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BRAUER ALGEBRAS, SYMPLECTIC SCHUR ALGEBRAS AND SCHUR-WEYL DUALITY

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Abstract. In this paper we prove the Schur-Weyl duality between the symplectic group and the Brauer algebra over an arbitrary infinite field K. We show that the natural homomorphism from the Brauer algebra $B_n(-2m)$ to the endomorphism algebra of the tensor space $(K^{2m})^{\otimes n}$ as a module over the symplectic similitude group $GSp_{2m}(K)$ (or equivalently, as a module over the symplectic group $Sp_{2m}(K)$) is always surjective. Another surjectivity, that of the natural homomorphism from the group algebra for $GSp_{2m}(K)$ to the endomorphism algebra of $(K^{2m})^{\otimes n}$ as a module over $B_n(-2m)$, is derived as an easy consequence of S. Oehms's results [S. Oehms, J. Algebra (1) 244 (2001), 19–44].

1. Introduction

Let K be an infinite field. Let $m, n \in \mathbb{N}$. Let U be an m-dimensional K-vector space. The natural left action of the general linear group $GL(U)$ on $U^{\otimes n}$ commutes with the right permutation action of the symmetric group $\mathfrak{S}_n$. Let φ, ψ be the natural representations

$$\varphi : (K\mathfrak{S}_n)^{\mathrm{op}} \to \mathrm{End}_K\big(U^{\otimes n}\big), \quad \psi : KGL(U) \to \mathrm{End}_K\big(U^{\otimes n}\big),$$

respectively. The well-known Schur-Weyl duality (see [Sc], [W], [CC], [CL]) says that

(a) $\varphi\big(K\mathfrak{S}_n\big) = \mathrm{End}_{KGL(U)}\big(U^{\otimes n}\big)$, and if $m \geq n$, then φ is injective, and hence an isomorphism onto $\mathrm{End}_{KGL(U)}\big(U^{\otimes n}\big)$,

(b) $\psi\big(KGL(U)\big) = \mathrm{End}_{K\mathfrak{S}_n}\big(U^{\otimes n}\big)$,

(c) if $\mathrm{char}\, K = 0$, then there is an irreducible $(KGL(U), (K\mathfrak{S}_n)^{\mathrm{op}})$-bimodule decomposition

$$U^{\otimes n} = \bigoplus_{\substack{\lambda=(\lambda_1,\lambda_2,\cdots)\vdash n \\ \ell(\lambda)\leq m}} \Delta_\lambda \otimes S^\lambda,$$

where Δ_λ (resp., S^λ) denotes the irreducible $KGL(U)$-module (resp., irreducible $K\mathfrak{S}_n$-module) associated to λ, and $\ell(\lambda)$ denotes the largest integer i such that $\lambda_i \neq 0$.

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Let τ be the automorphism of $K\mathfrak{S}_n$ which is defined on generators by $\tau(s_i) = -s_i$ for each $1 \le i \le n-1$. Then (by using this automorphism) it is easy to see that the same Schur-Weyl duality still holds if one replaces the right permutation action of $\mathfrak{S}_n$ by the right sign permutation action, i.e.,

$$(v_{i_1} \otimes \cdots \otimes v_{i_n})s_j := -(v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}),$$

for any $1 \le j \le n-1$ and any $v_{i_1}, \cdots, v_{i_n} \in U$.

In the case of $K = \mathbb{C}$, there are also Schur-Weyl dualities for other classical groups—symplectic groups and orthogonal groups. In this paper, we shall consider only the symplectic case.[1] Recall that symplectic groups are defined by certain bilinear forms $(\,,\,)$ on vector spaces. Let V be a $2m$-dimensional K-vector space equipped with a nondegenerate skew-symmetric bilinear form $(\,,\,)$. Then (see [Gri], [Dt, Section 4]) the symplectic similitude group (resp., the symplectic group) relative to $(\,,\,)$ is

$$GSp(V) := \Big\{ g \in GL(V) \;\Big|\; \exists\, 0 \ne d \in K, \text{ such that } (gv, gw) = d(v,w),\ \forall\, v, w \in V \Big\}$$

$$\Big(\text{resp., } Sp(V) := \Big\{ g \in GL(V) \;\Big|\; (gv, gw) = (v,w),\ \forall\ v, w \in V \Big\}\Big).$$

By restriction from $GL(V)$, we get natural left actions of $GSp(V)$ and $Sp(V)$ on $V^{\otimes n}$. Again we denote by ψ the natural K-algebra homomorphism

$$\psi : KGSp(V) \to \mathrm{End}_K\big(V^{\otimes n}\big),$$
$$\psi : KSp(V) \to \mathrm{End}_K\big(V^{\otimes n}\big).$$

Note that if $0 \ne d \in K$ is such that $(gv, gw) = d(v,w)$ for any $v, w \in V$, then $\big((\sqrt{d^{-1}}g)v, (\sqrt{d^{-1}}g)w\big) = (v,w)$ for any $v, w \in V$. Therefore, if K is large enough such that $\sqrt{d} \in K$ for any $d \in K$, then $g \in GSp(V)$ implies that $(a\,\mathrm{id}_V)g \in Sp(V)$ for some $0 \ne a \in K$. In that case,

$$\begin{aligned}\psi(g) &= \psi\big((a^{-1}\,\mathrm{id}_V)(a\,\mathrm{id}_V)g\big) = \psi\big(a^{-1}\,\mathrm{id}_V\big)\psi\big((a\,\mathrm{id}_V)g\big)\\ &= \big(a^{-n}\,\mathrm{id}_{V^{\otimes n}}\big)\psi\big((a\,\mathrm{id}_V)g\big) = a^{-n}\psi\big((a\,\mathrm{id}_V)g\big).\end{aligned}$$

It follows that

$$\psi\big(KSp(V)\big) = \psi\big(KGSp(V)\big) \tag{1.1}$$

provided K is large enough.

In the setting of Schur-Weyl duality for the symplectic group, the symmetric group $\mathfrak{S}_n$ should be replaced by Brauer algebras (introduced in [B]). Recall that the Brauer algebra $B_n(x)$ over a Noetherian integral domain R (with parameter $x \in R$) is a unital R-algebra with generators $s_1, \cdots, s_{n-1}, e_1, \cdots, e_{n-1}$ and relations (see [E]):

$$\begin{aligned}
&s_i^2 = 1,\ e_i^2 = xe_i,\ e_is_i = e_i = s_ie_i, && \forall\, 1 \le i \le n-1,\\
&s_is_j = s_js_i,\ s_ie_j = e_js_i,\ e_ie_j = e_je_i, && \forall\, 1 \le i < j-1 \le n-2,\\
&s_is_{i+1}s_i = s_{i+1}s_is_{i+1},\ e_ie_{i+1}e_i = e_i,\ e_{i+1}e_ie_{i+1} = e_{i+1}, && \forall\, 1 \le i \le n-2,\\
&s_ie_{i+1}e_i = s_{i+1}e_i,\ e_{i+1}e_is_{i+1} = e_{i+1}s_i, && \forall\, 1 \le i \le n-2.
\end{aligned}$$

Note that $B_n(x)$ was originally defined as the linear space with basis the set of all Brauer n-diagrams, graphs on $2n$ vertices and n edges with the property that

[1] In this paper, we will use the results by Oehms and also by Donkin on symplectic Schur algebras. To deal with the orthogonal case, one needs analogous results for orthogonal Schur algebras, which are not presently available.

every vertex is incident to precisely one edge. One usually thinks of the vertices as arranged in two rows of n each, the top and bottom rows. Label the vertices in each row of an n-diagram by the indices $1, 2, \cdots, n$ from left to right. Then s_i corresponds to the n-diagram with edges connecting vertices i (resp., $i+1$) on the top row with $i+1$ (resp., i) on the bottom row, and all other edges are vertical, connecting vertex k on the top and bottom rows for all $k \neq i, i+1$. e_i corresponds to the n-diagram with horizontal edges connecting vertices $i, i+1$ on the top and bottom rows, and all other edges are vertical, connecting vertex k on the top and bottom rows for all $k \neq i, i+1$. The multiplication is given by the linear extension of a product defined on diagrams. For more details, see [B], [GW].

There are right actions of Brauer algebras (with certain parameters) on tensor space. The definition of the actions depends on the choice of an orthogonal basis with respect to the defining bilinear form. Let δ_{ij} denote the value of the usual Kronecker delta. For any $1 \leq i \leq 2m$, set $i' := 2m+1-i$. We fix an ordered basis $\{v_1, v_2, \cdots, v_{2m}\}$ of V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}), \ \ (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall\ 1 \leq i, j \leq m.$$

For any $i, j \in \{1, 2, \cdots, 2m\}$, let

$$\epsilon_{ij} := \begin{cases} 1 & \text{if } j = i' \text{ and } i < j, \\ -1 & \text{if } j = i' \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

The right action of $B_n(-2m)$ on $V^{\otimes n}$ is defined on generators by

$$(v_{i_1} \otimes \cdots \otimes v_{i_n}) s_j := -(v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}),$$

$$(v_{i_1} \otimes \cdots \otimes v_{i_n}) e_j := \epsilon_{i_j i_{j+1}} v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \left(\sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}) \right) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_n}.$$

Let φ be the natural K-algebra homomorphism

$$\varphi : (B_n(-2m))^{\mathrm{op}} \to \mathrm{End}_K\big(V^{\otimes n}\big).$$

The following results are well known.

Theorem 1.2 ([B], [B1], [B2]). *1) The natural left action of $GSp(V)$ on $V^{\otimes n}$ commutes with the right action of $B_n(-2m)$. Moreover, if $K = \mathbb{C}$, then*

$$\varphi\big(B_n(-2m)\big) = \mathrm{End}_{\mathbb{C}GSp(V)}\big(V^{\otimes n}\big) = \mathrm{End}_{\mathbb{C}Sp(V)}\big(V^{\otimes n}\big),$$
$$\psi\big(\mathbb{C}GSp(V)\big) = \psi\big(\mathbb{C}Sp(V)\big) = \mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big).$$

2) If $K = \mathbb{C}$ and $m \geq n$, then φ is injective, and hence an isomorphism onto $\mathrm{End}_{\mathbb{C}GSp(V)}\big(V^{\otimes n}\big)$.

3) If $K = \mathbb{C}$, then there is an irreducible $(\mathbb{C}GSp(V), (B_n(-2m))^{\mathrm{op}})$-bimodule decomposition

$$V^{\otimes n} = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\substack{\lambda \vdash n-2f \\ \ell(\lambda) \leq m}} \Delta(\lambda) \otimes D(\lambda'),$$

where $\Delta(\lambda)$ (resp., $D(\lambda')$) denotes the irreducible $\mathbb{C}GSp(V)$-module (resp., the irreducible $B_n(-2m)$-module) corresponding to λ (resp., corresponding to λ'), and $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ denotes the conjugate partition of λ.

The aim of this work is to remove the restriction on K in part 1) and part 2) of the above theorem. We shall see that the following holds for any infinite field K.

Proposition 1.3. *For any infinite field K, $\psi\big(KGSp(V)\big) = \mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big)$.*

In fact, this is an easy consequence of [Oe, (6.1), (6.2), (6.3)] and [Dt, (3.2(b))]. The proof is given in Section 2. The main result of this paper is

Theorem 1.4. *Let K be an arbitrary infinite field. Then*

$$\varphi\big(B_n(-2m)\big) = \mathrm{End}_{KGSp(V)}\big(V^{\otimes n}\big) = \mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big),$$

and if $m \geq n$, then φ is also injective, and hence an isomorphism onto

$$\mathrm{End}_{KGSp(V)}\big(V^{\otimes n}\big).$$

Remark 1.5. 1) Note that when $m < n$, φ is in general not injective. For example, let $m = 2, n = 3, U = K^2, G = Sp_4(K)$. Then it is easy to check that the element $\alpha := (1 + s_1)(1 + s_2 + s_2s_1) + (1 + s_2 + s_1s_2)e_1(1 + s_2 + s_2s_1)$ lies in the kernel of $\varphi : B_3(-4) \to \mathrm{End}_{KSp_4(K)}(V^{\otimes 3})$. In fact, $\ker(\varphi) = K\alpha$.

2) It would be interesting to know if the quantized versions of Proposition 1.3 and Theorem 1.4 hold (see [BW], [CP] and [M]).

3) Although the paper [CC] should apply in the symplectic case (see [T]), our results provide an alternative approach (one which is rather more detailed).

2. The algebra $A^s_R(m)$

In this section, we shall show how Proposition 1.3 follows from results of [Oe, (6.1), (6.2), (6.3)] and [Dt, (3.2(b))].

We shall first introduce (following [Oe, Section 6]) a $\mathbb{Z}$-graded R-algebra $A^s_R(m)$ for any Noetherian integral domain R. Over an algebraically closed field, this algebra is isomorphic to the coordinate algebra of the symplectic monoid, and the dual of its n-th homogeneous summand is isomorphic to the symplectic Schur algebra introduced by S. Donkin ([Do2]).

Let R be a Noetherian integral domain. Let $x_{i,j}, 1 \leq i, j \leq 2m$ be $4m^2$ commuting indeterminates over R. Let $A_R(2m)$ be the free commutative R-algebra (i.e., polynomial algebra) in these $x_{i,j}, 1 \leq i, j \leq 2m$. Let I_R be the ideal of $A_R(2m)$ generated by elements of the form

$$\begin{cases} \sum_{k=1}^{2m} \epsilon_k x_{i,k} x_{j,k'}, & 1 \leq i \neq j' \leq 2m; \\ \sum_{k=1}^{2m} \epsilon_k x_{k,i} x_{k',j}, & 1 \leq i \neq j' \leq 2m; \\ \sum_{k=1}^{2m} \epsilon_k (x_{i,k} x_{i',k'} - x_{k,j} x_{k',j'}), & 1 \leq i, j \leq m. \end{cases} \tag{2.1}$$

The R-algebra $A_R(2m)/I_R$ will be denoted by $A^s_R(m)$. Write $c_{i,j}$ for the canonical image $x_{i,j} + I_R$ of $x_{i,j}$ in $A^s_R(m)$ $(1 \leq i, j \leq 2m)$. Then in $A^s_R(m)$ we have the relations

$$\begin{cases} \sum_{k=1}^{2m} \epsilon_k c_{i,k} c_{j,k'} = 0, & 1 \leq i \neq j' \leq 2m; \\ \sum_{k=1}^{2m} \epsilon_k c_{k,i} c_{k',j} = 0, & 1 \leq i \neq j' \leq 2m; \\ \sum_{k=1}^{2m} \epsilon_k (c_{i,k} c_{i',k'} - c_{k,j} c_{k',j'}) = 0, & 1 \leq i, j \leq m. \end{cases} \tag{2.2}$$

Note that $A_R(2m)$ is a graded algebra, $A_R(2m) = \bigoplus_{n \geq 0} A_R(2m, n)$, where the $A_R(2m, n)$ is the subspace spanned by the monomials of the form $x_{\underline{i},\underline{j}}$ for $(\underline{i}, \underline{j}) \in$

$I^2(2m,n)$, where

$$I(2m,n) := \{\underline{i} = (i_1, \cdots, i_n) \mid 1 \le i_j \le 2m, \forall\, j\},$$
$$I^2(2m,n) = I(2m,n) \times I(2m,n), \quad x_{\underline{i},\underline{j}} := x_{i_1,j_1} \cdots x_{i_n,j_n}.$$

Since I_R is a homogeneous ideal, $A^s_R(m)$ is graded too and $A^s_R(m) = \bigoplus_{n\ge 0} A^s_R(m,n)$, where $A^s_R(m,n)$ is the subspace spanned by the monomials of the form $c_{\underline{i},\underline{j}}$ for $(\underline{i},\underline{j}) \in I^2(2m,n)$, where

$$c_{\underline{i},\underline{j}} := c_{i_1,j_1} \cdots c_{i_n,j_n}.$$

By convention, throughout this paper, we identify the symmetric group $\mathfrak{S}_n$ with the set of maps acting on their arguments on the right. In other words, if $\sigma \in \mathfrak{S}_n$ and $a \in \{1,\ldots,n\}$ we write $(a)\sigma$ for the value of a under σ. This convention carries the consequence that, when considering the composition of two symmetric group elements, the leftmost map is the first to act on its argument. For example, we have $(1,2,3)(2,3) = (1,3)$ in the usual cycle notation.

Note that the symmetric group $\mathfrak{S}_n$ acts on the right on the set $I(2m,n)$ by the rule[2]

$$\underline{i}\sigma := (i_{(1)\sigma^{-1}}, \cdots, i_{(n)\sigma^{-1}}), \quad \sigma \in \mathfrak{S}_n.$$

It is clear (see [Dt]) that $A^s_R(m,n) \cong A_R(2m,n)/I_R(n)$, where $I_R(1) = 0$, and for $n \ge 2$, $I_R(n)$ is the R-submodule of $A_R(2m,n)$ generated by elements of the form

$$\begin{cases} \sum_{k=1}^{2m} \epsilon_k x_{(i_1,\cdots,i_n),(k,k',k_3,\cdots,k_n)}, \\ \sum_{k=1}^{2m} \epsilon_k x_{(k,k',i_3,\cdots,i_n),(j_1,\cdots,j_n)}, \\ \sum_{k=1}^{2m} \epsilon_k \big(x_{(i,i',i_3,\cdots,i_n),(k,k',j_3,\cdots,j_n)} - x_{(k,k',i_3,\cdots,i_n),(j,j',j_3,\cdots,j_n)}\big), \end{cases} \tag{2.3}$$

where $1 \le i,j \le m$, $\underline{i},\underline{j} \in I(2m,n)$ such that $i_1 \ne i'_2, j_1 \ne j'_2$.

Furthermore, if one defines

$$\Delta(x_{\underline{i},\underline{j}}) = \sum_{\underline{k}\in I(2m,n)} x_{\underline{i},\underline{k}} \otimes x_{\underline{k},\underline{j}}, \quad \varepsilon(x_{\underline{i},\underline{j}}) = \delta_{\underline{i},\underline{j}}, \ \forall\, \underline{i},\underline{j} \in I(2m,n), \forall\, n,$$

then the algebra $A_R(2m)$ becomes a graded bialgebra, and each $A_R(2m,n)$ is a subcoalgebra of $A_R(2m)$. Its linear dual $S_R(2m,n) := \mathrm{Hom}_R(A_R(2m,n),R)$ is the so-called *Schur algebra* over R (see [Gr]). Let $S^s_R(m,n) := \mathrm{Hom}_R(A^s_R(m,n),R)$. By [Oe, Section 6], $A^s_R(m,n)$ is in fact a quotient coalgebra of $A_R(2m,n)$; hence $S^s_R(m,n)$ is a subalgebra of $S_R(2m,n)$.

We define $(\underline{i},\underline{j}) \sim (\underline{u},\underline{v})$ if there exists some $\sigma \in \mathfrak{S}_n$ with $\underline{i}\sigma = \underline{u}, \underline{j}\sigma = \underline{v}$. Let $I^2(2m,n)/{\sim}$ be the set of orbits for the action of $\mathfrak{S}_n$ on $I^2(2m,n)$. For each $(\underline{i},\underline{j}) \in I^2(2m,n)/{\sim}$, we define $\xi_{\underline{i},\underline{j}} \in S_R(2m,n)$ by

$$\xi_{\underline{i},\underline{j}}(x_{\underline{u},\underline{v}}) = \begin{cases} 1, & \text{if } (\underline{i},\underline{j}) \sim (\underline{u},\underline{v}), \\ 0, & \text{otherwise}, \end{cases} \qquad \forall\, (\underline{u},\underline{v}) \in I^2(2m,n)/{\sim}.$$

The set $\{\xi_{\underline{i},\underline{j}} \mid (\underline{i},\underline{j}) \in I^2(2m,n)/{\sim}\}$ forms an R-basis of $S_R(2m,n)$. The natural action of $S_R(2m,n)$ on $V^{\otimes n}$ is given as follows:

$$\xi_{\underline{i},\underline{j}}\colon \quad V^{\otimes n} \to V^{\otimes n}$$
$$v_{\underline{a}} := v_{a_1} \otimes \cdots \otimes v_{a_n} \mapsto \sum_{\substack{\underline{b}\in I(2m,n),\\ (\underline{a},\underline{b})\sim(\underline{i},\underline{j})}} v_{\underline{b}}, \quad \forall\, \underline{a} := (a_1, \cdots, a_n) \in I(2m,n).$$

[2]This action is the so-called right place permutation action.

Let $\xi = \sum_{(\underline{i},\underline{j})\in I^2(2m,n)/\sim} a_{\underline{i},\underline{j}}\xi_{\underline{i},\underline{j}} \in S_R(2m,n)$. By (2.3), it is easy to see that $\xi \in S_R^s(m,n)$ if and only if

$$\begin{cases} \sum_{k=1}^{2m} \epsilon_k a_{(i_1,\cdots,i_n),(k,k',k_3,\cdots,k_n)} = 0, \\ \sum_{k=1}^{2m} \epsilon_k a_{(k,k',i_3,\cdots,i_n),(j_1,\cdots,j_n)} = 0, \\ \sum_{k=1}^{2m} \epsilon_k (a_{(i,i',i_3,\cdots,i_n),(k,k',j_3,\cdots,j_n)} - a_{(k,k',i_3,\cdots,i_n),(j,j',j_3,\cdots,j_n)}) = 0, \end{cases} \tag{2.4}$$

where $1 \le i,j \le m$, $\underline{i},\underline{j} \in I(2m,n)$ such that $i_1 \ne i_2', j_1 \ne j_2'$.

Let $R = K$ be an infinite field. Recall the ordered basis $\{v_1, v_2, \cdots, v_{2m}\}$ of V. Let $(\,,)$ be the unique (nondegenerate) skew-symmetric bilinear form on V such that

$$(v_i, v_j) = 0 = (v_{i'}, v_{j'}),\ \ (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall\ 1 \le i,j \le m.$$

This form is given (relative to the above ordered basis) by the block matrix

$$J := \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix},$$

where J_m is the unique anti-diagonal $m \times m$ permutation matrix. With respect to the above ordered basis of V, the group $GSp(V)$ may be identified with the group $GSp_{2m}(K)$ given by

$$GSp_{2m}(K) := \Big\{ A \in GL_{2m}(K) \ \Big|\ \exists\, 0 \ne d(A) \in K, \text{ such that } A^T JA = d(A)J \Big\}.$$

Let $M_{2m}(K)$ denote the affine algebraic monoid of $n \times n$ matrices over K. With respect to the above basis of V, the symplectic monoid $SpM(V)$, which by definition consists of the linear endomorphisms of V preserving the bilinear form up to any scalar (see [Dt, Section 4.2]), may be identified with

$$SpM_{2m}(K) := \Big\{ A \in M_{2m}(K) \ \Big|\ \exists\, d(A) \in K, \text{ such that } A^T JA = d(A)J \Big\}.$$

Let $\overline{K}$ be the algebraic closure of K. The coordinate algebra $\overline{K}[M_{2m}(\overline{K})]$ is isomorphic to $A_{\overline{K}}(2m) := A_K(2m) \otimes \overline{K}$. The coordinate algebra of $GL_{2m}(\overline{K})$ is isomorphic to $\overline{K}[\det^{-1}(x_{i,j})_{2m\times 2m}; x_{i,j}]_{1\le i,j\le 2m}$. The embedding $GSp_{2m}(K) \hookrightarrow GL_{2m}(K)$ induces a surjective map $\overline{K}[GL_{2m}(\overline{K})] \twoheadrightarrow \overline{K}[GSp_{2m}(\overline{K})]$. Denote by $A^{sy}_{\overline{K}}(m)$ (resp., $A^{sy}_{\overline{K}}(m,n)$) the image of $A_{\overline{K}}(2m)$ (resp., of $A_{\overline{K}}(2m,n)$) under this map. Then, by [Do2],

(1) $A^{sy}_{\overline{K}}(2m)$ is isomorphic to the coordinate algebra of $SpM_{2m}(\overline{K})$,
(2) $A^{sy}_{\overline{K}}(2m) = \bigoplus_{0\le n\in\mathbb{Z}} A^{sy}_{\overline{K}}(m,n)$, and the dimension of $A^{sy}_{\overline{K}}(m,n)$ is independent of the field K,
(3) the linear dual of $A^{sy}_{\overline{K}}(m,n)$, say, $S^{sy}_{\overline{K}}(m,n)$ is a generalized Schur algebra in the sense of [Do1].

The algebra $S^{sy}_{\overline{K}}(m,n)$ is called by S. Donkin the *symplectic Schur algebra*.

We define $A^{sy}_K(m)$ (resp., $A^{sy}_K(m,n)$) to be the image of $A_K(2m)$ (resp., of $A_K(2m,n)$) under the surjective map $\overline{K}[GL_{2m}(\overline{K})] \twoheadrightarrow \overline{K}[GSp_{2m}(\overline{K})]$. It is clear that

$$A^{sy}_K(m) \otimes \overline{K} = A^{sy}_{\overline{K}}(m), \quad A^{sy}_K(m,n) \otimes \overline{K} = A^{sy}_{\overline{K}}(m,n),$$

and hence $A^{sy}_K(2m) = \bigoplus_{0\le n\in\mathbb{Z}} A^{sy}_K(m,n)$.

On the other hand, by definition of $SpM_{2m}(K)$, it is easy to check that the defining relations (2.1) vanish on every matrix in $SpM_{2m}(K)$. It follows that there is an epimorphism of graded bialgebras from $A^s_K(m)$ onto $A^{sy}_K(m)$. Note that for each

$0 \leq n \in \mathbb{Z}$, the dimensions of both $A_K^s(m,n)$ (see [Oe, (6.1)]) and $A_K^{sy}(m,n)$ are independent of the field K. By [Dt, (9.5)], $A_{\mathbb{C}}^s(m,n) \cong A_{\mathbb{C}}^{sy}(m,n)$. So the two coalgebras always have the same dimension. It follows that $A_K^s(m,n) \cong A_K^{sy}(m,n)$ and $A_K^s(m) \cong A_K^{sy}(m)$. In particular, we have that $S_K^s(m,n) \cong S_K^{sy}(m,n)$. Therefore we have

Theorem 2.5 ([Oe, (6.2)][3])**.** *For any infinite field K, there is an isomorphism of graded bialgebras from $A_K^s(m)$ onto $A_K^{sy}(m)$. In particular, $A_K^s(m,n) \cong A_K^{sy}(m,n)$ and $S_K^s(m,n) \cong S_K^{sy}(m,n)$ for each $n \in \mathbb{N}$.*

As a $\mathbb{Z}$-submodule of $\mathrm{End}_{\mathbb{Z}}\big(V_{\mathbb{Z}}^{\otimes n}\big)$, the algebra $\mathrm{End}_{B_n(-2m)_{\mathbb{Z}}}\big(V_{\mathbb{Z}}^{\otimes n}\big)$ is a free $\mathbb{Z}$-module of finite rank. Oehms proved in [Oe, (6.3)] that the symplectic Schur algebra $S^s(m,n)$ is isomorphic to the centralizer algebra $\mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big)$ over any Noetherian integral domain. The following two results follow directly from the construction of his isomorphism.

Theorem 2.6 ([Oe, (6.3)])**.** *For any field K, under the natural homomorphism $S_K(2m,n) \to \mathrm{End}\big(V^{\otimes n}\big)$, the subalgebra $S_K^s(m,n)$ is mapped isomorphically onto the subalgebra $\mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big)$.*

Corollary 2.7 ([Oe, (6.3)])**.** *For any field K, the map which sends $f \otimes a$ to af naturally extends to a K-algebra isomorphism*

$$\mathrm{End}_{B_n(-2m)_{\mathbb{Z}}}\big(V_{\mathbb{Z}}^{\otimes n}\big) \otimes_{\mathbb{Z}} K \cong \mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big).$$

Now we can prove Proposition 1.3. By Theorem 2.6 and the canonical isomorphism $S_K^{sy}(m,n) \cong S_K^s(m,n)$ from Theorem 2.5, we know that the natural homomorphism from $S_K^{sy}(m,n)$ to $\mathrm{End}\big(V^{\otimes n}\big)$ maps $S_K^{sy}(m,n)$ isomorphically onto $\mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big)$. Now the second author showed in [Dt, (3.2(b))] that the images of $KGSp(V)$ and of $S_K^{sy}(m,n)$ (which is denoted by $S_d(G)$ in [Dt, (3.2(b))]) in $\mathrm{End}\big(V^{\otimes n}\big)$ are the same when K is infinite. It follows that for any infinite field K,

$$\psi(KGSp(V)) = \mathrm{End}_{B_n(-2m)}\big(V^{\otimes n}\big).$$

Note that this is also equivalent to the fact that the natural evaluation map

$$KGSp(V) \to S_K^{sy}(m,n) \cong S_K^s(m,n) \tag{2.8}$$

is surjective. This completes the proof of Proposition 1.3.

3. The action of $B_n(-2m)$ on $V^{\otimes n}$ for $m \geq n$

In this section, we shall give the proof of Theorem 1.4 in the case where $m \geq n$.

Let R be a Noetherian integral domain with $q \in R$ a fixed invertible element. It is well known that the Hecke algebra $\mathcal{H}_{R,q}(\mathfrak{S}_n)$ associated with the symmetric group $\mathfrak{S}_n$, and hence the group algebra of the symmetric group $\mathfrak{S}_n$ itself, are cellular algebras. An important cellular basis of $\mathcal{H}_{R,q}(\mathfrak{S}_n)$ is the Murphy basis, introduced in [Mu]. Another cellular basis is the Kazhdan-Lusztig basis [KL]. The latter one was extended by Graham and Lehrer to a cellular basis of the Brauer algebra. Xi extended this in [Xi] to the Birman-Murakami-Wenzl algebra, a quantization of the Brauer algebra; this algebra is also cellular. It is known that ([GL], [Xi], [E])

[3]Note that though Oehms assumed in [Oe, (6.2)] that K is an algebraically closed field, the validity over an arbitrary infinite field is an immediate consequence (as shown in our previous discussion).

any cellular basis of the Hecke algebra $\mathcal{H}_{R,q}(\mathfrak{S}_k)$ ($k \in \mathbb{N}$) can be extended to a cellular basis of the Birman-Murakami-Wenzl algebra. We shall follow Enyang's formulation in [E], which describes the basis explicitly in terms of the generators. We will use the Murphy basis of $R\mathfrak{S}_k$ ($k \in \mathbb{N}$), extended to a cellular basis of $B_n(-2m)$. We now describe this basis.

For a composition $\lambda = (\lambda_1, \cdots, \lambda_s)$ of k (i.e., $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\sum \lambda_i = k$), let

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\cdots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\cdots,\lambda_1+\lambda_2\}} \times \cdots$$

be the corresponding Young subgroup of $\mathfrak{S}_k$, and set $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} w \in R\mathfrak{S}_k$. The Young diagram associated with λ consists of an array of nodes in the plane with λ_i many nodes in row i. A λ-tableau $\mathfrak{t}$ is such a diagram in which the nodes are replaced by the numbers $1, \cdots, k$, in some order. The *initial* λ-tableau $\mathfrak{t}^\lambda$ is the one obtained by filling in the numbers $1, \cdots, k$ in order along successive rows. For example,

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}$$

is the initial $(3,2)$-tableau. The symmetric group $\mathfrak{S}_k$ acts naturally on the set of λ-tableaux (on the right), and for any λ-tableau $\mathfrak{t}$ we define $d(\mathfrak{t})$ to be the unique element of $\mathfrak{S}_k$ with $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$. A λ-tableau $\mathfrak{t}$ is called *row standard* if the numbers increase along rows. If $\lambda_1 \geq \cdots \geq \lambda_s$, i.e., λ is a partition of k, then $\mathfrak{t}$ is called *column standard* if the numbers increase down columns, and *standard* if it is both row and column standard. The set $\mathcal{D}_\lambda = \{d(\mathfrak{t}) \mid \mathfrak{t} \text{ is a row standard } \lambda\text{-tableau}\}$ is a set of right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_k$; its elements are known as distinguished coset representatives. For any standard λ-tableaux $\mathfrak{s}, \mathfrak{t}$, we define $m_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s})^{-1} x_\lambda d(\mathfrak{t})$. Murphy [Mu] showed

Theorem 3.1 ([Mu]). *$\{m_{\mathfrak{s}\mathfrak{t}} \mid \lambda \vdash k,\ \mathfrak{s}, \mathfrak{t}$ are standard λ-tableaux$\}$ is a cellular basis of $R\mathfrak{S}_k$ for any Noetherian integral domain R.*

To describe Enyang's cellular basis of the Brauer algebra $B_n(x)$, we need some more notation. First we fix certain bipartitions of n, namely $\nu = \nu_f := ((2^f), (n-2f))$, where $(2^f) := (\underbrace{2, 2, \cdots, 2}_{f \text{ copies}})$ and $(n-2f)$ are considered as partitions of $2f$ and $n-2f$ respectively, and $0 \leq f \leq [n/2]$. Here $[n/2]$ is the largest non-negative integer not greater than $n/2$. In general, a bipartition of n is a pair $(\lambda^{(1)}, \lambda^{(2)})$ of partitions of numbers n_1 and n_2 with $n_1 + n_2 = n$. The notions of Young diagram, bitableaux, etc., carry over easily. Let $\mathfrak{t}^\nu$ be the standard ν-bitableau in which the numbers $1, 2, \cdots, n$ appear in order along successive rows of the first component tableau, and then in order along successive rows of the second component tableau. We define

$$\mathcal{D}_\nu := \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{matrix} (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) = \mathfrak{t}^\nu d \text{ is row standard and the first column of } \mathfrak{t}^{(1)} \text{ is} \\ \text{an increasing sequence when read from top to bottom} \end{matrix} \right\}.$$

For each partition λ of $n-2f$, we denote by $\mathrm{Std}(\lambda)$ the set of all the standard λ-tableaux with entries in $\{2f+1, \cdots, n\}$. The initial tableau $\mathfrak{t}^\lambda$ in this case has the numbers $2f+1, \cdots, n$ in order along successive rows. Again, for each $\mathfrak{t} \in \mathrm{Std}(\lambda)$, let $d(\mathfrak{t})$ be the unique element in $\mathfrak{S}_{\{2f+1,\cdots,n\}} \subseteq \mathfrak{S}_n$ with $\mathfrak{t}^\lambda d(\mathfrak{t}) = \mathfrak{t}$.

For each integer f with $0 \leq f \leq [n/2]$, we denote by $B^{(f)}$ the two-sided ideal of $B_n(-2m)$ generated by $e_1 e_3 \cdots e_{2f-1}$. Note that $B^{(f)}$ is spanned by all Brauer

diagrams with at least $2f$ horizontal edges (f edges in each of the top and the bottom rows in the diagram).

Let f be an integer with $0 \le f \le [n/2]$. Let $\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$ and $d_1, d_2 \in \mathcal{D}_\nu$, where again ν is the bipartition $((2^f),(n-2f))$ of n. Then $d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2$ corresponds to the Brauer diagram where the top horizontal edges connect $(2i-1)d_1$ and $(2i)d_1$, the bottom horizontal edges connect $(2i-1)d_2$ and $(2i)d_2$, for $i = 1,2,\cdots,f$, and the vertical edges are determined by $d_1^{-1}\sigma d_2$. By [Xi, (3.5)], every Brauer diagram d can be written in this way.

Theorem 3.2 ([E]). *Let R be a Noetherian integral domain with $x \in R$. Let $B_n(x)_R$ be the Brauer algebra with parameter x over R. Then the set*

$$\left\{ d_1^{-1}e_1e_3\cdots e_{2f-1}m_{\mathfrak{s}\mathfrak{t}}d_2 \;\middle|\; \begin{array}{c} 0 \le f \le [n/2],\ \lambda \vdash n-2f,\ \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda), \\ d_1, d_2 \in \mathcal{D}_\nu,\ \text{where } \nu := ((2^f),(n-2f)) \end{array} \right\}$$

is a cellular basis of the Brauer algebra $B_n(x)_R$.

As a consequence, by combining Theorems 3.1 and 3.2, we get that

Corollary 3.3. *With the above notation, the set*

$$\left\{ d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2 \;\middle|\; \begin{array}{c} 0 \le f \le [n/2],\ \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}},\ d_1, d_2 \in \mathcal{D}_\nu, \\ \text{where } \nu := ((2^f),(n-2f)) \end{array} \right\}.$$

is a basis of the Brauer algebra $B_n(x)_R$, which coincides with the natural basis given by Brauer n-diagrams.

We now specialize R to be a field K, assume $m \ge n$, $V = K^m$ and consider the special Brauer algebra $B_n(-2m) = B_n(-2m \cdot 1_K)_K$. As pointed out in Section 1, this algebra acts on the tensor space $V^{\otimes n}$, centralizing the action of the symplectic similitude group $GSp(V)$ and hence that of the symplectic group $Sp(V)$ as well.

The proof of the next result will be given at the end of the section, after a series of preparatory lemmas.

Theorem 3.4. *Let K be a field. If $m \ge n$, then the natural homomorphism $\varphi : B_n(-2m) \to \mathrm{End}_K\big(V^{\otimes n}\big)$ is injective; if furthermore K is infinite, then it is in fact an isomorphism onto $\mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big)$.*

Suppose that $m \ge n$. Our first goal here is to show that the action of $B_n(-2m)$ on $V^{\otimes n}$ is faithful; that is, the annihilator $\mathrm{ann}_{B_n(-2m)}(V^{\otimes n})$ is (0). Note that

$$\mathrm{ann}_{B_n(-2m)}(V^{\otimes n}) = \bigcap_{v \in V^{\otimes n}} \mathrm{ann}_{B_n(-2m)}(v).$$

Thus it is enough to calculate $\mathrm{ann}_{B_n(-2m)}(v)$ for some set of chosen vectors $v \in V^{\otimes n}$ such that the intersection of annihilators is (0). We write

$$\mathrm{ann}(v) = \mathrm{ann}_{B_n(-2m)}(v) := \big\{ x \in B_n(-2m) \;\big|\; vx = 0 \big\}.$$

Recall that $(v_1,\cdots,v_{2m})$ denotes an ordered basis of V, and $I(2m,n)$ denotes the set of multi-indices $\underline{i} := (i_1,\cdots,i_n)$ with $i_j \in \big\{1,\cdots,2m\big\}$ for $j = 1,\cdots,n$. We write $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}$ for $\underline{i} := (i_1,\cdots,i_n) \in I(2m,n)$. Thus $\big\{ v_{\underline{i}} \;\big|\; \underline{i} \in I(2m,n) \big\}$ is a K-basis of $V^{\otimes n}$. Consider the action of the symmetric group $\mathfrak{S}_n$ on $I(2m,n)$ given by $\underline{i}\pi = (i_{(1)\pi^{-1}},\cdots,i_{(n)\pi^{-1}})$ for $\underline{i} := (i_1,\cdots,i_n) \in I(2m,n)$ and $\pi \in \mathfrak{S}_n$. Thus, in particular, by definition, $v_{\underline{i}}\pi = (-1)^{\ell(\pi)}v_{\underline{i}\pi}$. For $\underline{i} \in I(2m,n)$, an ordered pair (s,t) $(1 \le s < t \le n)$ is called a *symplectic pair* in $\underline{i}$ if $i_s = i'_t$. Two ordered pairs

(s,t) and (u,v) are called disjoint if $\{s,t\} \cap \{u,v\} = \emptyset$. We define the *symplectic length* $\ell_s(v_{\underline{i}})$ to be the maximal number of disjoint symplectic pairs (s,t) in $\underline{i}$. For $\sigma, \pi \in \mathfrak{S}_n$ and $1 \le j \le n-1$, it is easy to see that $v_{\underline{i}}\sigma e_j \pi$ is zero or a linear combination of tensors $v_{\underline{j}}$ with $\ell_s(v_{\underline{i}}) = \ell_s(v_{\underline{j}})$. Moreover, for $f > \ell_s(v_{\underline{i}})$ we have $B^{(f)} \subseteq \operatorname{ann}(v_{\underline{i}})$. Note that $\pi \mapsto (-1)^{\ell(\pi)}\pi$ for $\pi \in \mathfrak{S}_n$ defines an automorphism τ of the group algebra $K\mathfrak{S}_n$, and that our action of $\mathfrak{S}_n$ on tensor space is precisely the standard place-permutation action (see Section 2) twisted by this automorphism. In particular, this shows that $K\mathfrak{S}_n$ acts faithfully on $V^{\otimes n}$ for $m \ge n$. Moreover, for $\pi \in \mathfrak{S}_n$ and $\underline{i} \in I(2m,n)$, $\operatorname{ann}(v_{\underline{i}}\pi) = \operatorname{ann}(v_{\underline{i}\pi}) = \pi^{-1}\operatorname{ann}(v_{\underline{i}})$.

Now suppose again that $m \ge n$. We shall prove by induction on f that $B^{(f)} \supseteq \operatorname{ann}_{B_n(-2m)}\big(V^{\otimes n}\big)$ for all f. Since $B^{(f)} = 0$ for $f > [n/2]$, this shows the main result of this section, that is, $B_n(-2m)$ acts faithfully on $V^{\otimes n}$ if $m \ge n$. The start of the induction is the following.

Lemma 3.5. $\operatorname{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(1)}$.

Proof. Since $m \ge n$, the tensor $v := v_1 \otimes v_2 \otimes \cdots \otimes v_n$ is defined. Then $v\pi = (-1)^{\ell(\pi)} v_{(1)\pi^{-1}} \otimes \cdots \otimes v_{(n)\pi^{-1}}$ for $\pi \in \mathfrak{S}_n$. Now $B^{(1)}$ is contained in the annihilator of $v\pi$, hence is contained in the intersection of all annihilators of $v\pi$, as π ranges over $\mathfrak{S}_n$. Hence $B^{(1)}$ annihilates the subspace S spanned by the $v\pi$, where π runs through $\mathfrak{S}_n$. Then the subspace S becomes a $B_n(-2m)$-submodule of tensor space (since $B^{(1)}$ acts as zero).

On the other hand, since S, as a module for the symmetric group part, which is isomorphic with $B_n(-2m)$ modulo the ideal $B^{(1)}$, is faithful, it follows that the annihilator of S must be in $B^{(1)}$. Hence $\operatorname{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(1)}$. $\square$

Suppose that we have already shown $\operatorname{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(f)}$ for some natural number $f \ge 1$. We want to show $\operatorname{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(f+1)}$. If $f > [n/2]$, we are done already. Thus we may assume $f \le [n/2]$.

For $\underline{i} := (i_1, \cdots, i_n) \in I(2m,n)$, we define the *weight* $\lambda(v_{\underline{i}}) = \lambda$ to be the composition $\lambda = (\lambda_1, \cdots, \lambda_{2m})$ of n into $2m$ parts, where λ_j is the number of times v_j occurs as a tensor factor in $v_{\underline{i}}$, $j = 1, \cdots, 2m$. Note that the tensors of weight λ for a given composition λ of n span a $K\mathfrak{S}_n$-submodule M^λ of $V^{\otimes n}$; thus

$$V^{\otimes n} = \bigoplus_{\lambda \in \Lambda(2m,n)} M^\lambda$$

as a $K\mathfrak{S}_n$-module, where $\Lambda(2m,n)$ denotes the set of compositions of n into $2m$ parts. It is well known that M^λ is isomorphic to the sign permutation representation of $\mathfrak{S}_n$ on the cosets of the Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$.

As a consequence, each element $v \in V^{\otimes n}$ can be written as a sum

$$v = \sum_{\lambda \in \Lambda(2m,n)} v_\lambda$$

for uniquely determined $v_\lambda \in M^\lambda$.

Fix an index $\underline{c} \in I(2m,2f)$ of the form $(i_1, i_1', i_2, i_2', \cdots, i_f, i_f')$ with $1 \le i_s \le 2m$ for $1 \le s \le f$; for example, $\underline{c} = (1, 1', 2, 2', \cdots, f, f')$. Since $e_1 e_3 \cdots e_{2f-1}$ acts only on the first $2f$ parts of any simple tensor $v_{\underline{i}}$, $\underline{i} \in I(2m,n)$, we may consider these operators as acting on $V^{\otimes 2f}$.

Let $\nu = \nu_f := ((2^f), (n-2f))$. Consider the subgroup Π of $\mathfrak{S}_{\{1,\cdots,2f\}} \le \mathfrak{S}_n$ permuting the rows of $\mathfrak{t}^{\nu^{(1)}}$ but keeping the entries in the rows fixed. Obviously, Π

normalizes the stabilizer $\mathfrak{S}_{(2^f)}$ of $\mathfrak{t}^{\nu^{(1)}}$ in $\mathfrak{S}_{2f}$, where $\mathfrak{S}_{2f} := \mathfrak{S}_{\{1,2,\cdots,2f\}}$. In fact, it is well known that the semi-direct product $\Psi := \mathfrak{S}_{(2^f)} \rtimes \Pi$ is the normalizer of $\mathfrak{S}_{(2^f)}$ in $\mathfrak{S}_{2f}$.

Let $\lambda^{(1)} \in \Lambda(2m, 2f)$ be the weight of $v_{\underline{\widehat{c}}}$ with $\underline{\widehat{c}} = (f+1, (f+1)', \cdots, 2f, (2f)') \in I(2m, 2f)$. Note, if $\underline{j} = (j_1, \cdots, j_{n-2f}) \in I(2m, n-2f)$ satisfies $2f+1 \le j_s \le m$ for $s = 1, \cdots, n-2f$, and if $\lambda^{(2)} \in \Lambda(2m, n-2f)$ denotes the weight of $v_{\underline{j}} \in V^{\otimes n-2f}$, then we obtain the weight $\lambda \in \Lambda(2m, n)$ of $v_{\underline{\widehat{c}}} \otimes v_{\underline{j}}$ by adding $\lambda^{(1)}$ to $\lambda^{(2)}$ componentwise. Note that $\{s \mid \lambda_s^{(1)} \neq 0\} \cap \{s \mid \lambda_s^{(2)} \neq 0\} = \emptyset$. We write for this weight $\lambda = \lambda^{(1)} \otimes \lambda^{(2)}$. We define $E_f \in B_n(-2m)$ to be $e_1 e_3 \cdots e_{2f-1}$.

Lemma 3.6. *The weight component of $v_{\underline{c}} e_1 e_3 \cdots e_{2f-1}$ to weight $\lambda^{(1)}$ is*

$$\big(v_{\underline{c}} E_f\big)_{\lambda^{(1)}} = (-1)^f \sum_{\psi \in \Psi} v_{\underline{\widehat{c}}} \psi = (-1)^f \sum_{\psi \in \Psi} (-1)^{\ell(\psi)} v_{\underline{\widehat{c}}\psi}.$$

Proof. By definition,

$$v_{\underline{c}} E_f = \Big(\sum_{j=1}^{m} \big(v_{j'} \otimes v_j - v_j \otimes v_{j'}\big)\Big)^{\otimes f} = (-1)^f \Big(\sum_{j=1}^{m} \big(v_j \otimes v_{j'} - v_{j'} \otimes v_j\big)\Big)^{\otimes f}.$$

To obtain the components in the weight space $(V^{\otimes 2f})^{\lambda^{(1)}}$, we have to consider all occurring simple tensors which are obtained from $v_{\underline{\widehat{c}}} = w_1 \otimes \cdots \otimes w_f$ with $w_i = v_{f+i} \otimes v_{(f+i)'}$ by first permuting the tensors w_i, which is done by a permutation $\pi \in \Pi$, and then replacing (for some $i \in \{1, \cdots, f\}$) w_i by $w_{i'} = v_{(f+i)'} \otimes v_{f+i}$, which amounts to applying a permutation $\sigma \in \mathfrak{S}_{(2^f)}$. On the other hand, each such tensor occurs exactly once, and the sign $(-1)^{\ell(\psi)}$ is calculated taking in account that if we factor out $(-1)^f$, the w_i carry a positive sign and the $w_{i'}$ carry a negative sign, the elements of Π all have even length and the action of $\mathfrak{S}_n$ on $V^{\otimes n}$ considered here carries a sign as well. This proves the lemma. $\square$

Recall $\nu = \nu_f := ((2^f), (n-2f)) = (\nu^{(1)}, \nu^{(2)})$ and the definition of the set $\mathcal{D}_{\nu_f}$ in the beginning of this section. We set

$$\mathcal{D}_f = \mathcal{D}_{\nu_f} \cap \mathfrak{S}_\mu \quad \text{where } \mu = ((2f), (n-2f)) \in \Lambda(2, n).$$

Since $\mathfrak{t}^{\nu^{(2)}} d$ is row standard for any $d \in \mathcal{D}_{\nu_f}$, thus $\mathcal{D}_f$ consists of all $d \in \mathcal{D}_{\nu_f}$ which fix every element in the set $\{2f+1, \cdots, n\}$. That is, $\mathcal{D}_f = \mathcal{D}_{\nu_f} \cap \mathfrak{S}_{2f}$.

Lemma 3.7. *We have the equality*

$$\mathfrak{S}_{2f} = \bigsqcup_{d \in \mathcal{D}_f} \Psi d,$$

where "$\sqcup$" means a disjoint union.

Proof. Let $\mathfrak{t} = \mathfrak{t}^{\nu^{(1)}} w$, where $w \in \mathfrak{S}_{2f}$, be a $\nu^{(1)}$-tableau. Then $w^{-1}\mathfrak{S}_{(2^f)} w$ is its row stabilizer and $w^{-1}\Pi w$ is the subgroup of $\mathfrak{S}_{2f}$ permuting the rows of $\mathfrak{t}$. We therefore find a $\rho \in \mathfrak{S}_{(2^f)}$ such that $\mathfrak{t} w^{-1} \rho w$ is row standard, and then a $\pi \in \Pi$ such that $\mathfrak{t} w^{-1} \rho w w^{-1} \pi w = \mathfrak{t}^{\nu^{(1)}} \rho \pi w$ is row standard and has increasing first column. Thus $\mathfrak{t}^{\nu^{(1)}} \rho \pi w = \mathfrak{t}^{\nu^{(1)}} d$ for some $d \in \mathcal{D}_{\nu_f} \cap \mathfrak{S}_{2f} = \mathcal{D}_f$. Thus we have shown $\psi w = d$ with $\psi := \rho\pi \in \Psi$, and hence $w \in \Psi d$. To show that the union is disjoint, let $d_1, d_2 \in \mathcal{D}_f$ and suppose $d_1 = \psi d_2$ for some $\psi \in \Psi$. Consider $\mathfrak{t}_i = \mathfrak{t}^{\nu^{(1)}} d_i$, $i = 1, 2$. We see from $d_1 = \psi d_2$ that $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have the same numbers in their rows, in fact up to

a permutation the same rows, since they are row standard. But the first column has to be increasing, by definition of $\mathcal{D}_{\nu_f}$; hence the orders of the rows in $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have to be the same as well. This proves $d_1 = d_2$ and the union is disjoint. $\square$

We now turn to the full set $\mathcal{D}_{\nu_f}$. Fix $d \in \mathcal{D}_{\nu_f}$ and let $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ be the corresponding ν_f-bitableau. Since $\mathfrak{t}^{(2)}$ consists of a single row with increasing entries, it is completely determined by those entries. On the other hand, taking an arbitrary set partition $\{1, \cdots, n\} = \{i_1, \cdots, i_{2f}\} \sqcup \{i_{2f+1}, \cdots, i_{2n}\}$, and inserting the entries of the first set in increasing order along successive rows in $\mathfrak{t}^{\nu^{(1)}}$, and the numbers in the second set in increasing order into $\mathfrak{t}^{\nu^{(2)}}$, we obtain a ν_f-bitableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ such that obviously $d(\mathfrak{t}) \in \mathcal{D}_{\nu_f}$. Thus we may index those elements of $\mathcal{D}_{\nu_f}$ by the set $\mathcal{P}_f$ of subsets of $\{1, \cdots, n\}$ of size $2f$. Writing d_J for $J \in \mathcal{P}_f$, for an arbitrary $d \in \mathcal{D}_{\nu_f}$ with $\mathfrak{t}^{\nu_f} d = \mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$, the subset J of $\{1, \cdots, n\}$ of entries of $\mathfrak{t}^{(1)}$ is an element of $\mathcal{P}_f$, and one sees by direct inspection that there is an element $d_1 \in \mathcal{D}_f = \mathcal{D}_{\nu_f} \cap \mathfrak{S}_{2f}$ such that $\mathfrak{t} = \mathfrak{t}^{\nu_f} d_J (d_J^{-1} d_1 d_J) = d_1 d_J$. That is, $d = d_1 d_J$. Note also that each element d_J is a distinguished right coset representative of $\mathfrak{S}_{(2f, n-2f)}$ in $\mathfrak{S}_n$. Thus we have shown

Lemma 3.8.

$$\mathcal{D}_{\nu_f} = \bigsqcup_{J \in \mathcal{P}_f} \mathcal{D}_f d_J.$$

We define I_f to be the set of multi-indices $(i_{2f+1}, \cdots, i_n)$ of length $n - 2f$ with $2f + 1 \leq i_\rho \leq m$ for $\rho = 2f + 1, \cdots, n$, where we choose the position index ρ to run from $2f + 1$ to n in order to keep the notation straight, when we act by an element of $\mathfrak{S}_n$. Note that for $2f + 1 \leq i \leq m$, we have $i' > m$; hence $\ell_s(v_{\underline{k}}) = 0$ for all $\underline{k} \in I_f$.

For an arbitrary element $v \in V^{\otimes n}$, we say the simple tensor $v_{\underline{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}$ is involved in v if $v_{\underline{i}}$ has a nonzero coefficient in writing v as a linear combination $\sum_{\underline{j} \in I(2m,n)} k_{\underline{j}} v_{\underline{j}}$ of the basis $\{v_{\underline{j}} \mid \underline{j} \in I(2m, n)\}$ of $V^{\otimes n}$.

Lemma 3.9. *Let $\underline{k} \in I_f$, $v = v_{\underline{c}} \otimes v_{\underline{k}} \in V^{\otimes n}$. Let $1 \neq d \in \mathfrak{S}_n$. If either $d \notin \mathfrak{S}_{(2f,n-2f)}$ or $d \in \mathcal{D}_f = \mathcal{D}_{\nu_f} \cap \mathfrak{S}_{(2f,n-2f)}$, then $d^{-1} z E_f \in \operatorname{ann}(v)$ for any $z \in \Psi$.*

Proof. Write $v_{\underline{c}} = w_1 \otimes \cdots \otimes w_f$ with $w_j = v_j \otimes v_{j'}, j = 1, \cdots, f$. If $d \notin \mathfrak{S}_{(2f,n-2f)}$, then d^{-1} is not contained in $\mathfrak{S}_{(2f,n-2f)}$ either. In particular, there is some j, $2f + 1 \leq j \leq n$, such that $1 \leq jd^{-1} \leq 2f$, and hence the basis vector v_{k_j} with $2f + 1 \leq k_j \leq m$ appears at position jd^{-1} in vd^{-1}. However, $m < k_j' \leq 2m - 2f$; hence $v_{k_j'}$ does not occur as a factor in vd^{-1} at all and hence for any $z \in \Psi$, $0 = vd^{-1} z e_{jd^{-1}-1}$ if jd^{-1} is even, $0 = vd^{-1} z e_{jd^{-1}}$ if jd^{-1} is odd. As the e_i's in $E_f = e_1 e_3 \cdots e_{2f-1}$ commute we have $vd^{-1} z E_f = 0$ in this case. If $d \in \mathcal{D}_f = \mathcal{D}_{\nu_f} \cap \mathfrak{S}_{2f}$, then d and hence d^{-1} as well is not contained in the subgroup Ψ of $\mathfrak{S}_{2f}$ defined above. Therefore there exists $j \in \{1, 3, \cdots, 2f - 1\}$ such that $jd^{-1}, (j + 1)d^{-1}$ are not in the same row of $\mathfrak{t}^{(2^f)} d^{-1}$. Now we see similarly as above that zE_f annihilates vd^{-1} for any $z \in \Psi$. $\square$

We are now ready to prove the key lemma from which our main result in this section will follow easily.

Lemma 3.10. *Let S be the subset*

$$\left\{ d_1^{-1} E_f \sigma d_2 \;\middle|\; \begin{array}{c} d_1, d_2 \in \mathcal{D}_{\nu_f},\ d_1 \neq 1, \\ \sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}} \end{array} \right\}$$

of the basis (3.3) of $B_n(-2m)$, and let U be the subspace spanned by S. Then

$$B^{(f)} \cap \Big(\bigcap_{\underline{k} \in I_f} \operatorname{ann}(v_{\underline{c}} \otimes v_{\underline{k}}) \Big) = B^{(f+1)} \oplus U.$$

Proof. Since $\ell_s(v_{\underline{k}}) = 0$, by definition of I_f, hence $\ell_s(v_{\underline{c}} \otimes v_{\underline{k}}) = f$, it follows that $B^{(f+1)} \subseteq \operatorname{ann}(v_{\underline{c}} \otimes v_{\underline{k}})$. This, together with Lemma 3.9, shows that the right-hand side is contained in the left-hand side.

Now let $x \in B^{(f)} \cap \big(\bigcap_{\underline{k} \in I_f} \operatorname{ann}(v_{\underline{c}} \otimes v_{\underline{k}})\big)$. Using Lemma 3.9 and the basis (3.3) of $B_n(-2m)$, we may assume that $x = E_f \sum_{d \in \mathcal{D}_\nu} z_d d$, where $\nu = \nu_f = ((2^f), (n-2f))$ and the coefficients z_d, $d \in \mathcal{D}_\nu$ are taken from $K\mathfrak{S}_{\{2f+1,\cdots,n\}} \subseteq K\mathfrak{S}_n$. We then have to show $x = 0$.

Fix $\underline{k} \in I_f$ and write $v = v_{\underline{c}} \otimes v_{\underline{k}}$. As in Lemma 3.6, choose the weight $\lambda^{(1)} \in \Lambda(2m, 2f)$ to be the weight of $v_{\widehat{\underline{c}}} = w_1 \otimes \cdots \otimes w_f$, where $w_i = v_{f+i} \otimes v_{(f+i)'}, i = 1, \cdots, f$, and let $\lambda^{(2)}$ be the weight of $v_{\underline{k}}$; thus $\lambda = \lambda^{(1)} \otimes \lambda^{(2)}$ is the weight of $v_{\underline{c}} \otimes v_{\underline{k}}$. Since $V^{\otimes n}$ is the direct sum of its weight spaces M^λ, we conclude $(vx)_\mu = 0$ for all $\mu \in \Lambda(2m, n)$. In particular,

$$\begin{aligned} 0 = (vx)_\lambda &= \big((v_{\underline{c}} \otimes v_{\underline{k}})x\big)_\lambda = \sum_{d \in \mathcal{D}_\nu} \Big(v_{\underline{c}} E_f \otimes v_{\underline{k}} \Big)_\lambda z_d d \\ &= \sum_{d \in \mathcal{D}_\nu} \Big((v_{\underline{c}} E_f)_{\lambda^{(1)}} \otimes v_{\underline{k}} \Big) z_d d. \end{aligned}$$

The latter equality holds, since the action of $\mathfrak{S}_n$ preserves weight spaces.

By Lemma 3.6 we have $\big(v_{\underline{c}} E_f\big)_{\lambda^{(1)}} = (-1)^f \sum_{\psi \in \Psi} v_{\widehat{\underline{c}}} \psi = \widehat{v}$, where again Ψ is the normalizer of the Young subgroup $\mathfrak{S}_{(2^f)}$ in $\mathfrak{S}_{2f}$. Thus we have to investigate $\sum_{d \in \mathcal{D}_\nu} \big(\widehat{v} \otimes v_{\underline{k}}\big) z_d d = 0$ for the unknown element $z_d \in K\mathfrak{S}_{\{2f+1,\cdots,n\}}$. Note that $(\widehat{v} \otimes v_{\underline{k}}) z_d = \widehat{v} \otimes (v_{\underline{k}} z_d)$.

We fix $d \in \mathcal{D}_{\nu_f}$. By Lemma 3.8 we find a $2f$-element subset J of $\{1, \cdots, n\}$ and $d_1 \in \mathcal{D}_f \subseteq \mathfrak{S}_{2f}$ such that $d = d_1 d_J$. Thus

$$\big(\widehat{v} \otimes v_{\underline{k}}\big) z_d d = \big(\widehat{v} \otimes v_{\underline{k}} z_d\big) d = \big(\widehat{v} \otimes v_{\underline{k}} z_d\big) d_1 d_J = \big(\widehat{v} d_1 \otimes v_{\underline{k}} z_{d_1 d_J}\big) d_J,$$

since $d_1 \in \mathfrak{S}_{2f}$ and $z_d \in K\mathfrak{S}_{\{2f+1,\cdots,n\}}$.

If $J, L \in \mathcal{P}_f, J \neq L$, choose $1 \leq l \leq n$ with $l \in J$ but $l \notin L$. Thus there exists a $j \in \{1, 2, \cdots, 2f\}$ which is mapped by d_J to l, but $(l)d_L^{-1} > 2f$. Note that for any $d \in \mathcal{D}_f$ all basis vectors v_i occurring in $\widehat{v}d$ as factors have index in the set $\{f+1, f+2, \cdots, 2f, (2f)', \cdots, (f+2)', (f+1)'\}$, and all those v_i occurring in $v_{\underline{k}} z_{dd_J}$, respectively in $v_{\underline{k}} z_{dd_L}$, have index i between $2f+1$ and m. Let $v_{i_1} \otimes \cdots \otimes v_{i_n}$ be a simple tensor involved in $\big(\widehat{v} d_1 \otimes v_{\underline{k}} z_{d_1 d_J}\big) d_J$ and $v_{j_1} \otimes \cdots \otimes v_{j_n}$ be a simple tensor involved in $\big(\widehat{v} d_2 \otimes v_{\underline{k}} z_{d_2 d_L}\big) d_L$ for $d_1, d_2 \in \mathcal{D}_f$. Then, by the above, we have that $2f+1 \leq j_l \leq m$, and either $v_{i_l} = v_k$ or $v_{i_l} = v_{k'}$ for some $f+1 \leq k \leq 2f$. Consequently the simple tensors $v_{\underline{i}}, \underline{j} \in I(2m, n)$ involved in $\big\{(\widehat{v} d_1 \otimes v_{\underline{k}} z_{d_1 d_J}) d_J\big\}$ and in $\big\{(\widehat{v} d_2 \otimes v_{\underline{k}} z_{d_2 d_L}) d_L\big\}$ are disjoint; hence both sets are linearly independent. We conclude that $\sum_{d \in \mathcal{D}_f} \big(\widehat{v} d \otimes v_{\underline{k}} z_{dd_J}\big) d_J = 0$ for each $J \in \mathcal{P}_f$; hence $\sum_{d_1 \in \mathcal{D}_f} \widehat{v} d_1 \otimes v_{\underline{k}} z_{d_1 d_J} = 0$, since d_J is invertible.

Lemma 3.7 says in particular that $\widehat{v}d_1$ is a linear combination of basis tensors $v_{\underline{i}} = v_{i_1}\otimes\cdots\otimes v_{i_{2f}}$, with $\underline{i}\in\widehat{\underline{c}}\Psi d_1$, and that we obtain by varying d_1 through $\mathcal{D}_f$ precisely the partition of $\mathfrak{S}_{2f}$ into Ψ-cosets. These are mutually disjoint. This is because $\mathfrak{S}_{2f}$ acts faithfully on the K-span of $\{v_{\widehat{\underline{c}}}\sigma \mid \sigma\in\mathfrak{S}_{2f}\}$, and hence all the basis vectors $v_{\underline{i}}, 1\le i\le n$ appearing as factors in $\widehat{v}d_1$ are pairwise distinct. Consequently the cosets of $\Psi d_1, d_1\in\mathcal{D}_f$, partition the basis vectors in this set into mutually disjoint subsets, and we conclude that the basic tensors involved in $\widehat{v}d_1$ are disjoint for different choices of $d_1\in\mathcal{D}_f$. Therefore, the equality $\sum_{d_1\in\mathcal{D}_f}\widehat{v}d_1\otimes v_{\underline{k}}z_{d_1d_J}=0$ implies that $\widehat{v}d_1\otimes v_{\underline{k}}z_{d_1d_J}=0$ for each fixed $d_1\in\mathcal{D}_f$. Now we vary $\underline{k}\in I_f$. The K-span of $\{v_{\underline{k}}\mid \underline{k}\in I_f\}$ is isomorphic to the tensor space $V^{\otimes n-2f}$ for the symmetric group $\mathfrak{S}_{\{2f+1,\cdots,n\}}\cong\mathfrak{S}_{n-2f}$. Since $m-2f\ge n-2f$, hence $\mathfrak{S}_{\{2f+1,\cdots,n\}}$ acts faithfully on it. This implies $z_{d_1d_J}=0$ for all $d_1\in\mathcal{D}_f, J\in\mathcal{P}_f$. Thus $x=0$ and the lemma is proved. □

Corollary 3.11. *Let $d\in\mathcal{D}_\nu, \nu=\nu_f$. Then*

$$B^{(f)}\cap\Big(\bigcap_{\underline{k}\in I_f}\operatorname{ann}\big((v_{\underline{c}}\otimes v_{\underline{k}})d\big)\Big)=B^{(f+1)}\oplus\Bigg(\bigoplus_{\substack{d\neq\tilde{d}_1,d_2\in\mathcal{D}_\nu\\ \sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}}}K\tilde{d}_1^{-1}E_f\sigma d_2\Bigg).$$

Hence $B^{(f)}\cap\Big(\bigcap_{d\in\mathcal{D}_\nu}\bigcap_{\underline{k}\in I_f}\operatorname{ann}\big((v_{\underline{c}}\otimes v_{\underline{k}})d\big)\Big)=B^{(f+1)}$.

Proof. First, we claim that for any $\tilde{d},d_2\in\mathcal{D}_\nu$ with $\tilde{d}\neq d$,

$$\big((v_{\underline{c}}\otimes v_{\underline{k}})d\big)\tilde{d}^{-1}E_f=0,$$

and thus the right-hand side of the above equality is contained in the left-hand side.

By Lemma 3.9, it suffices to consider the case where $d\tilde{d}^{-1}=w\in\mathfrak{S}_{(2f,n-2f)}$. By Lemma 3.8, we can write $d=d_1d_J, \tilde{d}=\tilde{d}_1d_L$, where $d_1,\tilde{d}_1\in\mathcal{D}_f, J,L\in\mathcal{P}_f$. Since d_J,d_L are distinguished right coset representatives of $\mathfrak{S}_{(2f,n-2f)}$ in $\mathfrak{S}_n$, we deduce that $J=L$. Hence $d_1=w\tilde{d}_1$ and hence $w\in\mathfrak{S}_{2f}$. By Lemma 3.7, we can write $w=zd_3$, where $z\in\Psi, d_3\in\mathcal{D}_f$. Since $d\neq\tilde{d}$, it follows that $d_1\neq\tilde{d}_1$. Therefore, by the decomposition given in Lemma 3.7, we know that $d_1=w\tilde{d}_1=zd_3\tilde{d}_1$ implies that $d_3\neq 1$. Therefore, by Lemma 3.9, $v_{\underline{c}}d\tilde{d}^{-1}E_f=v_{\underline{c}}d_3^{-1}z^{-1}E_f=0$. This proves our first claim.

Second, note that the annihilator of $\big(v_{\underline{c}}\otimes v_{\underline{k}}\big)d$ ($\underline{k}\in I_f, d\in\mathcal{D}_{\nu_f}$) in $B^{(f)}$ is precisely $d^{-1}\operatorname{ann}(v_{\underline{c}}\otimes v_{\underline{k}})\cap B^{(f)}$. By Lemma 3.10, to complete the proof of the corollary, it suffices to show that each $d^{-1}d_1^{-1}E_f\sigma d_2$, where $1\neq d_1,d_2\in\mathcal{D}_\nu,\sigma\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$, can be written in the form $\tilde{d}_1^{-1}E_f\tilde{\sigma}\tilde{d}_2$, where $d\neq\tilde{d}_1,\tilde{d}_2\in\mathcal{D}_\nu,\tilde{\sigma}\in\mathfrak{S}_{\{2f+1,\cdots,n\}}$. In fact, assume that $d_1d=zd_3d_J$, where $z\in\Psi, d_3\in\mathcal{D}_f, J\in\mathcal{P}_f$. Then by the decomposition given in Lemma 3.7, $d_1\neq 1$ implies that $d_3d_J\neq d$. Note that Ψ is generated by $s_1, s_{2i}s_{2i+1}s_{2i-1}s_{2i}, i=1,\cdots,f$. Using the fact that $s_1e_1=e_1$ and

$$\begin{aligned}s_{2i}s_{2i+1}s_{2i-1}s_{2i}e_{2i-1}e_{2i+1}&=s_{2i}s_{2i+1}e_{2i}e_{2i-1}e_{2i+1}=e_{2i+1}e_{2i}e_{2i-1}e_{2i+1}\\&=e_{2i+1}e_{2i}e_{2i+1}e_{2i-1}=e_{2i-1}e_{2i+1},\end{aligned}$$

it is easy to see that $zE_f=E_f$ for any $z\in\Psi$. It follows that

$$d^{-1}d_1^{-1}E_f\sigma d_2=(d_3d_J)^{-1}z^{-1}E_f\sigma d_2=(d_3d_J)^{-1}E_f\sigma d_2,$$

as required. This completes the proof of the corollary. □

Proof of Theorem 3.4. We have seen in Lemma 3.5 that $\mathrm{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(1)}$, and Corollary 3.11 implies that $\mathrm{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(f+1)}$ provided that $\mathrm{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(f)}$. Thus by induction on f we have $\mathrm{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) \subseteq B^{(f)}$ for all natural numbers f. Since $B^{(f+1)} = 0$ for $f > [n/2]$ it follows that $\mathrm{ann}_{B_n(-2m)}\big(V^{\otimes n}\big) = 0$. In other words, φ is injective if $m \geq n$.

Suppose furthermore K is an infinite field. By (2.8) the natural homomorphism from the group algebra $KGSp(V)$ to the symplectic Schur algebra $S_K^{sy}(m,n)$ is surjective. Note that since $S_K^{sy}(m,n)$ is a quasi-hereditary algebra and $V \cong L(\varepsilon_1) \cong \triangle(\varepsilon_1) \cong \bigtriangledown(\varepsilon_1)$, it follows that $V^{\otimes n}$ is also a tilting module over $S_K^{sy}(m,n)$. By the general theory from tilting modules (e.g. [DPS, Lemma 4.4 (c)]),

$$\begin{aligned}\mathrm{End}_{KGSp(V)}\big(V^{\otimes n}\big) \otimes_K \overline{K} &= \mathrm{End}_{S_K^{sy}(m,n)}\big(V^{\otimes n}\big) \otimes_K \overline{K} \\ &= \mathrm{End}_{S_{\overline{K}}^{sy}(m,n)}\big(V_{\overline{K}}^{\otimes n}\big) = \mathrm{End}_{\overline{K}GSp(V_{\overline{K}})}\big(V_{\overline{K}}^{\otimes n}\big),\end{aligned}$$

where $V_{\overline{K}} := V \otimes_K \overline{K}$, and $\dim \mathrm{End}_{S_{\overline{K}}^{sy}(m,n)}\big(V_{\overline{K}}^{\otimes n}\big) = \dim \mathrm{End}_{S_{\mathbb{C}}^{sy}(m,n)}\big(V_{\mathbb{C}}^{\otimes n}\big)$.

Therefore

$$\begin{aligned}&\dim \mathrm{End}_{KGSp(V)}\big(V^{\otimes n}\big) \\ &= \dim \mathrm{End}_{\mathbb{C}GSp_{2m}(\mathbb{C})}\big((\mathbb{C}^{2m})^{\otimes n}\big) \\ &= \sum_{\substack{0\leq f\leq [n/2] \\ \lambda\vdash n-2f}} (\dim \widetilde{S}^{\lambda})^2 \quad \text{(by the fact that } m \geq n \text{ and [GW, (10.3.3)])} \\ &= \dim B_n(-2m),\end{aligned}$$

where $\widetilde{S}^{\lambda}$ is the cell module for $B_n(-2m)$ associated to λ. By comparing dimensions, we see that φ is in fact an isomorphism. This completes the proof of Theorem 3.4 and hence the proof of Theorem 1.4 in the case $m \geq n$. □

4. The case $m < n$

We shall now embark on the case where $m < n$. Our proof will use the result for $m \geq n$, which was done in the previous section.

Recall that for $m < n$ the algebra $B_n(-2m)$ does not in general act faithfully on $V^{\otimes n}$. To prove Theorem 1.4, it suffices to show that the dimension of $\mathrm{im}(\varphi)$ is independent of the choice of the infinite field K. From now on, unless otherwise stated, we assume that K is algebraically closed. In particular, by (1.1) we can work with $Sp(V)$ instead of $GSp(V)$.

We fix $m_0 \in \mathbb{N}$ such that $m_0 \geq m$ and $m_0 - m$ is even. Let $\widetilde{V}$ be an m_0-dimensional symplectic K-vector space with ordered basis $\tilde{v}_1, \cdots, \tilde{v}_{m_0}, \tilde{v}_{m_0'}, \cdots, \tilde{v}_{1'}$ and the symplectic form given by $(\tilde{v}_i, \tilde{v}_j) = \widetilde{\epsilon}_{ij},\ \forall\, 1 \leq i, j \leq 1'$, where

$$\widetilde{\epsilon}_{ij} := \begin{cases} 1 & \text{if } j = i' \text{ and } i < j, \\ -1 & \text{if } j = i' \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

We make the convention that $1 < 2 < \cdots < m_0 < m_0' < \cdots < 2' < 1'$. Identifying v_i with $\tilde{v}_i$ and $v_{i'}$ with $\tilde{v}_{i'}$ for each $1 \leq i \leq m$, we embed V into $\widetilde{V}$ as a K-subspace. In the following we shall construct objects and maps with respect to $\widetilde{V}$ and V, which will without further notice carry a symbol "$\sim$" if they are constructed

with respect to $\widetilde{V}$ and without this symbol for V. The notion of the signs $\widetilde{\epsilon}_{ij}$ for $i, j \in \{1, \cdots, m_0, m_0', \cdots, 1'\}$ extends the ϵ_{ij} defined in the beginning for V.

We have a natural embedding of $Sp(V)$ into $Sp(\widetilde{V})$, that is,

$$Sp(V) = \Big\{ g \in Sp(\widetilde{V}) \;\Big|\; g\tilde{v}_j = \tilde{v}_j, \text{ for each } m+1 \le j \le (m+1)' \Big\}. \tag{4.1}$$

The tensor space $V^{\otimes n}$ is a direct summand of $\widetilde{V}^{\otimes n}$; let $\pi_K : \widetilde{V}^{\otimes n} \to V^{\otimes n}$ be the corresponding projection. That is, π_K sends all simple tensors which contain a tensor factor $\tilde{v}_i$ or $\tilde{v}_{i'}$ for $m+1 \le i \le m_0$ to zero.

The symplectic form defines a $KSp(V)$-isomorphism ι from V onto $V^* := \mathrm{Hom}_K(V, K)$, taking $v \in V$ to $\iota(v) := (v, -) \in V^*$; thus V and hence $V^{\otimes n}$ are self-dual $KSp(V)$-modules. The analogous statement holds for $\widetilde{V}$ and $KSp(\widetilde{V})$.

We identify $\mathrm{End}_K(V)$ with $V \otimes V^*$ in the standard way. If we represent a K-endomorphism of V as a matrix $(d_{i,j})$ $(i, j \in \{1, \cdots, m, m' \cdots, 1'\})$, relative to a basis (v_i), then the corresponding vector of $V \otimes V^*$ is

$$\sum_{i,j} d_{ij} \big(v_i \otimes \iota(v_{j'}) \big).$$

Note that $\iota(v_{j'})(v_s) = \delta_{s,j}$ for any $1 \le s \le 1'$. This construction extends easily to a tensor product by

$$\mathrm{End}_K\big(V^{\otimes n}\big) \cong V^{\otimes n} \otimes \big(V^{\otimes n}\big)^* \cong V^{\otimes n} \otimes (V^*)^{\otimes n}$$

and works similarly for $\widetilde{V}$. If $g \in Sp(V)$, $\rho : KSp(V) \to \mathrm{End}_K\big(V^{\otimes n}\big)$ is the representation afforded by the tensor space, then $\rho(g)$ acts on $\mathrm{End}_K\big(V^{\otimes n}\big)$ by conjugation. Hence $\mathrm{End}_K\big(V^{\otimes n}\big)$ is naturally a $KSp(V)$-module and the isomorphisms above are $KSp(V)$-module maps. In particular

$$\mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big) \cong \big(V^{\otimes n} \otimes (V^*)^{\otimes n}\big)^{Sp(V)},$$

where the latter denotes the invariants of $V^{\otimes n} \otimes (V^*)^{\otimes n}$ under the left diagonal action of $KSp(V)$. Using the fact that $V \cong V^*$ as a $KSp(V)$-module, we obtain $\mathrm{End}_K\big(V^{\otimes n}\big) \cong V^{\otimes 2n}$, and hence

$$\mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big) \cong \big(V^{\otimes 2n}\big)^{Sp(V)}.$$

Note that the above isomorphism sends a homomorphism represented by the matrix $(d_{\underline{i},\underline{j}})$ $(\underline{i}, \underline{j} \in I(2m, n))$, relative to a basis $(v_{\underline{i}})$, to the vector

$$\sum_{\underline{i},\underline{j} \in I(2m,n)} d_{\underline{i},\underline{j}} \big(v_{\underline{i}} \otimes v_{\underline{j}'} \big), \tag{4.2}$$

where $\underline{i} = (i_1, \cdots, i_n), \underline{j} = (j_1, \cdots, j_n), \underline{j}' = (j_1', \cdots, j_n')$. Therefore, we can express our problem in terms of invariants. A similar construction works for $\widetilde{V}$ and $Sp(\widetilde{V})$.

Since $Sp(V) \le Sp(\widetilde{V})$ we may restrict $\widetilde{V}^{\otimes 2n}$ to $Sp(V)$, and it is easy to see that the projection $\pi_K : \widetilde{V}^{\otimes 2n} \to V^{\otimes 2n}$ is $KSp(V)$-linear. In particular, $\pi_K\big(\widetilde{V}^{\otimes 2n}\big)^{Sp(\widetilde{V})} \subseteq \big(V^{\otimes 2n}\big)^{Sp(V)}$.

Lemma 4.3. *Let $\theta : B_n(-2m_0) \to B_n(-2m)$ be the K-linear isomorphism which is defined on the common basis of these algebras, consisting of Brauer diagrams, as*

the identity. Then the following diagram

$$\begin{array}{ccccc} B_n(-2m_0) & \xrightarrow{\widetilde{\varphi}} & \mathrm{End}_{KSp(\widetilde{V})}\big(\widetilde{V}^{\otimes n}\big) & \xrightarrow{\sim} & \big(\widetilde{V}^{\otimes 2n}\big)^{Sp(\widetilde{V})} \\ \theta\big\downarrow & & \pi'_K\big\downarrow & & \pi_K\big\downarrow \\ B_n(-2m) & \xrightarrow{\varphi} & \mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big) & \xrightarrow{\sim} & \big(V^{\otimes 2n}\big)^{Sp(V)} \end{array}$$

is commutative, where π'_K maps an endomorphism of $\widetilde{V}^{\otimes n}$ to its restriction to $V^{\otimes n} \subseteq \widetilde{V}^{\otimes n}$ followed by the projection π_K.

Proof. We use the same symbols to denote the standard generators for the two Brauer algebras $B_n(-2m_0), B_n(-2m)$. By definition,

$$\theta\big(d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2\big) = d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2,$$

for any $0 \le f \le [n/2]$, $\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$, $d_1, d_2 \in \mathcal{D}_\nu$, where $\nu := ((2^f),(n-2f))$. Note that θ is a K-linear map, but does not respect multiplication, since $\theta(e_1e_1) = -2m_0e_1 \neq -2me_1 = \theta(e_1)\theta(e_1)$. The same is true for π'_K.

Let $\widehat{I} = \{m+1,\cdots,m_0,m'_0,\cdots,(m+1)'\}$. We identify endomorphisms of $\widetilde{V}^{\otimes n}$ (resp., of $V^{\otimes n}$) with their matrices relative to the basis $(\tilde{v}_{\underline{i}})$ (resp., the basis $(v_{\underline{i}})$). The map π'_K just sends a matrix $(d_{\underline{i},\underline{j}})$ to its submatrix obtained by deleting those rows and columns indexed by elements in $\widehat{I}$, while π_K sends all simple tensors which contain a tensor factor $\tilde{v}_i$ for $i \in \widehat{I}$ to zero. Using (4.2), one sees easily that the right square diagram is commutative. It remains to show that $\pi'_K\widetilde{\varphi} = \varphi\theta$.

We identify π'_K with π_K. We have to show that for any $0 \le f \le [n/2]$, $\sigma \in \mathfrak{S}_{\{2f+1,\cdots,n\}}$, $d_1, d_2 \in \mathcal{D}_\nu$, where $\nu := ((2^f),(n-2f))$,

$$\begin{aligned} \pi'_K\widetilde{\varphi}\big(d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2\big) &= \varphi\theta\big(d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2\big) \\ &= \varphi\big(d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2\big), \end{aligned}$$

or equivalently,

$$\pi'_K\Big(\widetilde{\varphi}\big(d_1^{-1}\big)\widetilde{\varphi}\big(e_1e_3\cdots e_{2f-1}\big)\widetilde{\varphi}\big(\sigma d_2\big)\Big) = \varphi\big(d_1^{-1}\big)\varphi\big(e_1e_3\cdots e_{2f-1}\big)\varphi\big(\sigma d_2\big). \tag{4.4}$$

Note that for any $w \in \mathfrak{S}_n$ where both $\widetilde{\varphi}(w)$ and $\varphi(w)$ are given by right place permutation, it is trivial that $\pi'_K\widetilde{\varphi}(w) = \varphi(w)$. Now

$$\begin{aligned} \widetilde{\varphi}(e_i) = \sum_{\underline{j}\in I(2m,n)} &\widetilde{\epsilon}_{j_ij_{i+1}}\tilde{v}_{j_1}\otimes\cdots\otimes\tilde{v}_{j_{i-1}}\otimes\Big(\sum_{k=1}^{m_0}(\tilde{v}_{k'}\otimes\tilde{v}_k - \tilde{v}_k\otimes\tilde{v}_{k'})\Big)\otimes\tilde{v}_{j_{i+2}}\otimes \\ &\cdots\otimes\tilde{v}_{j_n}\otimes\tilde{v}_{j'_1}\otimes\cdots\otimes\tilde{v}_{j'_i}\otimes\tilde{v}_{j'_{i+1}}\otimes\cdots\otimes\tilde{v}_{j'_n}, \\ \varphi(e_i) = \sum_{\underline{j}\in I(2m,n)} &\epsilon_{j_ij_{i+1}}v_{j_1}\otimes\cdots\otimes v_{j_{i-1}}\otimes\Big(\sum_{k=1}^{m}(v_{k'}\otimes v_k - v_k\otimes v_{k'})\Big)\otimes v_{j_{i+2}}\otimes \\ &\cdots\otimes v_{j_n}\otimes v_{j'_1}\otimes\cdots\otimes v_{j'_i}\otimes v_{j'_{i+1}}\otimes\cdots\otimes v_{j'_n}. \end{aligned}$$

It is also easy to see that $\pi'_K\widetilde{\varphi}(e_1e_3\cdots e_{2f-1}) = \varphi(e_1e_3\cdots e_{2f-1})$.

Therefore, to prove (4.4), it suffices to show that for any $x, y \in \mathfrak{S}_n$,

$$\pi'_K\Big(\widetilde{\varphi}(x)\widetilde{\varphi}\big(e_1e_3\cdots e_{2f-1}\big)\widetilde{\varphi}(y)\Big) = \pi'_K\widetilde{\varphi}(x)\pi'_K\widetilde{\varphi}\big(e_1e_3\cdots e_{2f-1}\big)\pi'_K\widetilde{\varphi}(y).$$

But this follows from direct verification (although π'_K is in general not an algebra homomorphism). This completes the proof of the lemma. $\square$

Henceforth we assume that $m_0 \geq n$. By Theorem 3.4, $\widetilde{\varphi}$ is an isomorphism; hence φ is surjective if and only if $\pi'_K\Big(\mathrm{End}_{KSp(\widetilde{V})}\big(\widetilde{V}^{\otimes n}\big)\Big) = \mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big)$, or equivalently, $\pi_K\Big(\big(\widetilde{V}^{\otimes 2n}\big)^{Sp(\widetilde{V})}\Big) = \big(V^{\otimes 2n}\big)^{Sp(V)}$. This means that every $KSp(V)$-endomorphism f of $V^{\otimes n}$ can be extended to a $KSp(\widetilde{V})$-endomorphism $\widetilde{f}$ of $\widetilde{V}^{\otimes n}$ such that $\pi'_K\big(\widetilde{f}\big) = f$. It also means that every $Sp(V)$-invariant v of $V^{\otimes 2n}$ can be extended to a $Sp(\widetilde{V})$-invariant $\widetilde{v}$ of $\widetilde{V}^{\otimes 2n}$ such that $\pi_K(\widetilde{v}) = v$.

To accomplish this we replace the groups $Sp(V)$ and $Sp(\widetilde{V})$ by their Lie algebras $\mathfrak{g} = sp_{2m}$ and $\widetilde{\mathfrak{g}} = sp_{2m_0}$. Let $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate over $\mathbb{Z}$, and let $\mathbb{Q}(v)$ be its quotient field. Let $\mathbb{U}_{\mathcal{A}}$ respectively $\widetilde{\mathbb{U}}_{\mathcal{A}}$ be Lusztig's $\mathcal{A}$-form (see [Lu3]) in the quantized enveloping algebra of $\mathfrak{g}$ respectively $\widetilde{\mathfrak{g}}$. For any commutative integral domain R and any invertible $q \in R$ we write $\mathbb{U}_R := \mathbb{U}_{\mathcal{A}} \otimes_A R$, where we consider R as an $\mathcal{A}$-module by the specialization $v \mapsto q$. Furthermore, taking $q = 1 \in \mathbb{Z}$ and taking the quotient by the ideal generated by the $K_i - 1$ for $i = 1, \cdots, m$, one gets the Kostant's $\mathbb{Z}$-form (see [Ko], [Lu2, (8.15)] and the proof of [Lu1, (6.7)(c), (6.7)(d)])

$$\begin{aligned}\mathbf{U}_{\mathbb{Z}} &\cong \big(\mathbb{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Z}\big)/\langle K_1 - 1, \cdots, K_m - 1\rangle \cong \mathbb{U}_{\mathbb{Z}}/\langle K_1 - 1, \cdots, K_m - 1\rangle \\ &\cong \Big(\mathbb{U}_{\mathcal{A}}/\langle K_1 - 1, \cdots, K_m - 1\rangle\Big) \otimes_{\mathcal{A}} \mathbb{Z}\end{aligned}$$

in the ordinary enveloping algebra of the complex Lie algebra $sp_{2m}(\mathbb{C})$, and the hyperalgebra

$$\begin{aligned}\mathbf{U}_K &\cong \mathbf{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K \cong \big(\mathbb{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Z}\big)/\langle K_1 - 1, \cdots, K_m - 1\rangle \otimes_{\mathbb{Z}} K \\ &\cong \mathbb{U}_K/\langle K_1 - 1, \cdots, K_m - 1\rangle\end{aligned}$$

of the simply connected simple algebraic group $Sp_{2m}(K)$. Similarly we define $\widetilde{\mathbb{U}}_R$, $\widetilde{\mathbf{U}}_{\mathbb{Z}}$ and $\widetilde{\mathbf{U}}_K$.

It is well known that (see [Ja]) there is an equivalence of categories between {rational $Sp_{2m}(K)$-modules} and {locally finite $\mathbf{U}_K$-modules} such that the trivial $Sp_{2m}(K)$-module corresponds to the trivial $\mathbf{U}_K$-module, where the trivial $\mathbf{U}_K$-module is the one dimensional module which affords the counit map of the Hopf algebra $\mathbf{U}_K$. The $Sp_{2m}(K)$-action on tensor space gives rise to a locally finite $\mathbf{U}_K$-action on tensor space. Therefore

$$\mathrm{End}_{KSp(V)}\big(V^{\otimes n}\big) = \mathrm{End}_{\mathbf{U}_K}\big(V^{\otimes n}\big) \cong \big(V^{\otimes 2n}\big)^{\mathbf{U}_K} = \big(V^{\otimes 2n}\big)^{Sp(V)}.$$

This works in the same way for $\widetilde{V}$. Hence π_K is a $\mathbf{U}_K$-linear map which maps the invariants $\big(\widetilde{V}^{\otimes 2n}\big)^{\widetilde{\mathbf{U}}_K}$ into $\big(V^{\otimes 2n}\big)^{\mathbf{U}_K}$.

Our goal is to show that $\pi_K\Big(\big(\widetilde{V}^{\otimes 2n}\big)^{\widetilde{\mathbf{U}}_K}\Big) = \big(V^{\otimes 2n}\big)^{\mathbf{U}_K}$. For this purpose, we have to investigate certain nice bases of $\big(V^{\otimes 2n}\big)^{\mathbf{U}_K}$ respectively $\big(\widetilde{V}^{\otimes 2n}\big)^{\widetilde{\mathbf{U}}_K}$. Let $\widetilde{V}_{\mathcal{A}}$ (resp., $V_{\mathcal{A}}$) be the free $\mathcal{A}$-module generated by $v_1, \cdots, v_{m_0}, v_{m_0'}, \cdots, v_{1'}$ (resp., by $v_1, \cdots, v_m, v_{m'}, \cdots, v_{1'}$). Recall that there is an action of $\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}$ on $\widetilde{V}_{\mathbb{Q}(v)} :=$

$\widetilde{V}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(v)$ which is defined on generators as follows.

$$E_i\tilde{v}_j := \begin{cases} \tilde{v}_i, & \text{if } j = i+1, \\ \tilde{v}_{(i+1)'}, & \text{if } j = i', \\ 0, & \text{otherwise;} \end{cases} \quad E_{m_0}\tilde{v}_j := \begin{cases} \tilde{v}_{m_0}, & \text{if } j = m_0', \\ 0, & \text{otherwise,} \end{cases}$$

$$F_i\tilde{v}_j := \begin{cases} \tilde{v}_{i+1}, & \text{if } j = i, \\ \tilde{v}_{i'}, & \text{if } j = (i+1)', \\ 0, & \text{otherwise;} \end{cases} \quad F_{m_0}\tilde{v}_j := \begin{cases} \tilde{v}_{m_0'}, & \text{if } j = m_0, \\ 0, & \text{otherwise,} \end{cases}$$

$$K_i\tilde{v}_j := \begin{cases} v\tilde{v}_j, & \text{if } j = i \text{ or } j = (i+1)', \\ v^{-1}\tilde{v}_j, & \text{if } j = i+1 \text{ or } j = i', \\ \tilde{v}_j, & \text{otherwise;} \end{cases}$$

$$K_{m_0}\tilde{v}_j := \begin{cases} v^2\tilde{v}_j, & \text{if } j = m_0, \\ v^{-2}\tilde{v}_j, & \text{if } j = m_0', \\ \tilde{v}_j, & \text{otherwise,} \end{cases}$$

where $1 \le i < m_0$, $1 \le j \le 1'$, and we replace $\tilde{v}_{i'}$ in the usual natural representation of $\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}$ with $(-1)^{m_0-i}\tilde{v}_{i'}$ for each $1 \le i \le m_0$. This works in the same way for $\mathbb{U}_{\mathbb{Q}(v)}$ and $V_{\mathcal{A}}$. That is, we replace $v_{i'}$ in the usual natural representation of $\mathbb{U}_{\mathbb{Q}(v)}$ with $(-1)^{m-i}v_{i'}$ for each $1 \le i \le m$. The action of the generators of $\mathbb{U}_{\mathbb{Q}(v)}$ on $V_{\mathbb{Q}(v)} := V_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(v)$ is as follows.

$$E_iv_j := \begin{cases} v_i, & \text{if } j = i+1, \\ v_{(i+1)'}, & \text{if } j = i', \\ 0, & \text{otherwise;} \end{cases} \quad E_mv_j := \begin{cases} v_m, & \text{if } j = m', \\ 0, & \text{otherwise,} \end{cases}$$

$$F_iv_j := \begin{cases} v_{i+1}, & \text{if } j = i, \\ v_{i'}, & \text{if } j = (i+1)', \\ 0, & \text{otherwise;} \end{cases} \quad F_mv_j := \begin{cases} v_{m'}, & \text{if } j = m, \\ 0, & \text{otherwise,} \end{cases}$$

$$K_iv_j := \begin{cases} vv_j, & \text{if } j = i \text{ or } j = (i+1)', \\ v^{-1}v_j, & \text{if } j = i+1 \text{ or } j = i', \\ v_j, & \text{otherwise;} \end{cases}$$

$$K_mv_j := \begin{cases} v^2v_j, & \text{if } j = m, \\ v^{-2}v_j, & \text{if } j = m', \\ v_j, & \text{otherwise,} \end{cases}$$

where $1 \le i < m$, $j \in \{1, \cdots, m\} \cup \{m', \cdots, 1'\}$. Our hypothesis that $m_0 - m$ is even ensures that the new basis of $V_{\mathcal{A}}$ is still a part of the new basis of $\widetilde{V}_{\mathcal{A}}$. By [Lu3, (19.3.5)], our new basis $\{\tilde{v}_i, \tilde{v}_{i'}\}_{1 \le i \le m_0}$ (resp., $\{v_i, v_{i'}\}_{1 \le i \le m}$) is a canonical basis of $\widetilde{V}_{\mathbb{Q}(v)}$ (resp., of $V_{\mathbb{Q}(v)}$) in the sense of [Lu3].

For any field k and any specialization $v \mapsto q \in k^\times$, $V_k \cong L_k(\varepsilon_1) \cong \triangle_k(\varepsilon_1) \cong \bigtriangledown_k(\varepsilon_1)$, it follows that V_k, hence $V_k^{\otimes n}$, is a tilting module over $\mathbb{U}_k$. By [DPS, (4.4)], we have that $\mathrm{End}_{\mathbb{U}_k}\left(V_k^{\otimes n}\right) \cong \mathrm{End}_{\mathbb{U}_{\mathcal{A}}}\left(V_{\mathcal{A}}^{\otimes n}\right) \otimes_{\mathcal{A}} k$, and the dimension of $\mathrm{End}_{\mathbb{U}_k}\left(V_k^{\otimes n}\right)$ is independent of k. The same is true for $\widetilde{V}_k$ and $\widetilde{\mathbb{U}}_k$.

For each $1 \le i \le 1'$, $\iota(\tilde{v}_i) = (\tilde{v}_i, -) \in \widetilde{V}_{\mathcal{A}}^* := \mathrm{Hom}_{\mathcal{A}}(\widetilde{V}_{\mathcal{A}}, \mathcal{A})$. Then $\iota(\tilde{v}_1)$ is a highest weight vector of weight ε_1. The map $\tilde{v}_1 \mapsto \iota(\tilde{v}_1)$ extends naturally to a $\widetilde{\mathbb{U}}_{\mathcal{A}}$-module isomorphism $\iota' : \widetilde{V}_{\mathcal{A}} \cong \widetilde{V}_{\mathcal{A}}^* = \widetilde{\mathbb{U}}_{\mathcal{A}}\iota(\tilde{v}_1)$. One checks easily that

$$\iota'(\tilde{v}_i) = v^{i-1}\iota(\tilde{v}_i), \;\; \iota'(\tilde{v}_{i'}) = v^{2m_0+1-i}\iota(\tilde{v}_{i'}), \;\; \forall\, 1 \le i \le m_0.$$

Using the isomorphism ι', we get that

$$\begin{aligned}\mathrm{End}_{\widetilde{\mathbb{U}}_{\mathcal{A}}}(\widetilde{V}_{\mathcal{A}}^{\otimes n}) &\cong \Big(\mathrm{End}(\widetilde{V}_{\mathcal{A}}^{\otimes n})\Big)^{\widetilde{\mathbb{U}}_{\mathcal{A}}} \cong \Big(\widetilde{V}_{\mathcal{A}}^{\otimes n} \otimes (\widetilde{V}_{\mathcal{A}}^{\otimes n})^*\Big)^{\widetilde{\mathbb{U}}_{\mathcal{A}}} \\ &\cong \Big(\widetilde{V}_{\mathcal{A}}^{\otimes n} \otimes (\widetilde{V}_{\mathcal{A}}^{*})^{\otimes n}\Big)^{\widetilde{\mathbb{U}}_{\mathcal{A}}} \cong (\widetilde{V}_{\mathcal{A}}^{\otimes 2n})^{\widetilde{\mathbb{U}}_{\mathcal{A}}}.\end{aligned}$$

Similarly, $\mathrm{End}_{\mathbb{U}_{\mathcal{A}}}(V_{\mathcal{A}}^{\otimes n}) \cong (V_{\mathcal{A}}^{\otimes 2n})^{\mathbb{U}_{\mathcal{A}}}$. Consequently, for any field k and any specialization $v \mapsto q \in k^\times$,

$$(\widetilde{V}_k^{\otimes 2n})^{\widetilde{\mathbb{U}}_k} \cong \mathrm{End}_{\widetilde{\mathbb{U}}_k}(\widetilde{V}_k^{\otimes n}) \cong \mathrm{End}_{\widetilde{\mathbb{U}}_{\mathcal{A}}}(\widetilde{V}_{\mathcal{A}}^{\otimes n}) \otimes_{\mathcal{A}} k \cong (\widetilde{V}_{\mathcal{A}}^{\otimes 2n})^{\widetilde{\mathbb{U}}_{\mathcal{A}}} \otimes_{\mathcal{A}} k.$$

Similarly, $(V_k^{\otimes 2n})^{\mathbb{U}_k} \cong \mathrm{End}_{\mathbb{U}_k}(V_k^{\otimes n}) \cong \mathrm{End}_{\mathbb{U}_{\mathcal{A}}}(V_{\mathcal{A}}^{\otimes n}) \otimes_{\mathcal{A}} k \cong (V_{\mathcal{A}}^{\otimes 2n})^{\mathbb{U}_{\mathcal{A}}} \otimes_{\mathcal{A}} k$. Note that when specializing q to 1, each K_i acts as the identity on the tensor space $V^{\otimes 2n}$. It follows that

$$\begin{aligned}(V_{\mathbb{Z}}^{\otimes 2n})^{\mathbf{U}_{\mathbb{Z}}} &\cong \mathrm{End}_{\mathbf{U}_{\mathbb{Z}}}(V_{\mathbb{Z}}^{\otimes n}) \cong \mathrm{End}_{\mathbb{U}_{\mathbb{Z}}}(V_{\mathbb{Z}}^{\otimes n}) \\ &\cong \mathrm{End}_{\mathbb{U}_{\mathcal{A}}}(V_{\mathcal{A}}^{\otimes n}) \otimes_{\mathcal{A}} \mathbb{Z} \cong (V_{\mathcal{A}}^{\otimes 2n})^{\mathbb{U}_{\mathcal{A}}} \otimes_{\mathcal{A}} \mathbb{Z}\end{aligned}$$

and

$$\begin{aligned}(V_K^{\otimes 2n})^{\mathbf{U}_K} &\cong \mathrm{End}_{\mathbf{U}_K}(V_K^{\otimes n}) \cong \mathrm{End}_{\mathbb{U}_K}(V_K^{\otimes n}) \cong \mathrm{End}_{\mathbb{U}_{\mathcal{A}}}(V_{\mathcal{A}}^{\otimes n}) \otimes_{\mathcal{A}} K \\ &\cong (V_{\mathcal{A}}^{\otimes 2n})^{\mathbb{U}_{\mathcal{A}}} \otimes_{\mathcal{A}} K \cong (V_{\mathcal{A}}^{\otimes 2n})^{\mathbb{U}_{\mathcal{A}}} \otimes_{\mathcal{A}} \mathbb{Z} \otimes_{\mathbb{Z}} K \cong (V_{\mathbb{Z}}^{\otimes 2n})^{\mathbf{U}_{\mathbb{Z}}} \otimes_{\mathbb{Z}} K.\end{aligned}$$

Similar results hold for $\widetilde{V}$, $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbb{U}}$.

In [Lu3, (27.1.2)], Lusztig introduced the notion of a based module and by [Lu3, (27.3)], the $\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}$-module $\widetilde{M} := (\widetilde{V}_{\mathbb{Q}(v)})^{\otimes 2n}$ is a based module. That is, there is a canonical basis $\widetilde{B}$ of $\widetilde{M}$, in Lusztig's notation ([Lu3, (27.3.2)]), each element in $\widetilde{B}$ is of the form $\tilde{v}_{i_1}\tilde{\diamond}\tilde{v}_{i_2}\tilde{\diamond}\cdots\tilde{\diamond}\tilde{v}_{i_{2n}}$, and $\tilde{v}_{i_1}\tilde{\diamond}\cdots\tilde{\diamond}\tilde{v}_{i_{2n}}$ is equal to $\tilde{v}_{i_1}\otimes\cdots\otimes\tilde{v}_{i_{2n}}$ plus a linear combination of elements $\tilde{v}_{j_1}\otimes\cdots\otimes\tilde{v}_{j_{2n}}$ with $(\tilde{v}_{j_1},\cdots,\tilde{v}_{j_{2n}}) < (\tilde{v}_{i_1},\cdots,\tilde{v}_{i_{2n}})$ and with coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$, where "$<$" is a partial order defined in [Lu3, (27.3.1)]. In particular, $\widetilde{B}$ is an $\mathcal{A}$-basis of $\widetilde{V}_{\mathcal{A}}^{\otimes 2n}$. Similarly, we define $M := (V_{\mathbb{Q}(v)})^{\otimes 2n}$ as a module over $\mathbb{U}_{\mathbb{Q}(v)}$, and we have a canonical basis B of M. Each element of B is of the form $v_{i_1}\diamond v_{i_2}\diamond\cdots\diamond v_{i_{2n}}$.

Let $\widetilde{X}_+$ be the set of all the dominant weights of $\widetilde{\mathfrak{g}}$. For $\lambda \in \widetilde{X}_+$, we denote by $\Delta_{\mathbb{Q}(v)}(\lambda)$ the irreducible $\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}$-module of highest weight λ. We define

$$\widetilde{M}[\lambda] := \sum_{\substack{\widetilde{N}\subseteq\widetilde{M} \\ \widetilde{N}\cong\Delta_{\mathbb{Q}(v)}(\lambda)}} \widetilde{N}.$$

Then

$$\widetilde{M} = \bigoplus_{\lambda\in\widetilde{X}_+} \widetilde{M}[\lambda].$$

For each $\lambda \in \widetilde{X}_+$, let $\widetilde{M}[>\lambda] := \bigoplus_{\lambda<\mu\in\widetilde{X}_+} \widetilde{M}[\mu]$ and define $\widetilde{B}[>\lambda] := \widetilde{B}\cap\widetilde{M}[>\lambda]$. By [Lu3, (27.1.8)(b)], $\widetilde{B}[>\lambda]$ is a $\mathbb{Q}(v)$-basis of $\widetilde{M}[>\lambda]$. We define $\widetilde{M}[>\lambda]_{\mathcal{A}} :=$

$\sum_{b\in\widetilde{B}[>\lambda]}\mathcal{A}b$. By [Lu3, (27.1.2)(b), (27.1.8)], it is easy to see that $\widetilde{M}[>\lambda]_{\mathcal{A}}$ is stable under $\widetilde{\mathbb{U}}_{\mathcal{A}}$. Hence for any specialization $v\mapsto q\neq 0$ in some field K, $\widetilde{M}[>\lambda]_K := \sum_{b\in\widetilde{B}[>\lambda]}Kb$ is $\widetilde{\mathbb{U}}_K$-stable and the set $\{b \mid b\in\widetilde{B}[>\lambda]\}$ forms a K-basis of $\widetilde{M}[>\lambda]_K$. Let $\widetilde{M}[\neq 0] := \bigoplus_{\lambda\neq 0}\widetilde{M}[\lambda]$. By [Lu3, (27.2.5)],

$$\widetilde{M}[\neq 0] = \sum_{\mu\in\widetilde{X}_+-\{0\}}\widetilde{M}[\geq\mu].$$

In particular, $\widetilde{B}[\neq 0] := \bigsqcup_{\lambda\neq 0}\widetilde{B}[\lambda]$ forms a $\mathbb{Q}(v)$-basis of $\widetilde{M}[\neq 0]$. We define $\widetilde{M}[\neq 0]_{\mathcal{A}} := \sum_{b\in\widetilde{B}[\neq 0]}\mathcal{A}b$. Then $\widetilde{M}[\neq 0]_{\mathcal{A}}$ is stable under $\widetilde{\mathbb{U}}_{\mathcal{A}}$. Hence for any specialization $v\mapsto q\neq 0$ in some field K, $\widetilde{M}[\neq 0]_K := \sum_{b\in\widetilde{B}[\neq 0]}Kb$ is $\widetilde{\mathbb{U}}_K$-stable and the set $\{b \mid b\in\widetilde{B}[\neq 0]\}$ forms a K-basis of $\widetilde{M}[\neq 0]_K$. The isomorphism ι' induces a natural isomorphism $\widetilde{V}_{\mathcal{A}}^{\otimes 2n}\cong\left(\widetilde{V}_{\mathcal{A}}^{\otimes 2n}\right)^*$, which we still denote by ι'. It is clear that ι' maps $\left(\widetilde{V}_{\mathbb{Q}(v)}^{\otimes 2n}\right)^{\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}}$ isomorphically onto $\left(\widetilde{V}_{\mathbb{Q}(v)}^{\otimes 2n}/\widetilde{M}[\neq 0]\right)^*$. In particular, $\iota'(a)$ vanishes on $\widetilde{B}[\neq 0]$ for every $a\in\left(\widetilde{V}_{\mathcal{A}}^{\otimes 2n}\right)^{\widetilde{\mathbb{U}}_{\mathcal{A}}}$. Therefore, ι' maps $\left(\widetilde{V}_{\mathcal{A}}^{\otimes 2n}\right)^{\widetilde{\mathbb{U}}_{\mathcal{A}}}$ into $\left(\widetilde{V}_{\mathcal{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathcal{A}}\right)^*$. By comparing dimensions, we conclude that for each field K which is an $\mathcal{A}$-algebra, ι' maps $\left(\widetilde{V}_K^{\otimes 2n}\right)^{\widetilde{\mathbb{U}}_K}$ isomorphically onto $\left(\widetilde{V}_K^{\otimes 2n}/\widetilde{M}[\neq 0]_K\right)^*$. As a consequence, ι' also maps $\left(\widetilde{V}_{\mathcal{A}}^{\otimes 2n}\right)^{\widetilde{\mathbb{U}}_{\mathcal{A}}}$ isomorphically onto $\left(\widetilde{V}_{\mathcal{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathcal{A}}\right)^*$. Similarly, one can define X_+ (the set of all the dominant weights of $\mathfrak{g}$), and for each $\lambda\in X_+$, one can define $M[\lambda]$, $M[>\lambda], B[>\lambda], M[\neq 0]$ and $B[\neq 0]$. One has that $M=\bigoplus_{\lambda\in X_+}M[\lambda]$, and $\left(V_{\mathcal{A}}^{\otimes 2n}\right)^{\mathbb{U}_{\mathcal{A}}}$ is canonically isomorphic to $\left(V_{\mathcal{A}}^{\otimes 2n}/M[\neq 0]_{\mathcal{A}}\right)^*$.

Recall that (see [Lu3, (27.2.1)]),

$$B=\bigsqcup_{\lambda\in X_+}B[\lambda],\quad \widetilde{B}=\bigsqcup_{\lambda\in\widetilde{X}_+}\widetilde{B}[\lambda].$$

By [Lu3, (27.2.5)], the image of $\widetilde{B}[0]$ (resp., $B[0]$) in $\widetilde{V}_{\mathcal{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathcal{A}}$ (resp., in $V_{\mathcal{A}}^{\otimes 2n}/M[\neq 0]_{\mathcal{A}}$) forms an $\mathcal{A}$-basis of $\widetilde{V}_{\mathcal{A}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathcal{A}}$ (resp., of $V_{\mathcal{A}}^{\otimes 2n}/M[\neq 0]_{\mathcal{A}}$). Let

$$J_0 := \{(i_1,\cdots,i_{2n})\in I(2m,2n) \mid v_{i_1}\diamond\cdots\diamond v_{i_{2n}}\in B[0]\},$$
$$\widetilde{J}_0 := \{(i_1,\cdots,i_{2n})\in I(2m_0,2n) \mid \tilde{v}_{i_1}\tilde{\diamond}\cdots\tilde{\diamond}\tilde{v}_{i_{2n}}\in\widetilde{B}[0]\}.$$

Corollary 4.5. *With the above notation the set*

$$\{v_{i_1}\otimes\cdots\otimes v_{i_{2n}}+M[>0]_{\mathcal{A}} \mid (i_1,\cdots,i_{2n})\in J_0\}$$

forms an $\mathcal{A}$-basis of $V_{\mathcal{A}}^{\otimes 2n}/M[>0]_{\mathcal{A}}$.

Proof. This is clear, by the fact that the image of $B[0]$ in $V_{\mathcal{A}}^{\otimes 2n}/M[>0]_{\mathcal{A}}$ is an $\mathcal{A}$-basis and each $v_{i_1}\diamond\cdots\diamond v_{i_{2n}}$ is equal to $v_{i_1}\otimes\cdots\otimes v_{i_{2n}}$ plus a linear combination of elements $v_{j_1}\otimes\cdots\otimes v_{j_{2n}}$ with $(v_{j_1},\cdots,v_{j_{2n}})<(v_{i_1},\cdots,v_{i_{2n}})$ and with coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$. $\square$

Similarly, the set

$$\left\{\tilde{v}_{i_1}\otimes\cdots\otimes\tilde{v}_{i_{2n}}+\widetilde{M}[>0]_{\mathcal{A}}\;\middle|\;(i_1,\cdots,i_{2n})\in\widetilde{J}_0\right\} \tag{4.6}$$

forms an $\mathcal{A}$-basis of $\widetilde{V}_{\mathcal{A}}^{\otimes 2n}/\widetilde{M}[>0]_{\mathcal{A}}$.

Theorem 4.7. *With the above notation, $J_0\subseteq\widetilde{J}_0$.*

Proof. For each $1\le i\le m_0$, let $\widetilde{e}_i,\widetilde{f}_i$ (resp., e_i, f_i) be the Kashiwara operators of $\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}$ (resp., of $\mathbb{U}_{\mathbb{Q}(v)}$). The $\widetilde{\mathbb{U}}_{\mathbb{Q}(v)}(sp_{2m_0})$-crystal structure on $\widetilde{V}_{\mathbb{Q}(v)}$ is given below:

$$\boxed{1}\xrightarrow{1}\boxed{2}\xrightarrow{2}\cdots\xrightarrow{m_0-1}\boxed{m_0}\xrightarrow{m_0}\boxed{m_0'}\xrightarrow{m_0-1}\cdots\xrightarrow{2}\boxed{2'}\xrightarrow{1}\boxed{1'},$$

where

$$\boxed{j}\xrightarrow{i}\boxed{k}\Leftrightarrow\tilde{f}_i\bar{v}_j\equiv\tilde{v}_k\ (\mathrm{mod}\,v^{-1}\widetilde{M})\Leftrightarrow\tilde{v}_j\equiv\tilde{e}_i\tilde{v}_k\ (\mathrm{mod}\,v^{-1}\widetilde{M}).$$

Similarly, the $\mathbb{U}_{\mathbb{Q}(v)}$-crystal structure on $V_{\mathbb{Q}(v)}$ is as below:

$$\boxed{1}\xrightarrow{1}\boxed{2}\xrightarrow{2}\cdots\xrightarrow{m-1}\boxed{m}\xrightarrow{m}\boxed{m'}\xrightarrow{m-1}\cdots\xrightarrow{2}\boxed{2'}\xrightarrow{1}\boxed{1'}.$$

Comparing with the two crystal graphs, it is easy to see that for each $1\le i\le m$ and each $j\in\{1,\cdots,m\}\cup\{m',\cdots,1'\}$,

$$\max\left\{k\ge 0\;\middle|\;\widetilde{e}_i^k\tilde{v}_j\notin v^{-1}\widetilde{M}\right\}=\max\left\{k\ge 0\;\middle|\;e_i^k v_j\notin v^{-1}M\right\},$$
$$\max\left\{k\ge 0\;\middle|\;\widetilde{f}_i^k\tilde{v}_j\notin v^{-1}\widetilde{M}\right\}=\max\left\{k\ge 0\;\middle|\;f_i^k v_j\notin v^{-1}M\right\}.$$

Moreover, for each $m+1\le i\le m_0$ and each $j\in\{1,\cdots,m\}\cup\{m',\cdots,1'\}$,

$$\widetilde{e}_i\tilde{v}_j\in v^{-1}\widetilde{M},\quad \widetilde{f}_i\tilde{v}_j\in v^{-1}\widetilde{M}.$$

Let B' (resp., $\widetilde{B}'$) be the canonical basis of $V^{\otimes n}$ (resp., of $\widetilde{V}^{\otimes n}$) constructed from the canonical basis of V (resp., of $\widetilde{V}$); see [Lu3, (27.3.1)]. For each $\lambda\in X_+$ (resp., $\lambda\in\widetilde{X}_+$), let $B'[\lambda]^{lo},B'[\lambda]^{hi}$ (resp., $\widetilde{B'}[\lambda]^{lo},\widetilde{B'}[\lambda]^{hi}$) be as defined in [Lu3, (27.2.3)].

Now [Lu3, (17.2.4)] gave the rules for the action of the Kashiwara operators $\widetilde{e}_i,\widetilde{f}_i,e_j,f_j$ on tensor products. As a consequence, our previous discussion shows that for any $1\le i_1,\cdots,i_n\le 2m$,

$$v_{i_1}\diamond\cdots\diamond v_{i_n}\in B'[\lambda]^{hi}\Longleftrightarrow\tilde{v}_{i_1}\tilde{\diamond}\cdots\tilde{\diamond}\tilde{v}_{i_n}\in\widetilde{B'}[\lambda]^{hi},$$
$$v_{i_1}\diamond\cdots\diamond v_{i_n}\in B'[\lambda]^{lo}\Longleftrightarrow\tilde{v}_{i_1}\tilde{\diamond}\cdots\tilde{\diamond}\tilde{v}_{i_n}\in\widetilde{B'}[\lambda]^{lo}.$$

Now applying [Lu3, (27.3.8)], which said that

$$\widetilde{B}[0]=\bigcup_{\lambda\in\widetilde{X}_+}\left\{b\,\tilde{\diamond}\,b'\;\middle|\;b\in\widetilde{B}'[-w_0(\lambda')]^{lo},b'\in\widetilde{B}'[\lambda']^{hi}\right\},$$
$$B[0]=\bigcup_{\lambda\in X_+}\left\{b\diamond b'\;\middle|\;b\in B'[-w_0(\lambda')]^{lo},b'\in B'[\lambda']^{hi}\right\},$$

our theorem follows immediately. □

Proof of Theorem 1.4. We regard $\mathbb{Z}$ as an $\mathcal{A}$-algebra by specializing v to $1\in\mathbb{Z}$, and regard K as a $\mathbb{Z}$-algebra as usual. Then it is easy to see that $\iota'\otimes_{\mathcal{A}}1_K$ coincides with the canonical Sp_{2m}-module isomorphism $V\to V^*$, $v\mapsto\iota(v):=(v,-)$ for any $v\in V$. Let $\widetilde{V}_{\mathbb{Z}}:=\widetilde{V}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathbb{Z}$, $\widetilde{M}[\neq 0]_{\mathbb{Z}}:=\widetilde{M}[\neq 0]_{\mathcal{A}}\otimes_{\mathcal{A}}\mathbb{Z}$. We have similar notation for

$V_{\mathbb{Z}}$, $M[\neq 0]_{\mathbb{Z}}$. We claim that the natural projection map $\left(\widetilde{V}^{\otimes 2n}\right)^{\widetilde{\mathbf{U}}_K} \to \left(V^{\otimes 2n}\right)^{\mathbf{U}_K}$ is surjective.

In fact, we have the following commutative diagram:

$$\begin{array}{ccccc}
\left(\widetilde{V}^{\otimes 2n}\right)^{\widetilde{\mathbf{U}}_K} & \xrightarrow{\sim} & \left(\frac{\widetilde{V}^{\otimes 2n}}{\widetilde{M}[\neq 0]_K}\right)^* & \xrightarrow{\sim} & \left(\frac{\widetilde{V}_{\mathbb{Z}}^{\otimes 2n}}{\widetilde{M}[\neq 0]_{\mathbb{Z}}}\right)^* \otimes_{\mathbb{Z}} K \\
{\scriptstyle \pi_K}\downarrow & & {\scriptstyle j_K^*}\downarrow & & {\scriptstyle j_{\mathbb{Z}}^* \otimes_{\mathbb{Z}} 1}\downarrow \\
\left(V^{\otimes 2n}\right)^{\mathbf{U}_K} & \xrightarrow{\sim} & \left(\frac{V^{\otimes 2n}}{M[\neq 0]_K}\right)^* & \xrightarrow{\sim} & \left(\frac{V_{\mathbb{Z}}^{\otimes 2n}}{M[\neq 0]_{\mathbb{Z}}}\right)^* \otimes_{\mathbb{Z}} K
\end{array}$$

where the rightmost vertical homomorphism is induced from the canonical homomorphism $j_{\mathbb{Z}} : V_{\mathbb{Z}}^{\otimes 2n}/M[\neq 0]_{\mathbb{Z}} \to \widetilde{V}_{\mathbb{Z}}^{\otimes 2n}/\widetilde{M}[\neq 0]_{\mathbb{Z}}$. Note that $j_{\mathbb{Z}}$ is well defined as $M[\neq 0]_{\mathbb{Z}} \subseteq \widetilde{M}[\neq 0]_{\mathbb{Z}}$ (which follows from the fact that for each $\lambda \in X_+$ with $\lambda \neq 0$, $M_{\mathbb{C}}[\lambda]$ should be contained in $\widetilde{M}_{\mathbb{C}}[\neq 0]$).

By (4.6) and Theorem 4.7, the image of

$$\left\{ v_{i_1} \otimes \cdots \otimes v_{i_{2n}} + M[>0]_K \;\middle|\; (i_1, \cdots, i_{2n}) \in J_0 \right\}$$

under $j_K := j_{\mathbb{Z}} \otimes_{\mathbb{Z}} 1_K$ is always linearly independent, which shows that j_K is injective. Hence $j_K^* := j_{\mathbb{Z}}^* \otimes_{\mathbb{Z}} 1_K$ is surjective. It follows that

$$\pi_K\left(\left(\widetilde{V}^{\otimes 2n}\right)^{\widetilde{\mathbf{U}}_K}\right) = \left(V^{\otimes 2n}\right)^{\mathbf{U}_K},$$

as required. Now using Lemma 4.3 and Theorem 3.4, we complete the proof of Theorem 1.4 when K is algebraically closed.

Now suppose that K is an arbitrary infinite field. Let $\overline{K}$ denote the algebraic closure of K. Note that the image of φ is generated (as an algebra) by

$$\left\{\varphi(e_1), \cdots, \varphi(e_{n-1}), \varphi(s_1), \cdots, \varphi(s_{n-1})\right\},$$

and the canonical homomorphism

$$\begin{aligned}
\mathrm{End}_{KSp(V_K)}\left(V_K^{\otimes n}\right) \otimes_K \overline{K} &= \mathrm{End}_{\mathbf{U}_K}\left(V_K^{\otimes n}\right) \otimes_K \overline{K} \\
&\to \mathrm{End}_{\mathbf{U}_{\overline{K}}}\left(V_{\overline{K}}^{\otimes n}\right) = \mathrm{End}_{\overline{K}Sp(V_{\overline{K}})}\left(V_{\overline{K}}^{\otimes n}\right)
\end{aligned}$$

is an isomorphism, where $\mathbf{U}_K = \mathbf{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$, $\mathbf{U}_{\overline{K}} = \mathbf{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{K} \cong \mathbf{U}_K \otimes_K \overline{K}$. It follows that the dimension of $\mathrm{im}(\varphi)$ is constant under field extensions $K \subseteq \overline{K}$. The proof is completed. □

Remark 4.8. The argument above in the proof of Theorem 1.4 actually shows that

$$\pi_{\mathbb{Z}}\left(\left(\widetilde{V}_{\mathbb{Z}}^{\otimes 2n}\right)^{\widetilde{\mathbf{U}}_{\mathbb{Z}}}\right) = \left(V_{\mathbb{Z}}^{\otimes 2n}\right)^{\mathbf{U}_{\mathbb{Z}}},$$

or equivalently, $\pi_{\mathbb{Z}}\left(\mathrm{End}_{\widetilde{\mathbf{U}}_{\mathbb{Z}}}\left(\widetilde{V}_{\mathbb{Z}}^{\otimes n}\right)\right) = \mathrm{End}_{\mathbf{U}_{\mathbb{Z}}}\left(V_{\mathbb{Z}}^{\otimes n}\right)$.

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