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LATTICE-ORDERED ABELIAN GROUPS AND SCHAUDER BASES OF UNIMODULAR FANS

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ABSTRACT. Baker-Beynon duality theory yields a concrete representation of any finitely generated projective Abelian lattice-ordered group G in terms of piecewise linear homogeneous functions with integer coefficients, defined over the support $|\Sigma|$ of a fan Σ. A unimodular fan Δ over $|\Sigma|$ determines a Schauder basis of G: its elements are the minimal positive free generators of the pointwise ordered group of Δ-linear support functions. Conversely, a Schauder basis $\mathbf{H}$ of G determines a unimodular fan over $|\Sigma|$: its maximal cones are the domains of linearity of the elements of $\mathbf{H}$. The main purpose of this paper is to give various representation-free characterisations of Schauder bases. The latter, jointly with the De Concini-Procesi starring technique, will be used to give novel characterisations of finitely generated projective Abelian lattice ordered groups. For instance, G is finitely generated projective iff it can be presented by a purely lattice-theoretical word.

1. Background: ℓ-groups and fans

We assume familiarity with lattice-ordered Abelian groups (for short, ℓ-groups [4, 7]) and fans. By a *fan* we shall always understand a finite rational polyhedral fan, as defined in [6, 15].

Throughout the paper, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}$ is the set of integers, and $\mathbb{R}$ is the set of reals. If G is an ℓ-group, we let $G^+ = \{g \in G \mid g \geq 0\}$. By *$\ell$-homomorphisms* we mean homomorphisms of ℓ-groups; the symbol '$\cong_\ell$' denotes ℓ-isomorphism. Kernels of ℓ-homomorphisms are precisely *ℓ-ideals*, always denoted by Gothic letters $\mathfrak{a}, \mathfrak{m}, \mathfrak{p}, \ldots$. An ℓ-ideal is *principal* iff it is finitely generated (which for ℓ-groups is equivalent to being singly generated). *Maximal* ℓ-ideals are defined in the obvious manner. A finitely generated ℓ-group G is *Archimedean* iff it has no "infinitesimal elements": thus, whenever $0 < x \leq y$ holds, there is $n \in \mathbb{N}$ such that $nx \not\leq y$; equivalently, the intersection of all maximal ℓ-ideals in G is $\{0\}$; see [7, 4.1] and [4, 10.2] for details. An ℓ-ideal $\mathfrak{m}$ is maximal in G iff $G/\mathfrak{m}$ is Archimedean totally ordered. As explained in [4, 13.2.6], Archimedean ℓ-groups with a *strong order unit* (that is, an element $u \in G$ such that for every $x \in G$ there is $n \in \mathbb{N}$ with $nu \geq x$) are precisely those representable as ℓ-groups of real-valued continuous functions over some compact Hausdorff space (operations being defined pointwise); finitely generated ℓ-groups may always be endowed with a strong order unit.

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Suppose G is a finitely generated ℓ-group. Equipped with the spectral topology, the set $\mathbf{MaxSpec}\,(G)$ of maximal ℓ-ideals of G is a (nonempty) compact Hausdorff space. The closed sets in $\mathbf{MaxSpec}\,(G)$ are given by the *zero sets* $\mathfrak{Z}(S)$ of arbitrary subsets S of G, where $\mathfrak{Z}(S) = \bigcap_{g\in S}\{\mathfrak{m} \in \mathbf{MaxSpec}\,(G) \mid g \in \mathfrak{m}\}$.

We let $|\Sigma| \subseteq \mathbb{R}^n$ denote the (closed) support of the fan Σ.[1] A continuous function $f\colon |\Sigma| \to \mathbb{R}$ is said to be an (*integral*) *ℓ-function* over $|\Sigma|$ iff it is (always finitely) piecewise linear homogeneous, each piece having the form $c_1x_1 + \cdots + c_nx_n$, for suitable integers $c_1, \ldots, c_n$.

By $\mathcal{A}_n$ we denote the ℓ-group of all ℓ-functions over $\mathbb{R}^n$, with pointwise operations. As is well known, $\mathcal{A}_n$ is free in the equational class of ℓ-groups. The (coordinate) projection functions $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ are free generators of $\mathcal{A}_n$. For Σ a fan in $\mathbb{R}^n$, we let

$$\mathcal{A}_n\restriction |\Sigma|$$

denote the ℓ-group of all ℓ-functions over $|\Sigma|$. This notation is in agreement with the fact that ℓ-functions over $|\Sigma|$ are the same as restrictions to $|\Sigma|$ of ℓ-functions over $\mathbb{R}^n$.

Elements of fans are called *cones*; thus, a cone is always a rational polyhedral cone. A fan Θ is *simplicial* iff every one of its cones is simplicial (i.e., taking an affine section in general position yields a simplex). A simplicial cone $\sigma \in \Theta$ is *unimodular* iff its 1-dimensional faces are spanned (over the positive reals) by primitive vectors[2] that are columns of some unimodular matrix (an integral square matrix with determinant ± 1). A fan is *unimodular* iff all of its cones are unimodular.[3]

Given two fans Σ and Δ, if $|\Delta| = |\Sigma|$ and all the cones of Σ are unions of cones of Δ, then we say that Δ *refines* Σ.

For Δ a unimodular fan in $\mathbb{R}^n$, let $\sigma \in \Delta$ be a 1-dimensional cone. Following [12, 13] (also see [17]), the *Schauder hat of Δ at σ* is the unique ℓ-function h_σ that is linear over each cone of Δ, has value 1 at the primitive vector along σ, and value 0 at every point of every other 1-dimensional cone of Δ. The *Schauder basis over* Δ, denoted Hats_Δ, is the collection of all Schauder hats of Δ. Integrality of the linear pieces is guaranteed by unimodularity of the fan.

Every Schauder hat of Δ is a particular case of a Δ-linear support function [15, page 66]. Specifically, the Schauder basis over Δ is a free generating set in the free group $\mathcal{F}$ of Δ-linear support functions. If, in addition, $\mathcal{F}$ is equipped with the pointwise order, then the Schauder hats over Δ are just the minimal positive free generators; see [13].

2. Schauder bases and De Concini-Procesi refinements

A nontrivial exercise shows that there are ℓ-automorphisms of, say, $\mathcal{A}_3$, carrying a Schauder basis to a set of ℓ-functions which is not a Schauder basis. Thus, the property of being a Schauder basis is *not* invariant under ℓ-automorphisms.

A useful representation-independent generalisation is the following.

Definition 2.1. Let G be an ℓ-group and $B = \{b_1, \ldots, b_t\} \subseteq G$, where $t \in \mathbb{N}$. We say that B is an *abstract Schauder basis of G* iff for some $n \in \mathbb{N}$ there exist a unimodular fan Δ in $\mathbb{R}^n$, and an ℓ-isomorphism $\phi : \mathcal{A}_n \restriction |\Delta| \cong_\ell G$, such that,

[1] All fans are denoted by capital Greek letters.

[2] An integral vector is *primitive* iff the greatest common divisor of its coordinates is 1.

[3] Unimodular fans are also known as *nonsingular* fans, as they correspond precisely to smooth toric varieties under the fan-toric dictionary. In [6], they are called *regular* fans.

letting $\mathrm{Hats}_\Delta = \{h_1, \dots, h_t\}$ be the Schauder basis over Δ, we have $\phi(h_i) = b_i$ for every $i \in \{1, \dots, t\}$.[4]

When we wish to emphasise that B is a Schauder basis, we sometimes write that it is a *concrete* Schauder basis. Trivially, any concrete Schauder basis is an abstract Schauder basis.

Proposition 2.2. *If the ℓ-group G has an abstract Schauder basis B, then*
(1) $B \subseteq G^+ \setminus \{0\}$,
(2) *B is linearly independent in the $\mathbb{Z}$-module G, and*
(3) *G is Archimedean.*

Proof. In view of the existence of the above ℓ-isomorphism $\phi\colon \mathcal{A}_n \restriction |\Delta| \cong_\ell G$, it is enough to prove the proposition for $\mathcal{A}_n \restriction |\Delta|$ equipped with the concrete Schauder basis $\mathrm{Hats}_\Delta = \{h_1, \dots, h_t\}$ corresponding to B via ϕ. Then (1) and (2) are immediate. Property (3) follows upon observing that $\mathcal{A}_n \restriction |\Delta|$ is an ℓ-group of real-valued functions. $\square$

In Proposition 2.5 below we shall prove that B generates G. The proof requires some background material on stellar subdivisions [6, 15]. Barycentric stellar subdivisions (*starrings* for short) can be used to subdivide a unimodular fan, *preserving* unimodularity.[5] If Δ is obtained from Σ via a finite number (possibly zero) of starrings, we write $\Delta \preceq \Sigma$; and if all such subdivisions are along 2-dimensional cones only (*binary* starrings for short), we write $\Delta \preceq_2 \Sigma$.

Lemma 2.3 (The De Concini-Procesi Lemma)**.** *Let Σ_1 and Σ_2 be unimodular fans with the same support. There exists a unimodular fan Δ such that $\Delta \preceq_2 \Sigma_1$ and Δ refines Σ_2.*

Proof. See [5]; for a short elementary proof, see [16] or [13]. $\square$

For our purposes, it is crucial to realise that starrings of unimodular fans are definable in the language of ℓ-groups. Although this is really a remark, we record it for easier reference.

Lemma 2.4. *Let Σ be a unimodular fan and $\mathbf{H} = \mathrm{Hats}_\Sigma$ its associated Schauder basis. Let $h_1, h_2 \in \mathbf{H}$ be hats at the 1-cones σ_1 and σ_2, respectively. Suppose σ_1 and σ_2 span a 2-cone τ of Σ (equivalently, $h_1 \wedge h_2 \neq 0$). Let $\Delta \preceq_2 \Sigma$ be the unimodular fan obtained from Σ by starring along τ. Then $\mathbf{K} = \mathrm{Hats}_\Delta$ is the Schauder basis obtained from $\mathbf{H}$ by removing h_1 and h_2, and adjoining the three hats*

$$h_1 - (h_1 \wedge h_2),\ h_2 - (h_1 \wedge h_2),\ h_1 \wedge h_2\ .$$

Conversely, if $\mathbf{K}$ is obtained from $\mathbf{H}$ by such substitutions, then there is a unique unimodular fan Δ, obtained from Σ by starring along τ, such that $\mathbf{K} = \mathrm{Hats}_\Delta$.

Proof. A straightforward computation. $\square$

Proposition 2.5. *If an ℓ-group G has an abstract Schauder basis B, then B generates G.*

[4] In Proposition 2.5 below we shall prove that if G has a Schauder basis, then G is finitely generated.

[5] Starrings correspond to (equivariant) blow-ups of toric varieties via the fan-toric dictionary [15, 6].

Proof. Using the ℓ-isomorphism $\phi\colon \mathcal{A}_n \restriction |\Delta| \cong_\ell G$, without loss of generality we can assume $G = \mathcal{A}_n \restriction |\Delta|$ equipped with the concrete Schauder basis $B = \mathrm{Hats}_\Delta = \{h_1, \ldots, h_t\}$. By [6, Theorem 8.5], for every ℓ-function f over $|\Delta|$, there exists a unimodular refinement Δ_f of Δ such that f is linear over every cone of Δ_f. By Lemma 2.3, an appropriate sequence of binary starrings transforms Δ into a refinement Δ^* of Δ_f. By Lemma 2.4, all hats of Δ^* are in the ℓ-group H of real-valued functions over $|\Delta|$ generated by B. Since, by the construction of Δ^*, f must take an integer value at each primitive generating vector of Δ^*, then f is a linear combination of the hats of Δ^* with integer coefficients. This shows that $H = \mathcal{A}_n \restriction |\Delta|$. $\square$

The concept of concrete Schauder bases only makes sense for ℓ-groups of the form $G = \mathcal{A}_n \restriction |\Delta|$, where $|\Delta|$ is a unimodular fan. The main question we shall address is whether one can characterise abstract Schauder bases without reference to their geometric realisations.

3. Topological characterisation of abstract Schauder bases

Let G be an ℓ-group, and D be a finite subset of G. Then by an *abstract k-simplex of D* we mean a set $Y \subseteq D$ of cardinality $k+1$ such that $\bigwedge Y \neq 0$.

Theorem 3.1. *Let G be an ℓ-group, and let $B = \{b_1, \ldots, b_t\} \subseteq G^+ \setminus \{0\}$ be a finite subset. Then the following conditions are equivalent:*

(1) *B is an abstract Schauder basis of G.*
(2) *G is Archimedean, B generates G, and for every abstract k-simplex Y of B, the zero set $\mathfrak{Z}(B \setminus Y)$ is homeomorphic to the closed k-dimensional ball.*

Proof.[6] $(2 \to 1)$ Let the ℓ-homomorphism $\eta\colon \mathcal{A}_t \to G$ be the unique extension of the map $\pi_i \mapsto b_i$ $(i = 1, \ldots, t)$. Then η is surjective, because B generates G. Since G is Archimedean, G is ℓ-isomorphic to the ℓ-group $\mathcal{A}_t \restriction W$ of restrictions of the ℓ-functions of $\mathcal{A}_t$ to a suitable homogeneous[7] closed subset W of $\mathbb{R}^t$. We shall show that W is the support of a unimodular fan Θ, and B is the η-image of Hats_Θ.

The assumption that each b_i is positive implies that W is contained in the positive orthant $\mathcal{O}$ of $\mathbb{R}^t$. Let L be the affine hyperplane determined by the unit basic vectors $\mathbf{e}_i$,

$$L = \left\{(\xi_1, \ldots, \xi_t) \in \mathbb{R}^t \mid \sum \xi_i = 1\right\}. \tag{1}$$

Let $W' = W \cap L$. Via the canonical ℓ-isomorphism of $\mathcal{A}_t \restriction W$ onto $\mathcal{A}_t \restriction W'$, we shall identify G with the ℓ-group of restrictions to W' of the functions of $\mathcal{A}_t$; in symbols,

$$G = \mathcal{A}_t \restriction W'. \tag{2}$$

Under this identification, η is just the restriction map, and every element $b_i \in B$ coincides with the restriction to W' of the ith coordinate function π_i of $\mathcal{A}_t$. Since the rays of W are in one-one correspondence with the maximal ℓ-ideals of G, there also is a natural one-one correspondence

$$\mathfrak{m} \in \mathbf{MaxSpec}\,(G) \mapsto \mathbf{z}_\mathfrak{m} \in W'$$

[6]The techniques for the proof of Theorem 3.1 were developed, in a different but related context, in [14]. For partitions of unity, a similar result is stated in [9].

[7]Meaning that whenever W contains a point $\mathbf{z}$, then W also contains the ray through 0 and $\mathbf{z}$.

between maximal ℓ-ideals $\mathfrak{m}$ and points in W'.[8] This correspondence is also a homeomorphism of $\mathbf{MaxSpec}\,(G)$ (equipped with the spectral topology; see [4, Chapter 13]) onto W' (with the natural topology of $\mathbb{R}^t$). Accordingly, we shall identify $\mathbf{MaxSpec}\,(G)$ and W' as topological spaces. The quotient map $f \in G \mapsto f/\mathfrak{m} \in G/\mathfrak{m}$ coincides with the evaluation map of every $f \in G$ at the point $\mathbf{z}_{\mathfrak{m}} = (\xi_1, \ldots, \xi_t)$ corresponding to the maximal ℓ-ideal $\mathfrak{m}$; in symbols,

$$f/\mathfrak{m} = f(\mathbf{z}_{\mathfrak{m}}), \quad \text{for all} \quad \mathfrak{m} \in \mathbf{MaxSpec}\,(G). \tag{3}$$

Claim 1. Let $Y = \{b_{i_1}, \ldots, b_{i_k}\}$ be an abstract simplex of B. Then the zero set $\mathfrak{Z}(B \setminus Y)$ coincides with the set $\{\mathbf{z} \in W' \mid b_{i_1}(\mathbf{z}) + \cdots + b_{i_k}(\mathbf{z}) = 1\}$.

As a matter of fact, by (3), $\mathfrak{Z}(B \setminus Y)$ is the set of points of W' where all functions of $B \setminus Y$ vanish. On the other hand, the sum of all functions b_j of B (equivalently, the sum of the corresponding π_j) constantly equals 1 over W'.

Claim 2. For each $i = 1, \ldots, t$, the singleton $\{b_i\}$ forms an abstract 0-simplex of B, and the zero set $\mathfrak{Z}(B \setminus \{b_i\})$ is the (singleton) standard basis vector $\mathbf{e}_i$. Thus $\mathbf{e}_i \in W'$.

By assumption $\{b_i\}$ is nonzero. By definition it forms an abstract 0-simplex of B. By our topological assumption, $\mathfrak{Z}(B \setminus \{b_i\})$ is a singleton in the compact Hausdorff space $\mathbf{MaxSpec}\,(G) = W'$. The only possible point $\mathbf{z}$ in $\mathcal{O} \cap L$ where $b_i(\mathbf{z}) = 1$ is the basis vector $\mathbf{e}_i$. Thus, by Claim 1, $\mathbf{e}_i \in W'$.

Claim 3. For every abstract $(k-1)$-simplex $Y = \{b_{i_1}, \ldots, b_{i_k}\}$ of B, the zero set $\mathfrak{Z}(B \setminus Y)$ is the convex hull $[\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}]$ of the basis vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$. Thus, in particular, $[\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}] \subseteq W'$.

The proof is by induction on k.

Basis. Suppose $\{b_i, b_j\}$ forms an abstract 1-simplex Y of B. By Claim 1, $\mathfrak{Z}(B \setminus Y)$ coincides with the set $V = \{\mathbf{z} \in W' \mid b_i(\mathbf{z}) + b_j(\mathbf{z}) = 1\}$. By definition of W', together with (3), V is a subset of the closed segment $[\mathbf{e}_i, \mathbf{e}_j]$. By Claim 2, both $\mathbf{e}_i$ and $\mathbf{e}_j$ belong to V, because both b_i and b_j form an abstract 0-simplex of B. If V were a *proper* subset of $[\mathbf{e}_i, \mathbf{e}_j]$, then it would not be connected, against our topological assumption. Thus, $\mathfrak{Z}(B \setminus Y) = [\mathbf{e}_i, \mathbf{e}_j]$.

Induction step. Let $D = \{b_{i_1}, \ldots, b_{i_{k+1}}\}$ be an abstract k-simplex of B. A fortiori, every subset $D' = \{b_{j_1}, \ldots, b_{j_k}\}$ of D is an abstract $(k-1)$-simplex of B. By induction hypothesis, the zero set $\mathfrak{Z}(B \setminus D')$ coincides with the (topological) $(k-1)$-simplex $[\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}]$. Thus, by (3), $\mathfrak{Z}(B \setminus D)$ is a certain subset $V \subseteq [\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{k+1}}]$, containing the union of S^{k-1} all $(k-1)$-dimensional faces of $[\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{k+1}}]$. Suppose V were a *proper* subset of $[\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{k+1}}]$ (*absurdum hypothesis*). Write

$$V = [\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{k+1}}] \setminus U$$

for a suitable nonempty subset U of the relative interior of $[\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{k+1}}]$. As in [13], the intuitively obvious fact that V cannot be homeomorphic to the k-ball is confirmed by the verification that the singular homology groups [11] of V and of $[\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{k+1}}]$ are not isomorphic.

[8] One passes from $\mathfrak{m}$ to $\mathbf{z}_{\mathfrak{m}}$ by taking the intersection of all zero sets of functions in $\mathfrak{m}$. Conversely, one passes from any point $\mathbf{z} \in W'$ to its corresponding maximal ℓ-ideal $\mathfrak{m}$ by taking all functions of G that vanish at $\mathbf{z}$.

On the one hand, $H_{k-1}([\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_{k+1}}])$ is zero, because $[\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_{k+1}}]$ is contractible. On the other hand, letting $\mathbf{u}$ be any point in set U, the inclusion maps

$$S^{k-1} \to V \to \mathbb{R}^n \setminus \{\mathbf{u}\}$$

induce homomorphisms of homology groups

$$\mathbb{Z} = H_{k-1}(S^{k-1}) \to H_{k-1}(V) \to H_{k-1}(\mathbb{R}^n \setminus \{\mathbf{u}\}) = \mathbb{Z}.$$

Moreover, the composition of these two homomorphisms is the homomorphism induced by the inclusion

$$S^{k-1} \to \mathbb{R}^n \setminus \{\mathbf{u}\}.$$

Because the latter map is a homotopy equivalence, it induces an isomorphism on homology groups. We have shown that $H_{k-1}(V)$ contains the group of integers as a summand. This contradicts our topological assumption, and also settles our claim.

To conclude the proof of $(2 \to 1)$, for each abstract $(k-1)$-simplex Y of B let σ_Y be an abbreviation of the zero set $\mathfrak{Z}(B \setminus Y)$. By Claim 3, σ_Y is a $(k-1)$-dimensional simplex. For any two abstract simplices Y', Y'' of B, their corresponding simplices $\sigma_{Y'}, \sigma_{Y''}$ intersect in a common face, namely the simplex $\sigma_{Y' \cap Y''}$. Here it is convenient to write $\sigma_\emptyset = \emptyset$.

The abstract simplices of B then determine a (concrete, topological) simplicial complex $\mathcal{S}$ in $\mathbb{R}^t$, and W' is the support of $\mathcal{S}$.[9] The t vertices of $\mathcal{S}$ are the standard basis vectors of $\mathbb{R}^t$. Further, for each $k = 1,\ldots,t$, letting $\mathcal{S}^{(k)}$ denote the set of k-dimensional simplices of $\mathcal{S}$, we have the correspondence

$$\mathcal{S}^{(k)} \ \longleftrightarrow \ \text{abstract } k\text{-simplices of } B.$$

Replacing each point $\mathbf{z}$ in W' by the ray passing through 0 and $\mathbf{z}$, we obtain from $\mathcal{S}$ a unimodular fan Θ whose support coincides with W. The primitive generating vectors of Θ are the standard basis vectors $\mathbf{e}_i$. The Schauder hats of Θ are precisely the restrictions $\pi_i \restriction W$, and hence they correspond to the elements of B, via our standing ℓ-isomorphism $b_i \cong_\ell \pi_i \restriction W$.

This completes the proof of $(2 \to 1)$.

$(1 \to 2)$ For some $n \in \mathbb{N}$ and unimodular fan Δ there is an ℓ-isomorphism

$$\phi\colon \mathcal{A}_n \restriction |\Delta| \cong_\ell G$$

that geometrically realises B by the Schauder basis over Δ. By Propositions 2.2 and 2.5, G is Archimedean and B generates G. Without loss of generality we may assume $|\Delta|$ to lie entirely in the positive orthant of $\mathbb{R}^n$.[10] Let L be the hyperplane

$$\{(\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \ \mid \ \xi_1 + \cdots + \xi_n = 1\}.$$

Let $L_\Delta = |\Delta| \cap L$ equipped with the natural topology. Then $\mathbf{MaxSpec}\,(\mathcal{A}_n \restriction |\Delta|)$ is canonically homeomorphic to L_Δ; let us identify these topological spaces. Since Δ is a complex of simplicial cones, its affine section L_Δ is a simplicial complex. For every abstract simplex $Y = \{f_1,\ldots,f_k\}$ of B, let $\mathbf{v}_i \in \mathbb{R}^n$ be the point of L_Δ where the Schauder hat f_i attains its maximum value. Then direct inspection shows that the zero set $T = \mathfrak{Z}(B \setminus Y)$ is the convex hull in $\mathbb{R}^n$ of the points $\mathbf{v}_1,\ldots,\mathbf{v}_k$. Therefore, T is homeomorphic to the $(k-1)$-dimensional ball.

The proof is complete. □

[9] That is, W' is the set-theoretical union of all simplices of $\mathcal{S}$.

[10] Indeed, we can further assume that the primitive generating vectors of the rays of Δ coincide with the standard basis vectors $\mathbf{e}_1,\ldots,\mathbf{e}_n$ of $\mathbb{R}^n$.

4. Characterising finitely generated projective ℓ-groups

As we have seen, abstract Schauder bases are in one-one correspondence with unimodular fans. In this section they are used to characterise projective ℓ-groups.

By definition, an ℓ-group G is *projective* if, whenever $\varphi\colon H \to K$ is a surjective ℓ-homomorphism and $\alpha\colon G \to K$ is an ℓ-homomorphism, there is an ℓ-homomorphism $\theta\colon G \to H$ such that $\varphi \circ \theta = \alpha$.

It follows that an ℓ-group G is *finitely generated projective* iff it is a retract of some $\mathcal{A}_n$. In other words, there are maps $\iota\colon G \to \mathcal{A}_n$ and $\sigma\colon \mathcal{A}_n \to G$ such that $\sigma \circ \iota$ is the identity over G.

The following well-known characterisation is due to Baker and Beynon [1, 2, 3].

Theorem 4.1 ([7, Corollary 5.2.2], [3, Theorem 3.1])**.** *For any ℓ-group G the following conditions are equivalent:*

(1) *G is finitely generated projective;*
(2) *for some $n \in \mathbb{N}$, G is ℓ-isomorphic to an ℓ-group of ℓ-functions over the support of some fan Σ in $\mathbb{R}^n$, in symbols, $G \cong_\ell \mathcal{A}_n \restriction |\Sigma|$; and*
(3) *for some $n \in \mathbb{N}$, G is ℓ-isomorphic to the quotient $\mathcal{A}_n/\mathfrak{p}$ for some principal ℓ-ideal $\mathfrak{p}$ of $\mathcal{A}_n$.*

Hence:

Proposition 4.2. *For any ℓ-group G the following are equivalent:*

(1) *G is finitely generated projective.*
(2) *G has an abstract Schauder basis.*

Proof. ($1 \to 2$) By Theorem 4.1 we can write $\psi\colon G \cong_\ell \mathcal{A}_n \restriction |\Sigma|$ for some fan Σ in $\mathbb{R}^n$. Now, every fan can be unimodularised (see, e.g., [6, Theorem 8.5]), hence its support is also the support of a unimodular fan. Accordingly, we shall assume that Σ is unimodular. Let Hats_Σ be the concrete Schauder basis over Σ. The ℓ-isomorphism ψ^{-1} induces a one-one correspondence between Hats_Σ and an abstract Schauder basis of G.

($2 \to 1$) If G has an abstract Schauder basis, then $G \cong_\ell \mathcal{A}_n \restriction |\Delta|$ for some unimodular fan Δ. By Theorem 4.1, $\mathcal{A}_n \restriction |\Delta|$ is finitely generated projective, hence so is G. $\square$

Now recalling Theorem 3.1 we immediately obtain from the proposition above:

Corollary 4.3. *For any ℓ-group G the following are equivalent:*

(1) *G is finitely generated projective.*
(2) *G is Archimedean and is generated by a finite set $B \subseteq G^+ \setminus \{0\}$ such that for every abstract k-simplex Y of B, the zero set $\mathfrak{Z}(B \setminus Y)$ is homeomorphic to the k-ball.*

The *proof* of Theorem 3.1, in conjunction with Proposition 4.2, actually yields the following interesting variant of ($1 \leftrightarrow 2$) in Theorem 4.1.

Corollary 4.4. *For any ℓ-group G the following are equivalent:*

(1) *G is finitely generated projective.*
(2) *For some positive integer n, G is ℓ-isomorphic to $\mathcal{A}_n \restriction |\Theta|$, where Θ is a (necessarily unimodular) fan in $\mathbb{R}^n$ whose 1-dimensional cones are generated by the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$.*

5. Refinements of abstract Schauder bases

As shown in [6, 15], refinements of unimodular fans are fundamental to the fan-theoretic treatment of birational toric geometry. In this section we derive abstract versions of two basic refinement procedures for unimodular fans.

Let Σ and Δ be unimodular fans in $\mathbb{R}^n$ with the same support. Let $[[\mathrm{Hats}_\Sigma]]$ and $[[\mathrm{Hats}_\Delta]]$ be the submonoids of $\mathcal{A}_n \upharpoonright |\Sigma|$ generated by the Schauder bases over Σ and Δ, respectively. Direct inspection shows that Σ refines Δ, henceforth written $\Sigma \le \Delta$, iff $[[\mathrm{Hats}_\Delta]] \subseteq [[\mathrm{Hats}_\Sigma]]$ iff $\mathrm{Hats}_\Delta \subseteq [[\mathrm{Hats}_\Sigma]]$. Thus the following proposition shows that, in a natural sense, any two abstract Schauder bases of the same ℓ-group admit a joint refinement.

Proposition 5.1 (Joint refinability)**.** *Let B_1, B_2 be abstract Schauder bases of an ℓ-group G. There exists an abstract Schauder basis C of G such that $C \le B_1, B_2$, where $C \le B_i$ stands for $B_i \subseteq [[C]]$, and $[[C]]$ is the submonoid of G generated by C.*

Proof. Let $\phi\colon G \cong_\ell \mathcal{A}_n \upharpoonright |\Sigma|$ be an ℓ-isomorphism carrying B_1 to Hats_Σ, for some unimodular fan Σ in $\mathbb{R}^n$. Then ϕ carries B_2 to a collection of ℓ-functions $S = \phi(B_2)$ defined over $|\Sigma|$.

Claim. There exists a unimodular fan Δ with $|\Delta| = |\Sigma|$ such that every ℓ-function in $S \cup \mathrm{Hats}_\Sigma$ is linear over each cone of Δ.

To prove this, let $\mathfrak{L}_1$ be the collection of all linear pieces of all ℓ-functions in $S \cup \mathrm{Hats}_\Sigma$, regarded by linear extension[11] as linear ℓ-functions from $\mathbb{R}^n$ into itself. Further, let $\mathfrak{L}_2$ be a collection of rational homogeneous hyperplanes in $\mathbb{R}^n$ defining the rational homogeneous polyhedral set $|\Sigma|$, again regarded as linear ℓ-functions from $\mathbb{R}^n$ into itself. Set $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2$, say $|\mathfrak{L}| = m$; display the elements of $\mathfrak{L}$ as $\mathfrak{L} = \{l_1, \ldots, l_m\}$. We construct a complete unimodular fan Ξ in $\mathbb{R}^n$ as follows. For every permutation σ of $\{1, \ldots, m\}$, set

$$E_\sigma = \{\mathbf{x} \in \mathbb{R}^n \mid l_{\sigma(1)}(\mathbf{x}) \ge l_{\sigma(2)}(\mathbf{x}) \ge \cdots \ge l_{\sigma(m)}(\mathbf{x})\} \ .$$

Then $E_\sigma \subseteq \mathbb{R}^n$ is a rational homogeneous polyhedral set; we do not exclude the trivial case $E_\sigma = \{0\}$, in which case there is nothing to prove. In all remaining cases, the dimension of E_σ is n. It is an exercise to check that there exists a unique refinement-maximal (not necessarily unimodular, nor even simplicial) fan Ξ in $\mathbb{R}^n$ whose n-dimensional cones are precisely the nontrivial E_σ, as σ ranges over all permutations of $\{1, \ldots, m\}$. Furthermore, it is clear that $\bigcup_\sigma E_\sigma = \mathbb{R}^n$, whence Ξ is complete. By e.g. [6, Theorem 8.5], every fan can be unimodularised; let Θ be a unimodular refinement of Ξ. Now $|\Sigma|$ is inscribed in Θ, meaning that $|\Sigma|$ is a union of cones of Θ, because $\mathfrak{L}$ includes $\mathfrak{L}_2$, a collection of hyperplanes defining Σ. Let Δ be the fan obtained from Ξ by selecting all those cones $\delta \in \Theta$ that intersect $|\Sigma|$ nontrivially, i.e. $\delta \cap |\Sigma| \ne \{0\}$. Then clearly $|\Delta| = |\Sigma|$, Δ is unimodular, and every ℓ-function in $S \cup \mathrm{Hats}_\Sigma$ is linear over each cone of Δ; the latter condition holds because $\mathfrak{L}$ includes $\mathfrak{L}_2$, the collection of all linear pieces of all ℓ-functions in $S \cup \mathrm{Hats}_\Sigma$. This settles our first claim.

[11] Such an extension is generally not unique, but the choice of the extension is immaterial.

Claim. $\mathrm{Hats}_\Delta = \{h_1, \ldots, h_t\}$ spans $S \cup \mathrm{Hats}_\Sigma$ positively.

Indeed, let $f\colon |\Sigma| \to \mathbb{R}$ be an ℓ-function that is linear over each cone of Δ. For each 1-cone $\delta_i \in \Delta$, $i \in \{1, \ldots, t\}$, f attains an integral value $z_i \in \mathbb{Z}$ at the primitive vector along δ_i. The ℓ-function $\tilde{f} = z_1 h_1 + \cdots + z_t h_t$ coincides with f: it indeed does by construction over 1-cones, hence it does over any cone because both f and $\tilde{f}$ are linear over each cone of Δ. This settles our second claim.

We thus have $S, \mathrm{Hats}_\Sigma \subseteq [[\mathrm{Hats}_\Delta]]$. Set $C = \phi^{-1}(\mathrm{Hats}_\Delta)$. Since ϕ^{-1} is an ℓ-isomorphism, $B_1, B_2 \subseteq [[C]]$ holds, and the proof is complete. □

If Σ refines Δ, there is generally no way of obtaining Σ from Δ via a sequence of starring operations; thus, $\Sigma \preceq_2 \Delta$ implies $\Sigma \leq \Delta$, but not conversely. The stronger $\Sigma \preceq_2 \Delta$ condition that Σ be a starring refinement of Δ also has an algebraic counterpart via Lemma 2.4. Specifically, let us write $[[\mathrm{Hats}_\Sigma]] \preceq_2 [[\mathrm{Hats}_\Delta]]$ iff Hats_Σ can be obtained from Hats_Δ via a finite number of transformations of the type

$$\{h_1, h_2\} \mapsto \{h_1 - (h_1 \wedge h_2), h_2 - (h_1 \wedge h_2), h_1 \wedge h_2\}, \tag{4}$$

with the *proviso* that $h_1 \wedge h_2 \neq 0$. We can then prove the following purely algebraic version of the De Concini-Procesi Lemma.

Proposition 5.2 (The abstract De Concini-Procesi Lemma)**.** *Let B_1, B_2 be abstract Schauder bases of an ℓ-group G. There exists an abstract Schauder basis C of G such that:*

1. *$C \leq B_2$, in the sense that $B_2 \subseteq [[C]]$.*
2. *$C \preceq_2 B_1$, in the sense that C is obtainable from B_1 via a finite number of transformations of the type* (4)*, where $h_1 \wedge h_2 \neq 0$.*

Proof. Let $\phi\colon G \cong_\ell \mathcal{A}_n \restriction |\Sigma|$ be an ℓ-isomorphism carrying B_1 to Hats_Σ, for some unimodular fan Σ in $\mathbb{R}^n$, and let $S = \phi(B_2)$. By the same argument as the one in Proposition 5.1, there is a unimodular fan Δ with $|\Delta| = |\Sigma|$ such that $S \cup \mathrm{Hats}_\Sigma \subseteq [[\mathrm{Hats}_\Delta]]$. By the De Concini-Procesi Lemma, there is a unimodular fan Θ such that $\Theta \preceq_2 \Sigma$, and Θ refines Δ. It follows that $\mathrm{Hats}_\Sigma, S \subseteq [[\mathrm{Hats}_\Theta]]$, because every ℓ-function in $\mathrm{Hats}_\Sigma \cup S$ is linear over each cone of Δ, hence over each cone of its refinement Θ. Thus, if $C = \phi^{-1}(\mathrm{Hats}_\Theta)$, we have $B_1, B_2 \subseteq [[C]]$. Since $\Theta \preceq_2 \Sigma$, Lemma 2.4 shows Hats_Θ is obtainable from Hats_Σ via a finite number of transformations of type (4), with $h_1 \wedge h_2 \neq 0$. Since these transformations are preserved by ℓ-isomorphism, we obtain $C \preceq_2 B_1$, which completes the proof. □

The above result should be compared with Lemma 2.3, to which it reduces when B_1 and B_2 are concrete Schauder bases.

6. Finitely generated projective = presentable by a lattice word

We refer to [7, Section 5.2], [13, Section 4] and [8] for background on finite presentations of ℓ-groups. As an immediate consequence of Corollary 4.4 we have:

Proposition 6.1. *Finitely presented ℓ-groups $G \cong \langle \pi_1, \ldots, \pi_m \colon v = 0\rangle$ are exactly the same as finitely generated projective ℓ-groups.*

We are in a position to prove:[12]

Theorem 6.2. *For any ℓ-group G the following are equivalent:*

(1) *G is finitely generated projective.*
(2) *G is finitely presented by $\langle \pi_1, \ldots, \pi_n \colon l = 0 \rangle$, where the ℓ-function $l \in \mathcal{A}_n$ is obtainable from $\pi_1, \ldots, \pi_n, 0$ by applying the lattice operations only.*

Proof. ($2 \to 1$) Follows immediately from Proposition 6.1.

($1 \to 2$) In light of Corollary 4.4 it is sufficient to prove the theorem for any ℓ-group G of the form $G = \mathcal{A}_n \restriction |\Theta|$, where Θ is a (necessarily) unimodular fan in $\mathbb{R}^n$ whose 1-dimensional cones are generated by the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. In order to get the desired presentation of such a G it is enough to exhibit an ℓ-function $l \in \mathcal{A}_n$ having the following two properties:

- the vanishing locus $l^{-1}(0) \subseteq \mathbb{R}^n$ of l coincides with $|\Theta|$, and
- l is obtainable from $\pi_1, \ldots, \pi_n, 0$ by applying the lattice operations only.

To this purpose one first notes that $|\Theta|$ is contained in the positive orthant $\mathcal{O}$ of $\mathbb{R}^n$, and in addition, $\mathrm{Hats}_\Theta = \{\pi_1 \restriction |\Theta|, \ldots, \pi_n \restriction |\Theta|\}$.

Let the ℓ-function $w \in \mathcal{A}_n$ be defined by

$$w = \bigvee_{i=1}^{n} \pi_i \wedge 0.$$

The vanishing locus of w coincides with the intersection $\mathcal{O}$ of the vanishing loci of the ℓ-functions $\pi_i \wedge 0$. If $|\Theta| = \mathcal{O}$ we are done. Otherwise, for every cone θ of Θ, let $\mathbf{e}_{\theta,1}, \ldots, \mathbf{e}_{\theta,r}$ be the primitive generating vectors of θ, and let $\mathbf{e}_{\theta,r+1}, \ldots, \mathbf{e}_{\theta,n}$ be the remaining basis vectors of $\mathbb{R}^n$. Note that this latter set is nonempty. Let the ℓ-function l_θ be defined by

$$l_\theta = w \vee \pi_{\theta,r+1} \vee \cdots \vee \pi_{\theta,n}.$$

Then the vanishing locus of l_θ coincides with θ, as is immediately seen. Finally, let the ℓ-function l be defined by

$$l = \bigwedge_{\theta \in \Theta} l_\theta.$$

The vanishing locus of l coincides with the union $|\Theta|$ of the cones of Θ. This completes the proof. $\square$

Final remark. Suppose the finitely generated projective ℓ-group G is presented by $\langle \pi_1, \ldots, \pi_m \colon v = 0 \rangle$. Suppose the ℓ-function v is explicitly written as an ℓ-group word ω, i.e., as a string of symbols over the alphabet $\{0, +, -, \wedge, \vee\}$ of ℓ-groups, plus symbols $X_1, \ldots, X_m$ for the variables,[13] using the familiar composition rules. Then the proof of the theorem above yields an alternative presentation $G \cong \langle \pi_1, \ldots, \pi_n \colon l = 0 \rangle$, where the ℓ-function l can be explicitly written as a $\{0, \wedge, \vee\}$-word λ in the variables $Y_1, \ldots, Y_n$, *and the map $\omega \mapsto \lambda$ is effective.*[14]

As a matter of fact, from $\omega(X_1, \ldots, X_m)$ one can effectively[15] construct a unimodular fan Δ whose support $|\Delta| \subseteq \mathbb{R}^m$ coincides with the vanishing locus of v.

[12] Compare with [10, page 47].
[13] Also, if needed, such orthographic symbols as parentheses and commas.
[14] Here, by abuse of notation, π_j may denote a free generator both in $\mathcal{A}_m$ and in $\mathcal{A}_n$.
[15] See [13, Proposition 4.3] for details.

Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ display the primitive generating vectors of the rays in Δ. By sending each $\mathbf{p}_i$ to the standard basis vector $\mathbf{e}_i$ of $\mathbb{R}^n$ one effectively obtains from Δ a unimodular fan Θ in $\mathbb{R}^n$: the cones in Δ precisely correspond to the cones in Θ via the map $\mathbf{p}_i \mapsto \mathbf{e}_i$. The desired lattice-word λ is now effectively obtainable from Θ as in the proof of Theorem 6.2. The ℓ-isomorphisms

$$\mathcal{A}_m \upharpoonright |\Delta| \cong \mathcal{A}_n \upharpoonright |\Theta| \cong \langle \pi_1, \ldots, \pi_m \colon v = 0 \rangle \cong \langle \pi_1, \ldots, \pi_n \colon l = 0 \rangle$$

follow from the trivial piecewise homogeneous linear homeomorphism existing between $|\Delta|$ and $|\Theta|$ in the light of Baker-Beynon duality (see [7, 5.2]).

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