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A GENERALIZATION OF MACMAHON'S FORMULA

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Abstract. We generalize the generating formula for plane partitions known as MacMahon's formula as well as its analog for strict plane partitions. We give a 2-parameter generalization of these formulas related to Macdonald's symmetric functions. The formula is especially simple in the Hall-Littlewood case. We also give a bijective proof of the analog of MacMahon's formula for strict plane partitions.

1. Introduction

A plane partition is a Young diagram filled with positive integers that form nonincreasing rows and columns. Each plane partition can be represented as a finite two sided sequence of ordinary partitions $(\dots, \lambda^{-1}, \lambda^0, \lambda^1, \dots)$, where λ^0 corresponds to the ordinary partition on the main diagonal and λ^k corresponds to the diagonal shifted by k. A plane partition where all of its diagonal partitions are strict ordinary partitions (i.e. partitions with all distinct parts) is called a *strict* plane partition. Figure 1 shows two standard ways of representing a plane partition. Diagonal partitions are marked on the figure on the left.

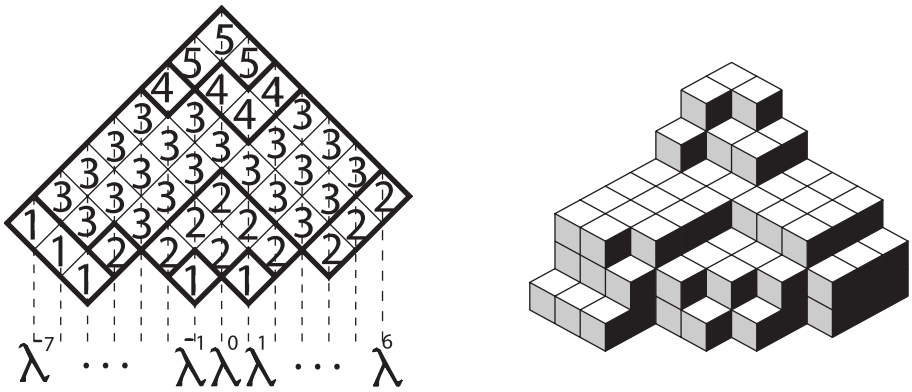

Figure 1. A plane partition

For a plane partition π one defines the weight $|\pi|$ to be the sum of all entries. A *connected component* of a plane partition is the set of all rookwise connected boxes of its Young diagram that are filled with the same number. We denote the number

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of connected components of π with $k(\pi)$. For the example from Figure 1 we have $k(\pi) = 10$, and its connected components are shown in Figure 1 (on the left, bold lines represent boundaries of these components, while on the right, white terraces are connected components).

Denote the set of all plane partitions with $\mathcal{P}$, and with $\mathcal{P}(r,c)$ we denote those that have zero (i,j)th entry for $i > r$ and $j > c$. Denote the set of all strict plane partitions with $\mathcal{SP}$.

A generating function for plane partitions is given by the famous MacMahon formula (see e.g. 7.20.3 of [S]):

$$\sum_{\pi\in\mathcal{P}} s^{|\pi|} = \prod_{n=1}^{\infty}\left(\frac{1}{1-s^n}\right)^n. \tag{1.1}$$

Recently, a generating formula for the set of strict plane partitions was found in [FW] and [V]:

$$\sum_{\pi\in\mathcal{SP}} 2^{k(\pi)} s^{|\pi|} = \prod_{n=1}^{\infty}\left(\frac{1+s^n}{1-s^n}\right)^n. \tag{1.2}$$

We refer to it as the shifted MacMahon formula.

In this paper we generalize both formulas (1.1) and (1.2). Namely, we define a polynomial $A_\pi(t)$ that gives a generating formula for plane partitions of the form

$$\sum_{\pi\in\mathcal{P}(r,c)} A_\pi(t) s^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1-ts^{i+j-1}}{1-s^{i+j-1}}$$

with the property that $A_\pi(0) = 1$ and

$$A_\pi(-1) = \begin{cases} 2^{k(\pi)}, & \pi \text{ is a strict plane partition,} \\ 0, & \text{otherwise.} \end{cases}$$

We further generalize this and find a rational function $F_\pi(q,t)$ that satisfies

$$\sum_{\pi\in\mathcal{P}(r,c)} F_\pi(q,t) s^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty},$$

where

$$(s;q)_\infty = \prod_{n=0}^{\infty}(1-sq^n)$$

and $F_\pi(0,t) = A_\pi(t)$. We describe $A_\pi(t)$ and $F_\pi(q,t)$ below.

In order to describe $A_\pi(t)$ we need more notation. If a box (i,j) belongs to a connected component C, then we define its *level* $h(i,j)$ as the smallest positive integer such that $(i+h, j+h)$ does not belong to C. In other words, levels represent the distance from the "rim", this distance being measured diagonally. A *border component* is a rookwise connected subset of a connected component where all boxes have the same level. We also say that this border component is of this level. For the example above, border components and their levels are shown in Figure 2.

For each connected component C we define a sequence $(n_1, n_2, \dots)$, where n_i is the number of i-level border components of C. We set

$$P_C(t) = \prod_{i\geq 1}(1-t^i)^{n_i}.$$

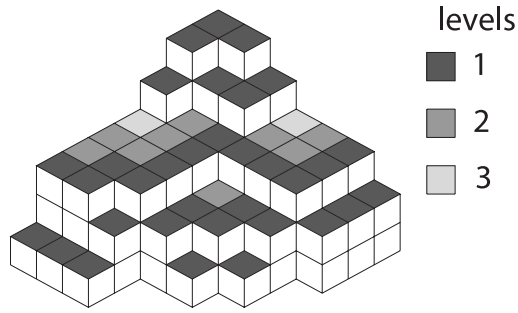

FIGURE 2. Border components

Let $C_1, C_2, \dots, C_{k(\pi)}$ be connected components of π. We define

$$A_\pi(t) = \prod_{i=1}^{k(\pi)} P_{C_i}(t).$$

For the example above $A_\pi(t) = (1-t)^{10}(1-t^2)^3(1-t^3)^2$.

$F_\pi(q,t)$ is defined as follows. For nonnegative integers n and m let

$$f(n,m) = \begin{cases} \displaystyle\prod_{i=0}^{n-1} \frac{1-q^i t^{m+1}}{1-q^{i+1}t^m}, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Here q and t are parameters.

Let $\pi \in \mathcal{P}$ and let (i,j) be a box in its support (where the entries are nonzero). Let λ, μ and ν be ordinary partitions defined by

$$\begin{aligned} \lambda &= (\pi(i,j), \pi(i+1,j+1), \dots), \\ \mu &= (\pi(i+1,j), \pi(i+2,j+1), \dots), \\ \nu &= (\pi(i,j+1), \pi(i+1,j+2), \dots). \end{aligned} \tag{1.3}$$

To the box (i,j) of π we associate

$$F_\pi(i,j)(q,t) = \prod_{m=0}^{\infty} \frac{f(\lambda_1 - \mu_{m+1}, m) f(\lambda_1 - \nu_{m+1}, m)}{f(\lambda_1 - \lambda_{m+1}, m) f(\lambda_1 - \lambda_{m+2}, m)}.$$

Only finitely many terms in this product are different from 1.

To a plane partition π we associate a function $F_\pi(q,t)$ defined by

$$F_\pi(q,t) = \prod_{(i,j)\in\pi} F_\pi(i,j)(q,t). \tag{1.4}$$

For the example above

$$F_\pi(0,0)(q,t) = \frac{1-q}{1-t} \cdot \frac{1-q^3t^2}{1-q^2t^3} \cdot \frac{1-q^5t^4}{1-q^4t^5} \cdot \frac{1-q^3t^5}{1-q^4t^4}.$$

Two main results of our paper are

Theorem A (Generalized MacMahon formula; Macdonald's case)**.**

$$\sum_{\pi\in\mathcal{P}(r,c)} F_\pi(q,t) s^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty}.$$

In particular,

$$\sum_{\pi\in\mathcal{P}} F_\pi(q,t)s^{|\pi|} = \prod_{n=1}^{\infty}\left[\frac{(ts^n;q)_\infty}{(s^n;q)_\infty}\right]^n.$$

Theorem B (Generalized MacMahon formula; Hall-Littlewood's case)**.**

$$\sum_{\pi\in\mathcal{P}(r,c)} A_\pi(t)s^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1-ts^{i+j-1}}{1-s^{i+j-1}}.$$

In particular,

$$\sum_{\pi\in\mathcal{P}} A_\pi(t)s^{|\pi|} = \prod_{n=1}^{\infty}\left(\frac{1-ts^n}{1-s^n}\right)^n.$$

Clearly, the second formulas (with summation over $\mathcal{P}$) are limiting cases of the first ones as $r, c \to \infty$.

The proof of Theorem A was inspired by [OR] and [V]. It uses a special class of symmetric functions called skew Macdonald functions. For each $\pi \in \mathcal{P}$ we introduce a weight function depending on several specializations of the algebra of symmetric functions. For a suitable choice of these specializations the weight functions become $F_\pi(q,t)$.

We first prove Theorem A, and then Theorem B is obtained as a corollary of Theorem A after we show that $F_\pi(0,t) = A_\pi(t)$.

Proofs of formula (1.2) appeared in [FW] and [V]. Both these proofs rely on skew Schur functions and a Fock space corresponding to strict plane partitions. In this paper we also give a bijective proof of (1.2) that does not involve symmetric functions.

We were informed by Sylvie Corteel that a generalization of (1.2) was found in her joint work with Cyrille Savelief. This generalization gives a generating formula for strict plane partitions with bounded entries. It will appear in the master thesis of Cyrille Savelief, Combinatoire des overpartitions planes, Universite Paris 7, 2007.

There are other generalizations of MacMahon's formula in literature. Note that the paper [C] with a title similar to ours gives a different generalization.

The paper is organized as follows. Section 2 consists of two subsections. In Subsection 2.1 we prove Theorem A. In Subsection 2.2 we prove Theorem B by showing that $F_\pi(0,t) = A_\pi(t)$. In Section 3 we give a bijective proof of (1.2).

2. Generalized MacMahon formula

2.1. **Macdonald's case.** We recall a definition of a plane partition. For basics, such as ordinary partitions and Young diagrams, see Chapter I of [Mac].

A plane partition π can be viewed in different ways. One way is to fix a Young diagram, the support of the plane partition, and then to associate a positive integer to each box in the diagram such that integers form nonincreasing rows and columns. Thus, a plane partition is a diagram with rowwise and columnwise nonincreasing integers. It can also be viewed as a finite two-sided sequence of ordinary partitions, since each diagonal in the support diagram represents a partition. We write $\pi = (\ldots,\lambda^{-1},\lambda^0,\lambda^1,\ldots)$, where the partition λ^0 corresponds to the main diagonal and λ^k corresponds to the diagonal that is shifted by k; see Figure 1. Every such

two-sided sequence of partitions represents a plane partition if and only if

$$\cdots \subset \lambda^{-1} \subset \lambda^0 \supset \lambda^1 \supset \cdots \text{ and}$$

$$[\lambda^{n-1}/\lambda^n] \text{ is a horizontal strip (a skew diagram with at most one square in each column) for every } n, \tag{2.1}$$

where

$$[\lambda/\mu] = \begin{cases} \lambda/\mu & \text{if } \lambda \supset \mu, \\ \mu/\lambda & \text{if } \mu \supset \lambda. \end{cases}$$

The weight of π, denoted by $|\pi|$, is the sum of all entries of π.

We denote the set of all plane partitions with $\mathcal{P}$, and its subset containing all plane partitions with at most r nonzero rows and c nonzero columns with $\mathcal{P}(r,c)$. Similarly, we denote the set of all ordinary partitions (Young diagrams) with $\mathcal{Y}$ and those with at most r parts with $\mathcal{Y}(r) = \mathcal{P}(r,1)$.

We use the definitions of $f(n,m)$ and $F_\pi(q,t)$ from the Introduction. To a plane partition π we associate a rational function $F_\pi(q,t)$ that is related to Macdonald symmetric functions (for a reference see Chapter VI of [Mac]).

In this section we prove Theorem A. The proof consists of a few steps. We first define weight functions on sequences of ordinary partitions (Section 2.1.1). These weight functions are defined using Macdonald symmetric functions. Second, for suitably chosen specializations of these symmetric functions we obtain that the weight functions vanish for every sequence of partitions except if the sequence corresponds to a plane partition (Section 2.1.2). Finally, we show that for $\pi \in \mathcal{P}$ the weight function of π is equal to $F_\pi(q,t)$ (Section 2.1.3).

Before showing these steps we first comment on a corollary of Theorem A.

Fix $c = 1$. Then, Theorem A gives a generating function formula for ordinary partitions since $\mathcal{P}(r,1) = \mathcal{Y}(r)$. For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{Y}(r)$ we define $d_i = \lambda_i - \lambda_{i+1}$, $i = 1, \dots, r$. Then

$$F_\lambda(q,t) = \prod_{i=1}^{r} f(d_i, 0) = \prod_{i=1}^{r} \prod_{j=1}^{d_i} \frac{1 - tq^{j-1}}{1 - q^j}.$$

Note that $F_\lambda(q,t)$ depends only on the set of distinct parts of λ.

Corollary 2.1.

$$\sum_{\lambda \in \mathcal{Y}(r)} F_\lambda(q,t) s^{|\lambda|} = \prod_{i=1}^{r} \frac{(ts^i;q)_\infty}{(s^i;q)_\infty}.$$

In particular,

$$\sum_{\lambda \in \mathcal{Y}} F_\lambda(q,t) s^{|\lambda|} = \prod_{i=1}^{\infty} \frac{(ts^i;q)_\infty}{(s^i;q)_\infty}.$$

This corollary is easy to show directly.

Proof. First, we expand $(ts;q)_\infty/(s;q)_\infty$ into the power series in s. Let $a_d(q,t)$ be the coefficient of s^d. Observe that

$$\frac{(ts;q)_\infty}{(s;q)_\infty} := \sum_{d=0}^{\infty} a_d(q,t) s^d = \frac{1-ts}{1-s} \sum_{d=0}^{\infty} a_d(q,t) s^d q^d.$$

By identifying coefficients of s^d and doing induction on d, this implies that

$$a_d(q,t) = f(d,0).$$

Every $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{Y}(r)$ is uniquely determined by $d_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, r$, where $d_i = \lambda_i - \lambda_{i+1}$. Then $\lambda_i = \sum_{j \geq 0} d_{i+j}$ and $|\lambda| = \sum_{i=1}^r i d_i$. Therefore,

$$\prod_{i=1}^r \frac{(ts^i;q)_\infty}{(s^i;q)_\infty} = \prod_{i=1}^r \sum_{d_i=0}^\infty a_{d_i}(q,t) s^{id_i}$$
$$= \sum_{d_1,\dots,d_r} \left[\prod_{i=1}^r a_{d_i}(q,t)\right] \cdot \left[s^{\sum_{i=1}^r i d_i}\right] = \sum_{\lambda \in \mathcal{Y}(r)} F_\lambda(q,t) s^{|\lambda|}. \qquad \square$$

2.1.1. *The weight functions.* The weight function is defined as a product of Macdonald symmetric functions P and Q. We assume familiarity with Chapter VI of [Mac], and we follow the notation used there.

Recall that Macdonald symmetric functions $P_{\lambda/\mu}(x;q,t)$ and $Q_{\lambda/\mu}(x;q,t)$ depend on two parameters q and t and are indexed by pairs of ordinary partitions λ and μ. In the case when $q = t = 0$ they are equal to ordinary Schur functions, in the case when $q = 0$ and $t = -1$ to Schur P and Q functions, and for $q = 0$ to Hall-Littlewood symmetric functions.

For an ordinary partition λ and a box $s = (i,j)$, we write $s \in \lambda$ if s is a box in the Young diagram of λ. Let

$$b_\lambda(s) = b_\lambda(s;q,t) = \begin{cases} \dfrac{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}}{1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i}}, & s \in \lambda, \\ 1 & \text{otherwise}, \end{cases}$$

where λ' is the conjugate partition of λ and

$$b_\lambda(q,t) = \prod_{s \in \lambda} b_\lambda(s). \tag{2.2}$$

The relationship between P and Q functions is given by (see (7.8) of [Mac, Chapter VI])

$$Q_{\lambda/\mu} = \frac{b_\lambda}{b_\mu} P_{\lambda/\mu}. \tag{2.3}$$

Recall that (by (7.7(i)) of [Mac, Chapter VI] and (2.3))

$$P_{\lambda/\mu} = Q_{\lambda/\mu} = 0 \quad \text{unless } \lambda \supset \mu. \tag{2.4}$$

We set $P_\lambda = P_{\lambda/\emptyset}$ and $Q_\lambda = Q_{\lambda/\emptyset}$. Recall that ((2.5) and (4.13) of [Mac, Chapter VI])

$$\Pi(x,y;q,t) = \sum_{\lambda \in \mathcal{Y}} Q_\lambda(x;q,t) P_\lambda(y;q,t) = \prod_{i,j} \frac{(tx_i y_j;q)_\infty}{(x_i y_j;q)_\infty}.$$

Let Λ be the algebra of symmetric functions. A specialization of Λ is an algebra homomorphism $\Lambda \to \mathbb{C}$. If ρ and σ are specializations of Λ, then we write $P_{\lambda/\mu}(\rho;q,t)$, $Q_{\lambda/\mu}(\rho;q,t)$ and $\Pi(\rho,\sigma;q,t)$ for the images of $P_{\lambda/\mu}(x;q,t)$, $Q_{\lambda/\mu}(x;q,t)$ and $\Pi(x,y;q,t)$ under ρ, respectively $\rho \otimes \sigma$. Every map $\rho : (x_1, x_2, \dots) \to (a_1, a_2, \dots)$, where $a_i \in \mathbb{C}$ and only finitely many a_i's are nonzero, defines a specialization.

Let

$$\varphi_{\lambda/\mu}(q,t) = \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}, \tag{2.5}$$

where $C_{\lambda/\mu}$ is the union of all columns that intersect λ/μ. If ρ is a specialization of Λ where $x_1 = a$, $x_2 = x_3 = \cdots = 0$, then by (7.14) of [Mac, Chapter VI]

$$Q_{\lambda/\mu}(\rho; q, t) = \begin{cases} \varphi_{\lambda/\mu}(q,t) a^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Let $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \dots, \rho_T^-)$ be a finite sequence of specializations. For two sequences of partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ and $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$ we set the weight function $W(\lambda, \mu; q, t)$ to be

$$W(\lambda, \mu; q, t) = \prod_{n=1}^{T} Q_{\lambda^n/\mu^{n-1}}(\rho_{n-1}^+; q, t) P_{\lambda^n/\mu^n}(\rho_n^-; q, t),$$

where $\mu^0 = \mu^T = \emptyset$. Note that by (2.4) it follows that $W(\lambda, \mu; q, t) = 0$ unless

$$\emptyset \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \cdots \supset \mu^{T-1} \subset \lambda^T \supset \emptyset.$$

Proposition 2.2. *The sum of the weights $W(\lambda, \mu; q, t)$ over all sequences of partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ and $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$ is equal to*

$$Z(\rho; q, t) = \prod_{0 \le i < j \le T} \Pi(\rho_i^+, \rho_j^-; q, t). \tag{2.7}$$

Proof. We use

$$\sum_{\lambda \in \mathcal{Y}} Q_{\lambda/\mu}(x) P_{\lambda/\nu}(y) = \Pi(x, y) \sum_{\tau \in \mathcal{Y}} Q_{\nu/\tau}(x) P_{\mu/\tau}(y).$$

The proof of this is analogous to the proof of Proposition 5.1 that appeared in our earlier paper [V]. Also, see Example 26 of Chapter I, Section 5 of [Mac].

We prove (2.7) by induction on T. Using the formula above we substitute sums over λ^i's with sums over τ^{i-1}'s as in the proof of Proposition 2.1 of [BR]. This gives

$$\prod_{i=0}^{T-1} \Pi(\rho_i^+, \rho_{i+1}^-) \sum_{\mu, \tau} Q_{\mu^1}(\rho_0^+) P_{\mu^1/\tau^1}(\rho_2^-) Q_{\mu^2/\tau^1}(\rho_1^+) \dots P_{\mu^{T-1}}(\rho_T^-).$$

This is the sum of $W(\mu, \tau)$ with $\mu = (\mu^1, \dots, \mu^{T-1})$ and $\tau = (\tau^1, \dots, \tau^{T-2})$. Inductively, we obtain (2.7). □

2.1.2. *Specializations.* For $\pi = (\dots, \lambda^{-1}, \lambda^0, \lambda^1, \dots) \in \mathcal{P}$ we define a function $\Phi_\pi(q, t)$ by

$$\Phi_\pi(q, t) = \frac{1}{b_{\lambda^0}(q, t)} \prod_{n=-\infty}^{\infty} \varphi_{[\lambda^{n-1}/\lambda^n]}(q, t), \tag{2.8}$$

where b and φ are given by (2.2) and (2.5). Only finitely many terms in the product are different than 1 because only finitely many λ^n are nonempty partitions.

We show that for a suitably chosen specialization the weight function vanishes for every sequence of ordinary partitions unless this sequence represents a plane partition, in which case it becomes (2.8). This, together with Proposition 2.2, will imply the following.

Proposition 2.3.

$$\sum_{\pi\in\mathcal{P}(r,c)} \Phi_\pi(q,t)s^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty}.$$

Proof. Let

$$\begin{array}{ll} \rho_n^+ : x_1 = s^{-n-1/2},\ x_2 = x_3 = \ldots = 0, & -r \le n \le -1, \\ \rho_n^- : x_1 = x_2 = \ldots = 0, & -r+1 \le n \le -1, \\ \rho_n^- : x_1 = s^{n+1/2},\ x_2 = x_3 = \ldots = 0, & 0 \le n \le c-1, \\ \rho_n^+ : x_1 = x_2 = \ldots = 0, & 0 \le n \le c-2. \end{array}$$

Then for any two sequences $\lambda = (\lambda^{-r+1},\ldots,\lambda^{-1},\lambda^0,\lambda^1,\ldots,\lambda^{c-1})$ and $\mu = (\mu^{-r+1},\ldots,\mu^{-1},\mu^0,\mu^1,\ldots,\mu^{c-2})$ the weight function is given by

$$W(\lambda,\mu) \quad = \quad \prod_{n=-r+1}^{c-1} Q_{\lambda^n/\mu^{n-1}}(\rho_{n-1}^+)P_{\lambda^n/\mu^n}(\rho_n^-),$$

where $\mu^{-r} = \mu^{c-1} = \emptyset$. By (2.3), (2.4) and (2.6) we have that $W(\lambda,\mu) = 0$ unless

$$\mu^n = \begin{cases} \lambda^n & n < 0, \\ \lambda^{n+1} & n \ge 0, \end{cases}$$

$$\cdots \lambda^{-1} \subset \lambda^0 \supset \lambda^1 \supset \cdots,$$

$$[\lambda^{n-1}/\lambda^n] \text{ is a horizontal strip for every } n,$$

i.e. $\lambda \in \mathcal{P}$ and in that case

$$\begin{aligned} W(\lambda,\mu) \quad &= \quad \prod_{n=-r+1}^{0} \varphi_{\lambda^n/\lambda^{n-1}}(q,t)s^{(-2n+1)(|\lambda^n|-|\lambda^{n-1}|)/2} \\ &\quad \cdot \prod_{n=1}^{c} \frac{b_{\lambda^n}(q,t)}{b_{\lambda^{n-1}}(q,t)} \varphi_{\lambda^{n-1}/\lambda^n}(q,t)s^{(2n-1)(|\lambda^{n-1}|-|\lambda^n|)/2} \\ &= \quad \frac{1}{b_{\lambda^0}(q,t)} \prod_{n=-r+1}^{c} \varphi_{[\lambda^{n-1}/\lambda^n]}(q,t)s^{|\lambda|} = \Phi_\lambda(q,t)s^{|\lambda|}. \end{aligned}$$

If ρ^+ is $x_1 = s$, $x_2 = x_3 = \cdots = 0$ and ρ^- is $x_1 = r$, $x_2 = x_3 = \cdots = 0$, then

$$\Pi(\rho^+,\rho^-) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\bigg|_{x=\rho^+,\, y=\rho^-} = \frac{(tsr;q)_\infty}{(sr;q)_\infty}.$$

Then, by Proposition 2.2, for the given specializations of ρ_i^+ and ρ_j^- we have

$$Z = \prod_{i=-1}^{-r}\prod_{j=0}^{c-1} \Pi(\rho_i^+,\rho_j^-) = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty}. \qquad \square$$

2.1.3. *Final step.* We show that $F_\pi(q,t) = \Phi_\pi(q,t)$. Then Proposition 2.3 implies Theorem A.

Proposition 2.4. *Let $\pi \in \mathcal{P}$. Then*

$$F_\pi(q,t) = \Phi_\pi(q,t).$$

Proof. We show this by induction on the number of boxes in the support of π. Denote the last nonzero part in the last row of the support of π by x. Let λ be a diagonal partition containing it, and let x be its kth part. Because of the symmetry with respect to the transposition we can assume that λ is one of diagonal partitions on the left.

Let π' be the plane partition obtained from π by removing x. We want to show that F_π and $F_{\pi'}$ satisfy the same recurrence relation as Φ_π and $\Phi_{\pi'}$. The verification uses the explicit formulas for b_λ and $\varphi_{\lambda/\mu}$ given by (2.2) and (2.5).

We divide the problem into several cases depending on the position of the box containing x. Let I, II and III be the cases shown in Figure 3.

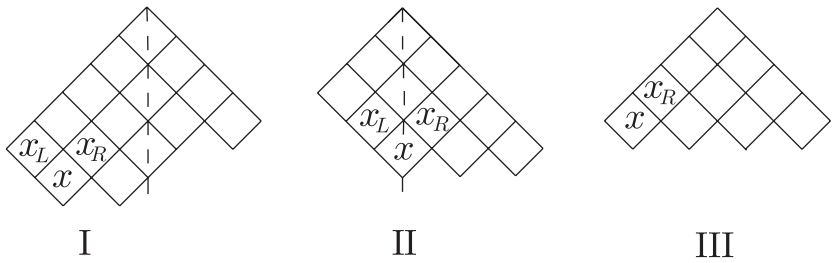

FIGURE 3. Cases I, II and III

Let (i,j) be the box containing x. We denote the numbers in boxes $(i,j-1)$ and $(i-1,j)$ by x_L and x_R, respectively. Let λ^L and λ^R be the diagonal partitions of π containing x_L and x_R, respectively. Let λ' be the partition obtained from λ by removing x.

If III, then $k=1$, and one checks easily that

$$\frac{F_{\pi'}}{F_\pi} = \frac{\Phi_{\pi'}}{\Phi_\pi} = \frac{f(\lambda_1^R, 0)}{f(\lambda_1^R - \lambda_1, 0) f(\lambda_1, 0)}.$$

Assume I or II. Then

$$\Phi_{\pi'} = \Phi_\pi \cdot \frac{\varphi_{[\lambda'/\lambda^L]}}{\varphi_{[\lambda/\lambda^L]}} \cdot \frac{\varphi_{[\lambda'/\lambda^R]}}{\varphi_{[\lambda/\lambda^R]}} \cdot \frac{b_{\lambda^0(\pi)}}{b_{\lambda^0(\pi')}} = \Phi_\pi \cdot \Phi_L \cdot \Phi_R \cdot \Phi_0.$$

Thus, we need to show that

$$\Phi_L \cdot \Phi_R \cdot \Phi_0 = F := \frac{F_{\pi'}}{F_\pi}. \tag{2.9}$$

If I, then $\lambda_{k-1}^L = x_L$ and $\lambda_k^R = x_R$. From the definition of φ we have that

$$\Phi_L = \prod_{i=0}^{k-1} \frac{f(\lambda_{k-i} - \lambda_k, i)}{f(\lambda_{k-i}, i)} \cdot \prod_{i=0}^{k-2} \frac{f(\lambda_{k-1-i}^L, i)}{f(\lambda_{k-1-i}^L - \lambda_k, i)}. \tag{2.10}$$

Similarly,

$$\Phi_R = \prod_{i=0}^{k-2} \frac{f(\lambda_{k-1-i} - \lambda_k, i)}{f(\lambda_{k-1-i}, i)} \cdot \prod_{i=0}^{k-1} \frac{f(\lambda_{k-i}^R, i)}{f(\lambda_{k-i}^R - \lambda_k, i)}.$$

If II, then $\lambda^L_{k-1} = x_L$ and $\lambda^R_{k-1} = x_R$, and both Φ_L and Φ_R are given with (2.10), substituting L with R for Φ_R, while

$$\Phi_0 = \prod_{i=0}^{k-1} \frac{f(\lambda_{k-i}, i)}{f(\lambda_{k-i} - \lambda_k, i)} \cdot \prod_{i=0}^{k-2} \frac{f(\lambda_{k-1-i} - \lambda_k, i)}{f(\lambda_{k-1-i}, i)}.$$

From the definition of F one can verify that (2.9) holds. □

2.2. **Hall-Littlewood's case.** We analyze the generalized MacMahon formula in Hall-Littlewood's case, i.e. when $q = 0$, in more detail. Namely, we describe $F_\pi(0,t)$.

We use the definition of $A_\pi(t)$ from the Introduction. In Proposition 2.6 we show that $F_\pi(0,t) = A_\pi(t)$. This, together with Theorem A, implies Theorem B.

Note that the result implies the following simple identities. If

$$\lambda \in \mathcal{Y} = \bigcup_{r \geq 1} \mathcal{P}(r,1),$$

then $k(\lambda)$ becomes the number of distinct parts of λ.

Corollary 2.5.

$$\sum_{\lambda \in \mathcal{Y}(r)} (1-t)^{k(\lambda)} s^{|\lambda|} = \prod_{i=1}^{r} \frac{1-ts^i}{1-s^i}.$$

In particular,

$$\sum_{\lambda \in \mathcal{Y}} (1-t)^{k(\lambda)} s^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1-ts^i}{1-s^i}.$$

These formulas are easily proved by the argument used in the proof of Corollary 2.1.

We now prove

Proposition 2.6. *Let $\pi \in \mathcal{P}$. Then*

$$F_\pi(0,t) = A_\pi(t).$$

Proof. Let B be an h-level border component of π. Let $F(i,j) = F_\pi(i,j)(0,t)$. It is enough to show that

$$\prod_{(i,j) \in B} F(i,j) = 1 - t^h. \tag{2.11}$$

Let

$$c(i,j) = \chi_B(i+1,j) + \chi_B(i,j+1),$$

where χ_B is the characteristic function of B taking value 1 on the set B and 0 elsewhere. If there are n boxes in B, then

$$\sum_{(i,j) \in B} c(i,j) = n - 1. \tag{2.12}$$

Let $(i,j) \in B$. We claim that

$$F(i,j) = (1-t^h)^{1-c(i,j)}. \tag{2.13}$$

Then (2.12) and (2.13) imply (2.11).

To show (2.13) we observe that

$$f(l,m)(0,t) = \begin{cases} 1, & l = 0, \\ 1 - t^{m+1}, & l \geq 1. \end{cases}$$

With the same notation as in (1.3) we have that μ_m, ν_m, λ_m, λ_{m+1} are all equal to λ_1 for every $m < h$, while for every $m > h$ they are all different from λ_1. Then

$$\begin{aligned} F(i,j) &= \prod_{m=0}^{\infty} \frac{f(\lambda_1 - \mu_{m+1}, m)(0,t) f(\lambda_1 - \nu_{m+1}, m)(0,t)}{f(\lambda_1 - \lambda_{m+1}, m)(0,t) f(\lambda_1 - \lambda_{m+2}, m)(0,t)} \\ &= \frac{f(\lambda_1 - \mu_h, h-1)(0,t) f(\lambda_1 - \nu_h, h-1)(0,t)}{f(\lambda_1 - \lambda_h, h-1)(0,t) f(\lambda_1 - \lambda_{h+1}, h-1)(0,t)} \\ &= \frac{(1-t^h)^{1-\chi_B(i+1,j)}(1-t^h)^{1-\chi_B(i,j+1)}}{1 \cdot (1-t^h)} = (1-t^h)^{1-c(i,j)}. \qquad \square \end{aligned}$$

3. A bijective proof of the shifted MacMahon formula

In this section we are going to give another proof of the shifted MacMahon formula (1.2). More generally, we prove

Theorem 3.1.

$$\sum_{\pi \in \mathcal{SP}(r,c)} 2^{k(\pi)} x^{\mathrm{tr}(\pi)} s^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 + x s^{i+j-1}}{1 - x s^{i+j-1}}.$$

Here $\mathcal{SP}(r,c)$ is the set of strict plane partitions with at most r rows and c columns. The trace of π, denoted by $\mathrm{tr}(\pi)$, is the sum of diagonal entries of π.

The x-version of the shifted MacMahon formula given in Theorem 3.1 is, as far as we know, a new result. Its proof is mostly independent of the rest of the paper. It is similar in spirit to the proof of MacMahon's formula given in Section 7.20 of [S]. It uses two bijections. One correspondence is between strict plane partitions and pairs of shifted tableaux. The other one is between pairs of marked shifted tableaux and marked matrices, and it is obtained by the shifted Knuth algorithm.

We recall the definitions of a marked tableau and a marked shifted tableau (see, e.g. Chapter XIII of [HH]).

Let P be a totally ordered set

$$P = \{1 < 1' < 2 < 2' < \cdots\}.$$

We distinguish elements in P as marked and unmarked, the former being the one with a prime. We use $|p|$ for the unmarked number corresponding to $p \in P$.

A marked (shifted) tableau is a (shifted) Young diagram filled with row and column nonincreasing elements from P such that any given unmarked element occurs at most once in each column, whereas any marked element occurs at most once in each row. Examples of a marked tableau and a marked shifted tableau are given in Figure 4.

An unmarked (shifted) tableau is a tableau obtained by deleting primes from a marked (shifted) tableau. We can also define it as a (shifted) diagram filled with row and column nonincreasing positive integers such that no 2×2 square is filled with the same number. Unmarked tableaux are strict plane partitions.

We define connected components of a marked or unmarked (shifted) tableau in a similar way as for plane partitions. Namely, a connected component is a set

5	3'	1'	1	1
3	1'	1		
2	1'			

5	3'	2'	1'	1
	3	2'	1'	
		1	1	

FIGURE 4. A marked tableau and a marked shifted tableau

of rookwise connected boxes filled with p or p'. By the definition of a tableau all connected components are border strips (a set of rookwise connected boxes containing no 2×2 block of boxes). Connected components for the examples above are shown in Figure 4 (bold lines represent boundaries of these components).

We use $k(S)$ to denote the number of components of a marked or unmarked (shifted) tableau S. For every marked (shifted) tableau there is a corresponding unmarked (shifted) tableau obtained by deleting all the primes. The number of marked (shifted) tableaux corresponding to the same unmarked (shifted) tableau S is equal to $2^{k(S)}$ because there are exactly two possible ways to mark each border strip.

For a tableau S, we use $\mathrm{sh}(S)$ to denote the shape of S that is an ordinary partition with parts equal to the lengths of rows of S. We define $\ell(S) = \ell(\mathrm{sh}(S))$ and $\max(S) = |p_{\max}|$, where $p_{\max}$ is the maximal element in S. For both examples $\mathrm{sh}(S) = (5, 3, 2)$, $\ell(S) = 3$ and $\max(S) = 5$.

A marked matrix is a matrix with entries from $P \cup \{0\}$. Let $\mathcal{M}(r, c)$ be the set of $r \times c$ marked matrices.

Let $\mathcal{ST}^M(r, c)$, respectively $\mathcal{ST}^U(r, c)$, be the set of ordered pairs (S, T) of marked, respectively unmarked, shifted tableaux of the same shape, where $\max(S) = c$, $\max(T) = r$ and T has no marked letters on its main diagonal. There is a natural mapping $i : \mathcal{ST}^M(r, c) \to \mathcal{ST}^U(r, c)$ obtained by deleting primes and for $(S, T) \in \mathcal{ST}^U(r, c)$

$$|i^{-1}[(S, T)]| = 2^{k(S)+k(T)-l(S)}. \tag{3.1}$$

The shifted Knuth algorithm (see Chapter XIII of [HH]) establishes the following correspondence.

Theorem 3.2. *There is a bijective correspondence between matrices $A = [a_{ij}]$ over $P \cup \{0\}$ and ordered pairs (S, T) of marked shifted tableaux of the same shape such that T has no marked elements on its main diagonal. The correspondence has the property that $\sum_i a_{ij}$ is the number of entries s of S for which $|s| = j$ and $\sum_j a_{ij}$ is the number of entries t of T for which $|t| = i$.*

In particular, this correspondence maps $\mathcal{M}(r, c)$ onto $\mathcal{ST}^M(r, c)$ and

$$|\,\mathrm{sh}(S)| = \sum_{i,j} |a_{ij}|, \qquad |S| = \sum_{i,j} j|a_{ij}|, \qquad |T| = \sum_{i,j} i|a_{ij}|.$$

Remark. The shifted Knuth algorithm described in Chapter XIII of [HH] establishes a correspondence between marked matrices and pairs of marked shifted tableaux with row and column *nondecreasing* elements. This algorithm can be adjusted to work for marked shifted tableaux with row and column *nonincreasing* elements.

Namely, one needs to change the encoding of a matrix over $P \cup \{0\}$ and two algorithms BUMP and EQBUMP, while INSERT, UNMARK, CELL and *unmix* remain unchanged.

One encodes a matrix $A \in \mathcal{P}(r,c)$ into a two-line notation E with pairs $\begin{smallmatrix} i \\ j \end{smallmatrix}$ repeated $|a_{ij}|$ times, where i is going from r to 1 and j from c to 1. If a_{ij} was marked, then we mark the leftmost j in the pairs $\begin{smallmatrix} i \\ j \end{smallmatrix}$. The example from p. 246 of [HH],

$$A = \begin{pmatrix} 1' & 0 & 2 \\ 2 & 1 & 2' \\ 1' & 1' & 0 \end{pmatrix},$$

would be encoded as

$$E = \begin{matrix} 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2' & 1' & 3' & 3 & 2 & 1 & 1 & 3 & 3 & 1' \end{matrix}.$$

Algorithms BUMP and EQBUMP insert $x \in P \cup \{0\}$ into a vector v over $P \cup \{0\}$. By BUMP (resp. EQBUMP) one inserts x into v by removing (bumping) the leftmost entry of V that is less (resp. less or equal) than x and replacing it by x, or if there is no such entry, then x is placed at the end of v.

For the example from above this adjusted shifted Knuth algorithm would give

$$S = \begin{matrix} 3' & 3 & 3 & 3 & 1' \\ & 2' & 2 & 1 & 1 \\ & & 1' & & \end{matrix} \quad \text{and} \quad T = \begin{matrix} 3 & 3 & 2' & 2 & 2 \\ & 2 & 2 & 1 & 1 \\ & & 1 & & \end{matrix}.$$

The other correspondence between pairs of shifted tableaux of the same shape and strict plane partitions is described in the following theorem. It is parallel to the correspondence from Section 7.20 of [S].

Theorem 3.3. *There is a bijective correspondence* Π *between strict plane partitions* π *and ordered pairs* (S,T) *of shifted tableaux of the same shape. This correspondence maps* $\mathcal{SP}(r,c)$ *onto* $\mathcal{ST}^U(r,c)$ *and if* $(S,T) = \Pi(\pi)$*, then*

$$|\pi| = |S| + |T| - |\operatorname{sh}(S)|,$$
$$\operatorname{tr}(\pi) = |\operatorname{sh}(S)| = |\operatorname{sh}(T)|,$$
$$k(\pi) = k(S) + k(T) - l(S).$$

Proof. Every $\lambda \in \mathcal{Y}$ is uniquely represented by Frobenius coordinates $(p_1, \dots, p_d \mid q_1, \dots, q_d)$, where d is the number of diagonal boxes in the Young diagram of λ and p's and q's correspond to the arm length and the leg length, i.e. $p_i = \lambda_i - i + 1$ and $q_i = \lambda'_i - i + 1$, where $\lambda' \in \mathcal{Y}$ is the transpose of λ.

Let $\pi \in \mathcal{SP}$. Let $(\mu_1, \mu_2, \dots)$ be a sequence of ordinary partitions whose diagrams are obtained by the horizontal slicing of the 3-dimensional diagram of π (see Figure 5). The Young diagram of μ_1 corresponds to the first slice and is the same as the support of π, μ_2 corresponds to the second slice, etc. More precisely, the Young diagram of μ_i consists of all boxes of the support of π filled with numbers

greater than or equal to i. For example, if

$$\pi = \begin{array}{ccccc} 5 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 & \\ 3 & 3 & 2 & & \\ 2 & 2 & 1 & & \end{array},$$

then $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ are

$$\mu_1 = \begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & \\ * & * & * & & \\ * & * & * & & \end{array}, \quad \mu_2 = \begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & \end{array}, \quad \mu_3 = \begin{array}{cc} * & * \\ * & * \\ * & * \end{array}, \quad \mu_4 = \begin{array}{c} * \\ * \end{array}, \quad \mu_5 = \begin{array}{c} * \end{array}.$$

Let S, respectively T, be an unmarked shifted tableau whose ith diagonal is equal to the p, respectively q, Frobenius coordinate of μ_i. For the example above

$$S = \begin{array}{ccccc} 5 & 3 & 2 & 1 & 1 \\ & 3 & 2 & 1 & \\ & & 1 & 1 & \end{array} \quad \text{and} \quad T = \begin{array}{ccccc} 4 & 4 & 3 & 2 & 1 \\ & 3 & 3 & 2 & \\ & & 2 & 1 & \end{array}.$$

It is not hard to check that Π is a bijection between pairs of unmarked shifted tableaux of the same shape and strict plane partitions.

We only verify that

$$k(\pi) = k(S) + k(T) - l(S). \tag{3.2}$$

Other properties are straightforward implications of the definition of Π.

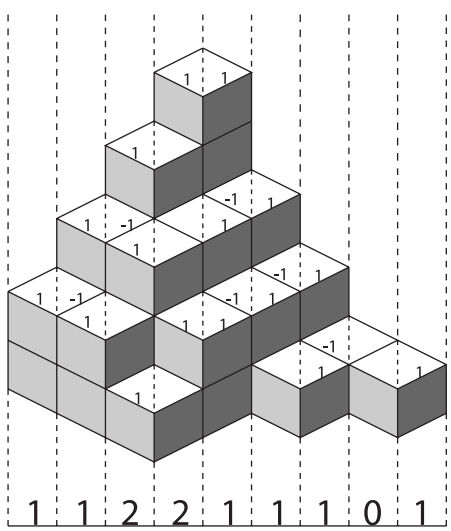

FIGURE 5. 3-dimensional diagram of a plane partition

Consider the 3-dimensional diagram of π (see Figure 5) and fix one of its vertical columns on the right (with respect to the main diagonal). A rhombus component consists of all black rhombi that are either directly connected or that have one

white space between them. For the columns on the left we use gray rhombi instead of black ones. The number at the bottom of each column in Figure 5 is the number of rhombus components for that column. Let b, respectively g, be the number of rhombus components for all right, respectively left, columns. For the given example $b = 4$ and $g = 6$.

One can obtain b by a different counting. Consider edges on the right side. Mark all the edges with 0 except the following ones. Mark a common edge for a white rhombus and a black rhombus, where the black rhombus is below the white rhombus with 1. Mark a common edge for two white rhombi that is perpendicular to the plane of black rhombi with -1. See Figure 5. One obtains b by summing these numbers over all edges on the right side of the 3-dimensional diagram. One recovers c in a similar way by marking edges on the left.

Now, we restrict to a connected component (one of the white terraces; see Figure 5) and sum all the numbers associated to its edges. If a connected component does not intersect the main diagonal, then the sum is equal to 1. Otherwise this sum is equal to 2. This implies that

$$k(\pi) = b + g - l(\lambda^0).$$

Since $l(S) = l(\lambda^0)$ it is enough to show that $k(S) = b$ and $k(T) = g$, and (3.2) follows.

Each black rhombus in the right ith column of the 3-dimensional diagram corresponds to an element of a border strip of S filled with i, and each rhombus component corresponds to a border strip component. If two adjacent boxes from the same border strip are in the same row, then the corresponding rhombi from the 3-dimensional diagram are directly connected, and if they are in the same column, then there is exactly one white space between them. This implies $k(S) = b$. Similarly, we get $k(T) = g$. □

Now, using the described correspondences sending $\mathcal{SP}(r,c)$ to $\mathcal{ST}^U(r,c)$ and $\mathcal{ST}^M(r,c)$ to $\mathcal{M}(r,c)$, we can prove Theorem 3.1.

Proof.

$$\begin{aligned}
\sum_{\pi\in\mathcal{SP}(r,c)} 2^{k(\pi)} x^{\mathrm{tr}(\pi)} s^{|\pi|} &\overset{\text{Thm } 3.3}{=} \sum_{(S,T)\in\mathcal{ST}^U(r,c)} 2^{k(S)+k(T)-l(S)} x^{|\mathrm{sh}S|} s^{|S|+|T|-|\mathrm{sh}S|} \\
&\overset{\text{by } (3.1)}{=} \sum_{(S,T)\in\mathcal{ST}^M(r,c)} x^{|\mathrm{sh}S|} s^{|S|+|T|-|\mathrm{sh}S|} \\
&\overset{\text{Thm } 3.2}{=} \sum_{A\in\mathcal{M}(r,c)} x^{\sum_{i,j}|a_{ij}|} s^{\sum_{i,j}(i+j-1)|a_{ij}|} \\
&= \prod_{i=1}^{r}\prod_{j=1}^{c} \sum_{a_{ij}\in P\cup 0} x^{|a_{ij}|} s^{(i+j-1)|a_{ij}|} \\
&= \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1+xs^{i+j-1}}{1-xs^{i+j-1}}.
\end{aligned}$$

□

Letting $r \to \infty$ and $c \to \infty$ we get

Corollary 3.4.

$$\sum_{\pi \in \mathcal{SP}} 2^{k(\pi)} x^{tr(\pi)} s^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1 + xs^n}{1 - xs^n} \right)^n .$$

At $x = 1$ we recover the shifted MacMahon formula.

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