

TWISTED K -THEORY AND POINCARÉ DUALITY

JEAN-LOUIS TU

ABSTRACT. Using methods of KK -theory, we generalize Poincaré K -duality to the framework of twisted K -theory.

INTRODUCTION

In [4], Connes and Skandalis showed, using Kasparov's KK -theory, that given a compact manifold M , the K -theory of M is isomorphic to the K -homology of TM and vice-versa. It is well known to experts that a similar result holds in twisted K -theory, although this is apparently written nowhere in the literature. In this paper, using Kasparov's more direct approach, we show that given any (graded) locally trivial bundle \mathcal{A} of elementary C^* -algebras over M , the C^* -algebras of continuous sections $C(M, \mathcal{A})$ and $C(M, \mathcal{A}^{op} \hat{\otimes} \text{Cliff}(TM \otimes \mathbb{C}))$ are K -dual to each other. When $\mathcal{A} = M$ is the trivial bundle, we recover Poincaré duality between $C(M)$ and $C_\tau(M) := C(M, \text{Cliff}(TM \otimes \mathbb{C}))$ [14], which is equivalent to Poincaré duality between $C(M)$ and $C_0(TM)$ since $C_\tau(M)$ and $C_0(TM)$ are KK -equivalent to each other.

Let us review some of the literature related to the present paper. Thom isomorphism, which is closely related to Poincaré duality, was established by Karoubi in [11] in the setting of non-twisted K -theory, and by Carey and Wang [5] and Karoubi [12] in twisted K -theory. Let us also mention that (a slight variation) of our main result was obtained independently and simultaneously by Echterhoff, Emerson and Kim [7].

1. TWISTED K -THEORY

In this section, we review the basic theory of twisted K -theory in the graded setting (see [6, 12, 1, 8, 17, 16, 3]).

1.1. Graded Dixmier-Douady bundles. Let M be a (finite dimensional) manifold endowed with an action of a locally compact group G . A *graded, G -equivariant Dixmier-Douady bundle* of type 0 (resp. of type 1) \mathcal{A} over M is a locally trivial bundle of $\mathbb{Z}/2\mathbb{Z}$ -graded C^* -algebras over M , endowed with a continuous action of G , such that for all $x \in M$, the fiber \mathcal{A}_x is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$ -graded algebra $\mathcal{K}(\hat{H}_x)$ of compact operators over a $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert space \hat{H}_x (resp. to the $\mathbb{Z}/2\mathbb{Z}$ -graded algebra $\mathcal{K}(H_x) \oplus \mathcal{K}(H_x) \cong \mathcal{K}(H_x) \hat{\otimes} \mathbb{C}\ell_1$, where H_x is some Hilbert space and $\mathbb{C}\ell_1$ is the first complex Clifford algebra). Beware that H_x or \hat{H}_x does

Received by the editors November 29, 2006.

2000 *Mathematics Subject Classification.* Primary 46L80; Secondary 19K35.

Key words and phrases. Twisted K -theory, KK -theory, Poincaré duality.

©2008 American Mathematical Society
Reverts to public domain 28 years from publication

not necessarily depend continuously on x . Of course, the action of G is required to preserve the degree.

Note that the terminology “Dixmier-Douady bundle” is not standard, but used in this paper for convenience. Also, [16] considers more general bundles, which are not necessarily locally trivial but are stably isomorphic to locally trivial ones; hence the notion of graded Brauer group introduced in [16] coincides with ours.

Denote by \hat{H} the graded Hilbert space $H^0 \oplus H^1$, where $H^i = \ell^2(\mathbb{N})$, and $\hat{H}_G = L^2(G) \otimes \hat{H}$.

Two graded D-D bundles \mathcal{A} and \mathcal{A}' are said to be Morita equivalent if (they have the same type and) $\mathcal{A} \hat{\otimes} \mathcal{K}(\hat{H}_G) \cong \mathcal{A}' \hat{\otimes} \mathcal{K}(\hat{H}_G)$. The set of Morita equivalence classes of graded D-D bundles forms a group $\widehat{Br}_{*,G}(M) = \widehat{Br}_{0,G}(M) \oplus \widehat{Br}_{1,G}(M)$, the graded equivariant Brauer group of \mathcal{M} . The sum of \mathcal{A} and \mathcal{A}' is $\mathcal{A} \hat{\otimes} \mathcal{A}'$ (note that the types do add up), and the opposite \mathcal{A}^{op} of \mathcal{A} is the bundle whose fiber at $x \in \mathcal{M}^{(0)}$ is the conjugate algebra of \mathcal{A}_x . In other words, $(\mathcal{A}^{op})_x = \mathcal{K}(\hat{H}_x^*)$ (resp. $(\mathcal{A}^{op})_x = \mathcal{K}(H_x^*) \oplus \mathcal{K}(H_x^*)$) in the even (resp. odd) case.

Moreover, $\sigma : \mathcal{A} \mapsto \mathcal{A} \hat{\otimes} \mathcal{C}\ell_1$ is an isomorphism from $\widehat{Br}_{i,G}(M)$ to $\widehat{Br}_{1-i,G}(M)$ such that $\sigma^2 = \text{Id}$; hence $\widehat{Br}_{*,G}(M) \cong \widehat{Br}_{0,G}(M) \times \mathbb{Z}/2\mathbb{Z}$. Therefore, to study $\widehat{Br}_{*,G}(M)$ it suffices to study $\widehat{Br}_{0,G}(M)$.

Let us recall the relation with the ordinary Brauer group $Br_G(M)$ of Morita equivalence classes of ungraded D-D bundles. Recall that $Br_G(M) \cong H_G^2(M, \underline{S}^1)$, where \underline{S}^1 is the sheaf of smooth S^1 -valued functions, and that $Br_G(M) \cong H_G^3(M, \mathbb{Z})$ when the action of G is proper.

There is a split exact sequence [6, 1, 8]

$$(1) \quad 0 \rightarrow Br_G(M) \rightarrow \widehat{Br}_{0,G}(M) \rightarrow H_G^1(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow 0.$$

Furthermore, in the decomposition $\widehat{Br}_{0,G}(M) \cong H_G^1(M, \mathbb{Z}/2\mathbb{Z}) \times Br_G(M)$, the sum becomes

$$(\delta_1, \mathcal{A}_1) + (\delta_2, \mathcal{A}_2) = (\delta_1 + \delta_2, \mathcal{A}_1 + \mathcal{A}_2 + \delta_1 \cdot \delta_2).$$

1.2. Graded S^1 -central extensions. Although the material in this subsection is not used directly in the main theorem, it might be useful to include it since twistings are naturally constructed from central extensions (as in [9]). Since this is a minor variation of e.g. [17], we include no proofs.

In this subsection, \mathcal{M} denotes the transformation groupoid $M \rtimes G$ (actually the theory extends in an obvious way to arbitrary Lie groupoids).

Definition 1.1. A graded S^1 -central extension of a groupoid $\Gamma \rightrightarrows \Gamma^{(0)}$ is a central extension $S^1 \rightarrow \tilde{\Gamma} \xrightarrow{\pi} \Gamma$, together with a groupoid morphism $\delta : \Gamma \rightarrow \mathbb{Z}/2\mathbb{Z}$.

One defines the sum of two graded central extensions $(\tilde{\Gamma}_1, \delta_1)$ and $(\tilde{\Gamma}_2, \delta_2)$ as $(\tilde{\Gamma}, \delta)$, where $\delta(g) = \delta_1(g) + \delta_2(g)$ and

$$\tilde{\Gamma} = (\tilde{\Gamma}_1 \times_{\Gamma} \tilde{\Gamma}_2) / S^1 = \{(g_1, g_2) \in \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 \mid \pi_1(g_1) = \pi_2(g_2)\} / \sim,$$

where \sim is the equivalence relation

$$(g_1, g_2) \sim (g'_1, g'_2) \iff \exists \lambda \in S^1, (g'_1, g'_2) = (\lambda g_1, \lambda^{-1} g_2).$$

The multiplication for the groupoid $\tilde{\Gamma}$ is $(\tilde{g}_1, \tilde{g}_2)(\tilde{h}_1, \tilde{h}_2) = (-1)^{\delta_2(g)\delta_1(h)}(\tilde{g}_1\tilde{h}_1, \tilde{g}_2\tilde{h}_2)$, where $g = \pi_i(\tilde{g}_i)$, $h = \pi_i(\tilde{h}_i)$.

Note that the set of isomorphism classes of graded S^1 -central extensions of Γ forms an abelian group. To see that the product is commutative, if $\tilde{\Gamma}' = (\tilde{\Gamma}_2 \times_{\Gamma} \tilde{\Gamma}_1)/S^1$ is endowed with the product $(\tilde{g}_2, \tilde{g}_1)(\tilde{h}_2, \tilde{h}_1) = (-1)^{\delta_1(g)\delta_2(h)}(\tilde{g}_2\tilde{h}_2, \tilde{g}_1\tilde{h}_1)$, then

$$\begin{aligned} \tilde{\Gamma} &\rightarrow \tilde{\Gamma}' \\ (\tilde{g}_1, \tilde{g}_2) &\mapsto (-1)^{\delta_1(g)\delta_2(g)}(\tilde{g}_2, \tilde{g}_1) \end{aligned}$$

is an S^1 -equivariant isomorphism.

To see that $(\tilde{\Gamma}, \delta)$ has an inverse, let $\tilde{\Gamma}^{op}$ be equal to $\tilde{\Gamma}$ as a set, but the S^1 -principal bundle structure is replaced by the conjugate one, and the product $*_{op}$ in $\tilde{\Gamma}^{op}$ is

$$\tilde{g} *_{op} \tilde{h} = (-1)^{\delta(g)\delta(h)}\tilde{g}\tilde{h}.$$

Then

$$\begin{aligned} \Gamma \times S^1 &\rightarrow (\tilde{\Gamma} \times_{\Gamma} \tilde{\Gamma}^{op})/S^1 \\ (g, \lambda) &\mapsto [\lambda\tilde{g}, \tilde{g}] \end{aligned}$$

is an isomorphism ($\tilde{g} \in \tilde{\Gamma}$ is any lift of $g \in \Gamma$).

Let us define the group $\widehat{\text{Ext}}(\mathcal{M}, S^1)$. We recall first the notion of generalized morphism and of Morita equivalence (see e.g. [10]). Suppose that $G \rightrightarrows G^{(0)}$ and $\Gamma \rightrightarrows \Gamma^{(0)}$ are two Lie groupoids. Then a generalized morphism from G to Γ is given by a space P , two maps $G^{(0)} \xleftarrow{\tau} P \xrightarrow{\sigma} \Gamma^{(0)}$, a left action of G on P with respect to τ , a right action of Γ on P with respect to σ , such that the two actions commute, and such that $P \rightarrow G^{(0)}$ is a right Γ -principal bundle. The set of isomorphism classes of generalized morphisms from G to Γ is denoted by $H^1(G, \Gamma)$. There is a category whose objects are Lie groupoids and arrows are isomorphism classes of generalized morphisms; isomorphisms in this category are called Morita equivalences.

If $f : G \rightarrow \Gamma$ is a map such that $f(gh) = f(g)f(h)$ whenever g and h are composable, then f is called a (strict) morphism. Then f determines a generalized morphism $P_f = G^{(0)} \times_{\Gamma^{(0)}} \Gamma$. Two strict morphisms f and f' determine the same element of $H^1(G, \Gamma)$ if there exists $\lambda : G^{(0)} \rightarrow \Gamma$ such that $f'(g) = \lambda(t(g))f(g)\lambda(s(g))^{-1}$.

Finally, we recall that any element of $H^1(G, \Gamma)$ is given by the composition of a Morita equivalence with a strict morphism.

Consider the collection of triples $(S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma, \delta, P)$ where $(S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma, \delta)$ is a graded central extension and P is a Morita equivalence from $\Gamma \rightarrow \mathcal{M}$. Two such triples $(S^1 \rightarrow \tilde{\Gamma}_1 \rightarrow \Gamma_1, \delta_1, P_1)$ and $(S^1 \rightarrow \tilde{\Gamma}_2 \rightarrow \Gamma_2, \delta_2, P_2)$ are said to be Morita equivalent if there exists a Morita equivalence $\tilde{Q} : \tilde{\Gamma}_1 \rightarrow \tilde{\Gamma}_2$ which is S^1 -equivariant, such that the diagrams of isomorphism classes of generalized morphisms

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{[Q]} & \Gamma_2 \\ & \searrow [P_1] & \downarrow [P_2] \\ & & \mathcal{M} \end{array}$$

and

$$\begin{array}{ccc}
 \Gamma_1 & \xrightarrow{[Q]} & \Gamma_2 \\
 & \searrow [\delta_1] & \downarrow [\delta_2] \\
 & & \mathbb{Z}/2\mathbb{Z}
 \end{array}$$

commute, where $Q : \Gamma_1 \rightarrow \Gamma_2$ is the Morita equivalence induced by \tilde{Q} . Then the group $\widehat{\text{Ext}}(\mathcal{M}, S^1)$ is the quotient of the collection of triples by Morita equivalence.

Then $\widehat{Br}_0(\mathcal{M}) \cong \widehat{\text{Ext}}(\mathcal{M}, S^1)$. Let us explain the map $\widehat{Br}_0(\mathcal{M}) \rightarrow \widehat{\text{Ext}}(\mathcal{M}, S^1)$.

Any element of $\widehat{Br}_0(\mathcal{M})$ is given by a bundle \mathcal{A} with fiber $\mathcal{K}(\hat{H})$; hence by a $P\hat{U}(\hat{H})$ -principal bundle P over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$, i.e. by a generalized morphism $f : \mathcal{M} \rightarrow P\hat{U}(\hat{H})$. Let \mathcal{E} be the graded central extension $(S^1 \rightarrow \hat{U}(\hat{H}) \rightarrow P\hat{U}(\hat{H}), \delta)$, where $\delta : P\hat{U}(\hat{H}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ is the degree map. Then $f^*\mathcal{E}$ is an element of $\widehat{\text{Ext}}(\mathcal{M}, S^1)$.

Remark 1.2. Let $f_1^*\mathcal{E}, f_2^*\mathcal{E} \in \widehat{\text{Ext}}(\mathcal{M}, S^1)$. Then the sum of $f_1^*\mathcal{E}$ and $f_2^*\mathcal{E}$ (where $f_i : \Gamma \rightarrow P\hat{U}(\hat{H})$ is a strict morphism and $\Gamma \rightrightarrows \Gamma^{(0)}$ is a groupoid which is Morita-equivalent to \mathcal{M}) is given by $f^*\mathcal{E}$, where

$$\begin{aligned}
 f : \Gamma &\rightarrow P\hat{U}(\hat{H} \hat{\otimes} \hat{H}) \cong P\hat{U}(\hat{H}) \\
 g &\mapsto f_1(g) \hat{\otimes} f_2(g).
 \end{aligned}$$

(Note that f is indeed a homomorphism, since $f(gh) = (-1)^{\delta(g)\delta(h)} f(g)f(h)$ agrees with $f(g)f(h)$ up to a scalar in S^1 .)

1.3. Twisted K -theory. Given any bundle \mathcal{A} of Banach spaces over any manifold M , we denote by $C_{\mathcal{A}}(M)$ its space of continuous sections vanishing at infinity. We will also write $C_{\mathcal{A}}$ whenever there is no ambiguity.

Let $\mathcal{A} \rightarrow M$ be a G -equivariant graded D-D bundle over M . We define $K_{\mathcal{A},G}^*(M)$ as $K_*(C_{\mathcal{A}}(M) \rtimes_r G)$, the K -theory of the reduced crossed-product of the graded C^* -algebra $C_{\mathcal{A}}(M)$ by the action of G .

Note that it suffices to study $K_{\mathcal{A},G}^0(M)$, since $K_{\mathcal{A},G}^1(M) = K_{\mathcal{A} \hat{\otimes} \mathbb{C}\ell_1, G}^0(M)$.

Suppose that E is a G -equivariant Euclidean vector bundle over M .

Consider the G -equivariant, graded D-D bundle $\mathcal{A}_E = \text{Cliff}(E \otimes_{\mathbb{R}} \mathbb{C})$. Its type is the same as $\dim E \pmod 2$.

From Kasparov's interpretation of Thom isomorphism [13] which says that $C_0(E)$ and $C_{\mathcal{A}}(M)$ are KK_G -equivalent, one gets:

$$K_{\mathcal{A}_E, G}^*(M) \cong K_*(C_0(E) \rtimes_r G).$$

Remark 1.3. E is given by an $O(n)$ -principal bundle over $\mathcal{M} \rightrightarrows \mathcal{M}^{(0)}$; hence by a morphism $f : \Gamma \rightarrow O(n)$ together with a Morita equivalence from Γ to \mathcal{M} .

Let \mathcal{E} be the graded S^1 -central extension

$$S^1 \rightarrow \text{Pin}^c(n) \rightarrow O(n),$$

where $\text{Pin}^c(n) = \text{Pin}(n) \times_{\{\pm 1\}} S^1$, and $\delta : O(n) \rightarrow \mathbb{Z}/2\mathbb{Z}$ is the map such that $\det A = (-1)^{\delta(A)}$. Then $f^*\mathcal{E}$ is a graded central extension of Γ ; hence determines an even graded D-D bundle \mathcal{A}'_E . One can show that its class in the graded Brauer group is the same as the class of \mathcal{A}_E or $\mathcal{A}_E \hat{\otimes} \mathbb{C}\ell_1$, according to the parity of $\dim E$.

2. POINCARÉ DUALITY

2.1. Kasparov’s constructions. Let M be a compact manifold (actually, Poincaré duality can be generalized to arbitrary manifolds [14], but in this paper we confine ourselves to compact ones for simplicity). We suppose that M is endowed with a Riemannian metric which is invariant by the action of a locally compact group G . Recall that $C_{\mathcal{A}}(M)$ denotes the algebra of continuous sections of \mathcal{A} . We will also write $C_{\mathcal{A}}$ whenever there is no ambiguity. We denote by τ the complexified Clifford algebra of the cotangent bundle of M .

In [14], Kasparov constructed two elements

$$\theta \in KK_{M \rtimes G}(C(M), C(M) \otimes C_{\tau}(M)) = RKK_G(M; \mathbb{C}, C_{\tau}(M))$$

and $D \in KK_G(C_{\tau}(M), \mathbb{C})$ (in this paper, we will use Le Gall’s [15] notation $KK_{M \rtimes G}(\cdot, \cdot)$ for equivariant KK -theory with respect to the groupoid $M \rtimes G$, rather than Kasparov’s $RKK_G(M; \cdot, \cdot)$, but of course both are equivalent).

Let us recall the construction of θ and D .

Let $H = L^2(\Lambda^*M)$, and

$$\begin{aligned} \varphi : C_{\tau}(M) &\rightarrow \mathcal{L}(H) \\ \omega &\mapsto e(\omega) + e(\omega)^*, \end{aligned}$$

where $e(\omega)$ is the exterior multiplication, and let $F = \mathcal{D}(1 + \mathcal{D})^{-1/2}$ where $\mathcal{D} = d + d^*$. Then $D = [(H, \varphi, F)]$.

Let us explain the construction of θ . Denoting by ρ the distance function on M , let $r > 0$ be so small that for all (x, y) in $U = \{(x, y) \in M \times M \mid \rho(x, y) < r\}$, there exists a unique geodesic from x to y .

For every $C(M \times M)$ -algebra A , we denote by A_U the C^* -algebra $C_0(U)A$. Then the element θ is defined as $[(C_M \otimes C_{\tau}(M))_U, \Theta]$ where $\Theta = (\Theta_x)_{x \in M}$, $\Theta_x(y) = \frac{\rho(x,y)}{r}(d_y \rho)(x, y) \in T_y^*M \subset \text{Cliff}_{\mathbb{C}}(T_y^*M)$.

2.2. Constructions in twisted K -theory. In this subsection, we construct an element $\theta^{\mathcal{A}} \in KK_{M \rtimes G}(C(M), C_{\mathcal{A}} \hat{\otimes} C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}})$ for any G -equivariant graded D-D bundle over M . We may assume that \mathcal{A} is stabilized, i.e. that $\mathcal{A} \cong \mathcal{A} \hat{\otimes} \mathcal{K}(\hat{H} \otimes L^2(G))$. First, let us denote by $p_t(x, y)$ the geodesic segment joining x to y at constant speed ($0 \leq t \leq 1$).

Using p_t , we see that $p_t : U \rightarrow M$ is a G -equivariant homotopy equivalence. Unfortunately, this does not imply that $Br(U \rtimes G)$ and $Br(M \rtimes G)$ are isomorphic for arbitrary G ; hence we will make the following

Assumption. In the sequel of this paper, and unless stated otherwise, G will be a compact Lie group acting smoothly on a compact manifold M .

In that case, $H^2(U \rtimes G, \underline{S}^1) \cong H^3(U \rtimes G, \mathbb{Z}) = H^3(\frac{U \times EG}{G}, \mathbb{Z}) \cong H^3(\frac{M \times EG}{G}, \mathbb{Z}) \cong H^2(M \rtimes G, \underline{S}^1)$. As a consequence, there is a continuous, G -equivariant family of isomorphisms

$$u_{t,x,y} : \mathcal{A}_x \xrightarrow{\sim} \mathcal{A}_{p_t(x,y)}.$$

Of course, the u_t ’s are not unique, but this will not be important as far as K -theory is concerned as we will see.

Consider the canonical Morita-equivalence \mathcal{H}_x between \mathbb{C} and $\mathcal{A}_x \hat{\otimes} \mathcal{A}_x^{op}$. Let $\mathcal{H} = (\mathcal{H}_x)_{x \in M}$ be the corresponding Morita equivalence between $C(M)$ and

$C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}(M)$. Then $[(C(M) \otimes C_\tau(M))_U \hat{\otimes}_{C(M)} \mathcal{H}, \Theta \hat{\otimes} 1]$ is an element of $KK_{M \rtimes G}(C(M), C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} C_\tau(M))$.

Now, using the map $u_{1,x,y} : \mathcal{A}_x \xrightarrow{\sim} \mathcal{A}_y$, we get a Morita equivalence $\mathcal{E}_{x,y}$ from \mathcal{A}_x^{op} to \mathcal{A}_y^{op} , thus a Morita equivalence \mathcal{E} from $(C_{\mathcal{A}^{op}}(M) \hat{\otimes} C(M))_U$ to $(C(M) \hat{\otimes} C_{\mathcal{A}^{op}}(M))_U$. Tensoring over $C(M \times M)$ with $C_{\mathcal{A}}(M) \hat{\otimes} C_\tau(M)$, we get a Morita equivalence $\mathcal{E}' = (\mathcal{E}'_{x,y})_{(x,y) \in U}$ from $(C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} C_\tau(M))_U$ to $(C_{\mathcal{A}}(M) \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M))_U$.

We then define $\theta^{\mathcal{A}}$ as

$$\begin{aligned} \theta^{\mathcal{A}} &= [((C(M) \otimes C_\tau(M))_U \hat{\otimes}_{C(M)} \mathcal{H} \hat{\otimes}_{C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_\tau} \mathcal{E}', \Theta \hat{\otimes} 1)] \\ &\in KK_{M \rtimes G}(C(M), C_{\mathcal{A}}(M) \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M)). \end{aligned}$$

2.3. Twisted K -homology. Given a C^* -algebra A endowed with an action of a locally compact group G , the G -equivariant K -homology of A , $K_G^*(A)$, is defined by $KK_G^*(A, \mathbb{C})$. If \mathcal{A} is a G -equivariant graded D-D bundle over M , we define $K_*^{G, \mathcal{A}}(M)$ by $K_G^*(C_{\mathcal{A}}(M))$.

2.4. Maps between twisted K -theory and twisted K -homology. Let A and B be two separable graded G - C^* -algebras (recall that G is assumed to be a compact Lie group). We define two maps

$$\begin{aligned} \mu : KK_G(A, C_{\mathcal{A}} \hat{\otimes} B) &\rightarrow KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, B), \\ \nu : KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, B) &\rightarrow KK_G(A, C_{\mathcal{A}} \hat{\otimes} B). \end{aligned}$$

First, let us introduce some notation. Suppose that M is a locally compact space endowed with an action of a locally compact group. If A, B and D are G -equivariant graded $C(M)$ -algebras, and \mathcal{E} is an A - B - C^* -bimodule, then Kasparov defined an $A \hat{\otimes}_{C_0(M)} D$ - $B \hat{\otimes}_{C_0(M)} D$ - C^* -bimodule $\sigma_{M,D}(\mathcal{E})$, and thus a ‘‘suspension’’ map $\sigma_{M,D} : KK_{M \rtimes G}(A, B) \rightarrow KK_{M \rtimes G}(A \hat{\otimes}_{C_0(M)} D, B \hat{\otimes}_{C_0(M)} D)$. There is also a suspension map $\sigma_D : KK_{M \rtimes G}(A, B) \rightarrow KK_{M \rtimes G}(A \hat{\otimes} D, B \hat{\otimes} D)$ defined in a similar way.

Given an $(A_1, B_1 \hat{\otimes} D)$ - C^* -bimodule \mathcal{E}_1 and a $(D \hat{\otimes} A_2, B_2)$ - C^* -bimodule \mathcal{E}_2 , the $(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$ - C^* -bimodule $\mathcal{E}_1 \hat{\otimes}_D \mathcal{E}_2$ is defined by $\sigma_{A_2}(\mathcal{E}_1) \hat{\otimes}_{A_2 \hat{\otimes} B_1} \sigma_{B_1}(\mathcal{E}_2)$.

We introduce a similar notation when all tensor products are replaced by tensor products over a space M : given an $(A_1, B_1 \hat{\otimes}_{C(M)} D)$ - C^* -bimodule \mathcal{E}_1 and every $(D \hat{\otimes}_{C(M)} A_2, B_2)$ - C^* -bimodule \mathcal{E}_2 , $\mathcal{E}_1 \hat{\otimes}_D \mathcal{E}_2$ is an $(A_1 \hat{\otimes}_{C(M)} A_2, B_1 \hat{\otimes}_{C(M)} B_2)$ - C^* -bimodule.

Let us now define μ and ν . First, let us note that $KK_G(A, C_{\mathcal{A}} \hat{\otimes} B)$ is isomorphic to $KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} B)$.

The map μ is defined as the composition

$$\begin{aligned} KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} B) &\xrightarrow{\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}} KK_{M \rtimes G}(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A \hat{\otimes} B) \\ &\xrightarrow{\hat{\otimes} \mathcal{H}^{op}} KK_{M \rtimes G}(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, C_\tau \hat{\otimes} B) \\ &\xrightarrow{\otimes D} KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, B). \end{aligned}$$

The map ν is just $\theta^{\mathcal{A}} \otimes \cdot : KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, B) \rightarrow KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} B)$.

2.5. **The main theorem.**

Theorem 2.1. *Let G be a compact Lie group acting on a compact manifold M , and let A and B be two graded separable G - C^* -algebras. Let \mathcal{A} be a G -equivariant graded D - D bundle over M .*

Then the maps μ and ν defined above are inverse to each other:

$$KK_G^*(A, C_{\mathcal{A}} \hat{\otimes} B) \cong KK_G^*(C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} A, B).$$

Replacing \mathcal{A} by $\tau \hat{\otimes} \mathcal{A}^{op}$, we get:

$$KK_G^*(C_{\mathcal{A}} \hat{\otimes} A, B) \cong KK_G^*(A, C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} B).$$

In particular, for $A = B = \mathbb{C}$ we get

$$\begin{aligned} K_{G, \mathcal{A}}^*(M) &\cong K_*^{G, \tau \hat{\otimes} \mathcal{A}^{op}}(M), \\ K_*^{G, \mathcal{A}}(M) &\cong K_{G, \tau \hat{\otimes} \mathcal{A}^{op}}^*(M). \end{aligned}$$

Remark 2.2. This result (in the case when G is the trivial group) is observed in [2, Section 7].

Remark 2.3. The map μ does not depend on the choice of the isomorphisms $u_{t,x,y}$; hence ν doesn't either.

The rest of the paper is devoted to the proof of Theorem 2.1.

2.6. **Proof of $\mu \circ \nu = \text{Id}$.** For all $\alpha \in KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A, B)$, we have

$$\mu \circ \nu(\alpha) = \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \hat{\otimes}_{C_{\mathcal{A}^{op}} \hat{\otimes} A} \mathcal{H}^{op} \otimes_{C_{\tau}} D \otimes_{C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} A} \alpha.$$

Thus, we need to prove that

$$\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \hat{\otimes}_{C_{\mathcal{A}^{op}} \hat{\otimes} A} \mathcal{H}^{op} \otimes_{C_{\tau}} D = 1 \in KK_G(C_{\tau \hat{\otimes} \mathcal{A}^{op}}, C_{\tau \hat{\otimes} \mathcal{A}^{op}}).$$

Consider the element $\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \in KK_{M \times G}(C_{\tau \hat{\otimes} \mathcal{A}^{op}}, C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau})$. Denote the flip by

$$\begin{aligned} s : C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau} &\xrightarrow{\cong} C_{\tau} \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}} \\ x \otimes y &\mapsto (-1)^{\deg x \cdot \deg y} y \otimes x. \end{aligned}$$

Suppose it is proven that

$$(2) \quad \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \otimes [s] = \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \hat{\otimes}_{C_{\mathcal{A}^{op}} \hat{\otimes} A} \mathcal{H}^{op}.$$

Then

$$\begin{aligned} \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^{\mathcal{A}}) \otimes_{C_{\mathcal{A}^{op}} \hat{\otimes} A} \mathcal{H}^{op} \otimes_{C_{\tau}} D &= \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \otimes_{C_{\tau}} D \\ &= \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta \otimes_{C_{\tau}} D) \\ &= \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(1) = 1, \end{aligned}$$

since $\theta \otimes_{C_{\tau}} D = 1$ (from [14, Theorem 4.8]).

We postpone the proof of (2) until subsection 2.8.

2.7. **Proof of $\nu \circ \mu = \text{Id}$.** For all $\alpha \in KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} B)$, we have

$$\nu \circ \mu(\alpha) = \theta^A \otimes_{C_{\tau \hat{\otimes} \mathcal{A}^{op}}} (\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\alpha) \otimes_{C_{\mathcal{A}^{op} \otimes A}} \mathcal{H}^{op} \otimes_{C_{\tau}} D).$$

Suppose it is shown that

$$(3) \quad \theta^A \otimes_{C_{\tau \hat{\otimes} \mathcal{A}^{op}}} \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\alpha) \otimes_{C_{\mathcal{A}^{op} \otimes A}} \mathcal{H}^{op} = \alpha \otimes_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\theta) \in KK_{M \rtimes G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} C_{\tau} \hat{\otimes} B).$$

Then $\nu \circ \mu(\alpha) = \alpha \otimes_{C_{\mathcal{A}}} (\sigma_{M, C_{\mathcal{A}}}(\theta) \otimes_{C_{\tau}} D) = \alpha \otimes_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\theta \otimes_{C_{\tau}} D) = \alpha \otimes_{C_{\mathcal{A}}} 1 = \alpha$.

We postpone the proof of (3) until subsection 2.9

2.8. **Proof of (2).** Recall the proof when \mathcal{A} is the trivial bundle [14, Lemma 4.6]. We want to show that $\sigma_{M, C_{\tau}}(\theta)$ is flip-invariant. Denote by $p_t^* : T_{p_t(x, y)}^* M \hookrightarrow T_{(x, y)}^* U$ the pull-back map induced by p_t , and let q_t^* be the isometry $q_t^* = p_t^*(p_t p_t^*)^{-1/2}$. We denote again by $q_t^* : \Omega^1(M) \hookrightarrow \Omega^1(U)$ the corresponding map. Then q_t^* induces a map $\varphi_t : C_{\tau}(M) \rightarrow \mathcal{L}(C_{\tau}(U))$.

Let $\beta(s) = (p_{t(1-s)}(x, y), p_{t(1-s)+s}(x, y)) \in U$, and

$$\Theta_t(x, y) = \frac{\rho(x, y)}{r} \left\| \frac{d\beta}{ds} \Big|_{s=1} \right\|^{-1} \frac{d\beta}{ds} \Big|_{s=1} \in T_{x, y} U.$$

Then $(C_{\tau}(U), \varphi_t, \Theta_t)_{0 \leq t \leq 1}$ is a homotopy between $\sigma_{M, C_{\tau}}(\theta)$ and $\sigma_{M, C_{\tau}}(\theta) \otimes [s]$.

Now, consider the general case. $\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta)$ is the Kasparov triple

$$((C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau})_U, \varphi, \Theta)$$

where $\varphi : C_{\tau \hat{\otimes} \mathcal{A}^{op}} \rightarrow \mathcal{L}((C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau})_U)$ is the obvious map. Thus,

$$\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \otimes [s] = [((C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau})_U, \varphi, \Theta_1)]$$

with $\theta_1 = \frac{\rho(x, y)}{r}(d_x \rho)(x, y) \in T_x^* M \subset \text{Cliff}_{\mathbb{C}}(T_x^* M)$, while

$$\sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^A) \otimes_{C_{\mathcal{A}^{op} \otimes A}} \mathcal{H}^{op} = [((C_{\tau \hat{\otimes} \mathcal{A}^{op}}(M) \hat{\otimes} C_{\tau}(M))_U \hat{\otimes}_{C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau}} \mathcal{E}'', \varphi', \Theta \otimes 1)]$$

where \mathcal{E}'' is the Morita equivalence between $(C_{\tau \hat{\otimes} \mathcal{A}^{op}} \hat{\otimes} C_{\tau})_U$ and $(C_{\tau} \hat{\otimes} C_{\tau \hat{\otimes} \mathcal{A}^{op}})_U$ obtained from the Morita equivalence \mathcal{E} between $p_0^* C_{\mathcal{A}^{op}} = (C_{\mathcal{A}^{op}} \otimes C(M))_U$ and $p_1^* C_{\mathcal{A}^{op}} = (C(M) \otimes C_{\mathcal{A}^{op}})_U$.

Let $\mathcal{E}_t = (\mathcal{E}_{x, y, t})_{(x, y) \in U}$ be the Morita equivalence between $p_t^* C_{\mathcal{A}^{op}}$ and $p_1^* C_{\mathcal{A}^{op}}$ constructed in the same way as $\mathcal{E} = \mathcal{E}_0$.

Then

$$\begin{aligned} \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta) \otimes [s] &= (C_{\tau}(U) \hat{\otimes}_{C_0(U)} \mathcal{E}_1, \varphi, \Theta_1), \\ \sigma_{M, C_{\tau \hat{\otimes} \mathcal{A}^{op}}}(\theta^A) \hat{\otimes}_{C_{\mathcal{A}^{op} \otimes A}} \mathcal{H}^{op} &= (C_{\tau}(U) \hat{\otimes}_{C_0(U)} \mathcal{E}_0, \varphi', \Theta). \end{aligned}$$

Let Θ_t be as above. We produce a homotopy

$$(C_{\tau}(U) \hat{\otimes}_{C_0(U)} \mathcal{E}_t, \psi_t, \Theta_t)$$

between those two elements. Only $\psi_t : C_{\tau \hat{\otimes} \mathcal{A}^{op}} \rightarrow \mathcal{L}(C_{\tau}(U) \hat{\otimes}_{C_0(U)} \mathcal{E}_t)$ remains to be defined. We need two compatible maps

$$\begin{aligned} \psi_t' : C_{\tau} &\rightarrow \mathcal{L}(C_{\tau}(U) \hat{\otimes}_{C_0(U)} \mathcal{E}_t) \\ \text{and } \psi_t'' : C_{\mathcal{A}^{op}} &\rightarrow \mathcal{L}(C_{\tau}(U) \hat{\otimes}_{C_0(U)} \mathcal{E}_t). \end{aligned}$$

The map ψ'_t is just $\varphi_t \otimes 1$. The map ψ''_t is given by the composition

$$C_{\mathcal{A}^{op}} \xrightarrow{p_i^*} C_b(U, p_t^* \mathcal{A}^{op}) \rightarrow \mathcal{L}(\mathcal{E}_t).$$

2.9. Proof of (3). Let us first recall the proof when \mathcal{A} is trivial [14, Lemma 4.5]. We want to show that for all $\alpha \in KK_{M \times G}(C(M) \otimes A, C(M) \otimes B)$ we have

$$\alpha \otimes_{C(M)} \theta = \theta \otimes_{C_\tau(M)} \sigma_{M, C_\tau(M)}(\alpha) \in KK_{M \times G}(C(M) \otimes A, C(M) \otimes C_\tau(M) \hat{\otimes} B).$$

Write $\alpha = [(E, T)]$ where $\overline{C(M, A)E} = E$ and T is G -continuous. Then both products can be written as

$$[(E \hat{\otimes}_{C(M)} (C(M) \otimes C_\tau(M)))_U, \varphi_i, F_i]$$

where F_i is of the form $M_2^{1/2}(T \hat{\otimes} 1) + M_1^{1/2}(1 \hat{\otimes} \Theta)$ ($i = 0, 1$), and where the map $C(M) \rightarrow \mathcal{L}((C(M) \otimes C_\tau(M))_U)$ used to define φ_i is p_i^* .

Since p_0 and p_1 are homotopic, φ_0 and φ_1 are homotopic. One then constructs a homotopy between F_0 and F_1 using Kasparov's technical theorem as in [14, Lemma 4.5].

Let us now consider a general G -equivariant graded D-D bundle \mathcal{A} over M . Let $\alpha = [(E, T)] \in KK_{M \times G}(C(M) \otimes A, C_{\mathcal{A}} \hat{\otimes} B)$ where $\overline{C(M, A)E} = E$ and T is G -continuous.

We want to show that $\alpha \otimes_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\theta) = \theta^{\mathcal{A}} \hat{\otimes}_{C_\tau \hat{\otimes} \mathcal{A}^{op}}(\alpha) \hat{\otimes}_{C_{\mathcal{A}^{op}} \hat{\otimes} \mathcal{A}} \mathcal{H}^{op}$. Let us just explain the homotopy between the two modules, the homotopy between the F_i 's being obtained using Kasparov's technical theorem in the same way as in [14, Lemma 4.5].

The left-hand side is

$$(4) \quad E \hat{\otimes}_{C_{\mathcal{A}}} (C_{\mathcal{A}} \hat{\otimes} C_\tau)_U,$$

and the right-hand side is

$$(5) \quad E \hat{\otimes}_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\mathcal{F}_1) \hat{\otimes}_{C_0(U)} \mathcal{H}'_1 \hat{\otimes}_{C_{p_1^*(\mathcal{A} \hat{\otimes} \mathcal{A}^{op})}(U)} p_1^* \mathcal{H}^{op},$$

where we recall that $p_1 : U \rightarrow M$ is the second projection $(x, y) \mapsto y$. The $C(M)$ - $(C(M) \otimes C_\tau)_U$ -bimodule \mathcal{F}_1 is $(C(M) \otimes C_\tau)_U$, with the left action of $C(M)$ on \mathcal{F}_1 obtained via $C(M) \xrightarrow{p_1^*} C_b(U) \rightarrow \mathcal{L}((C(M) \otimes C_\tau)_U)$.

\mathcal{H}'_1 is the Morita equivalence between $C_0(U)$ and $p_0^* C_{\mathcal{A}} \hat{\otimes}_{C_0(U)} p_1^* C_{\mathcal{A}^{op}}$ obtained by composing the Morita equivalence $p_0^* \mathcal{H}$ between $C_0(U)$ and $p_0^*(C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}})$ with the isomorphism $p_0^* \mathcal{A}^{op} \cong p_1^* \mathcal{A}^{op}$.

Using the map $p_t : U \rightarrow M$ instead of p_1 , consider (with obvious notation) the homotopy $E \hat{\otimes}_{C_{\mathcal{A}}} \sigma_{M, C_{\mathcal{A}}}(\mathcal{F}_t) \hat{\otimes}_{C_0(U)} \mathcal{H}'_t \hat{\otimes}_{C_{p_t^*(\mathcal{A} \hat{\otimes} \mathcal{A}^{op})}(U)} p_t^* \mathcal{H}^{op}$.

For $t = 1$, we get (5).

For $t = 0$, we get $E \hat{\otimes}_{C_{\mathcal{A}}} (C_{\mathcal{A}} \hat{\otimes} C_\tau)_U \hat{\otimes}_{C_0(U)} p_0^* \mathcal{H} \hat{\otimes}_{C_{p_0^*(\mathcal{A} \hat{\otimes} \mathcal{A}^{op})}(U)} p_0^* \mathcal{H}^{op}$ where the right $C_{p_0^*(\mathcal{A} \hat{\otimes} \mathcal{A}^{op})}(U)$ -structure on $E \hat{\otimes}_{C_{\mathcal{A}}} (C_{\mathcal{A}} \hat{\otimes} C_\tau)_U \hat{\otimes}_{C_0(U)} p_0^* \mathcal{H}$ is defined as follows: $C_{p_0^* \mathcal{A}}$ acts on $(C_{\mathcal{A}} \hat{\otimes} C_\tau)_U$ by the obvious action, and $C_{p_0^* \mathcal{A}^{op}}$ acts on $p_0^* \mathcal{H}$. In other words, it is the tensor product of (4) with $\beta_{\mathcal{A}}$ over $C_{\mathcal{A}}$, where $\beta_{\mathcal{A}}$ is the $C_{\mathcal{A}}$ - $C_{\mathcal{A}}$ -bimodule

$$\beta_{\mathcal{A}} = (C_{\mathcal{A}} \hat{\otimes}_{C(M)} \mathcal{H}) \hat{\otimes}_{C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}} \mathcal{H}^{op}.$$

In the expression above, the right $C_{\mathcal{A} \hat{\otimes} \mathcal{A}^{op}}$ -module structure on $C_{\mathcal{A} \hat{\otimes} C(M)} \mathcal{H}$ is defined as follows: $\forall a \in C_{\mathcal{A}}, \forall b \in C_{\mathcal{A}^{op}}, \forall \xi \otimes \eta \in C_{\mathcal{A} \hat{\otimes} C(M)} \mathcal{H}$,

$$(\xi \otimes \eta) \cdot (a \otimes b) = (-1)^{|\eta||a|} \xi a \otimes \eta b.$$

To finish the proof, it remains to show that $\beta_{\mathcal{A}} \cong C_{\mathcal{A}}$. Suppose for instance that \mathcal{A} is an even graded D-D bundle. Let $x \in M$. Denoting by \hat{H}_x a Hilbert space such that $\mathcal{A}_x \cong \mathcal{K}(\hat{H}_x)$, we have $(\beta_{\mathcal{A}})_x = [\mathcal{K}(\hat{H}_x) \hat{\otimes} (\hat{H}_x^* \hat{\otimes} \hat{H}_x)] \hat{\otimes}_{\mathcal{A}_x \hat{\otimes} \mathcal{A}_x^{op}} (\hat{H}_x \hat{\otimes} \hat{H}_x^*)$, where in $\hat{H}_x \hat{\otimes} \hat{H}_x^*$, \hat{H}_x (resp. \hat{H}_x^*) is considered as an \mathcal{A}_x - \mathbb{C} (resp. an \mathcal{A}_x^{op} - \mathbb{C})-bimodule, and in $\hat{H}_x^* \hat{\otimes} \hat{H}_x$, \hat{H}_x (resp. \hat{H}_x^*) is considered as a \mathbb{C} - \mathcal{A}_x^{op} (resp. a \mathbb{C} - \mathcal{A}_x)-bimodule, and the right $\mathcal{A}_x \hat{\otimes} \mathcal{A}_x^{op}$ -module structure on $\mathcal{K}(\hat{H}_x) \hat{\otimes} (\hat{H}_x^* \hat{\otimes} \hat{H}_x)$ is $(\xi \otimes (\eta \otimes \zeta)) \cdot (a \otimes b) = (-1)^{|a|(|\xi|+|\eta|)} \xi a \otimes \eta \otimes \zeta b$. It follows that $(\beta_{\mathcal{A}})_x \cong \hat{H}_x \hat{\otimes} \hat{H}_x^*$ is the natural $\mathcal{K}(\hat{H}_x)$ -bimodule $\mathcal{K}(\hat{H}_x)$.

REFERENCES

- [1] Atiyah, M. and Segal, G. Twisted K -theory. *Ukr. Mat. Visn.* **1** (2004), no. 3, 287–330; translation in *Ukr. Math. Bull.* **1** (2004), no. 3, 291–334. MR2172633 (2006m:55017)
- [2] Brodzki, J., Mathai, V., Rosenberg, J. and Szabo, R. D -branes, RR-fields and duality on noncommutative manifolds. *Comm. Math. Phys.* **277** (2008), no. 3, 643–706. MR2365448
- [3] Bunke, U. and Schröder, I. Twisted K -theory and TQFT. Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005, 33–80, Universitätsdrucke Göttingen, Göttingen, 2005. MR2206878 (2007c:19008)
- [4] Connes, A. and Skandalis, G. The longitudinal index theorem for foliations. *Publ. Res. Inst. Math. Sci.* **20**, 1139–1183 (1984). MR775126 (87h:58209)
- [5] Carey, A.L. and Wang, B.L., Thom isomorphism and Push-forward map in twisted K -theory. Preprint. [arXiv:math/0512139](https://arxiv.org/abs/math/0512139)
- [6] Donovan, P. and Karoubi, M. Graded Brauer groups and K -theory with local coefficients. *Pub. Math. IHES* No. 38, p. 5–25 (1971). MR0282363 (43:8075)
- [7] Echterhoff, S., Emerson, H. and Kim, H.J. KK -theoretic duality for proper twisted actions. [arXiv:math/0610044](https://arxiv.org/abs/math/0610044) Preprint.
- [8] Freed, D., Hopkins, M. and Teleman, C. Twisted equivariant K -theory with complex coefficients. *J. Topol.* **1** (2008), no. 1, 16–44. MR2365650
- [9] Freed, D., Hopkins, M. and Teleman, C. Loop groups and twisted K -theory II. [ArXiv:math.AT/0206257](https://arxiv.org/abs/math.AT/0206257).
- [10] Hilsaum, M. and Skandalis, G. Morphismes K -orientés d’espaces de feuilles et functorialité en théorie de Kasparov. *Ann. Sci. École Norm. Sup. (4)* **20** (1987), no. 3, 325–390. MR925720 (90a:58169)
- [11] Karoubi, M. Algèbres de Clifford et K -théorie. *Ann. Sci. Ec. Norm. Sup.*, p. 161–270 (1968). MR0238927 (39:287)
- [12] Karoubi, M. Twisted K -theory, old and new. To appear in the Proceedings of the Valladolid conference (2007), European Mathematical Society.
- [13] Kasparov, G. The operator K -functor and extensions of C^* -algebras. *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), no. 3, 571–636, 719. MR582160 (81m:58075)
- [14] Kasparov, G. Equivariant KK -theory and the Novikov conjecture. *Inventiones Mathematicae*, **91** (1988), 147–201. MR918241 (88j:58123)
- [15] Le Gall, P.Y. Théorie de Kasparov équivariante et groupoïdes. *K-Theory* **16**, 361–390 (1999). MR1686846 (2000f:19006)
- [16] Parker, E. M. The Brauer group of graded continuous trace C^* -algebras. *Trans. Amer. Math. Soc.* **308** (1988), no. 1, 115–132. MR946434 (89h:46080)
- [17] Tu, J.L., Xu, P. and Laurent-Gengoux, C. Twisted K -theory of differentiable stacks. *Ann. Sci. Ens* **37** (2004), no. 6, 841–910. MR2119241 (2005k:58037)

DEPARTMENT OF MATHEMATICS, UNIVERSITY PAUL VERLAINE-METZ, LMAM-CNRS UMR 7122, ILE DU SAULCY, 57000 METZ, FRANCE

E-mail address: tu@univ-metz.fr