TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 9, September 2007, Pages 4107–4123
S 0002-9947(07)04002-0
Article electronically published on April 6, 2007

VERTEX OPERATOR ALGEBRAS, EXTENDED E_8 DIAGRAM, AND MCKAY'S OBSERVATION ON THE MONSTER SIMPLE GROUP

CHING HUNG LAM, HIROMICHI YAMADA, AND HIROSHI YAMAUCHI

Abstract. We study McKay's observation on the Monster simple group, which relates the $2A$-involutions of the Monster simple group to the extended E_8 diagram, using the theory of vertex operator algebras (VOAs). We first consider the sublattices L of the E_8 lattice obtained by removing one node from the extended E_8 diagram at each time. We then construct a certain coset (or commutant) subalgebra U associated with L in the lattice VOA $V_{\sqrt{2}E_8}$. There are two natural conformal vectors of central charge $1/2$ in U such that their inner product is exactly the value predicted by Conway (1985). The Griess algebra of U coincides with the algebra described in his Table 3. There is a canonical automorphism of U of order $|E_8/L|$. Such an automorphism can be extended to the Leech lattice VOA V_Λ, and it is in fact a product of two Miyamoto involutions. In the sequel (2005) to this article, the properties of U will be discussed in detail. It is expected that if U is actually contained in the Moonshine VOA $V^\natural$, the product of two Miyamoto involutions is in the desired conjugacy class of the Monster simple group.

1. Introduction

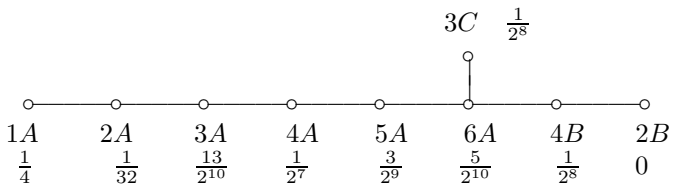
The Moonshine vertex operator algebra $V^\natural$ constructed by Frenkel-Lepowsky-Meurman [7] is one of the most important examples of vertex operator algebras (VOAs). Its full automorphism group is the Monster simple group. The weight 2 subspace $V_2^\natural$ of $V^\natural$ has a structure of commutative non-associative algebra and possesses a non-degenerate associative form, which coincides with the 196884-dimensional algebra investigated by Griess [9] in his construction of the Monster simple group (see also Conway [1]). The structure of this algebra, which is called the Monstrous Griess algebra, has been studied by group theorists. It is well known [1] that each $2A$-involution ϕ of the Monster simple group uniquely defines an idempotent e_ϕ called an axis in the Monstrous Griess algebra. The inner product $\langle e_\phi, e_\psi\rangle$ between any two axes e_ϕ and e_ψ was also calculated [1], and it was shown that the value $\langle e_\phi, e_\psi\rangle$ is uniquely determined by the conjugacy class of the product $\phi\psi$ of $2A$-involutions. Actually, $2A$-involutions of the Monster simple group satisfy a 6-transposition property, that is, $|\,\phi\psi\,| \leq 6$ for any two $2A$-involutions ϕ and ψ. In

Received by the editors April 4, 2004 and, in revised form, March 4, 2005.
2000 *Mathematics Subject Classification.* Primary 17B68, 17B69, 20D08.
The first author was partially supported by NSC grant 91-2115-M-006-014 of Taiwan, R.O.C.
The second author was partially supported by JSPS Grant-in-Aid for Scientific Research No. 15540015.

addition, the conjugacy class of $\phi\psi$ is one of $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $4B$, $2B$, or $3C$.

John McKay [15] observed that there is an interesting correspondence with the extended E_8 diagram. Namely, one can assign $1A, 2A, 3A, 4A, 5A, 6A, 4B, 2B$, and $3C$ to the nodes of the extended E_8 diagram as follows (cf. Conway [1], Glauberman and Norton [8]):

(1.1)



where the numerical labels are equal to the multiplicities of the corresponding simple roots in the highest root and the fractional values under or on the right of the labels denote the inner product $\langle 2e_\phi, 2e_\psi\rangle$ of $2e_\phi$ and $2e_\psi$ when the product of the corresponding $2A$-involutions $\phi\psi$ belongs to the conjugacy class nX associated with the node.

On the other hand, from the point of view of VOAs, Miyamoto [16, 18] showed that an axis is essentially a half of a conformal vector e of central charge $1/2$ which generates a Virasoro VOA $\mathrm{Vir}(e) \cong L(1/2, 0)$ inside the Moonshine VOA $V^\natural$. Moreover, an involutive automorphism τ_e can be defined by

$$\tau_e = \begin{cases} 1 & \text{on } W_0 \oplus W_{1/2}, \\ -1 & \text{on } W_{1/16}, \end{cases}$$

where W_h denotes the sum of all irreducible $\mathrm{Vir}(e)$-modules isomorphic to $L(1/2, h)$ inside $V^\natural$. In fact, τ_e is always of class $2A$ for any conformal vector e of central charge $1/2$ in $V^\natural$.

In this article, we try to give an interpretation of the McKay diagram (1.1) using the theory of VOAs. We first observe that there is a conformal vector $\hat{e}$ of central charge $1/2$ in the lattice VOA $V_{\sqrt{2}E_8}$ which is fixed by the action of the Weyl group of type E_8. Let Φ be the root system corresponding to the Dynkin diagram obtained by removing one node from the extended E_8 diagram and $L = L(\Phi)$ the root lattice associated with Φ. Then the Weyl group $W(\Phi)$ of Φ and the quotient group E_8/L both act naturally on $V_{\sqrt{2}E_8}$, and their actions commute with each other. The action of the quotient group E_8/L can be extended to the Leech lattice VOA V_Λ.

The main idea is to construct certain vertex operator subalgebras U of the lattice VOA $V_{\sqrt{2}E_8}$ corresponding to the nine nodes of the McKay diagram. In each case, U is constructed as a coset (or commutant) subalgebra of $V_{\sqrt{2}E_8}$ associated with Φ. In fact, U is chosen so that the Weyl group $W(\Phi)$ acts trivially on it. We show that in each of the nine cases U always contains $\hat{e}$ and another conformal vector $\hat{f}$ of central charge $1/2$ such that the inner product $\langle \hat{e}, \hat{f}\rangle$ is exactly the value listed in the McKay diagram. Both $\hat{e}$ and $\hat{f}$ are fixed by the Weyl group $W(\Phi)$. Thus the Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$ commute with the action of $W(\Phi)$. Furthermore, the quotient group E_8/L naturally induces some automorphism of U of order $n = |E_8/L|$, which is identical with the numerical label of the corresponding node in the McKay diagram. Such an automorphism can be extended to the Leech

lattice VOA V_Λ and it is in fact a product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$.

In the sequel [12] to this article we shall study the properties of the coset subalgebra U in detail. Except the $4A$ case, U always contains a set of mutually orthogonal conformal vectors such that their sum is the Virasoro element of U and the central charge of those conformal vectors are all coming from the unitary series

$$c = c_m = 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 1, 2, 3, \ldots.$$

Such a conformal vector generates a Virasoro VOA isomorphic to $L(c_m, 0)$ inside U. The structure of U as a module for a tensor product of those Virasoro VOAs is determined.

In the $4A$ case, U is isomorphic to the fixed point subalgebra $V_{\mathcal{N}}^+$ of θ for some rank two lattice $\mathcal{N}$, where θ is an automorphism of $V_{\mathcal{N}}$ induced from the -1 isometry of the lattice $\mathcal{N}$.

The VOA U is generated by $\hat{e}$ and $\hat{f}$. As a consequence we know that every element of U is fixed by the Weyl group $W(\Phi)$. The weight 1 subspace U_1 of U is 0. The Griess algebra U_2 of U is also generated by $\hat{e}$ and $\hat{f}$, and it has the same structure as the algebra studied in Conway [1, Table 3]. The automorphism group of U is a dihedral group of order $2n$ except the cases for $1A$, $2A$, and $2B$. It is a trivial group in the $1A$ case, a symmetric group of degree 3 in the $2A$ case, and of order 2 in the $2B$ case. The product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions should be in the desired conjugacy class of the Monster simple group, provided that the Moonshine VOA $V^\natural$ contains a subalgebra isomorphic to U.

Further mysteries concerning the McKay diagram can be found in Glauberman and Norton [8]. Among other things, some relation between the Weyl group $W(\Phi)$ and the centralizer of a certain subgroup generated by two $2A$-involutions and one $2B$-involution in the Monster simple group was discussed. That every element of U is fixed by $W(\Phi)$ seems quite suggestive.

Let us recall some terminology (cf. [7, 14]). A VOA is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with a linear map $Y(\,\cdot\,, z) : V \to (\operatorname{End} V)[[z, z^{-1}]]$ and two distinguished vectors; the vacuum vector $\mathbf{1} \in V_0$ and the Virasoro element $\omega \in V_2$ which satisfy certain conditions. For any $v \in V$, $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ is called a vertex operator and $v_n \in \operatorname{End} V$ a component operator. Each homogeneous subspace V_n is the eigenspace for the operator $L(0) = \omega_1$ with eigenvalue n. The eigenvalue for $L(0)$ is called a weight. Suppose $V = \bigoplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{C}\mathbf{1}$ and $V_1 = 0$. For $u, v \in V_2$, one can define a product $u \cdot v$ by $u_1 v$ and an inner product $\langle u, v\rangle$ by $u_3 v = \langle u, v\rangle \mathbf{1}$. The inner product is invariant, that is, $\langle u_1 v, w\rangle = \langle v, u_1 w\rangle$ for $u, v, w \in V_2$ (cf. [7, Section 8.9]). With the product and the inner product V_2 becomes an algebra, which is called the Griess algebra of V.

The organization of the article is as follows. In Section 2 we review some notation for lattice VOAs from [7] (see also [14]) and certain conformal vectors in the lattice VOA $V_{\sqrt{2}R}$ given by [5], where R is a root lattice of type A, D, or E. Moreover, we study some highest weight vectors in irreducible modules of $V_{\sqrt{2}R}$ with respect to those conformal vectors. In Section 3 we consider the sublattice L of E_8 and define the coset subalgebra U and two conformal vectors $\hat{e}$ and $\hat{f}$ of central charge $1/2$. We calculate the inner product $\langle \hat{e}, \hat{f}\rangle$ and verify that it is identical with the value given in the McKay diagram. A canonical automorphism σ of order $n = |E_8/L|$ induced

by the quotient group E_8/L is also discussed. Then in Section 4 we consider an embedding of an orthogonal sum $\sqrt{2}E_8{}^3$ of three copies of $\sqrt{2}E_8$ into the Leech lattice Λ and show that the product $\tau_{\hat{e}}\tau_{\hat{f}}$ of two Miyamoto involutions $\tau_{\hat{e}}$ and $\tau_{\hat{f}}$ is of order n as an automorphism of V_Λ. Finally, in Section 5 we give an explicit correspondence between the Griess algebra U_2 of U and the algebra in Conway [1, Table 3].

The authors thank Masaaki Kitazume and Masahiko Miyamoto for stimulating discussions and Kazuhiro Yokoyama for helping them to compute the conformal vectors for the cases of $5A$ and $6A$ by a computer algebra system Risa/Asir.

2. Conformal vectors in lattice VOAs

In this section, we review the construction of certain conformal vectors in the lattice VOA $V_{\sqrt{2}R}$ from [5], where R is a root lattice of type A_n, D_n, or E_n. The notation for lattice VOAs here is standard (cf. [7, 14]). Let N be a positive definite even lattice with inner product $\langle\cdot,\cdot\rangle$. Then the VOA V_N associated with N is defined to be $M(1)\otimes\mathbb{C}\{N\}$. More precisely, let $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}N$ be an abelian Lie algebra and $\hat{\mathfrak{h}}=\mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ its affine Lie algebra. Then $M(1)=\mathbb{C}[\alpha(n)\,|\,\alpha\in\mathfrak{h}, n<0]\cdot 1$ is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)\cdot 1=0$ for $\alpha\in\mathfrak{h}$, $n\geq 0$ and $K=1$, where $\alpha(n)=\alpha\otimes t^n$. Moreover, $\mathbb{C}\{N\}$ denotes a twisted group algebra of the additive group N. In the case for $N=\sqrt{2}R$, the twisted group algebra $\mathbb{C}\{\sqrt{2}R\}$ is isomorphic to the ordinary group algebra $\mathbb{C}[\sqrt{2}R]$ since $\sqrt{2}R$ is a doubly even lattice. The standard basis of $\mathbb{C}[\sqrt{2}R]$ is denoted by $e^{\sqrt{2}\alpha}$, $\alpha\in R$. Then the vacuum vector $\mathbf{1}$ is $1\otimes e^0$.

Let Φ be the root system of R and Φ^+ and Φ^- the set of all positive roots and negative roots, respectively. Then $\Phi=\Phi^+\cup\Phi^-=\Phi^+\cup(-\Phi^+)$. The Virasoro element ω of $V_{\sqrt{2}R}$ is given by

$$\omega=\omega(\Phi)=\frac{1}{2h}\sum_{\alpha\in\Phi^+}\alpha(-1)^2\cdot 1,$$

where h is the Coxeter number of Φ. Now define

$$\begin{aligned} s&=s(\Phi)=\frac{1}{2(h+2)}\sum_{\alpha\in\Phi^+}\left(\alpha(-1)^2\cdot 1-2(e^{\sqrt{2}\alpha}+e^{-\sqrt{2}\alpha})\right),\\ \tilde{\omega}&=\tilde{\omega}(\Phi)=\omega-s. \end{aligned}\tag{2.1}$$

It is shown in [5] that $\tilde{\omega}$ and s are mutually orthogonal conformal vectors, that is, $\tilde{\omega}_1\tilde{\omega}=2\tilde{\omega}, s_1 s=2s$, and $\tilde{\omega}_1 s=0$. The central charge of $\tilde{\omega}$ is $2n/(n+3)$ if R is of type A_n, 1 if R is of type D_n and $6/7, 7/10$ and $1/2$ if R is of type E_6, E_7 and E_8, respectively.

Let $W(\Phi)$ be the Weyl group of Φ. Any element $g\in W(\Phi)$ induces an automorphism of the lattice R and hence it defines an automorphism of the VOA $V_{\sqrt{2}R}$ by

$$g(u\otimes e^{\sqrt{2}\alpha})=gu\otimes e^{\sqrt{2}g\alpha}\qquad\text{for}\quad u\otimes e^{\sqrt{2}\alpha}\in M(1)\otimes e^{\sqrt{2}\alpha}\subset V_{\sqrt{2}R}.$$

Note that both s and $\tilde{\omega}$ are fixed by the Weyl group $W(\Phi)$.

We shall study certain highest weight vectors with respect to the subalgebra $\mathrm{Vir}(s)\otimes\mathrm{Vir}(\tilde{\omega})$, where $\mathrm{Vir}(s)$ and $\mathrm{Vir}(\tilde{\omega})$ denote the Virasoro VOAs generated by the conformal vectors s and $\tilde{\omega}$, respectively.

Let $R^*=\{\alpha\in\mathbb{Q}\otimes_{\mathbb{Z}}R\,|\,\langle\alpha,R\rangle\subset\mathbb{Z}\}$ be the dual lattice of R.

Lemma 2.1. *Let R be a root lattice of type A, D, or E and $\gamma + R$ a coset of R in R^*. Let $k = \min\{\langle\alpha,\alpha\rangle | \alpha \in \gamma + R\}$. For any $\eta \in \gamma + R$ with $\langle\eta,\eta\rangle = k$, we define*

$$X_\eta = \{(\alpha,\beta) \in R \times (\gamma + R) \mid \langle\alpha,\alpha\rangle = 2, \langle\beta,\beta\rangle = k \text{ and } \alpha + \beta = \eta\}.$$

Then $|X_\eta| = kh$, where h is the Coxeter number of R.

Proof. The proof is just by direct verification. We only discuss the case for $R = A_n$. The other cases can be proved similarly.

Let $R = A_n$. Then the Coxeter number h is $n+1$ and the roots of A_n are given by the vectors in the form $\pm(1,-1,0^{n-1}) \in \mathbb{R}^{n+1}$, that is, the vectors whose one entry is ± 1, another entry is ∓ 1, and the remaining $n-1$ entries are 0. Let $\mu = \frac{1}{n+1}(1,\dots,1,-n)$. Then $\mu + R$ is a generator of the group R^*/R. Denote $\gamma = j\mu$ for $j = 0,\dots,n$. Then

$$k = \min\{\langle\alpha,\alpha\rangle | \alpha \in \gamma + R\} = \frac{j(n+1-j)}{n+1},$$

and the elements of square norm k in $\gamma + R$ are of the form

$$\frac{1}{n+1}(j^{n+1-j}, (-n-1+j)^j).$$

Now it is easy to see that $|X_\eta| = (n+1-j)j = kh$ for any η with $\langle\eta,\eta\rangle = k$. □

Proposition 2.2. *Let $\gamma + R$ be a coset of R in R^* and $k = \min\{\langle\alpha,\alpha\rangle | \alpha \in \gamma + R\}$. Define*

$$v = \sum_{\substack{\alpha\in\gamma+R \\ \langle\alpha,\alpha\rangle=k}} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}(\gamma+R)}.$$

Then v is a highest weight vector of highest weight $(0,k)$ in $V_{\sqrt{2}(\gamma+R)}$ with respect to $\mathrm{Vir}(s) \otimes \mathrm{Vir}(\tilde{\omega})$, that is, $s_j v = \tilde{\omega}_j v = 0$ for all $j \geq 2$, $s_1 v = 0$, and $\tilde{\omega}_1 v = kv$.

Proof. Since k is the minimum weight of $V_{\sqrt{2}(\gamma+R)}$, it is clear that $s_j v = \tilde{\omega}_j v = 0$ for all $j \geq 2$. Since $\omega_1 v = kv$, it suffices to show that $s_1 v = 0$. By the definition (2.1) of s and the above lemma, we have

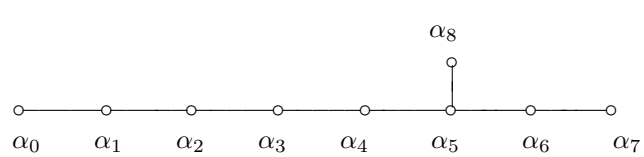
$$\begin{aligned} s_1 v &= \frac{1}{2(h+2)} \sum_{\alpha\in\Phi^+} \Big(\alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\Big)_1 v \\ &= \Big(\frac{h}{h+2}\omega - \frac{1}{h+2} \sum_{\alpha\in\Phi^+} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})\Big)_1 v \\ &= \frac{hk}{h+2} v - \frac{hk}{h+2} v = 0. \end{aligned}$$

Hence the assertion holds. □

3. Extended E_8 diagram and sublattices of the root lattice E_8

In this section, we consider certain sublattices of the root lattice E_8 by using the extended E_8 diagram

(3.1)



where $\alpha_1, \alpha_2, \ldots, \alpha_8$ are the simple roots of E_8 and

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0. \tag{3.2}$$

Thus $\langle \alpha_i, \alpha_i \rangle = 2$, $0 \le i \le 8$. Moreover, for $i \neq j$, $\langle \alpha_i, \alpha_j \rangle = -1$ if the nodes α_i and α_j are connected by an edge and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. Note that $-\alpha_0$ is the highest root.

For any $i = 0, 1, \ldots, 8$, let $L(i)$ be the sublattice generated by $\alpha_j, 0 \le j \le 8, j \neq i$. Then $L(i)$ is a rank 8 sublattice of E_8. In fact, $L(i)$ is the lattice associated with the Dynkin diagram obtained by removing the corresponding node α_i from the extended E_8 diagram (3.1). Note that the index $|E_8/L(i)|$ is equal to n_i, where n_i is the coefficient of α_i in the left hand side of (3.2). Actually, we have

$$\begin{aligned} &L(0) \cong E_8, && L(1) \cong A_1 \oplus E_7, && L(2) \cong A_2 \oplus E_6, \\ &L(3) \cong A_3 \oplus D_5, && L(4) \cong A_4 \oplus A_4, && L(5) \cong A_5 \oplus A_2 \oplus A_1, \\ &L(6) \cong A_7 \oplus A_1, && L(7) \cong D_8, && L(8) \cong A_8. \end{aligned} \tag{3.3}$$

Remark 3.1. If n_i is not a prime, there is an intermediate sublattice as follows:

$$\begin{aligned} &A_3 \oplus D_5 \subset D_8 \subset E_8, \\ &A_5 \oplus A_2 \oplus A_1 \subset A_2 \oplus E_6 \subset E_8, \qquad A_5 \oplus A_2 \oplus A_1 \subset A_1 \oplus E_7 \subset E_8, \\ &A_7 \oplus A_1 \subset A_1 \oplus E_7 \subset E_8. \end{aligned}$$

There are corresponding power maps between conjugacy classes of the Monster simple group, namely,

$$(4A)^2 = 2B, \qquad (6A)^2 = 3A, \qquad (6A)^3 = 2A, \qquad (4B)^2 = 2A,$$

where $(mX)^k = nY$ means that the k-th power g^k of an element g in the conjugacy class mX is in the conjugacy class nY (cf. [2]).

3.1. **Coset subalgebras of the lattice VOA $V_{\sqrt{2}E_8}$.** We shall construct some VOAs U corresponding to the nine nodes of the McKay diagram (1.1). In each case, we show that the VOA U contains some conformal vectors of central charge 1/2, and the inner products among these conformal vectors are the same as the numbers given in the McKay diagram.

Let us explain the details of our construction. First, we fix $i \in \{0, 1, \ldots, 8\}$ and denote $L(i)$ by L. In each case, $|E_8/L| = n_i$ and $\alpha_i + L$ is a generator of the quotient group E_8/L. Hence we have

$$E_8 = L \cup (\alpha_i + L) \cup (2\alpha_i + L) \cup \cdots \cup ((n_i - 1)\alpha_i + L). \tag{3.4}$$

Then the lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as

$$V_{\sqrt{2}E_8} = V_{\sqrt{2}L} \oplus V_{\sqrt{2}\alpha_i + \sqrt{2}L} \oplus \cdots \oplus V_{\sqrt{2}(n_i-1)\alpha_i + \sqrt{2}L},$$

where $V_{\sqrt{2}j\alpha_i + \sqrt{2}L}$, $j = 0, 1, \ldots, n_i - 1$, are irreducible modules of $V_{\sqrt{2}L}$ (cf. [4]).

The quotient group E_8/L induces an automorphism σ of $V_{\sqrt{2}E_8}$ such that

$$\sigma(u) = \xi^j u \qquad \text{for any} \quad u \in V_{\sqrt{2}j\alpha_i + \sqrt{2}L}, \tag{3.5}$$

where $\xi = e^{2\pi\sqrt{-1}/n_i}$ is a primitive n_i-th root of unity. More precisely, let

$$\mathbf{a} = \begin{cases} \alpha_1 & \text{if } i = 0, \\ -\frac{1}{i+1}(\alpha_0 + 2\alpha_1 + \cdots + i\alpha_{i-1}) & \text{if } 1 \le i \le 5, \\ -\frac{1}{8}(\alpha_0 + 2\alpha_1 + \cdots + 6\alpha_5 + 7\alpha_8) & \text{if } i = 6, \\ \frac{1}{2}(\alpha_6 + \alpha_8) & \text{if } i = 7, \\ -\frac{1}{9}(\alpha_0 + 2\alpha_1 + \cdots + 8\alpha_7) & \text{if } i = 8. \end{cases} \tag{3.6}$$

Then $\langle \mathbf{a}, \alpha_j \rangle \in \mathbb{Z}$ for $0 \le j \le 8$ with $j \ne i$ and $\langle \mathbf{a}, \alpha_i \rangle \equiv -1/n_i \pmod{\mathbb{Z}}$. The automorphism $\sigma : V_{\sqrt{2}E_8} \to V_{\sqrt{2}E_8}$ is in fact defined by

$$\sigma = e^{-\pi\sqrt{-1}\boldsymbol{\beta}(0)} \qquad \text{with} \qquad \boldsymbol{\beta} = \sqrt{2}\mathbf{a}. \tag{3.7}$$

For $u \in M(1) \otimes e^{\alpha} \subset V_{\sqrt{2}E_8}$, we have $\sigma(u) = e^{-\pi\sqrt{-1}\langle \boldsymbol{\beta}, \alpha \rangle} u$. Note that $\mathbf{a} + R$ is a generator of the quotient group R^*/R for the cases $i \ne 0, 7$, where R is an indecomposable component of the lattice L of type A and R^* is the dual lattice of R.

For any lattice VOA V_N associated with a positive definite even lattice N, there is a natural involution θ induced by the isometry $\alpha \to -\alpha$ for $\alpha \in N$. If $N = \sqrt{2}E_8$, which is doubly even, we may define $\theta : V_{\sqrt{2}E_8} \to V_{\sqrt{2}E_8}$ by

$$\alpha(-n) \to -\alpha(-n) \qquad \text{and} \qquad e^{\alpha} \to e^{-\alpha} \tag{3.8}$$

for $\alpha \in \sqrt{2}E_8$ (cf. [7]). Then $\theta\sigma\theta = \sigma^{-1}$, and the group generated by θ and σ is a dihedral group of order $2n_i$.

Let $R_1, \ldots, R_l$ be the indecomposable components of the lattice L and $\Phi_1, \ldots, \Phi_l$ the corresponding root systems of $R_1, \ldots, R_l$ (cf. (3.3)). Then $L = R_1 \oplus \cdots \oplus R_l$ and

$$V_{\sqrt{2}L} \cong V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_l}$$

(see [6] for tensor products of VOAs). By (2.1), one obtains $2l$ mutually orthogonal conformal vectors

$$s^k = s(\Phi_k), \quad \tilde{\omega}^k = \tilde{\omega}(\Phi_k), \quad k = 1, \ldots, l, \tag{3.9}$$

such that the Virasoro element ω of $V_{\sqrt{2}L}$, which is also the Virasoro element of $V_{\sqrt{2}E_8}$, can be written as a sum of these conformal vectors

$$\omega = s^1 + \cdots + s^l + \tilde{\omega}^1 + \cdots + \tilde{\omega}^l.$$

Now we define U to be a coset (or commutant) subalgebra

$$U = \{v \in V_{\sqrt{2}E_8} \mid (s^k)_1 v = 0 \text{ for all } k = 1, \ldots, l\}. \tag{3.10}$$

Note that U is a VOA with the Virasoro element $\omega' = \tilde{\omega}^1 + \cdots + \tilde{\omega}^l$, and the automorphism σ defined by (3.5) induces an automorphism of order n_i on U. By abuse of notation, we denote it by σ also.

Remark 3.2. In [11], it is shown that $\{v \in V_{\sqrt{2}A_n} \mid s(A_n)_1 v = 0\}$ is isomorphic to a parafermion algebra $W_{n+1}(2n/(n+3))$ of central charge $2n/(n+3)$. Thus, if L has some indecomposable component of type A_n, then U contains some subalgebra isomorphic to a parafermion algebra. It is well known [19] that the parafermion algebra $W_{n+1}(2n/(n+3))$ possesses a certain $\mathbb{Z}_{n+1}$ symmetry in the fusion rules among its irreducible modules. The automorphism σ is in fact related to such a symmetry. More details about the relation between coset subalgebra U and the parafermion algebra $W_{n+1}(2n/(n+3))$ can be found in [12].

3.2. **Conformal vectors of central charge** $1/2$**.** Next, we shall study some conformal vectors in $V_{\sqrt{2}E_8}$. We shall also show that the coset subalgebra U always contains some conformal vectors of central charge $1/2$. Moreover, the inner products among these conformal vectors will be discussed.

Recall that the lattice $\sqrt{2}E_8$ can be constructed by using the $[8,4,4]$ Hamming code H_8 and the Construction A (cf. [3]). That means

$$\sqrt{2}E_8 = \left\{(a_1,\dots,a_8)\in\mathbb{Z}^8 \mid (a_1,\dots,a_8)\in H_8 \mod 2\right\}. \tag{3.11}$$

We denote the vectors $(0,0,0,0,0,0,0,0)$ and $(1,1,1,1,1,1,1,1)$ by $\mathbf{0}$ and $\mathbf{1}$, respectively. For any $\gamma\in H_8$, we define

$$X^0_\gamma = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} (-1)^{\langle\alpha,\mathbf{0}\rangle/2} e^\alpha = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} e^\alpha,$$
$$X^1_\gamma = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} (-1)^{\langle\alpha,\mathbf{1}\rangle/2} e^\alpha,$$

and for any binary word $\delta\in{\mathbb{Z}_2}^8$, we define

$$\hat{e}^\epsilon_\delta = \frac{1}{16}\omega + \frac{1}{32}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma, \qquad \epsilon=0,1,$$

where ω is the Virasoro element of the VOA $V_{\sqrt{2}E_8}$. Note that $X^\epsilon_{\mathbf{1}}=0$ for any $\epsilon=0,1$ and that $\hat{e}^\epsilon_\delta=\hat{e}^\epsilon_\eta$ if and only if $\eta\in\delta+H_8$.

Lemma 3.3. *For any $\epsilon=0,1$ and $\delta\in{\mathbb{Z}_2}^8$, $\hat{e}^\epsilon_\delta$ is a conformal vector of central charge $1/2$. The inner product among them are as follows:*

$$\langle\hat{e}^\epsilon_\delta,\hat{e}^\epsilon_\eta\rangle = \begin{cases} 0 & \text{if } \delta+\eta \text{ is even,}\\ 1/32 & \text{if } \delta+\eta \text{ is odd}\end{cases}$$

for any $\eta\notin\delta+H_8$, and

$$\langle\hat{e}^0_\delta,\hat{e}^1_\eta\rangle = 0$$

for any $\delta,\eta\in{\mathbb{Z}_2}^8$.

Proof. We have

$$(X^\epsilon_\gamma)_1(X^\epsilon_\zeta) = 4X^\epsilon_{\gamma+\zeta} \qquad \text{if} \qquad |\gamma+\zeta|=4,$$
$$(X^\epsilon_{\mathbf{0}})_1(X^\epsilon_{\mathbf{0}}) = \sum_{\substack{\alpha\equiv\mathbf{0} \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1.$$

Moreover, for any $\gamma\in H_8$ with $|\gamma|=4$,

$$(X^\epsilon_\gamma)_1(X^\epsilon_\gamma)+(X^\epsilon_{\mathbf{1}+\gamma})_1(X^\epsilon_{\mathbf{1}+\gamma}) = \sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1 + \sum_{\substack{\alpha\equiv\mathbf{1}+\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1 + 8X^\epsilon_{\mathbf{0}}.$$

Note also that

$$\sum_{\gamma\in H_8}\sum_{\substack{\alpha\equiv\gamma \bmod 2\\ \langle\alpha,\alpha\rangle=4}} \frac{1}{2}\alpha(-1)^2\cdot 1 = \sum_{\beta\in\Phi(E_8)}\beta(-1)^2\cdot 1 = 2\sum_{\beta\in\Phi^+(E_8)}\beta(-1)^2\cdot 1.$$

In addition, we have

$$\langle X^\epsilon_\gamma, X^\epsilon_\zeta\rangle = \begin{cases} 16 & \text{if } \gamma=\zeta \text{ and } \langle\gamma,\gamma\rangle \neq 8,\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle X^0_\gamma, X^1_\zeta\rangle = \begin{cases} -16 & \text{if } \gamma=\zeta=0,\\ 0 & \text{otherwise.}\end{cases}$$

Then since $\omega_1\omega = 2\omega$ and $\langle\omega,\omega\rangle = 4$, it follows that

$$\begin{aligned}(\hat{e}^\epsilon_\delta)_1\hat{e}^\epsilon_\delta &= \Big(\frac{1}{16}\omega + \frac{1}{32}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\Big)_1\Big(\frac{1}{16}\omega + \frac{1}{32}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\Big)\\ &= \frac{1}{2^8}\times 2\omega + 2\times\frac{1}{16}\times\frac{1}{32}\times 2\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\\ &\quad + \frac{1}{2^{10}}\Big(\sum_{\beta\in\Phi^+(E_8)} 2\beta(-1)^2\cdot 1 + 56\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma\Big)\\ &= \frac{1}{8}\omega + \frac{1}{16}\sum_{\gamma\in H_8}(-1)^{\langle\delta,\gamma\rangle}X^\epsilon_\gamma = 2\hat{e}^\epsilon_\delta,\end{aligned}$$

and

$$\langle\hat{e}^\epsilon_\delta, \hat{e}^\epsilon_\delta\rangle = \frac{1}{2^8}\times 4 + \frac{1}{2^{10}}\times 240 = \frac{1}{4}.$$

Hence $\hat{e}^\epsilon_\delta$ is a conformal vector of central charge $1/2$.

For any $\eta \notin \delta + H_8$, we calculate that

$$\begin{aligned}\langle\hat{e}^\epsilon_\delta, \hat{e}^\epsilon_\eta\rangle &= \frac{1}{2^8}\times 4 + \frac{1}{2^{10}}\sum_{\gamma\in H_8}(-1)^{\langle\delta+\eta,\gamma\rangle}\langle X^\epsilon_\gamma, X^\epsilon_\gamma\rangle\\ &= \begin{cases} \frac{1}{64} + \frac{1}{2^{10}}\times 16\times(7-8) = 0 & \text{if } \delta+\eta \text{ is even,}\\ \frac{1}{64} + \frac{1}{2^{10}}\times 16\times(8-7) = \frac{1}{32} & \text{if } \delta+\eta \text{ is odd.}\end{cases}\end{aligned}$$

Note that there are exactly eight elements in H_8 which are orthogonal to $\delta+\eta$. Note also that $\delta+\eta$ is orthogonal to $(1,1,1,1,1,1,1,1)$ if and only if $\delta+\eta$ is even.

Finally, for any $\delta,\eta \in {\mathbb{Z}_2}^8$ we obtain

$$\langle\hat{e}^0_\delta, \hat{e}^1_\eta\rangle = \frac{1}{2^8}\times 4 - \frac{1}{2^{10}}\times 16 = 0.$$

□

In Miyamoto [17], certain conformal vectors of central charge $1/2$ are constructed inside the Hamming code VOA. Our construction of $\hat{e}^\epsilon_\delta$ is essentially a lattice analogue of Miyamoto's construction. In fact, take $\lambda_j = (0,\dots,2,\dots,0) \in \mathbb{Z}^8$ to be the element in $\sqrt{2}E_8$ such that the j-th entry is 2 and all other entries are zero. Then we have a set of 16 mutually orthogonal conformal vectors of central charge $1/2$ given by

$$\omega^\pm_{\lambda_j} = \frac{1}{16}\lambda_j(-1)^2\cdot 1 \pm \frac{1}{4}(e^{\lambda_j} + e^{-\lambda_j}), \quad j = 1,2,\dots,8.$$

A set of mutually orthogonal conformal vectors of central charge $1/2$ whose sum is equal to the Virasoro element in a VOA is called a Virasoro frame. Thus, $\{\omega^\pm_{\lambda_j} \mid 1 \le j \le 8\}$ is a Virasoro frame of $V_{\sqrt{2}E_8}$. With respect to this Virasoro frame, the lattice VOA $V_{\sqrt{2}E_8}$ is a code VOA (cf. [17]). Let $V^+_{\sqrt{2}E_8}$ be the fixed

point subalgebra of $V_{\sqrt{2}E_8}$ under the automorphism θ (cf. (3.8)). Then $\omega^{\pm}_{\lambda_j} \in V^+_{\sqrt{2}E_8}$ and $V^+_{\sqrt{2}E_8}$ is isomorphic to a code VOA M_D, where D is the second order Reed-Müller code $RM(4,2)$ of length 16. Note that $\dim RM(4,2) = 11$ and the dual code of $RM(4,2)$ is the first order Reed-Müller code $RM(4,1)$ with the generating matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Let H^+ and H^- be the subcodes of D whose supports are contained in the positions corresponding to $\{\omega^+_{\lambda_j} \mid 1 \le j \le 8\}$ and $\{\omega^-_{\lambda_j} \mid 1 \le j \le 8\}$, respectively. Then H^+ and H^- are both isomorphic to the $[8,4,4]$ Hamming code H_8. The conformal vectors $\hat{e}^0_\delta$ and $\hat{e}^1_\delta$ are actually the conformal vectors s_δ constructed by Miyamoto [17] using the Hamming code VOAs M_{H^+} and M_{H^-}, respectively.

Proposition 3.4. *The set $\{\hat{e}^0_\delta, \hat{e}^1_\zeta \mid \delta, \zeta \in {\mathbb{Z}_2}^8/H_8, \delta, \zeta$ are even$\}$ is a Virasoro frame of $V^+_{\sqrt{2}E_8}$. Moreover, $V^+_{\sqrt{2}E_8} \cong M_{RM(4,2)}$ with respect to this frame, where $M_{RM(4,2)}$ denotes the code VOA associated with the second order Reed-Müller code $RM(4,2)$.*

Proof. The first assertion follows from Lemma 3.3. As mentioned above, we know that $V^+_{\sqrt{2}E_8} \cong M_D$ with respect to the frame $\{\omega^{\pm}_{\lambda_j} \mid 1 \le j \le 8\}$, where $D \cong RM(4,2)$. It contains a subalgebra isomorphic to $M_{H^+} \otimes M_{H^-}$. For convenience, we arrange the positions of $\{\omega^{\pm}_{\lambda_j}\}$ so that the support $\operatorname{supp} H^+$ of H^+ is $(1^8, 0^8)$ and the support $\operatorname{supp} H^-$ of H^- is $(0^8, 1^8)$. Let $\{\beta_0, \beta_1, \dots, \beta_7\}$ with $\beta_0 = 0$ be a complete set of coset representatives of $D/(H^+ \oplus H^-)$. Then

$$V^+_{\sqrt{2}E_8} \cong M_{H^+ \oplus H^-} \oplus \bigoplus_{i=1}^{7} M_{\beta_i + (H^+ \oplus H^-)}.$$

By a result of Miyamoto [17], $M_{H^+ \oplus H^-}$ is still isomorphic to the code VOA $M_{H^+ \oplus H^-}$ associated with $H^+ \oplus H^-$ with respect to the frame $\{\hat{e}^0_\delta, \hat{e}^1_\zeta \mid \delta, \zeta \in {\mathbb{Z}_2}^8/H_8, \delta, \zeta$are even$\}$. Moreover, we know that $(1^8, 0^8)$ and $(0^8, 1^8)$ are contained in the dual code of D. Thus $\langle (1^8, 0^8), \beta_i \rangle = \langle (0^8, 1^8), \beta_i \rangle = 0$ for all i. Let β^+ and β^- be such that $\operatorname{supp}\beta^+ \subset \operatorname{supp} H^+$, $\operatorname{supp}\beta^- \subset \operatorname{supp} H^-$, and $\beta_i = \beta^+ + \beta^-$. Then $M_{\beta_i + (H^+ \oplus H^-)} \cong M_{\beta^+ + H^+} \otimes M_{\beta^- + H^-}$ and both $M_{\beta^+ + H^+}$ and $M_{\beta^- + H^-}$ are of integral weight. Hence, by [17], $M_{\beta_i + (H^+ \oplus H^-)}$ is again isomorphic to $M_{\beta_i + (H^+ \oplus H^-)}$ with respect to the frame $\{\hat{e}^0_\delta, \hat{e}^1_\zeta \mid \delta, \zeta \in {\mathbb{Z}_2}^8/H_8, \delta, \zeta$are even$\}$ and thus we still have $V^+_{\sqrt{2}E_8} \cong M_D$. □

Now let

$$\begin{aligned} \hat{e} &= \hat{e}^0_{\mathbf{0}} = \frac{1}{16}\omega + \frac{1}{32} \sum_{\alpha \in \Phi^+(E_8)} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}), \\ \hat{f} &= \sigma \hat{e}, \end{aligned} \tag{3.12}$$

where σ is the automorphism defined by (3.5). These conformal vectors of central charge 1/2 play an important role for the rest of the paper.

Let Φ be the root system of $L = L(i)$. Let $H_j = \{\alpha \in j\alpha_i + L \mid \langle \alpha, \alpha \rangle = 2\}$ be the set of all roots in the coset $j\alpha_i + L$ for $j = 1, \dots, n_i - 1$. Then

$$\Phi(E_8) = \Phi \cup \bigcup_{j=1}^{n_i-1} H_j.$$

We introduce weight 2 elements X^j, namely,

$$X^j = \sum_{\alpha \in H_j} e^{\sqrt{2}\alpha}, \quad j = 1, \dots, n_i - 1. \tag{3.13}$$

Then

$$\begin{aligned} \hat{e} &= \frac{1}{16}\omega + \frac{1}{32}\Big(\sum_{\alpha\in\Phi} e^{\sqrt{2}\alpha} + \sum_{j=1}^{n_i-1} X^j\Big), \\ \hat{f} &= \frac{1}{16}\omega + \frac{1}{32}\Big(\sum_{\alpha\in\Phi} e^{\sqrt{2}\alpha} + \sum_{j=1}^{n_i-1} \xi^j X^j\Big), \end{aligned} \tag{3.14}$$

where $\xi = e^{2\pi\sqrt{-1}/n_i}$ is a primitive n_i-th root of unity.

Lemma 3.5. (1) $X^j \in U$, $j = 1, \dots, n_i - 1$.
(2) $\hat{e}, \hat{f} \in U$.

Proof. Let s^k be defined as in (3.9). Then by a similar argument as in the proof of Proposition 2.2, we can verify that $(s^k)_1 X^j = 0$ and $(s^k)_1 \hat{e} = 0$ for $k = 1, \dots, l$. Thus $X^j, \hat{e} \in U$ by the definition (3.10) of U. Since σ leaves U invariant, we also have $\hat{f} \in U$. □

Remark 3.6. The Weyl group $W(E_8)$ of the root system of type E_8 acts naturally on the lattice VOA $V_{\sqrt{2}E_8}$ and $\hat{e}$ is the only conformal vector among $\hat{e}^0_\delta, \hat{e}^1_\zeta$ which is fixed by $W(E_8)$. The conformal vector $\hat{f}$ is fixed by the Weyl group $W(\Phi) = W(\Phi_1) \times \cdots \times W(\Phi_l)$ of the root system $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_l$ of $L = L(i)$. The conformal vector $\hat{e}$ is also fixed by the automorphism θ (cf. (3.8)). However, $\hat{f}$ is not fixed by θ in general.

Theorem 3.7. *Let $\hat{e}, \hat{f}$ be defined as in* (3.12). *Then*

$$\langle \hat{e}, \hat{f} \rangle = \begin{cases} 1/4 & \text{if } i = 0, \\ 1/32 & \text{if } i = 1, \\ 13/2^{10} & \text{if } i = 2, \\ 1/2^7 & \text{if } i = 3, \\ 3/2^9 & \text{if } i = 4, \\ 5/2^{10} & \text{if } i = 5, \\ 1/2^8 & \text{if } i = 6, \\ 0 & \text{if } i = 7, \\ 1/2^8 & \text{if } i = 8. \end{cases} \tag{3.15}$$

In other words, the values of $\langle \hat{e}, \hat{f} \rangle$ are exactly the values given in McKay's diagram (1.1).

Proof. By (3.14), we can easily obtain that

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}} \Big(|\Phi| + \sum_{j=1}^{n_i - 1} \xi^j |H_j| \Big),$$

where $H_j = \{\alpha \in j\alpha_i + L \mid \langle \alpha, \alpha \rangle = 2\}$.

If $i = 0$, then $n_0 = 1$ and $|\Phi| = 240$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{240}{2^{10}} = \frac{1}{4}.$$

If $i = 1$, then $n_1 = 2$, $|\Phi| = |\Phi(A_1)| + |\Phi(E_7)| = 128$, and $|H_1| = 112$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(128 - 112) = \frac{1}{32}.$$

If $i = 2$, then $n_2 = 3$, $|\Phi| = |\Phi(A_2)| + |\Phi(E_6)| = 78$, and $|H_1| = |H_2| = 81$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(78 - 81) = \frac{13}{2^{10}}.$$

If $i = 3$, then $n_3 = 4$, $|\Phi| = |\Phi(A_3)| + |\Phi(D_5)| = 52$, $|H_1| = |H_3| = 64$, and $|H_2| = 60$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(52 - 60) = \frac{1}{2^7}.$$

If $i = 4$, then $n_4 = 5$, $|\Phi| = |\Phi(A_4)| + |\Phi(A_4)| = 40$, and $|H_1| = |H_2| = |H_3| = |H_4| = 50$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(40 - 50) = \frac{3}{2^9}.$$

If $i = 5$, then $n_5 = 6$, $|\Phi| = |\Phi(A_1)| + |\Phi(A_2)| + |\Phi(A_5)| = 38$, $|H_1| = |H_5| = 36$, $|H_2| = |H_4| = 45$, and $|H_3| = 40$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(38 + 36 - 45 - 40) = \frac{5}{2^{10}}.$$

If $i = 6$, then $n_6 = 4$, $|\Phi| = |\Phi(A_1)| + |\Phi(A_7)| = 58$, $|H_1| = |H_3| = 56$, and $|H_2| = 70$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(58 - 70) = \frac{1}{2^8}.$$

If $i = 7$, then $n_7 = 2$, $|\Phi| = |\Phi(D_8)| = 112$, and $|H_1| = 128$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(112 - 128) = 0.$$

If $i = 8$, then $n_8 = 3$, $|\Phi| = |\Phi(A_8)| = 72$, and $|H_1| = |H_2| = 84$. Hence

$$\langle \hat{e}, \hat{f} \rangle = \frac{1}{2^6} + \frac{1}{2^{10}}(72 - 84) = \frac{1}{2^8}.$$

Thus we have proved the theorem. □

Remark 3.8. The same result still holds if we replace $\hat{e}$ by $\hat{e}^{\epsilon}_{\delta}$ and $\hat{f} = \sigma\hat{e}$ by $\sigma\hat{e}^{\epsilon}_{\delta}$ for any $\epsilon = 0, 1$ and $\delta \in {\mathbb{Z}_2}^8$.

4. Miyamoto's τ-involutions and the canonical automorphism σ

Let V be a VOA. If V contains a conformal vector w of central charge $1/2$ such that the subalgebra $\mathrm{Vir}(w)$ generated by w is isomorphic to the Virasoro VOA $L(1/2,0)$, then an automorphism τ_w of V with $(\tau_w)^2=1$ can be defined. Indeed, V is a direct sum of irreducible $\mathrm{Vir}(w)$-modules. Denote by W_h the sum of all irreducible direct summands which are isomorphic to $L(1/2,h)$. Then τ_w is defined to be 1 on $W_0\oplus W_{1/2}$ and -1 on $W_{1/16}$ (cf. [16, 18]). Thus τ_w is the identity if V has no irreducible direct summand isomorphic to $L(1/2,1/16)$. We call τ_w the Miyamoto involution or the τ-involution associated with w.

In this section, we shall study the relationship between the canonical automorphism σ and the Miyamoto involutions $\tau_{\hat{e}}$, $\tau_{\sigma\hat{e}},\dots$ and $\tau_{\sigma^{n_i-1}\hat{e}}$. Let us recall two conformal vectors $\hat{e}$ and $\hat{f}$ of central charge $1/2$ defined by (3.12) and two automorphisms σ and θ introduced in Subsection 3.1.

Lemma 4.1. *As automorphisms of $V_{\sqrt{2}E_8}$, $\tau_{\hat{e}}=\theta$.*

Proof. By Proposition 3.4, we know that $\{\hat{e}^0_\delta,\hat{e}^1_\zeta\,|\,\delta,\zeta\in {\mathbb{Z}_2}^8/H_8,\ \delta,\zeta \text{ even}\}$ is a Virasoro frame of $V^+_{\sqrt{2}E_8}$ and, with respect to this frame, $V^+_{\sqrt{2}E_8}$ is a code VOA isomorphic to $M_{RM(4,2)}$. Therefore, $\tau_{\hat{e}}|_{V^+_{\sqrt{2}E_8}}=\mathrm{id}$. On the other hand,

$$\begin{aligned}\hat{e}_1\gamma(-1)\cdot 1&=\frac{1}{16}\omega_1\gamma(-1)\cdot 1+\frac{1}{32}\sum_{\alpha\in\Phi^+(E_8)}(e^{\sqrt{2}\alpha}+e^{-\sqrt{2}\alpha})_1\gamma(-1)\cdot 1\\&=\frac{1}{16}\gamma(-1)\cdot 1\end{aligned}$$

for any $\gamma\in\sqrt{2}E_8$. By the definition of $\tau_{\hat{e}}$, this implies that $\tau_{\hat{e}}(\gamma(-1)\cdot 1)=-\gamma(-1)\cdot 1$. Then $\tau_{\hat{e}}|_{V^-_{\sqrt{2}E_8}}=-\mathrm{id}$, since $V^-_{\sqrt{2}E_8}$ is an irreducible $V^+_{\sqrt{2}E_8}$-module. Hence $\tau_{\hat{e}}=\theta$ as automorphisms of $V_{\sqrt{2}E_8}$. □

Theorem 4.2. *As automorphisms of $V_{\sqrt{2}E_8}$, $\tau_{\hat{e}}\tau_{\hat{f}}=(\sigma^{-1})^2=e^{2\pi\sqrt{-1}\boldsymbol{\beta}(0)}$ and thus $|\tau_{\hat{e}}\tau_{\hat{f}}|=n_i$ if n_i is odd and $|\tau_{\hat{e}}\tau_{\hat{f}}|=n_i/2$ if n_i is even.*

Proof. Since $\hat{f}=\sigma\hat{e}$, we have $\tau_{\hat{f}}=\sigma\tau_{\hat{e}}\sigma^{-1}$. By (3.5) and the preceding lemma, we also have $\tau_{\hat{e}}\sigma\tau_{\hat{e}}=\theta\sigma\theta=\sigma^{-1}$. Hence the assertion holds by (3.7). □

Next, we shall extend $\tau_{\hat{e}},\tau_{\hat{f}}$, and σ to the Leech lattice VOA V_Λ. According to the presentation (3.11) of $\sqrt{2}E_8$, the dual lattice $\mathcal{L}$ of $\sqrt{2}E_8$ is given by

$$\mathcal{L}=\{(a_1,\dots,a_8)\in\frac{1}{2}\mathbb{Z}^8\mid 2(a_1,\dots,a_8)\in H_8\mod 2\}.$$

Note that $|\mathcal{L}/\sqrt{2}E_8|=2^8$. Note also that

$$V_{\mathcal{L}}=S(\mathfrak{h}^-_{\mathbb{Z}})\otimes\mathbb{C}\{\mathcal{L}\}\cong\bigoplus_{\alpha+\sqrt{2}E_8\in\mathcal{L}/\sqrt{2}E_8}V_{\alpha+\sqrt{2}E_8}$$

as a module of $V_{\sqrt{2}E_8}$.

For any coset $\alpha+\sqrt{2}E_8$ of $\sqrt{2}E_8$ in $\mathcal{L}$, one can always find a coset representative α whose square norm is minimum in the coset such that α is in one of the following forms:

$$\begin{aligned}&(0^8),\quad (1,0^7),\quad (1^2,0^6),\quad ((1/2)^4,0^4),\\ &((1/2)^3,-1/2,0^4),\quad ((1/2)^2,(-1/2)^2,0^4),\quad ((1/2)^4,1,0^3),\\ &((1/2)^3,-1/2,1,0^3),\quad ((1/2)^8),\quad ((1/2)^7,-1/2),\quad ((1/2)^6,(-1/2)^2).\end{aligned} \tag{4.1}$$

The square norm $\langle\alpha,\alpha\rangle$ of such α is 0, 1, or 2. Moreover, if $\langle\alpha,\alpha\rangle=2$, then α can be written as a sum $\alpha=a+b$, where $a,b\in\mathcal{L}$ are in the above forms with $\langle a,a\rangle=\langle b,b\rangle=1$ and $\langle a,b\rangle=0$. In particular, the minimal weight of the irreducible module $V_{\alpha+\sqrt{2}E_8}$ is either 1/2 or 1 for $\alpha\notin\sqrt{2}E_8$.

Now $\sigma=e^{-\pi\sqrt{-1}\boldsymbol{\beta}(0)}$ (cf. (3.7)) acts on $V_{\mathcal{L}}$ as an automorphism of order $2n_i$. The τ-involution $\tau_{\hat{e}}$ also acts on $V_{\mathcal{L}}$. In fact, $V_{\alpha+\sqrt{2}E_8}$ is $\tau_{\hat{e}}$-invariant for any coset $\alpha+\sqrt{2}E_8$ of $\sqrt{2}E_8$ in $\mathcal{L}$.

Lemma 4.3. *For any $x\in\mathcal{L}$ with $\langle x,x\rangle=1$, $\tau_{\hat{e}}(e^x)=-e^{-x}$.*

Proof. If $\langle\gamma,\gamma\rangle=4$ and $\langle\gamma+x,\gamma+x\rangle=1$ for some $\gamma\in\sqrt{2}E_8$, then $\langle\gamma,x\rangle=-2$ and $\gamma+x=-x$. Thus, by the definition of $\hat{e}$ it follows that

$$\hat{e}_1e^x=\frac{1}{16}\Big(\frac{1}{2}e^x\Big)+\frac{1}{32}e^{-x}\quad\text{and}\quad \hat{e}_1e^{-x}=\frac{1}{16}\Big(\frac{1}{2}e^{-x}\Big)+\frac{1}{32}e^{x}.$$

Therefore, $\hat{e}_1(e^x+e^{-x})=\frac{1}{16}(e^x+e^{-x})$ and $\hat{e}_1(e^x-e^{-x})=0$. Hence

$$\tau_{\hat{e}}(e^x+e^{-x})=-(e^x+e^{-x})\text{ and }\tau_{\hat{e}}(e^x-e^{-x})=e^x-e^{-x}$$

by the definition of $\tau_{\hat{e}}$, and so $\tau_{\hat{e}}(e^x)=-e^{-x}$. □

Lemma 4.4. *Let $\alpha+\sqrt{2}E_8$ be a coset of $\sqrt{2}E_8$ in $\mathcal{L}$. Then for any $u\in V_{\alpha+\sqrt{2}E_8}$, $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(u)=\sigma^{-1}(u)$.*

Proof. We have $V_{\alpha+\sqrt{2}E_8}=\mathrm{span}_{\mathbb{C}}\{v_ne^\alpha\mid v\in V_{\sqrt{2}E_8},\,n\in\mathbb{Z}\}$, since $V_{\alpha+\sqrt{2}E_8}$ is an irreducible $V_{\sqrt{2}E_8}$-module. If $\langle\alpha,\alpha\rangle=1$, then we know that $\tau_{\hat{e}}(e^\alpha)=-e^{-\alpha}$ by Lemma 4.3. Thus $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(e^\alpha)=\sigma^{-1}(e^\alpha)$ and so

$$\begin{aligned}\tau_{\hat{e}}\sigma\tau_{\hat{e}}(v_ne^\alpha)&=(\tau_{\hat{e}}\sigma\tau_{\hat{e}}(v))_n\,(\tau_{\hat{e}}\sigma\tau_{\hat{e}}(e^\alpha))\\&=\sigma^{-1}(v)_n\sigma^{-1}(e^\alpha)\\&=\sigma^{-1}(v_ne^\alpha)\end{aligned}$$

for any $v\in V_{\sqrt{2}E_8}$ by Lemma 4.1.

If $\langle\alpha,\alpha\rangle=2$, then $\alpha=a+b$ for some vectors a,b in the forms of (4.1) with $\langle a,a\rangle=\langle b,b\rangle=1$ and $\langle a,b\rangle=0$. In this case, $e^\alpha=(e^a)_{-1}e^b$ and we still have $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(e^\alpha)=\sigma^{-1}(e^\alpha)$. Thus for any $v\in V_{\sqrt{2}E_8}$,

$$\tau_{\hat{e}}\sigma\tau_{\hat{e}}(v_ne^\alpha)=\sigma^{-1}(v)_n\sigma^{-1}(e^\alpha)=\sigma^{-1}(v_ne^\alpha)$$

as required. □

As a consequence, we have the following proposition.

Proposition 4.5. *For any $u\in V_{\mathcal{L}}$, $\tau_{\hat{e}}\sigma\tau_{\hat{e}}(u)=\sigma^{-1}(u)$. Hence $\tau_{\hat{e}}\tau_{\hat{f}}=(\sigma^{-1})^2=e^{2\pi\sqrt{-1}\boldsymbol{\beta}(0)}$ as automorphisms of $V_{\mathcal{L}}$.*

Now we discuss the situation in the Leech lattice VOA V_Λ. First let us recall the following theorem [5, Theorem 4.1] (see also [10, 13]).

Theorem 4.6. *For any even unimodular lattice N of rank 24, there is at least one (in general many) isometric embedding of $\sqrt{2}N$ into the Leech lattice Λ.*

It is well known (cf. [10]) that the Leech lattice Λ can be constructed by "Construction A" for $\mathbb{Z}_4$-codes of length 24. In fact,

$$\Lambda = A_4(\mathcal{C}) = \frac{1}{2}\{x \in \mathbb{Z}^{24} \mid x \equiv c \mod 4 \quad \text{for some } c \in \mathcal{C}\}$$

for some type II self-dual $\mathbb{Z}_4$-code $\mathcal{C}$ of length 24. By [10], $\mathcal{C}$ can be taken to be the $\mathbb{Z}_4$-code having the generating matrix (4.2):

$$\begin{pmatrix}
2222 & 0000 & 0000 & 0000 & 0000 & 0000 \\
0022 & 2200 & 0000 & 0000 & 0000 & 0000 \\
0000 & 0022 & 2020 & 0000 & 0000 & 0000 \\
0000 & 0000 & 0202 & 2020 & 0000 & 0000 \\
0000 & 0000 & 0000 & 0202 & 2002 & 0000 \\
2020 & 2020 & 0000 & 0000 & 0000 & 0000 \\
0000 & 0220 & 2200 & 0000 & 0000 & 0000 \\
0000 & 0000 & 2002 & 2002 & 0000 & 0000 \\
0000 & 0000 & 0000 & 0022 & 2020 & 0000 \\
2000 & 2000 & 2000 & 2000 & 2000 & 2000 \\
1111 & 1111 & 2000 & 2000 & 0000 & 0000 \\
2000 & 1111 & 1111 & 0000 & 2000 & 0000 \\
0000 & 0000 & 1111 & 1111 & 2000 & 2000 \\
2000 & 0000 & 2000 & 1111 & 1111 & 0000 \\
2000 & 2000 & 0000 & 0000 & 1111 & 1111 \\
3012 & 1010 & 1001 & 1001 & 1100 & 1100 \\
3201 & 1001 & 1100 & 1100 & 1010 & 1010
\end{pmatrix}. \tag{4.2}$$

For any $\mathbb{Z}_4$-code C of length n, one can obtain a binary code

$$B(C) = \{(b_1, \dots, b_n) \in {\mathbb{Z}_2}^n \mid (2b_1, \dots, 2b_n) \in C\},$$

where $2b_j$ should be considered as $0 \in \mathbb{Z}_4$ if $b_j = 0 \in \mathbb{Z}_2$ and $2 \in \mathbb{Z}_4$ if $b_j = 1 \in \mathbb{Z}_2$. Moreover, the lattice

$$L_{B(C)} = \{x \in \mathbb{Z}^n \mid x \in B(C) \mod 2\}$$

is a sublattice of $A_4(C)$. In the case for $C = \mathcal{C}$, the binary code $B(\mathcal{C})$ contains a subcode isomorphic to $H_8 \oplus H_8 \oplus H_8$. Thus by (3.11), we have an explicit embedding of $\sqrt{2}{E_8}^3$ into the Leech lattice Λ.

Now let $\sqrt{2}{E_8}^3 \longrightarrow \Lambda$ be any embedding of $\sqrt{2}{E_8}^3$ into the Leech lattice $\Lambda \subset \mathcal{L}^3$. Let $\tilde{\boldsymbol{\beta}} = \sqrt{2}(\mathbf{a}, 0, 0) \in \mathcal{L}^3$, where $\mathbf{a}$ is defined as in (3.6). Define $\tilde{\sigma} : (V_{\mathcal{L}})^{\otimes 3} \to (V_{\mathcal{L}})^{\otimes 3}$ by

$$\tilde{\sigma} = \sigma \otimes 1 \otimes 1 = e^{-\pi\sqrt{-1}\tilde{\boldsymbol{\beta}}(0)}.$$

Then $\tilde{\sigma}$ is an automorphism of V_Λ. Moreover, the following theorem holds.

Theorem 4.7. *Let $\tilde{\boldsymbol{\beta}}$ and $\tilde{\sigma}$ be defined as above. Then as automorphisms of V_Λ, $\tau_{\hat{e}}\tau_{\hat{f}} = (\tilde{\sigma}^{-1})^2 = e^{2\pi\sqrt{-1}\tilde{\boldsymbol{\beta}}(0)}$ and $|\tau_{\hat{e}}\tau_{\hat{f}}| = n_i$ for any $i = 0, 1, \dots, 8$.*

5. Correspondence with Conway's axes

Recall the elements $\tilde{\omega}^k$ and X^j defined by (3.9) and (3.13). It turns out that the Griess algebra U_2 of U is generated by $\hat{e}$ and $\hat{f}$, and is of dimension $l+n_i-1$ with basis $\tilde{\omega}^k, 1 \le k \le l$ and $X^j, 1 \le j \le n_i - 1$ (see [12] for details). We can verify that the Griess algebra U_2 coincides with the algebra described in Conway [1, Table 3]. In [1], it is shown that for each $2A$-involution of the Monster simple group, there is a unique idempotent in the Monstrous Griess algebra $V_2^\natural$ corresponding to the involution. Such an idempotent is called an axis. By Miyamoto [16], an axis is exactly half of a conformal vector of central charge $1/2$. Note that the product $t * t'$ and the inner product (t, t') of two axes t, t' in [1] are equal to $t \cdot t' = t_1 t'$ and $\langle t, t' \rangle / 2$, respectively in our notation. Let t_n be as in [1]. We denote t, u, v, and w of [1] by t_{2A}, u_{3A}, v_{4A}, and w_{5A}, respectively.

In each of the nine cases, we obtain an isomorphism of our Griess algebra U_2 to Conway's algebra generated by two axes through the following correspondence between our conformal vectors and Conway's axes:

$1A$ case. $\hat{e} \longleftrightarrow \frac{1}{32} t_0$.

$2A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{32} t_{2A}$.

$3A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1, 2, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{45} u_{3A}$.

$4A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 3, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{96} v_{4A}$.

$5A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 4, \quad \tilde{\omega}^1 - \tilde{\omega}^2 \longleftrightarrow -\frac{1}{35\sqrt{5}} w_{5A}$.

$6A$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 5, \quad \tilde{\omega}^2 \longleftrightarrow \frac{1}{32} t_{2A}, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{45} u_{3A}$.

$4B$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ 0 \le j \le 3, \quad \tilde{\omega}^1 \longleftrightarrow \frac{1}{32} t_{2A}$.

$2B$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1$.

$3C$ case. $\sigma^j \hat{e} \longleftrightarrow \frac{1}{32} t_j,\ j = 0, 1, 2$.

References

1. J. H. Conway, A simple construction for the Fisher-Griess Monster group, *Invent. Math.* **79** (1985), 513–540. MR0782233 (86h:20019)
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *ATLAS of Finite Groups*, Oxford Univ. Press, 1985. MR0827219 (88g:20025)
3. J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, Berlin-New York, 1988. MR0920369 (89a:11067)
4. C. Dong, Vertex algebras associated with even lattices, *J. Algebra* **161** (1993), 245–265. MR1245855 (94j:17023)
5. C. Dong, H. Li, G. Mason and S. P. Norton, Associative subalgebras of Griess algebra and related topics, *Proc. of the Conference on the Monster and Lie algebra at the Ohio State University*, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin-New York, 1998, pp. 27–42. MR1650629 (99k:17048)
6. I. B. Frenkel, Y. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104**, 1993. MR1142494 (94a:17007)
7. I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. **134**, Academic Press, New York, 1988. MR0996026 (90h:17026)
8. G. Glauberman and S. P. Norton, On McKay's connection between the affine E_8 diagram and the Monster, *CRM Proceedings and Lecture Notes*, Vol. **30**, Amer. Math. Soc., Providence, RI, 2001, pp. 37–42. MR1877755 (2002k:20024)
9. R. Griess, The Friendly Giant, *Invent. Math.* **69** (1982), 1–102. MR0671653 (84m:20024)

10. M. Harada and M. Kitazume, $\mathbb{Z}_4$-code constructions for the Niemeier lattices and their embeddings in the Leech lattice, *Europ. J. Combinatorics* **21** (2000), 473–485. MR1756153 (2001d:94042)
11. C. H. Lam and H. Yamada, Decomposition of the lattice vertex operator algebra $V_{\sqrt{2}A_l}$, *J. Algebra* **272** (2004), 614–624. MR2028073 (2004k:17049)
12. C. H. Lam, H. Yamada and H. Yamauchi, McKay's observation and vertex operator algebras generated by two conformal vectors of central charge 1/2, *IMRP Int. Math. Res. Pap.* **2005**, 117–181. MR2160172
13. J. Lepowsky and A. Meurman, An E_8-approach to the Leech lattice and the Conway group, *J. Algebra* **77** (1982), 484–504. MR0673130 (84a:10033)
14. J. Lepowsky and H.S. Li, Introduction to vertex operator algebras and their representations. Progress in Mathematics, 227. Birkhauser Boston, Inc., Boston, MA, 2004. MR2023933 (2004k:17050)
15. J. McKay, Graphs, singularities, and finite groups, *Proc. Symp. Pure Math.*, Vol. **37**, Amer. Math. Soc., Providence, RI, 1980, pp. 183–186. MR0604577 (82e:20014)
16. M. Miyamoto, Griess algebras and conformal vectors in vertex operator algebras, *J. Algebra* **179** (1996), 523–548. MR1367861 (96m:17052)
17. M. Miyamoto, Binary codes and vertex operator (super)algebras, *J. Algebra* **181** (1996), 207–222. MR1382033 (97a:17027)
18. M. Miyamoto, A new construction of the Moonshine vertex operator algebras over the real number field, *Ann. of Math.*, **159** (2004), no. 2, 535–596. MR2081435 (2005h:17052)
19. A. B. Zamolodchikov and V. A. Fateev, Nonlocal (parafermion) currents in two dimensional conformal quantum field theory and self-dual critical points in $\mathbb{Z}_N$-symmetric statistical systems, *Sov. Phys. JETP* **62** (1985), 215–225. MR0830910 (87f:82030)

Department of Mathematics, National Cheng Kung University, Tainan, Taiwan 701
E-mail address: chlam@mail.ncku.edu.tw

Department of Mathematics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan
E-mail address: yamada@math.hit-u.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, 153-8914, Japan
E-mail address: yamauchi@ms.u-tokyo.ac.jp