if $u \in C(\mathbf{G})$ is a viscosity solution to (1.13), then $u_a(x) = u(a\cdot x) : \mathbf{G} \to R$ is also a viscosity solution to (1.13). This enables us to extend the sup/inf convoluation construction by [JLS] to $\mathbf{G}$ to convert viscosity subsolutions (or supersolutions, resp.) of (1.13) into semiconvex subsolutions (or semiconcave supersolutions, resp.). (ii) We modify the original arguments from [J1] and [J2] to prove a comparison principle between semiconvex subsolutions and semiconcave *strict* supersolutions to any second order horizontally elliptic equations. (iii) We adopt the approximation scheme of p-Laplacians by [J2] to build viscosity solutions to two auxiliary equations having the property that any supersolution can be converted into *strict* supersolution

under small perturbations. (iv) Finally, we apply the comparison principle for the two auxiliary equations to prove the uniqueness of (1.13).

As a consequence of Theorems A and C, we have

Corollary D. *Let* $\mathbf{G}$ *be a Carnot group and let* $\Omega \subset \mathbf{G}$ *be a bounded domain. Assume that* $f \in C^2(R^m, R_+)$ *is strictly convex, homogeneous of degree* $\alpha \geq 1$*, and* $f(p) > 0$ *for* $p \neq 0$*. Then, for any* $\phi \in W^{1,\infty}_{\mathrm{X}}(\Omega)$*, there is a unique absolute minimizer* $u \in W^{1,\infty}_{\mathrm{X}}(\Omega)$*, with* $u|_{\partial\Omega} = \phi$*, to the functional* $F(v, \Omega) = \operatorname{ess\,sup}_{x\in\Omega} f(Xv(x))$*, and* (1.13) *has a unique viscosity solution in* $C(\bar{\Omega})$*. In particular,* ϕ *has a unique* AMLE *in* $W^{1,\infty}_{\mathrm{X}}(\Omega)$ *and the subelliptic* ∞*-Laplacian equation* (1.8) *has a unique viscosity solution.*

We would like to remark that our method is of euclidean nature, which is considerably different from that by [B1, B2] and [FM], where a subelliptic approach was employed.

The paper is written as follows. In §2, we outline the proof of Theorem A. In §3, we discuss the sup/inf convolution construction on any Carnot group $\mathbf{G}$. In §4, we discuss the comparison principle between semiconvex subsolutions and *strict* semiconcave supersolutions to any horizontally elliptic equations. In §5, we study two auxiliary equations to (1.13), with horizontal gradient constraints. In §6, we prove Theorem C.

§2. Proof of Theorem A

This section is devoted to the proof of Theorem A. It consists of two steps: (i) the construction of test functions by solving the Hamilton-Jacobi equation, motivated by [BJW] and [C], and (ii) the comparison between viscosity subsolution and classical strict supersolution of the Hamilton-Jacobi equation.

Proof of Theorem A. It suffices to prove that if u is not a viscosity subsolution of (1.12) at the point $x = 0 \in \Omega$, then u is not an absolute minimizer of $F(\cdot, \Omega)$. This assumption implies that there exists $r_0 > 0$ and $\phi \in C^2(\Omega)$ such that $B_{r_0}(0) \subset\subset \Omega$, and

$$0 = u(0) - \phi(0) \geq u(x) - \phi(x), \ \forall x \in \Omega, \tag{2.1}$$

but

$$-\sum_{i=1}^{m} X_i(f(x, X\phi(x)))f_{p_i}(x, X\phi(x))|_{x=0} = C_0 > 0. \tag{2.2}$$

Now we have

Lemma 2.1. *There exist a neighborhood* V *of* 0 *and a function* $\Phi \in C^2(V)$ *such that*

$$\Phi(0) = \phi(0), \quad D\Phi(0) = D\phi(0), \quad D^2\Phi(0) > D^2\phi(0), \tag{2.3}$$

and

$$f(x, X\Phi(x)) = f(0, X\phi(0)) > 0, \ \forall x \in V. \tag{2.4}$$

Proof. Since $\{X_i\}_{i=1}^m \subset C^2(\Omega, R^n)$, there is a matrix-valued function $(a_{ij}) \in C^2(\Omega, R^{mn})$ such that for $1 \le i \le m$,

$$X_i(x) = \sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j},\ \forall x \in \Omega.$$

Define $\bar{f} : \Omega \times R^n \to R$ by

$$\bar{f}(x, q_1, \cdots, q_n) = f(x, \sum_{j=1}^n a_{1j}(x)q_j, \cdots, \sum_{j=1}^n a_{mj}(x)q_j), \quad \forall (x, q_1, \cdots, q_n) \in \Omega \times R^n.$$

Since $f \in C^2(\Omega \times R^m)$ and $(a_{ij}) \in C^2(\Omega, R^{mn})$, it is easy to see that $\bar{f} \in C^2(\Omega \times R^n)$. Moreover, for any $(x, q) \in \Omega \times R^n$ and $1 \le i \le n$, we have

$$\frac{\partial \bar{f}}{\partial q_i}(x, q) = \sum_{k=1}^m a_{ki}(x)\frac{\partial f}{\partial p_k}(x, \sum_{j=1}^n a_{1j}(x)q_j, \cdots, \sum_{j=1}^n a_{mj}(x)q_j)$$

and

$$\bar{f}(x, D\phi(x)) = f(x, X_1\phi(x), \cdots, X_m\phi(x)) = f(x, X\phi(x)),\ \forall x \in \Omega. \tag{2.5}$$

Therefore, for any $x \in \Omega$, we have

$$\sum_{j=1}^m X_j(f(x, X\phi(x)))f_{p_j}(x, X\phi(x)) = \sum_{i=1}^n \frac{\partial}{\partial x_i}(\bar{f}(x, D\phi(x))\frac{\partial \bar{f}}{\partial q_i}(x, D\phi(x)). \tag{2.6}$$

This, combined with (2.2), implies

$$A(0, D\phi(0), D^2\phi(0)) := -\sum_{i=1}^n \frac{\partial}{\partial x_i}(\bar{f}(x, D\phi))\frac{\partial \bar{f}}{\partial q_i}(x, D\phi)(0) = C_0 > 0. \tag{2.7}$$

Now the argument in [C], pages 275-276, implies that there exist a neighborhood V of 0 and a function $\Phi \in C^2(V)$ such that

$$\Phi(0) = \phi(0), \quad D\Phi(0) = D\phi(0), \quad D^2\Phi(0) > D^2\phi(0), \tag{2.8}$$

$$\bar{f}(x, D\Phi(x)) = \bar{f}(0, D\phi(0)), \quad \forall x \in V. \tag{2.9}$$

Now (2.8) and (2.9), combined with (2.5), give (2.4). For the convenience of the readers, we briefly sketch Crandall's argument as follows. Let $E = D_x\bar{f}(0, D\phi(0))$, $F = -D_q\bar{f}(0, D\phi(0))$, and $G = E - D^2\phi(0)F$. Then (2.7) becomes $G \cdot F > 0$. It is easy to check that $X = P + D^2\phi(0)$, where P is given by

$$PZ = \frac{Z \cdot G}{F \cdot G}G + Z - \frac{Z \cdot F}{|F|^2}F, \quad \forall Z \in R^n,$$

has the property that $X > D^2\phi(0)$ and $E - XF = 0$. We now define

$$\psi(x) = \phi(0) + x \cdot D\phi(0) + \frac{1}{2}(Xx \cdot x)$$

and the hyperplane

$$\Sigma = \{x \in R^n \mid x \cdot \{D_x\bar{f}(0, D\phi(0)) + D^2\phi(0)D_q\bar{f}(0, D\phi(0))\} = 0\},$$

and solve the Cauchy problem

$$\bar{f}(x, D\Phi(x)) = \bar{f}(0, D\phi(0)) \text{ and } \Phi = \psi \text{ on } \Sigma$$

subject to

$$D\Phi(0) = D\phi(0)$$

in a neighhorhood of 0. Since Σ is not characteristic near 0, we have a local C^2 solution Φ to the Cauchy problem. The classical theory also implies that $D^2\Phi(0) = X > D^2\phi(0)$. This proves (2.8) and (2.9).

To see $f(0, X\phi(0)) > 0$, we observe that (2.2) implies

$$f_p(0, X\phi(0)) \equiv (\frac{\partial f}{\partial p_1}(0, X\phi(0)), \cdots, \frac{\partial f}{\partial p_m}(0, X\phi(0))) \neq 0.$$

This implies $X\phi(0) \neq 0$, for $f_p(0,0) = 0$. Note that the homogeneity of f implies that if $f(0,p) = 0$, then $p = 0$. Therefore, $f(0, X\phi(0)) > 0$. This finishes the proof of Lemma 2.1. □

It follows from Lemma 2.1 that there exists an open neighborhood $V_1 \subset V$ of 0 such that $\Phi(0) = \phi(0) = u(0)$, and $\Phi(x) > \phi(x) \geq u(x)$ for any $0 \neq x \in V_1$. Therefore, for any small $\epsilon > 0$, there exists another neighborhood $V_\epsilon \subset V_1$ of 0 such that

$$\Phi(x) - \epsilon < u(x), \quad \forall x \in V_\epsilon; \quad \Phi(x) - \epsilon = u(x), \quad \forall x \in \partial V_\epsilon. \tag{2.10}$$

It follows from the absolute minimality of u to $F(\cdot,\ \Omega)$ that

$$F(u, V_\epsilon) \leq F(\Phi - \epsilon, V_\epsilon) = \text{ess sup}_{x \in V_\epsilon} f(x, X\Phi(x)) = f(0, X\phi(0)). \tag{2.11}$$

Now we want to show that u is a viscosity subsolution of the Hamilton-Jacobi equation (2.4) on V_ϵ. For the reader's convenience, we first recall from [CIL] the definition of viscosity solution to (2.4).

Definition 2.2. A function $v \in C(V_\epsilon)$ is a viscosity subsolution to the Hamilton-Jacobian equation

$$\bar{f}(x, Dv(x)) = K, \quad x \in V_\epsilon, \tag{2.12}$$

where $K > 0$ is a constant, if for any $(x, \phi) \in V_\epsilon \times C^1(V_\epsilon)$ such that ϕ touches v at x from above, we have

$$f(x, D\phi(x)) \leq K.$$

We remark that if $v \in C^1(V_\epsilon)$ is a classical subsolution to (2.12), then v is also a viscosity subsolution to (2.12).

We also recall the following compactness theorem and a comparison principle for (2.12).

Proposition 2.3 ([CIL])**.** (1) *Assume that $\{v_k\} \subset C(V_\epsilon)$ are a family of viscosity subsolutions to* (2.12) *with $K = C_k$. If $v_k \to v$ uniformly on V_ϵ and $C_k \to C > 0$, then $v \in C(V_\epsilon)$ is a viscosity subsolution to* (2.12) *with $K = C$.*

(2) *If $v \in C(\bar{V}_\epsilon)$ is a viscosity subsolution to* (2.12) *and $w \in C^1(\bar{V}_\epsilon)$ is a strict, classical supersolution to* (2.12)*, i.e.*

$$\bar{f}(x, Dw(x)) > K, \quad \forall x \in V_\epsilon,$$

then we have

$$\sup_{V_\epsilon}(v - w) = \sup_{\partial V_\epsilon}(v - w). \tag{2.13}$$

Now we have

Lemma 2.4. *Under the same notations as above, $u \in W^{1,\infty}_{\mathrm{X}}(V_\epsilon)$ is a viscosity subsolution to the Hamilton-Jacobi equation* (2.4)*.*

Proof. For any subdomain $U \subset\subset V_\epsilon$ and $0 < \delta < \mathrm{dist}(U, \partial V_\epsilon) = \inf\{\|x-y\|, x \in U, y \in \partial V_\epsilon\}$, let $g_\delta : U \to R$ be the standard δ-mollifier of g for any function g on V_ϵ. Since $u \in W^{1,\infty}_{\mathrm{X}}(V_\epsilon)$, u_δ converges uniformly to u on U as $\delta \to 0$. Since (f1) implies $f(x,\cdot)$ is quasiconvex, it follows from the Jensen inequality for quasiconvex functions (cf. [BJW], Theorem 1.1) that for any $x \in U$, we have

$$\begin{aligned} f(x,(Xu)_\delta(x)) &\le \text{ess sup}_{y\in B_\delta(x)} f(x, Xu(y)) \\ &\le F(u, V_\epsilon) + C\delta \\ &\le f(0, X\phi(0)) + C\delta, \end{aligned} \tag{2.14}$$

where $C = \sup\{|D_x f|(x,p) : x \in U, |p| \le 2\|Xu\|_{L^\infty(\Omega)}\} < \infty$. Therefore we have

$$\max_{x\in U} f(x,(Xu)_\delta(x)) \le f(0, X\phi(0)) + C\delta. \tag{2.15}$$

On the other hand, for any $1 \le i \le m$ and $x \in U$, we estimate $(X_i u)_\delta(x) - X_i(u_\delta)(x)$ as follows:

$$\begin{aligned} &(X_i u)_\delta(x) - X_i(u_\delta)(x) \\ &= \int_{R^n} \eta_\delta(x-y)(\sum_{j=1}^n a_{ij}(y)\frac{\partial}{\partial y_j})(u(y)-u(x))\,dy \\ &- \int_{R^n} \sum_{j=1}^n a_{ij}(x)\frac{\partial \eta_\delta(x-y)}{\partial x_j}(u(y)-u(x))\,dy \\ &= \sum_{j=1}^n \int_{R^n} \{-\frac{\partial}{\partial y_j}(a_{ij}(y)\eta_\delta(x-y)) - a_{ij}(x)\frac{\partial(\eta_\delta(x-y))}{\partial x_j}\}(u(y)-u(x))\,dy \\ &= \sum_{j=1}^n \int_{R^n} (a_{ij}(y)-a_{ij}(x))\frac{\partial \eta_\delta(x-y)}{\partial x_j}(u(y)-u(x))\,dy \\ &+ \sum_{j=1}^n \int_{R^n} \frac{\partial a_{ij}(y)}{\partial y_j}\eta_\delta(x-y)(u(y)-u(x))\,dy. \end{aligned}$$

Therefore we have, for any $x \in U$,

$$\begin{aligned} |(X_i u)_\delta(x) - X_i(u_\delta)(x)| &\le C \max_{1\le j\le n} \|Da_{ij}\|_{L^\infty(\Omega)} \int_{R^n} \{\eta_\delta(x-y)|u(y)-u(x)| \\ &\quad + |y-x||D\eta_\delta(x-y)||u(y)-u(x)|\}\,dy \\ &\le C\|X_i\|_{C^1(\Omega)} \sup_{\|y-x\|\le\delta} |u(y)-u(x)| \\ &\le C\|X_i\|_{C^1(\Omega)}\|u\|_{W^{1,\infty}_{\mathrm{X}}(V_\epsilon)}\omega(\delta), \end{aligned}$$

where $\omega(\delta) = \sup\{d_{\mathrm{X}}(x,y) : \|y-x\| \le \delta\}$ is the modular of continuity of d_{X} with respect to the euclidean distance $\|\cdot\|$. This implies

$$\begin{aligned} f(x, X(u_\delta)(x)) &\le \text{ess sup}_{x\in U} f(x,(Xu)_\delta(x)) + \|f_p\|_{L^\infty}\|X(u_\delta) - (Xu)_\delta\|_{L^\infty(U)} \\ &\le f(0, X\phi(0)) + C(\delta + \omega(\delta)), \quad \forall x \in U. \end{aligned} \tag{2.16}$$

Hence u_δ is a classical subsolution to (2.12), with

$$K = K_\delta \equiv f(0, X\phi(0)) + C(\delta + \omega(\delta)),$$

on U. Since (1.4) implies $\lim_{\delta\to 0}\omega(\delta) = 0$, we have that $K_\delta \to f(0, X\phi(0))$ as $\delta \to 0$. Note also that $u_\delta \to u$ uniformly on U, and Proposition 2.3(1) implies

that u is a viscosity subsolution to (2.4) in U. Since U exhausts V_ϵ as $\delta \to 0$, we have that u is a viscosity subsolution of (2.4) in V_ϵ. The proof of Lemma 2.4 is complete. □

Now we return to the proof of Theorem A. It follows from (f2) that

$$f(x,(1+t)p) = (1+t)^\alpha f(x,p) = (1+g(t))f(x,p),\ \forall t>0,\ \forall (x,p)\in \Omega\times R^n,$$

where $g(t) \equiv (1+t)^\alpha - 1 > 0$ for $t>0$ and $\alpha \geq 1$. This, combined with (2.8), implies that for any $t>0$ we have

(2.17)
$$f(x, X((1+t)\Phi_\epsilon)(x)) = (1+g(t))f(0,X\phi(0)) = f(0,X\phi(0)) + \delta(t),\ \forall x\in V_\epsilon,$$

where $\Phi_\epsilon \equiv \Phi - \epsilon$ and $\delta(t) = g(t)f(0,X\phi(0)) > 0$. Therefore, for any $t>0$, $(1+t)\Phi_\epsilon$ is a *strict*, *classical* supersolution of (2.4). Therefore, Proposition 2.3(2) implies

$$\sup_{V_\epsilon}(u-(1+t)\Phi_\epsilon) \leq \sup_{\partial V_\epsilon}(u-(1+t)\Phi_\epsilon),\ \forall t>0. \tag{2.18}$$

Taking t into zero, we have

$$\sup_{V_\epsilon}(u-\Phi_\epsilon) \leq \sup_{\partial V_\epsilon}(u-\Phi_\epsilon) = 0.$$

This implies

$$u(x) \leq \Phi_\epsilon(x),\ \forall x\in V_\epsilon.$$

This clearly contradicts with (2.10). Therefore the proof of Theorem A is complete. □

§3. The construction of sup/inf convolutions on $\mathbf{G}$

This section is devoted to the construction of sup/inf convolutions on any Carnot group $\mathbf{G}$, which is the necessary extension of Jensen-Lions-Souganidis [JLS] and plays an important role in the proof of Theorem C.

Let $\Omega \subset \mathbf{G}$ be a bounded domain and let $d : \mathbf{G}\times\mathbf{G} \to R_+$ be the gauge distance defined in §1. For any $\epsilon > 0$, define

$$\Omega_\epsilon = \{x\in\Omega | \inf_{y\in \mathbf{G}\setminus\Omega} d(x^{-1},y^{-1})^{2r!} \geq \epsilon\}.$$

Definition 3.1. For any $u\in C(\bar\Omega)$ and $\epsilon>0$, the sup convolution u_ϵ of u is defined by

$$u^\epsilon(x) = \sup_{y\in\bar\Omega}\{u(y) - \frac{1}{2\epsilon}d(x^{-1},y^{-1})^{2r!}\},\ \forall x\in\Omega. \tag{3.1}$$

Similarly, the inf convolution v_ϵ of $v\in C(\bar\Omega)$ is defined by

$$v_\epsilon(x) = \inf_{y\in\bar\Omega}\{v(y) + \frac{1}{2\epsilon}d(x^{-1},y^{-1})^{2r!}\},\ \forall x\in\Omega. \tag{3.2}$$

For $x\in\mathbf{G}$, let $\|x\| := (\sum_{j=1}^r \sum_{i=1}^{m_j} x_{ij}^2)^{\frac{1}{2}}$ be the euclidean norm. We recall the notion of semiconvexity in the euclidean sense.

Definition 3.2. A function $u\in C(\bar\Omega)$ is semiconvex if there is a constant $C>0$ such that $u(x)+C\|x\|^2$ is convex; u is semiconcave if $-u$ is semiconvex. Note that if $u\in C^2(\Omega)$ satisfies $D^2u(x)\geq -C$ for $x\in\Omega$, then u is semiconvex. Here $D^2u = (\frac{\partial^2 u}{\partial x_i \partial x_j})_{1\leq i,j\leq n}$ is the euclidean hessian of u.

We now have the generalized version of [JLS].

Proposition 3.3. *For $u, v \in C(\bar{\Omega})$, denote $R_0 = 2\max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\}$. Then, for any $\epsilon > 0$, $u^\epsilon, v_\epsilon \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ satisfy:*

(1) *u^ϵ is semiconvex and v_ϵ is semiconcave.*

(2) *u^ϵ (or v_ϵ, resp.) is monotonically nondecreasing (or nonincreasing, resp.) w.r.t. ϵ and converges uniformly to u on $\Omega_{(1+4R_0)\epsilon}$.*

(3) *If u (or v, resp.) is a viscosity subsolution (or supersolution, resp.) to the horizontally elliptic equation*

$$B(Xu, (D^2u)^*) = 0 \ \text{ in } \ \Omega, \tag{3.3}$$

then u^ϵ (or v_ϵ, resp.) is a viscosity subsolution (or supersolution, resp.) to (3.3) in $\Omega_{(1+4R_0)\epsilon}$.

Proof. Since the proof for v_ϵ is parallel to that for u^ϵ, we only consider u^ϵ.

(1) Since $\Omega \subset \mathbf{G}$ is bounded, it is easy to see from the formula of d that

$$C_d(\Omega) \equiv \|D^2_x\{d(x^{-1}, y^{-1})^{2r!}\}\|_{L^\infty(\bar{\Omega}\times\bar{\Omega})} < \infty.$$

Therefore, for any $y \in \bar{\Omega}$, the function

$$\tilde{u}^\epsilon_y(x) := u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!} + \frac{C_d(\Omega)}{2\epsilon}|x|^2, \ \forall x \in \Omega,$$

has nonnegative hessian and is convex. Since the supremum of a family of convex functions is convex, we have that

$$u_\epsilon(x) + \frac{C_d(\Omega)}{2\epsilon}|x|^2 = \sup_{y\in\bar{\Omega}} \tilde{u}^\epsilon_y(x)$$

is convex so that u_ϵ is semiconvex. Since any semiconvex function is Lipschitz with respect to the euclidean metric, we have $u^\epsilon \in W^{1,\infty}_{\mathrm{X}}(\Omega)$.

(2) It is easy to see that for any $\epsilon_1 < \epsilon_2$, $u^{\epsilon_1}(x) \le u^{\epsilon_2}(x)$ and $u(x) \le u^\epsilon(x) \le R_0$ for any $x \in \Omega$. Observe that for any $x \in \Omega$ we have

$$u^\epsilon(x) = \sup_{\bar{\Omega}\cap\{d(x^{-1},y^{-1})^{2r!}\le 4R_0\epsilon\}} (u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!}).$$

Therefore, for any $x \in \Omega_{(1+4R_0)\epsilon}$, $u^\epsilon(x)$ is attained at a point $y \in \Omega$. To see $u^\epsilon \to u$ uniformly on $\Omega_{(1+4R_0)\epsilon}$, we observe that if $u^\epsilon(x)$ is attained by x_ϵ, then

$$u_{\frac{\epsilon}{2}}(x) \ge u(x_\epsilon) - \frac{1}{\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!} = u_\epsilon(x) - \frac{1}{2\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!}.$$

This implies

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!} = 0,$$

so that $x_\epsilon \to x$ and $\lim_{\epsilon\to 0} u^\epsilon(x) = u(x)$. Since

$$|u^\epsilon(x_1) - u^\epsilon(x_2)| \le |u(x_1) - u(x_2)|, \ \forall x_1, \ x_2 \in \Omega,$$

$u^\epsilon \to u$ uniformly on $\Omega_{(1+4R_0)\epsilon}$.

(3) For any $x_0 \in \Omega_{(1+4R_0)\epsilon}$, let $\phi \in C^2(\Omega_{(1+4R_0)\epsilon})$ be such that

$$u^\epsilon(x_0) - \phi(x_0) \ge u^\epsilon(x) - \phi(x), \quad \forall x \in \Omega_{(1+4R_0)\epsilon}.$$

It follows from the proof of (2) above that there exists a $y_0 \in \Omega$ such that

$$u^\epsilon(x_0) = u(y_0) - \frac{1}{2\epsilon} d(x_0^{-1}, y_0^{-1})^{2r!}.$$

Therefore we have

$$u(y_0)-\frac{1}{2\epsilon}d(x_0^{-1},y_0^{-1})^{2r!}-\phi(x_0)\geq u(y)-\frac{1}{2\epsilon}d(x^{-1},y^{-1})^{2r!}-\phi(x),\forall x,y\in\Omega_{(1+4R_0)\epsilon}.$$

For y near y_0, we have $x=x_0\cdot y_0^{-1}\cdot y\in\Omega_{(1+4R_0)\epsilon}$ and hence

$$u(y_0)-\phi(x_0\cdot y_0^{-1}\cdot y_0)\geq u(y)-\phi(x_0\cdot y_0^{-1}\cdot y).$$

Set $\tilde{\phi}(y)=\phi(x_0\cdot y_0^{-1}\cdot y)$ for $y\in\Omega_{(1+4R_0)\epsilon}$ near y_0. Then $\tilde{\phi}$ touches u from above at $y=y_0$ so that we have

$$B(X\tilde{\phi},(D^2\tilde{\phi})^*)(y_0)\leq 0. \tag{3.4}$$

It follows from the left invariance of X_i that

$$X\tilde{\phi}(y)=(X\phi)(x_0\cdot y_0^{-1}\cdot y),\quad (D^2(\tilde{\phi}))^*(y)=(D^2\phi)^*(x_0\cdot y_0^{-1}\cdot y).$$

This implies

$$B(X\phi(x_0),(D^2\phi)^*(x_0))\leq 0.$$

Hence u^ϵ is a viscosity subsolution of (3.3) on $\Omega_{(1+4R_0)\epsilon}$. The proof is complete. □

§4. Comparison principle between semiconvex subsolutions and semiconcave supersolutions

In this section, we establish the comparison principle between semiconvex subsolutions and semiconcave *strict* supersolutions for any second order horizontally elliptic PDE on the Carnot-Carathédory space. The argument is inspired by Jensen's maximum principle for semiconvex functions (see [J1], [J2]) on second order elliptic PDEs. Here we assume that $\{X_i=\sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j}\}_{i=1}^m\}$ are C^1 vector fields on R^n. The main proposition of this section is

Proposition 4.1. *For any bounded domain $\Omega\subset R^n$, suppose that $B\in C(\Omega\times\mathcal{S}^m)$ is horizontally elliptic. Assume that $u\in C(\bar{\Omega})$ is a semiconvex subsolution to*

$$B(Xw,(D^2w)^*)=0\ \text{ in }\ \Omega, \tag{4.1}$$

and $v\in C(\bar{\Omega})$ is a semiconcave supersolution to

$$B(Xw,(D^2w)^*)-\mu=0\ \text{ in }\ \Omega \tag{4.2}$$

for some $\mu>0$. Then

$$\sup_\Omega(u-v)\leq\sup_{\partial\Omega}(u-v). \tag{4.3}$$

Proof. Suppose that (4.3) were false. Then

$$\sup_\Omega(u-v)>\sup_{\partial\Omega}(u-v).$$

Hence $u-v$ achieves its maximum on $\bar{\Omega}$ at a point $x_0\in\Omega$. Since $u-v$ is semiconvex, it is well known (cf. [J2], page 67) that

$$Du(x_0),Dv(x_0)\text{ both exist and are equal,}$$

$$u(x)-u(x_0)-\langle Du(x_0),x-x_0\rangle=O(\|x-x_0\|^2), \tag{4.4}$$

$$v(x)-v(x_0)-\langle Dv(x_0),x-x_0\rangle=O(\|x-x_0\|^2), \tag{4.5}$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the Euclidean inner product and norm. Denote $R_0=\inf\{\|x_0-y\|:y\in\partial\Omega\}$ and let $R_1>0$ be such that both (4.4) and (4.5) hold with

$\|x - x_0\| < R_1$. Set $R_2 = \min\{R_0, R_1\} > 0$ and define, for any $\rho > 0$, the rescaled maps $u^\rho, v^\rho : B_{R_2\rho^{-1}} = \{x \in R^n | \|x\| \le R_2\rho^{-1}\} \to R$ by

$$u^\rho(x) = \frac{1}{\rho^2}(u(x_0 + \rho x) - u(x_0) - \rho\langle Du(x_0), x\rangle),$$
$$v^\rho(x) = \frac{1}{\rho^2}(v(x_0 + \rho x) - v(x_0) - \rho\langle Dv(x_0), x\rangle),$$

where the addition and scalar multiplication are the euclidean ones. Then it is easy to see that

$$0 = (u^\rho - v^\rho)(0) \ge (u^\rho - v^\rho)(x), \quad \forall x \in B_{R_2\rho^{-1}}.$$

It follows from (4.4) and (4.5) that, for any $R > 0$, there exists a $\rho_0 = \rho_0(R) > 0$ such that (i) $\{u^\rho\}_{\{0<\rho\le\rho_0\}}$ are uniformly bounded, uniformly semiconvex, and uniformly Lipschitz continuous in B_R; (ii) $\{v^\rho\}_{\{0<\rho\le\rho_0\}}$ are uniformly bounded, uniformly semiconcave, and uniformly Lipschitz continuous in B_R. Therefore, by the Cauchy diagonal process, we find that there is $\rho_i \downarrow 0$ such that $u^{\rho_i} \to u^*$, $v^{\rho_i} \to v^*$ locally uniformly in R^n. Moreover, it is not difficult to see that u^* is locally bounded, semiconvex in R^n, v^* is locally bounded, semiconcave in R^n, and

$$0 = (u^* - v^*)(0) \ge (u^* - v^*)(x), \quad \forall x \in R^n.$$

Now we need

Claim 4.2. *u^* is a viscosity subsolution to*

$$B_1(D^2w) = 0 \ \text{ in } \ R^n, \tag{4.6}$$

and v^* is a viscosity supersolution to

$$B_2(D^2w) + \mu = 0 \ \text{ in } \ R^n, \tag{4.7}$$

where $B_1, B_2 : \mathcal{S}^n \to R$ are defined by

$$B_1(M) = B(Xu(x_0), \{\sum_{k,l=1}^n (a_{ik}(x_0)a_{jl}(x_0)M_{kl} + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial u}{\partial x_l}(x_0))\}_{1\le i,j\le m}),$$
$$B_2(M) = B(Xv(x_0), \{\sum_{k,l=1}^n (a_{ik}(x_0)a_{jl}(x_0)M_{kl} + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial v}{\partial x_l}(x_0))\}_{1\le i,j\le m}).$$

Before we prove this claim, we would like to remark that the horizontal ellipticity (1.9) of B implies that $B_1, B_2 \in C(\mathcal{S}^n)$ is elliptic in the sense that

$$B_i(M) \le B_i(N), \ \ \forall M, N \in \mathcal{S}^n, \text{ with } M \ge N$$

for $i = 1, 2$. This is also a key observation we need to use in the following proof.

To prove it, we first observe that u^ρ is a viscosity subsolution to

$$B(Xu(x_0) + \rho X^\rho w, \{\sum_{k,l=1}^n a^\rho_{ik}a^\rho_{jl}\frac{\partial^2 w}{\partial x_k \partial x_l} + a^\rho_{ik}(\frac{\partial a_{jl}}{\partial x_k})^\rho(\frac{\partial u}{\partial x_l}(x_0) + \rho\frac{\partial w}{\partial x_l})\}_{1\le i,j\le m}) = 0 \tag{4.8}$$

and v^ρ is a viscosity supersolution to

$$B(Xv(x_0) + \rho X^\rho w, \{\sum_{k,l=1}^n a^\rho_{ik}a^\rho_{jl}\frac{\partial^2 w}{\partial x_k \partial x_l} + a^\rho_{ik}(\frac{\partial a_{jl}}{\partial x_k})^\rho(\frac{\partial v}{\partial x_l}(x_0) + \rho\frac{\partial w}{\partial x_l})\}_{1\le i,j\le m}) = 0, \tag{4.9}$$

where $X^\rho = (X_1^\rho, \cdots, X_m^\rho)$, $X_i^\rho(x) = X_i(x_0 + \rho x)$, $a_{ik}^\rho(x) = a_{ik}(x_0 + \rho x)$, and $(\frac{\partial a_{jl}}{\partial x_k})^\rho(x) = \frac{\partial a_{jl}}{\partial x_k}(x_0 + \rho x)$. For simplicity, we only indicate (4.8). Let $x_1 \in B_{R_2\rho^{-1}}$ and $\phi \in C^2(B_{R_2\rho^{-1}})$ be such that

$$0 = u^\rho(x_1) - \phi(x_1) \geq u^\rho(x) - \phi(x),\ \forall x \in B_{R_2\rho^{-1}}.$$

It is straightforward to see that

$$\phi^\rho(x) \equiv u(x_0) + \langle Du(x_0), x - x_0\rangle + \rho^2\phi(\frac{x - x_0}{\rho})$$

satisfies

$$0 = u(x_0 + \rho x_1) - \phi^\rho(x_0 + \rho x_1) \geq u(x) - \phi^\rho(x),\ \forall x \in B_{R_0}(x_0).$$

This, combined with the fact that u is a viscosity subsolution to (4.6), implies

$$B(X\phi^\rho, (D^2\phi^\rho)^*)(x_0 + \rho x_1) \geq 0. \tag{4.10}$$

Direct calculations yield

$$\frac{\partial \phi^\rho}{\partial x_k}(x) = \frac{\partial u}{\partial x_k}(x_0) + \rho\frac{\partial \phi}{\partial x_k}(\frac{x - x_0}{\rho}),\quad \frac{\partial^2 \phi^\rho}{\partial x_k \partial x_l}(x) = \frac{\partial^2 \phi}{\partial x_k \partial x_l}(\frac{x - x_0}{\rho}).$$

Hence (4.10) implies that u^ρ is a viscosity solution to (4.8). Since $X^\rho \to X(x_0)$, $a_{ij}^\rho \to a_{ij}(x_0)$, and $(\frac{\partial a_{ik}}{\partial x_l})^\rho \to \frac{\partial a_{ik}}{\partial x_l}(x_0)$ locally uniformly on R^n, we observe that the family of elliptic operator $B^\rho : R^n \times R^n \times \mathcal{S}^n \to R$ defined by

$$B^\rho(x, p, M) := B[(X_i u(x_0) + \rho\sum_{j=n}^n a_{ij}^\rho(x)p_j)_{1\leq i\leq n},$$
$$\{\sum_{k,l=1}^n a_{ik}^\rho(x)a_{jl}^\rho(x)M_{kl} + a_{ik}^\rho(x)(\frac{\partial a_{jl}}{\partial x_k})^\rho(x)(\frac{\partial u}{\partial x_l}(x_0) + \rho p_l)\}_{1\leq i,j\leq m}]$$

coverges to $B_1(M)$ locally uniformly as $\rho \to 0$. Therefore the compactness theorem for viscosity subsolutions (see, e.g. Caffarelli-Cabré [CC], proposition 2.9, page 14) implies that u^* is a viscosity subsolution of (4.6). This proves Claim 4.2.

Since $u^* - v^*$ is semiconvex and achieves its maximum at $x = 0$, we can apply Jensen's maximum principle for semiconvex functions (see [J1] and also [CIL], Lemma A.3, page 60) to conclude that there exists $x_* \in R^n$ such that $D^2u^*(x_*), D^2v^*(x_*)$ both exist and $D^2(u^* - v^*)(x_*) \leq 0$. Let $M_1, M_2 : \mathcal{S}^m \to R$ be given by

$$M_1^{ij} = \sum_{k,l=1}^n \{a_{ik}(x_0)a_{jl}(x_0)\frac{\partial^2 u^*}{\partial x_k \partial x_l}(x_*) + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial u}{\partial x_l}(x_0)\}, 1 \leq i, j \leq m,$$

$$M_2^{ij} = \sum_{k,l=1}^n \{a_{ik}(x_0)a_{jl}(x_0)\frac{\partial^2 v^*}{\partial x_k \partial x_l}(x_*) + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial v}{\partial x_l}(x_0)\}, 1 \leq i, j \leq m.$$

Since $Du(x_0) = Dv(x_0)$, we have $Xu(x_0) = Xv(x_0)$ and, for any $p \in R^m$,

$$\sum_{1\leq i,j\leq m}(M_1^{ij} - M_2^{ij})p_i p_j = \sum_{k,l=1}^n \{\sum_{i=1}^m p_i a_{ik}(x_0)\}\{\sum_{j=1}^m p_j a_{jl}(x_0)\}\frac{\partial^2(u^* - v^*)}{\partial x_k \partial x_l}(x_*) \leq 0.$$

Hence $M_1 \leq M_2$. Therefore the horizontal ellipticity of B implies

$$B(Xu(x_0), M_1) - B(Xv(x_0), M_2) \geq 0. \tag{4.11}$$

On the other hand, we have

$$B_1(D^2u^*(x_*)) - B_2(D^2v^*(x_*)) = B(Xu(x_0), M_1) - B(Xv(x_0), M_2) \le -\mu < 0. \tag{4.12}$$

This contradicts (4.11). Hence the proof of Proposition 4.1 is complete. □

§5. Auxiliary equations with horizontal gradient constraints

Due to the degenerancy of (1.13), we establish a comparison principle for its viscosity solutions via the L^p-approximation scheme by [J2] to construct two auxiliary equations with horizontal gradient constraints, to which supersolutions can be deformed into *strict* supersolutions under small perturbations. This section is valid for any Carnot-Carathédory metric space generated by a family of Hörmander's vector fields $\{X_i\}_{i=1}^m \subset C^\infty(\Omega, R^n)$.

Lemma 5.1. *Suppose that $f \in C^2(R^m, R_+)$ is homogeneous of degree $\alpha \ge 1$. Let $v \in C(\bar\Omega)$ be a viscosity supersolution to*

$$\min\{f(Xw) - \epsilon, \ -\sum_{i,j=1}^m f_{p_i}(Xw) f_{p_j}(Xw) X_i X_j w\} = 0, \ \text{ in } \Omega, \tag{5.1}$$

where $\epsilon > 0$. Then, for any $\delta > 0$, there exist $\mu = \mu(\alpha, \epsilon, \delta) > 0$ and $v_\delta \in C(\bar\Omega)$, with $\|v_\delta - v\|_{L^\infty(\Omega)} \le \delta$, such that v_δ is a viscosity supersolution to

$$\min\{f(Xw) - \epsilon, \ -\sum_{i,j=1}^m f_{p_i}(Xw) f_{p_j}(Xw) X_i X_j w\} - \mu = 0, \ \text{ in } \ \Omega. \tag{5.2}$$

Proof. It is similar to that of [J2] (see also [J] and [B1]). We sketch it here. We look for $v_\delta = g_\delta(v)$, where $g_\delta \in C^\infty(R)$ is monotonically increasing such that $g_\delta^{-1} \in C^\infty(R)$. To find g_δ, let $x_0 \in \Omega$, and $\phi \in C^2(\Omega)$ touches v_δ from below at x_0. Let $\phi_\delta = g_\delta^{-1}(\phi)$. Then ϕ_δ touches v from below at x_0 and

$$\min\{f(X\phi_\delta) - \epsilon, \ -\sum_{i,j=1}^m f_{p_i}(X\phi_\delta) f_{p_j}(X\phi_\delta) X_i X_j \phi_\delta\}|_{x=x_0} \ge 0.$$

Since

$$X_i\phi = g_\delta'(\phi_\delta) X_i \phi_\delta, \ X_i X_j \phi = g_\delta'(\phi_\delta) X_i X_j \phi_\delta + g_\delta''(\phi_\delta) X_i \phi_\delta X_j \phi_\delta,$$

we have, by the α-homogeneity of f,

$$f(X\phi(x_0)) = f(g_\delta'(\phi_\delta(x_0)) X\phi_\delta(x_0)) = (g_\delta'(\phi_\delta(x_0)))^\alpha f(X\phi_\delta(x_0)) \ge (g_\delta'(\phi_\delta(x_0)))^\alpha \epsilon \tag{5.3}$$

and

$$\begin{aligned} &-\sum_{i,j=1}^m f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi|_{x=x_0} \\ &= \{g_\delta'(\phi_\delta)^{3\alpha}(-\sum_{ij=1}^m f_{p_i}(X\phi_\delta) f_{p_i}(X\phi_\delta) X_i X_j \phi_\delta) \\ &- g_\delta'(\phi_\delta)^{2\alpha} g_\delta''(\phi_\delta)(\sum_{i=1}^m f_{p_i}(X\phi_\delta) X_i \phi_\delta)^2\}|_{x=x_0} \\ &\ge -g_\delta'(\phi_\delta(x_0))^{2\alpha} g_\delta''(\phi_\delta(x_0)) \alpha^2 \epsilon^2, \end{aligned} \tag{5.4}$$

provided that $g''_\delta(\phi_\delta) < 0$. Here we have used (5.3) and the identity $\sum_{i=1}^m f_{p_i}(p)p_i = \alpha f(p)$ in the last step. Let $C_0 = 4\|v\|_{L^\infty(\Omega)} < \infty$ and define

$$g_\delta(t) = (1+\delta)t - \frac{\delta}{4C_0}t^2,\ |t| \le 2C_0,$$

and extend it suitably to a monotonically increasing function on R. Since $g'(t) \ge 1 + \frac{\delta}{2}$ and $g''(t) = -\frac{\delta}{2C_0}$ for $|t| \le C_0$, we have

$$f(X\phi)(x_0) \ge (1+\frac{\delta}{2})\epsilon$$

and

$$-\sum_{i,j=1}^m f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi|_{x=x_0} \ge \frac{\delta\alpha^2\epsilon^2}{2C_0}.$$

Therefore, if we choose $\mu = \min\{\frac{\delta\epsilon}{2}, \frac{\delta\alpha^2\epsilon^2}{2C_0}\} > 0$, then

$$\min\{f(X\phi) - \epsilon,\ \ -\sum_{i,j=1}^m f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi\}|_{x=x_0} \ge \mu.$$

The proof of Lemma 5.1 is complete. □

Since the argument is similar, we state without proof the analogous lemma on viscosity subsolutions.

Lemma 5.2. *Suppose that $f \in C^2(R^m, R_+)$ is homogeneous of degree $\alpha \ge 1$. Let $u \in C(\bar\Omega)$ be a viscosity subsolution to*

$$\max\{\epsilon - f(Xw),\ \ -\sum_{i,j=1}^m f_{p_i}(Xw)f_{p_j}(Xw)X_iX_jw\} = 0,\ \text{ in }\ \Omega, \tag{5.5}$$

where $\epsilon > 0$. Then, for any $\delta > 0$, there are an $\mu = \mu(\alpha, \epsilon, \delta) > 0$ and $u_\delta \in C(\bar\Omega)$, with $\|u_\delta - u\|_{L^\infty(\Omega)} \le \delta$, such that u_δ is a viscosity subsolution to the equation

$$\max\{\epsilon - f(Xw),\ \ -\sum_{i,j=1}^m f_{p_i}(Xw)f_{p_j}(Xw)X_iX_jw\} = -\mu,\ \text{ in }\ \Omega. \tag{5.6}$$

We end this section with existences of viscosity solutions to (1.13), (5.3), and (5.5). For this, we need both convexity of f and $f(p) > 0$ for $p \ne 0$. More precisely,

Lemma 5.3. *Suppose that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \ge 1$, and $f(p) > 0$ for $p \ne 0$. Then, for any $g \in W^{1,\infty}_{\rm cc}(\Omega)$, we have*

(1) *There exists a viscosity solution $u \in W^{1,\infty}_{\rm cc}(\Omega)$ to (1.13) such that $u|_{\partial\Omega} = g$.*
(2) *There exists a viscosity solution $u_\epsilon \in W^{1,\infty}_{\rm cc}(\Omega)$ of (5.3) such that $u_\epsilon|_{\partial\Omega} = g$.*
(3) *There exists a viscosity solution $v_\epsilon \in W^{1,\infty}_{\rm cc}(\Omega)$ of (5.5) such that $v_\epsilon|_{\partial\Omega} = g$.*
(4) *There exists a continuous, nondecreasing function $\beta : R_+ \to R_+$, with $\beta(0) = 0$, such that*

$$\|u_\epsilon - v_\epsilon\|_{L^\infty(\Omega)} \le \beta(\epsilon). \tag{5.7}$$

Proof. The proof is inspired by the L^k-approximation scheme, which has been used by Bhatthacharya-Dibenedetto-Manfredi [BDM], [J2], [J], [B1] in the euclidean setting. For completeness, we outline it here. Since (1) follows from (2) with $\epsilon = 0$

and (3) can be done exactly in the way as (2), we only sketch (2) and (4) as follows. For $1 < k < \infty$, let $u_k \in W^{1,k}_{\mathrm{X}}(\Omega)$ be the unique minimizer to the functional

$$F_k(v) = \int_\Omega (f(Xv)^k - \epsilon^{k-1} v),\ \forall v \in W^{1,k}_{\mathrm{X}}(\Omega),\ \text{with } v|_{\partial\Omega} = g.$$

The existence of u_k can be obtained by the direct method, due to both the convexity of f and α-homogeneity of f, i.e. $f(p) = |p|^\alpha f(\frac{p}{|p|}) \geq |p|^\alpha \min_{|z|=1} f(z) \geq C|p|^\alpha$. It is easy to verify that u_k satisfies the (subelliptic) k-Laplacian: the equation

$$-\sum_{i=1}^m X_i^*(kf(Xu_k)^{k-1} f_{p_i}(Xu)) = \epsilon^{k-1}, \quad \text{in } \ \Omega \tag{5.8}$$

in the sense of distributions, where $X_i^* w = \sum_{j=1}^m \frac{\partial}{\partial x_j}(a_{ij}(x)w)$ is the adjoint operator of X_i. Let Q denote the homogeneous dimension of R^n, with respect to the vector fields $\{X_i\}_{i=1}^m$. Then it follows from the Sobolev inequality (see Hajlasz-Koskela [HK]) that $\{u_k\}_{k\geq Q+1}$ is bounded and equicontinuous. Therefore we may assume, after taking possible subsequences, that there exists a $u_\epsilon \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ such that

$$u_k \to u_\epsilon \ \text{ in } \ C^0(\bar\Omega) \cap_{Q+1\leq k<\infty} W^{1,k}_{\mathrm{X}}(\Omega).$$

It is easy to see that $u_\epsilon|_{\partial\Omega} = g$. To show that u_ϵ is a viscosity solution to (5.3), we need

Claim 5.4. *For $k \geq Q+1$, $u_k \in C(\bar\Omega)$ is a viscosity solution to* (5.8).

For simplicity, we only indicate that u_k is a viscosity subsolution. For, otherwise, there are $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$0 = u_k(x_0) - \phi(x_0) \geq u_k(x) - \phi(x),\ \forall x \in \Omega,$$

but

$$-\sum_{i=1}^m X_i^*(kf(X\phi)^{k-1} f_{p_i}(X\phi))(x_0) - \epsilon^{k-1} = C_0 > 0. \tag{5.9}$$

Then there exists $\delta_0 > 0$ such that

$$-\sum_{i=1}^m X_i^*(kf(X\phi)^{k-1} f_{p_i}(X\phi))(x) - \epsilon^{k-1} \geq \frac{C_0}{2}, \quad \forall x \in B_{\delta_0}(x_0). \tag{5.10}$$

For any small $\delta > 0$, there is a neighborhood $V_\delta (\subset B_{\delta_0}(x_0))$ of x_0 such that $\phi_\delta \equiv \phi - \delta$ satisfies

$$\phi_\delta(x) < u_k(x),\ \forall x \in V_\delta; \quad \phi_\delta(x) = u_k(x),\ \forall x \in \partial V_\delta.$$

Note that ϕ_δ also satisfies (5.10). Multiplying (5.8) by $u_k - \phi_\delta$ and integrating over V_δ, we have

$$\sum_{i=1}^m \int_{V_\delta} kf(Xu_k)^{k-1} f_{p_i}(Xu_k) X_i(u_k - \phi_\delta) = \epsilon^{k-1} \int_{V_\delta} (u_k - \phi_\delta). \tag{5.11}$$

On the other hand, multiplying (5.10) by $(u_k - \phi_\delta)(\geq 0)$ and integrating over V_δ, we have

$$\sum_{i=1}^m \int_{V_\delta} k(f(X\phi_\delta))^{k-1} f_{p_i}(X\phi_\delta) X_i(u_k - \phi_\delta) > \epsilon^{k-1} \int_{V_\delta} (u_k - V_\delta). \tag{5.12}$$

Subtracting (5.11) from (5.12), we obtain

$$0 > k \int_{V_\delta} \sum_{i=1}^{m} (f(Xu_k)^{k-1} f_{p_i}(Xu_p) - f(X\phi_\delta)^{k-1} f_{p_i}(X\phi_\delta)) X_i(u_k - \phi_\delta);$$

this contradicts with the convexity of f. This finishes the proof of Claim 5.4.

Now we show that u_ϵ is a viscosity subsolution to (5.3). Let $x \in \Omega$ and $\phi \in C^2(\Omega)$ be such that

$$0 = u_\epsilon(x) - \phi(x) \geq u_\epsilon(y) - \phi(y), \ \forall y \in \Omega.$$

We need to show that

$$\min\{f(X\phi) - \epsilon, \ -\sum_{i,j=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi\}|_x \leq 0.$$

Since this is true if $f(X\phi(x)) \leq \epsilon$, we may assume that $f(X\phi(x)) \geq (1+2\delta)\epsilon$ for some $\delta > 0$. We know that there exist $x_k \in \Omega$ such that $(u_k - \phi)$ achieves its maximum at x_k and $x_k \to x$. We may also assume that, for k sufficiently large,

$$f(X\phi(x_k)) \geq (1+\delta)\epsilon.$$

It follows from Claim 5.4 that

$$-\sum_{i=1}^{m} X_i^*(kf(X\phi)^{k-1} f_{p_i}(X\phi))(x_k) \leq \epsilon^{k-1}.$$

After expansion and dividing both sides by $k(k-1)f(X\phi)^{k-2}(x_k)$, this gives

$$\begin{aligned} -\sum_{i,j=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi(x_k) \leq [\frac{\epsilon}{k(k-1)} \{\frac{\epsilon}{f(X\phi(x_k))}\}^{k-2} \\ - \frac{f(X\phi(x_k))}{(k-1)} \sum_{i=1}^{m} X_i^*(f_{p_i}(X\phi))(x_k)]. \end{aligned}$$

This, after letting k tend to ∞, gives

$$-\sum_{i,j=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi(x) \leq 0.$$

One can show similarly that u_ϵ is also a viscosity supersolution to (5.3). This finishes the proof of (2).

Since v_ϵ is a limit, as $k \to \infty$, of the minimizers v_k to

$$G_k(v) = \int_\Omega f(Xv)^k + \epsilon^{k-1} v, \ \forall v \in W^{1,k}_{\mathrm{X}}(\Omega), \text{ with } u|_{\partial\Omega} = g,$$

v_k satisfies

$$-\sum_{i=1}^{m} X_i^*(kf(Xv_k)^{k-1} f_{p_i}(Xv_k)) = -\epsilon^{k-1}, \quad \text{in } \Omega. \tag{5.13}$$

Multiplying (5.11) and (5.13) by $(u_k - v_k)$, integrating over Ω, and subtracting each other, we get

$$\begin{aligned} &\int_\Omega k(f(Xu_k)^{k-1} f_{p_i}(Xu_k) - f(Xv_k)^{k-1} f_{p_i}(Xv_k)) X_i(u_k - v_k) \\ &\leq 4\epsilon^{k-1} \|u_k - v_k\|_{L^1(\Omega)}. \end{aligned} \tag{5.14}$$

Now we need

Claim 5.5. *If $f \in C^2(R^m)$ is strictly convex, then for any $p, q \in R^m$*

$$(f^{k-1}(p)f_p(p) - f^{k-1}(q)f_p(q)) \cdot (p-q) \geq C|p-q|^{\alpha(k-1)+2}. \tag{5.15}$$

To see (5.15), we observe that

$$\begin{aligned}
&(f^{k-1}(p)f_p(p) - f^{k-1}(q)f_p(q)) \cdot (p-q) \\
&= k^{-1}\int_0^1 \frac{d}{dt}(f^k)_p(tp+(1-t)q)\,dt \cdot (p-q) \\
&\geq \sum_{ij=1}^m \int_0^1 f^{k-1}(tp+(1-t)q) f_{p_ip_j}(tp+(1-t)q)\,dt(p_i-q_i)(p_j-q_j) \qquad (5.16)\\
&\geq C^{k-1}\int_0^1 |tp+(1-t)q|^{\alpha(k-1)}\,dt|p-q|^2,
\end{aligned}$$

where we have used the strict convexity of f:

$$\sum_{ij=1}^m f_{p_ip_j}(v)p_ip_j \geq C_0|p-q|^2,\ \forall p,q,v \in R^m,$$

the α-homogeneity of f and the fact that

$$f(v) = |v|^\alpha f(\frac{v}{|v|}) \geq \min_{|z|=1} f(z)|v|^\alpha \geq C|v|^\alpha,\ \forall v \in R^m,$$

for some $C > 0$ depending only on f. Since

$$\int_0^1 |tp+(1-t)q|^{\alpha(k-1)}\,dt \geq C|p-q|^{\alpha(k-1)},$$

(5.16) implies (5.15). Putting (5.15) into (5.14), we obtain

$$kC^{k-1}\int_\Omega |Xu_k - Xv_k|^{\alpha(k-1)+2} \leq C\epsilon^{k-1}.$$

This, combined with the Hölder inequality, implies

$$\int_\Omega |Xu_k - Xv_k| \leq k^{-\frac{1}{\alpha(k-1)+2}}(C\epsilon)^{\frac{k-1}{\alpha(k-1)+2}}|\Omega|^{\frac{\alpha(k-1)+1}{\alpha(k-1)+2}}.$$

Letting k tend to ∞, we have

$$\|Xu_\epsilon - Xv_\epsilon\|_{L^1(\Omega)} \leq C\epsilon^{\frac{1}{\alpha}}. \tag{5.17}$$

For $u_\epsilon, v_\epsilon \in W^{1,\infty}_{\mathrm{X}}(\Omega)$, (5.17), the usual interpolation inequality, and the Sobolev inequality yield that the function β must exist as asserted in (4). □

§6. Proof of Theorem C

This section is devoted to the proof of Theorem C. In this section, the $\{X_i\}_{i=1}^m$ are horizontal vector fields of a Carnot group $\mathbf{G}$. Since we can identify $\mathbf{G}$ with R^n, $n = \dim(\mathbf{G})$, via the exponential map, the results in §4 and §5 are all applicable to $\mathbf{G}$. The idea to prove Theorem C is based on the sup/inf convolution and the comparison principle for both equations (5.1) and (5.5).

Lemma 6.1. *Under the same assumptions as Theorem C, for any $\epsilon > 0$, if $v \in C(\bar{\Omega})$ is a viscosity subsolution to* (5.1) *and $w \in C(\bar{\Omega})$ is a viscosity supersolution to the* (5.1)*, then*

$$\sup_{x\in\Omega}(v-w)(x) = \sup_{x\in\partial\Omega}(v-w)(x). \tag{6.1}$$

Proof. Suppose that (6.1) were false. Then there is an $\delta_0 > 0$ such that

$$\sup_{x\in\Omega}(v-w)(x) \geq \sup_{x\in\partial\Omega}(v-w)(x) + \delta_0.$$

For any $\delta \in (0, \frac{\delta_0}{2})$, it follows from Lemma 5.1 that there are $w_\delta \in C(\bar{\Omega})$, with $\|w_\delta - w\|_{L^\infty(\Omega)} \leq \delta$, and $\mu = \mu(\delta, \epsilon, \alpha) > 0$, such that w_δ is a viscosity supersolution to (5.2). Moreover, we have

$$\sup_{x\in\Omega}(v-w_\delta) \geq \sup_{x\in\partial\Omega}(v-w_\delta)(x) + \frac{\delta_0}{4}. \tag{6.2}$$

Now we apply Proposition 3.3 to conclude that for any $\delta \in (0, \frac{\delta_0}{2})$ there are a semiconvex $v^\delta \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ and a semiconcave $\tilde{w_\delta} \in W^{1,\infty}_{\mathrm{X}}(\Omega)$ such that

$$\lim_{\delta\to 0}\max\{\|v^\delta - v\|_{L^\infty(\Omega_{C\delta})},\ \|\tilde{w_\delta} - w_\delta\|_{L^\infty(\Omega_{C\delta})}\} = 0, \tag{6.3}$$

where $\Omega_{C\delta}$ is defined in §3. Moreover, v^δ is a viscosity subsolution to (5.1) and $\tilde{w_\delta}$ is a viscosity supersolution to (5.2) on $\Omega_{C\delta}$. Therefore, we can apply Proposition 4.1 to conclude that

$$\sup_{\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}) = \sup_{\partial\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}). \tag{6.4}$$

Letting δ tend to zero, this yields

$$\begin{aligned}
\lim_{\delta\to 0}\sup_{\Omega_{C\delta}}(v-w) &= \lim_{\delta\to 0}\sup_{\Omega_{C\delta}}[(v-v^\delta) + (v^\delta - \tilde{w_\delta}) + (\tilde{w_\delta} - w_\delta) + (w_\delta - w)] \\
&= \lim_{\delta\to 0}\sup_{\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}) \\
&= \lim_{\delta\to 0}\sup_{\partial\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}) \\
&= \sup_{\partial\Omega}(v-w).
\end{aligned}$$

This yields the desired contradiction. The proof is complete. □

Similarly, we have the comparison principle for (5.5).

Lemma 6.2. *Under the same assumptions as Theorem C, for any $\epsilon > 0$, let $v \in C(\bar{\Omega})$ be a viscosity subsolution to* (5.5) *and $w \in C(\bar{\Omega})$ be a viscosity supersolution to* (5.5)*. Then*

$$\sup_{x\in\Omega}(v-w)(x) = \sup_{x\in\partial\Omega}(v-w)(x).$$

We are ready to prove the maximum principle for (1.13).

Lemma 6.3. *Under the same assumptions as Theorem C, for a given $\phi \in W^{1,\infty}_{\mathrm{X}}(\Omega)$, assume that $v \in C(\bar{\Omega})$ is a viscosity subsolution to* (1.13) *and $w \in C(\bar{\Omega})$ is a viscosity supersolution to* (1.13) *such that $v|_{\partial\Omega} = w|_{\partial\Omega} = \phi$. Then*

$$v(x) \leq w(x),\ \forall x \in \Omega. \tag{6.5}$$

Proof. Let v^+ be a viscosity solution to (5.1), and let w^- be a viscosity solution to (5.5), with $v^+|_{\partial\Omega} = w^-|_{\partial\Omega} = \phi$, obtained by Lemma 5.3. Since subsolutions of (1.13) are also subsolutions to (5.1), and supersolutions to (1.13) are also supersolutions to (5.5), we can apply Lemmas 6.1 and 6.2 to conclude that

$$\sup_{\Omega}(v - v^+) = \sup_{\partial\Omega}(v - v^+) = 0, \ \sup_{\Omega}(w^- - w) = \sup_{\partial\Omega}(w^- - w) = 0.$$

Hence we have

$$\sup_{\Omega}(v - w) \leq \sup_{\Omega}(v^+ - w^-) \leq \beta(\epsilon),$$

where β is given by Lemma 5.3. Since ϵ is arbitrary, this implies

$$\sup_{\Omega}(v - w) \leq 0.$$

This finishes the proof of Lemma 6.3. □

It is clear that Theorem C follows from Lemma 6.3 and hence the proof is complete. □

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References

[A1] G. Aronsson, *Extension of functions satisfying Lipschitz conditions*. Ark. Mat. 6 (1967) 551–561. MR0217665 (36:754)

[A2] G. Aronsson, *On the partial differential equation* ${u_x}^2 u_{xx} + 2u_x u_y u_{xy} + {u_y}^2 u_{yy} = 0$. Ark. Mat. 7 (1968) 395–425. MR0237962 (38:6239)

[ACJ] G. Aronsson, M. Crandall, P. Juutinen, *A Tour of the Theory of Absolutely Minimizing Functions*. Preprint (2003). MR2083637

[B] N. Barron, *Viscosity solutions and analysis in* L^∞. Nonlinear analysis, differential equations and control (Montreal, QC, 1998), Kluwer Acad. Publ. Dordrecht, 1999 (1-60). MR1695005 (2001c:49045)

[BDM] T. Bhatthacharya, E. DiBenedetto, J. Manfredi, *Limits as* $p \to \infty$ *of* $\Delta_p u_p = f$ *and Related Extremal Problems*. Rend. Sem. Mat. Univ. Politec Torino, 1989, 15-68. MR1155453 (93a:35049)

[B1] T. Bieske, *On* ∞*-harmonic functions on the Heisenberg group*. Comm. Partial Differential Equations 27 (2002), no. 3-4, 727–761. MR1900561 (2003g:35033)

[B2] T. Bieske, *Viscosity solutions on Grushin-type planes*. Illinois J. Math. 46 (2002), no. 3, 893–911. MR1951247 (2003k:35037)

[B3] T. Bieske, *Lipschitz extension on generalized Grushin type spaces*. To appear in Mich. Math. J. MR2125531

[BC] T. Bieske, L. Capogna, *The Aronsson-Euler equation for absolutely minimizing Lipschitz extensions with repsect to Carnot-Carathéory metrics*. Preprint (2002). MR2095631 (2005g:35036)

[BJW] N. Barron, R. Jensen, C. Y. Wang, *The Euler equation and absolute minimizers of* L^∞ *functionals*. Arch. Ration. Mech. Anal. 157 (2001), no. 4, 255–283. MR1831173 (2002m:49006)

[C] M. Crandall, *An Efficient Derivation of the Aronsson Equation*. Arch. Rational Mech. Anal. 167 (2003) 4, 271-279. MR1981858

[CC] L. Caffarelli, X. Cabré, Fully nonlinear elliptic equations. AMS colloquium publications 43, AMS, Providence, RI, 1995. MR1351007 (96h:35046)

[CE] M. Crandall, L. C. Evans, *A remark on infinity harmonic functions*. Proceedings of the USA-Chile Workshop on Nonlinear Analysis, 123–129 (electronic), Electron. J. Differ. Equ. Conf., 6, Southwest Texas State Univ., San Marcos, TX, 2001. MR1804769 (2001j:35076)

[CEG] M. Crandall, L. C. Evans, R. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian.* Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123–139. MR1861094 (2002h:49048)

[CIL] M. Crandall, H. Ishii, P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67. MR1118699 (92j:35050)

[CL] M. Crandall, P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations.* Trans. Amer. Math. Soc. 277 (1983), no. 1, 1–42. MR0690039 (85g:35029)

[E] L. C. Evans, *Estimates for smooth absolutely minimizing Lipschitz extensions.* Electron. J. Differential Equations 1993, No. 03, 9 pp. MR1241488 (94k:35117)

[FS] G. Folland, E. Stein, Hardy spaces on homogeneous groups. Mathematical Notes, 28. Princeton University Press, Princeton, N.J., 1982. MR0657581 (84h:43027)

[FSS] B. Franchi, R. Serapioni, F. Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields.* Houston J. Math. 22 (1996), no. 4, 859–890. MR1437714 (98c:49037)

[GN] N. Garofalo, D. Nhieu, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathodory spaces.* J. Anal. Math. 74 (1998), 67–97. MR1631642 (2000i:46025)

[HK] P. Hajlasz, P. Koskela, *Sobolev met Poincaré.* Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101 pp. MR1683160 (2000j:46063)

[I] H. Ishii, *On existence and uniqueness of viscosity solutions of fully nonlinear second-order elliptic PDEs.* Comm. Pure Appl. Math. 42 (1989), 14-45. MR0973743 (89m:35070)

[J] P. Juutinen, *Minimization problems for Lipschitz functions via viscosity solutions.* Ann. Acad. Sci. Fenn. Math. Diss. No. 115 (1998), MR1632063 (2000a:49055)

[J1] R. Jensen, *The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations.* Arch. Rational Mech. Anal. 101 (1988), no. 1, 1–27. MR0920674 (89a:35038)

[J2] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient.* Arch. Rational Mech. Anal. 123 (1993), no. 1, 51–74. MR1218686 (94g:35063)

[JLS] R. Jensen, P. L. Lions, P. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations.* Proc. Amer. Math. Soc. 102 (1988), no. 4, 975–978. MR0934877 (89f:35027)

[LM] P. Lindqvist, J. Manfredi, *The Harnack inequality for ∞-harmonic functions.* Electron. J. Differential Equations 1995, No. 04, 5 pp. MR1322829 (96b:35025)

[FM] J. Manfredi, *Fully nonlinear subelliptic equations.* Preprint (2005).

[NSW] A. Nagel, E. Stein, S. Wainger, *Balls and metrics defined by vector fields. I. Basic properties.* Acta Math. 155 (1985), no. 1-2, 103–147. MR0793239 (86k:46049)

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