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AUSLANDER-REITEN COMPONENTS CONTAINING MODULES WITH BOUNDED BETTI NUMBERS

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Abstract. Let R be a connected selfinjective Artin algebra, and M an indecomposable nonprojective R-module with bounded Betti numbers lying in a regular component of the Auslander-Reiten quiver of R. We prove that the Auslander-Reiten sequence ending at M has at most two indecomposable summands in the middle term. Furthermore we show that the component of the Auslander-Reiten quiver containing M is either a stable tube or of type $\mathbb{Z}A_\infty$. We use these results to study modules with eventually constant Betti numbers, and modules with eventually periodic Betti numbers.

1. Introduction

Auslander-Reiten sequences and Auslander-Reiten quivers have played a central role in the study of the representation theory of Artin algebras, in particular, of algebras that are finite dimensional over a field. We refer the reader to [2] for basic definitions and properties of Auslander-Reiten sequences and quivers. The paper deals primarily with selfinjective algebras R. In [15], Riedtmann classified the Auslander-Reiten quiver in case R is selfinjective and of finite representation type. Webb classified the Auslander-Reiten components in case R is a group algebra over a field; also see [7, 8]. Finally, the tame selfinjective case has also been thoroughly investigated (see [18]) and the case of selfinjective Koszul algebras of Loewy length greater than 3 has been studied in [13].

Assume that R is a selfinjective Artin algebra. Recall that if M is a finitely generated R-module, and if

$$\cdots \to P^2 \xrightarrow{\delta_2} P^1 \xrightarrow{\delta_1} P^0 \xrightarrow{\delta_0} M \to 0$$

is a minimal projective resolution of M, then the *i-th Betti number* of M, $\beta_i(M)$, equals the number of indecomposable summands of P^i. We also denote by $\Omega^i(M)$, the i-th syzygy of M; that is, $\mathrm{Im}(\delta_i) = \Omega^i(M)$. We say that the *complexity of a finitely generated R-module M is at most n*, and we write

$$\mathrm{cx}(M) \le n$$

if $\beta_i(M) \le ci^{n-1}$, for some $c \in \mathbb{Q}$ and $i \gg 0$ and that the *complexity of M is n*, $\mathrm{cx}(M) = n$ if $\mathrm{cx}(M) \le n$ but $\mathrm{cx}(M) \not\le n-1$.

If M is an R-module, note that $\mathrm{cx}(M) = 0$ is equivalent to the projective dimension of M being finite and hence a projective module, and $\mathrm{cx}(M) = 1$ is equivalent

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to M having infinite projective dimension and the existence of $b \in \mathbb{R}_{>0}$ such that $\beta_n(M) \leq b$, for all $n \geq 0$. The aim of this paper is to investigate the structure of regular components of the Auslander-Reiten quiver containing R-modules M whose Betti numbers are bounded, that is, $\operatorname{cx}(M) = 1$, and then to use this structure to study the modules having eventually constant and eventually periodic Betti numbers.

Recall first that if R is an Artin algebra and $0 \to A \to B \to C \to 0$ is an Auslander-Reiten sequence of R-modules, then the indecomposable R-module A is called the *Auslander-Reiten translate of* C and is denoted by τC. In Section 2 we present some preliminary results including the behavior of Auslander-Reiten sequences under taking syzygies (see Proposition 2.4). To this end we introduce for notational purposes the notion of Ω-perfect modules, and we prove in Proposition 2.10 that if C is an indecomposable module of complexity 1 over a selfinjective Artin algebra, lying in a regular component that is not a tube, then, for sufficiently large n, $\tau^n C$ is Ω-perfect. In Proposition 2.8, we set the stage for proving that

$$\beta_n(B) = \beta_n(A) + \beta_n(C), \text{ for all } n \geq 0,$$

whenever the exact sequence of R-modules $0 \to A \to B \to C \to 0$ has the property that $B \to C$ is an irreducible map and C is an indecomposable Ω-perfect R-module of complexity 1. In particular, the above identity holds for every Auslander-Reiten sequence ending at such an Ω-perfect module.

In Section 3 we turn our attention to modules having bounded Betti numbers. The main result in this section is Theorem 3.7. We prove that if $\mathcal{C}$ is a regular component of the Auslander-Reiten quiver of R containing a module of complexity 1, then $\mathcal{C}$ is either a stable tube or a $\mathbb{Z}A_\infty$-component.

In the last section, we study the structure of Auslander-Reiten sequences and components that contain modules whose Betti numbers are eventually constant or eventually periodic. Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R containing a module having eventually constant Betti numbers. Then, in Theorem 4.6, we prove that there is an infinite family $\{M^n\}_{n=1}^\infty$ of modules in $\mathcal{C}$ having constant Betti numbers $\{b_n\}_{n=1}^\infty$ such that the sequence $\{b_n\}$ is strictly increasing and the R-modules M^n lie on distinct τ-orbits.

We also prove, in Theorem 4.9, for a local selfinjective algebra $(R, \mathbf{m})$ that, if $\mathcal{C}$ is a regular component of the Auslander-Reiten quiver of R, then $\mathcal{C}$ contains a module with eventually constant Betti numbers if and only if there is a module on the boundary of $\mathcal{C}$ with eventually periodic Betti numbers.

Before beginning our preliminary results, we need to mention a convention that we use in this paper. We introduce this convention to simplify notation in later sections. In general, if R is an Artin algebra, then the syzygy function Ω is a functor from the category of R-modules modulo projective modules, $\underline{\operatorname{mod}}(R)$, to itself. Assume that

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence of R-modules such that the induced sequence

$$0 \to A/JA \xrightarrow{\overline{f}} B/JB \xrightarrow{\overline{g}} C/JC \to 0$$

is exact (and hence splits), where J denotes the Jacobson radical of R. Then we obtain the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(A) \arrow[r, "h|"] \arrow[d] & \Omega(B) \arrow[r, "k|"] \arrow[d] & \Omega(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_A \arrow[r, "h"] \arrow[d] & P_B \arrow[r, "k"] \arrow[d] & P_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

for some maps h and k, where $P_A \to A$, $P_B \to B$ and $P_C \to C$ are projective covers and $h|$ and $k|$ denote the restrictions of h and k. We abuse notation and will write Ωf instead of $h|$ and Ωg instead of $k|$, even though they are dependent on the choice of h and k. If the induced sequence

$$0 \to \Omega A/J\Omega A \xrightarrow{\overline{\Omega f}} \Omega B/J\Omega B \xrightarrow{\overline{\Omega g}} \Omega C/J\Omega C \to 0$$

is exact, then we again abuse notation and note that there is an exact sequence

$$0 \to \Omega^2 A \xrightarrow{\Omega^2 f} \Omega^2 B \xrightarrow{\Omega^2 g} \Omega^2 C \to 0$$

as above. We use this convention and, when applicable, its extension to Ω^n in the remainder of the paper. Finally, throughout this paper R will be a *connected* Artin algebra, that is, an Artin algebra that is indecomposable as an algebra.

2. Preliminary results

We begin this section by recalling both the definitions of the Nakayama functor, ν, and of the Auslander-Reiten translate, τ (see [2], Chapter IV). The *Nakayama functor,* $\nu\colon \mathrm{mod}R \to \mathrm{mod}R$, is defined by

$$\nu(-) = D \circ \mathrm{Hom}_R(-, R),$$

where K is the center of R and $D = \mathrm{Hom}_K(-, K)\colon \mathrm{mod}R \to \mathrm{mod}R^{op}$ is the usual duality. The Nakayama functor is an exact equivalence if R is selfinjective. Furthermore, in the selfinjective case, if M is an indecomposable, nonprojective R-module, the Auslander-Reiten translate is given by $\tau M = \nu \circ \Omega^2 M$. We also note that, for a selfinjective Artin algebra, τ and Ω commute; see [2], IV.3.7., for instance.

We have the following well-known result.

Lemma 2.1. *Let R be an Artin algebra, and let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of R-modules. Then, for each i, we have*

$$\mathrm{cx}(A_i) \leq \max\{\mathrm{cx}(A_j), \mathrm{cx}(A_k)\},$$

where $\{i, j, k\} = \{1, 2, 3\}$. □

It is well known that, if $M \in \text{mod}R$, then $\beta_n(M) = \beta_n(\nu M)$, for all $n \geq 0$. Hence, if M is a nonprojective R-module, $\beta_n(\tau M) = \beta_{n+2}(M)$, for $n \geq 0$. These observations, together with Lemma 2.1, yield the following well-known result.

Proposition 2.2. *If R is a selfinjective Artin algebra and $\mathcal{C}$ is a component of the Auslander-Reiten quiver of R, then all the nonprojective modules in $\mathcal{C}$ have the same complexity.* □

We see that the complexity is constant on a regular (or on a stable) component of the Auslander-Reiten quiver. In particular, if one module in such a component of the Auslander-Reiten quiver has bounded Betti numbers, then every module in that component has bounded Betti numbers.

In most of our applications R will be a selfinjective Artin algebra of infinite representation type since the finite representation type case is well-understood. If R is of finite represesentation type, every indecomposable R-module is both Ω and τ-periodic, and therefore has complexity zero or one. For example, in the local selfinjective case, R being of finite representation type implies that R is a Nakayama algebra and its representation theory is trivial. Namely, every nonprojective indecomposable module is uniserial and all its Betti numbers equal 1. However, note that there exist selfinjective tame algebras for which every finitely generated nonprojective module is τ-periodic (see [17, 19]).

We recall that for a selfinjective Artin algebra, if

$$0 \to A \to B \to C \to 0$$

is an Auslander-Reiten sequence, then

$$0 \to \Omega A \to \Omega B \oplus P \to \Omega C \to 0$$

is also an Auslander-Reiten sequence, for some indecomposable projective-injective module P; see [1]. In particular, if $f\colon X \to Y$ is an irreducible morphism with both X and Y having no projective summands and at least one of X or Y indecomposable, then $\Omega f\colon \Omega X \to \Omega Y$ is also an irreducible morphism.

From the above discussion, we see that it is important to know when the middle term of an Auslander-Reiten sequence has projective summands. To that end, we present a useful technical lemma.

Lemma 2.3. *Let R be a selfinjective Artin algebra. If*

$$0 \to A \to B \to C \to 0$$

is an Auslander-Reiten sequence of R-modules with

$$0 \to \Omega A \to \Omega B \oplus P \to \Omega C \to 0$$

an Auslander-Reiten sequence with P a nonzero indecomposable projective-injective R-module, then $\Omega^2 C$ and τC are simple R-modules.

Proof. From [2], V.5.5., we know that if P is an indecomposable injective module, then the only Auslander-Reiten sequence where P appears as a direct summand in the middle term, is of the form

$$0 \to JP \to P \oplus JP/S \to P/S \to 0,$$

where S is the simple socle of P. It follows that ΩC is isomorphic to P/S, where S is the simple socle of P. Hence $\Omega^2 C$ is isomorphic to S and $\tau C = \nu\Omega^2 C$ are simple modules. □

We now present the definition of Ω-perfect modules and maps. Let R be a selfinjective Artin algebra, and let $B \xrightarrow{g} C$ be an irreducible map with B and C indecomposable nonprojective R-modules. We say that the map g is *Ω-perfect* if the induced irreducible maps: $\Omega^n B \xrightarrow{\Omega^n g} \Omega^n C$ are either all monomorphisms, for every $n \geq 0$ or are all epimorphisms, for every $n \geq 0$. If C is an indecomposable nonprojective R-module, and if

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \cdots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

is the Auslander-Reiten sequence ending at C with each E_i indecomposable, then we say that C is *Ω-perfect* if

(1) the maps f_i and g_i are Ω-perfect for all $i = 1, \ldots, n$, and
(2) for each $n \geq 0$, $\Omega^n C$ is not a simple R-module.

If C is an indecomposable Ω-perfect module over a selfinjective Artin algebra, then, by repeating the argument given in the proof of the above lemma, no summand of the middle term of the almost split sequence ending at C can be projective-injective. In a similar fashion, it follows from τ and Ω commuting, that, for all $n \geq 0$, the middle term of an Auslander-Reiten sequence ending at $\tau^n C$ does not have a projective-injective summand, for all $n \geq 0$. Moreover, $\Omega^n C$ and $\tau^n C$ are also Ω-perfect for all $n \geq 0$, and so is νC.

The following result investigates the behavior of Auslander-Reiten sequences ending at Ω-perfect modules under syzygies.

Proposition 2.4. *Let R be a selfinjective Artin algebra. Let C be a nonprojective Ω-perfect indecomposable R-module, and let*

$$0 \longrightarrow \tau C \longrightarrow E \longrightarrow C \longrightarrow 0$$

be an Auslander-Reiten sequence ending at C. Then, for each $n \geq 1$,

$$0 \longrightarrow \Omega^n \tau C \longrightarrow \Omega^n E \longrightarrow \Omega^n C \longrightarrow 0$$

is also an Auslander-Reiten sequence.

Proof. Since C is Ω-perfect, $\Omega^n C$ is not simple, for all $n \geq 0$. Note that, for $m, n \geq 0$, $\Omega^n \tau^m C$ is simple if and only if $\Omega^{n+2m} C$ is simple since ν takes simple modules into simple modules. The result now follows from Lemma 2.3. $\square$

The next proposition and lemma apply to arbitrary Artin algebras, and the remainder of the results of the section deals with selfinjective Artin algebras. We recall the following proposition, which was applied in [13] and whose proof is immediate and is left to the reader.

Proposition 2.5. *Let R be an Artin algebra and $0 \to A \to B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules with g irreducible and A nonsimple. Then $JA = A \cap JB$, where J is the Jacobson radical of R.* $\square$

From these observations, we have the following immediate consequence of Proposition 2.5, in case the algebra is selfinjective:

Corollary 2.6. *Let R be a selfinjective Artin algebra and let*

$$0 \to A \to B \xrightarrow{g} C \to 0$$

be a short exact sequence of R-modules with g irreducible. Then the induced irreducible map $\Omega g\colon \Omega B \to \Omega C$ is an epimorphism if and only if A is not a simple module. If A is simple, then Ωg is an irreducible monomorphism and we have an induced exact sequence

$$0 \to \Omega B \xrightarrow{\Omega g} \Omega C \to A \to 0. \qquad \square$$

As an application of the above corollary, we see that many selfinjective Artin algebras have components of their Auslander-Reiten quiver in which every nonprojective module C is *eventually Ω-perfect*; that is, $\tau^n C$ is Ω-perfect, for $n \gg 0$.

Proposition 2.7. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a component of the Auslander-Reiten quiver containing a nonprojective indecomposable module whose complexity is less than the complexity of every simple R-module. Then the component is a regular component, and every module lying in $\mathcal{C}$ is eventually Ω-perfect.*

Proof. By Lemma 2.1, each module in $\mathcal{C}$ has complexity less than the complexity of every simple R-module. If the component contains a projective-injective module P, then there is an irreducible map $f\colon P \to M$ for some indecomposable module M. This implies that f is an epimorphism and that $\mathrm{Ker} f$ is a simple module, contradicting the assumption that the complexities of all the modules in $\mathcal{C}$ are less than the complexity of every simple module. Suppose now that $g\colon B \to C$ is an irreducible morphism with B and C indecomposable modules in $\mathcal{C}$. First assume that g is an epimorphism. Then the previous argument also shows that $\mathrm{Ker} g$ cannot be simple. Thus by Corollary 2.6, Ωg is an irreducible epimorphism, and its kernel cannot be simple either. But then $\Omega^m g$ is an epimorphism for every $m \geq 0$. Thus, g is eventually Ω-perfect.

Assume now that g is a monomorphism. Either $\Omega^n g$ is a monomorphism, for all $n \geq 0$, or, for some $n \geq 0$, $\Omega^n g$ is an irreducible epimorphism. Then, we repeat the above argument and we see that in either case, g is eventually Ω-perfect. It now follows that every indecomposable module in $\mathcal{C}$ is eventually Ω-perfect. $\square$

If $(R, \mathbf{m})$ is a commutative local selfinjective algebra, then it is known that the complexity of the residue field $\Bbbk$ is at least the embedding dimension of the algebra. Hence any component containing a module of complexity less than $\dim_{\Bbbk} \mathbf{m}/\mathbf{m^2}$ would satisfy the assumptions of the above proposition. We also present a noncommutative and nonlocal example in Section 4 where the hypothesis of the above proposition holds. The next result plays an important role in the rest of the paper.

Proposition 2.8. *Let R be a selfinjective Artin algebra and let*

$$0 \to A \to B \xrightarrow{g} C \to 0$$

be a short exact sequence of finitely generated R-modules, where C is an indecomposable module and the map g is irreducible. Assume further that either B is indecomposable and the map g is Ω-perfect, or that the sequence is an Auslander-Reiten sequence and C is Ω-perfect. Then

(1) *A is not simple and B has no projective summands.*
(2) *For every $n \geq 0$, there is an induced short exact sequence*

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0,$$

and the map $\Omega^n g$ is irreducible. Moreover, for each $n \geq 0$, $\tau^n g$ is an irreducible epimorphism.

(3) *For each $n \geq 0$, $\beta_n(B) = \beta_n(A) + \beta_n(C)$.*

Proof. (1) It is clear that A is also indecomposable being the kernel of an irreducible epimorphism [3, 12]. In either case, it follows immediately from the preceding remarks that B has no projective summand. If the sequence is an Auslander-Reiten sequence and C is Ω-perfect, then $\Omega^2 C$, and hence A, is not simple. Assume now that g is Ω-perfect. Then A is not simple by Corollary 2.6.

(2) follows now immediately by repeatedly applying Proposition 2.5 and Lemma 2.3: If C is Ω-perfect and the given sequence is an Auslander-Reiten sequence, or if g is Ω-perfect, we get that $0 \to \Omega A \to \Omega B \xrightarrow{\Omega g} \Omega C \to 0$ is exact with ΩA not simple. Applying Ω one more time, we obtain that $0 \to \Omega^2 A \to \Omega^2 B \xrightarrow{\Omega^2 g} \Omega^2 C \to 0$ is either an Auslander-Reiten sequence and $\Omega^2 C$ is Ω-perfect if C is Ω-perfect, or is an exact sequence and $\Omega^2 g$ is Ω-perfect with $\Omega^2 B$ indecomposable if g is Ω-perfect. We proceed by induction and obtain, for each positive integer n, a short exact sequence

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0,$$

where the map $\Omega^n g$ is irreducible and $\Omega^n A$ is not simple. Moreover, for each $n > 0$, we have $J\Omega^n A = \Omega^n A \cap J\Omega^n B$. To complete the proof of part (2), we note that $\tau = \nu\Omega^2$ and that ν preserves irreducibility of maps. Part (3) follows easily from part (2). □

We now turn our attention to components of Auslander-Reiten quivers of selfinjective Artin algebras that contain a module of complexity 1. We want to show that if $\mathcal{C}$ is a regular component of the Auslander-Reiten quiver and if C is an indecomposable non-Ω-periodic module in $\mathcal{C}$ of complexity 1, then there is a positive integer n such that $\tau^n C$ is Ω-perfect. We will need the following graph-theoretical lemma whose proof can be found in [4](Theorem 12.1.5).

Lemma 2.9. *Let $\mathcal{G}$ be a finite directed graph having n vertices, and assume that there is at least one arrow between any two vertices in $\mathcal{G}$. Then, there exists a directed path in $\mathcal{G}$ of length greater than or equal to $n-1$.* □

We can prove now the desired result about modules of complexity 1 being eventually Ω-perfect.

Proposition 2.10. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R that is not a tube. Let $C \in \mathcal{C}$ be a module of complexity* 1*. Then, there exists a positive integer n such that $\tau^n C$ is Ω-perfect.*

Proof. By applying Ω a finite number of times if necessary, we may assume that $\Omega^n C$ is not simple, for all $n \geq 0$. The reason for this is the following: if the modules $\Omega^n C$ are simple for arbitrarily large integers, then there would be two positive integers, i and j, such that the modules $\Omega^i C$ and $\Omega^{i+j} C$ are both isomorphic to some simple module S. It is obvious that S is Ω-periodic. Moreover, the Nakayama functor ν has finite order, say k, when applied to S, since there are only finitely many nonisomorphic simple R-modules. Therefore, $S = \Omega^i C$ is τ-periodic, of some period n (dividing kj) since ν and Ω commute. We have that $\tau^n \Omega^i C \cong \Omega^i C$; hence

$$\Omega^i \nu^n \Omega^{2n} C \cong \nu^n \Omega^{2n+i} C \cong \Omega^i C$$

so that $C \cong \tau^n C$ too, contradicting the fact that $\mathcal{C}$ is not a tube. Thus, we may assume that for each $n \geq 0$, $\Omega^n C$ is not simple.

Let

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \cdots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be the Auslander-Reiten sequence ending at C, with each E_i indecomposable. It is enough to show that for each i, the irreducible maps f_i and g_i are eventually Ω-perfect. Since there are a finite number of them, it is enough to prove that if $B \xrightarrow{g} C$ is an irreducible map with B and C indecomposable nonprojective modules, then there is an even positive integer n such that the irreducible map $\Omega^n B \xrightarrow{\Omega^n g} \Omega^n C$ is Ω-perfect whenever C has complexity 1. It is immediate that such an n exists if and only if $\Omega^n g$ is either an epimorphism, for $n \gg 0$ or a monomorphism, for $n \gg 0$. Assume that g is not Ω-perfect. Then, for infinitely many positive even integers j, $\Omega^j g$ is an epimorphism but $\Omega^{j+1} g$ is a monomorphism. As we have seen, the only way the syzygy functor takes an irreducible epimorphism into an irreducible monomorphism is if the kernel of that epimorphism is simple by Corollary 2.6. Hence, we have an infinite number of positive integers, $n_1, n_2, n_3, \ldots$, such that, for each i, we have a short exact sequence

$$0 \longrightarrow S \xrightarrow{k_i} \Omega^{n_i} B \xrightarrow{\Omega^{n_i} g} \Omega^{n_i} C \longrightarrow 0,$$

for some simple R-module S.

For each $i \neq j$ we have the following exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "k_i"] \arrow[d, equal] & \Omega^{n_i} B \arrow[r, "\Omega^{n_i} g"] & \Omega^{n_i} C \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r, "k_j"] & \Omega^{n_j} B \arrow[r, "\Omega^{n_j} g"] & \Omega^{n_j} C \arrow[r] & 0
\end{tikzcd}$$

and since the maps $\Omega^{n_j} g$ and $\Omega^{n_i} g$ are irreducible, there exist homomorphisms $f_i^j \colon \Omega^{n_j} B \to \Omega^{n_i} B$ or $f_j^i \colon \Omega^{n_i} B \to \Omega^{n_j} B$ commuting the left square of the above diagram. We may now invoke the graph-theoretical Lemma 2.9, so after relabeling if needed, we may assume that we have an arbitrary long chain of homomorphisms

$$\Omega^{n_1} B \xrightarrow{f_1^2} \Omega^{n_2} B \xrightarrow{f_2^3} \cdots \Omega^{n_m} B \xrightarrow{f_m^{m+1}}$$

where we have for each $i \leq j$ that $f_i^j k_i = k_j$. The compositions $f_m^{m+1} f_{m-1}^m \cdots f_1^2$ are all nonzero, since by composing them with k_1 we obtain k_{m+1}. Since B has bounded Betti numbers, there is a common bound for the lengths of all its syzygies, but we have just produced arbitrarily long chains of homomorphisms with nonzero compositions between (indecomposable) syzygies of B, and this cannot happen if each f_{i+1}^i is not an isomorphism; see [2],VI.1.3. It follows that for some $i \neq j$, $\Omega^{n_i} B$ is isomorphic to $\Omega^{n_j} B$. This means that B is Ω-periodic. We show now that B is also τ-periodic. This would imply that the component $\mathcal{C}$ is a tube by [11]. We will then have a contradiction and the proof of the proposition will be complete. To show that B is τ-periodic, we will prove that the Nakayama functor ν has finite order when applied to B. As we saw at the beginning of the proof, $\nu^k S \cong S$, for some positive integer k. Let n be a positive integer such that we have a short exact sequence

$$0 \longrightarrow S \xrightarrow{l} \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0.$$

But ν takes irreducible maps into irreducible maps, so for all integer multiples kt_i and kt_j of k, we have the induced exact diagrams

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, equal] & \nu^{kt_i}\Omega^n B \arrow[r, "\nu^{kt_i}\Omega^n g"] & \nu^{kt_i}\Omega^n C \arrow[r] & 0 \\ 0 \arrow[r] & S \arrow[r] & \nu^{kt_j}\Omega^n B \arrow[r, "\nu^{kt_j}\Omega^n g"] & \nu^{kt_j}\Omega^n C \arrow[r] & 0. \end{tikzcd}$$

Just as above we have for each i and j homomorphisms l_i^j or l_j^i commuting the left squares of these diagrams, and we obtain again chains of homomorphisms of arbitrary length with nonzero compositions between the indecomposable modules of the same length as $\nu^{kt_i}\Omega^n B$. Again invoking [2],VI.1.3., we obtain our desired result. $\square$

Note that we also proved in the previous result that if $B \to C$ is an irreducible epimorphism between indecomposable modules over a selfinjective algebra with simple kernel, then the Nakayama functor has finite order on B (and therefore on C too).

We have the following property of Ω-perfect modules of complexity 1:

Lemma 2.11. *Let R be a selfinjective Artin algebra and let C be an Ω-perfect module of complexity* 1. *Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \dots, f_t)^T} E_1 \oplus E_2 \oplus \cdots \oplus E_t \xrightarrow{(g_1, g_2, \dots, g_t)} C \longrightarrow 0$$

be an Auslander-Reiten sequence ending at C with each E_i indecomposable. Then for each $1 \leq i \leq t$, one of the maps f_i, g_i is a monomorphism, and the other is an epimorphism.

Proof. Suppose for some i that both f_i and g_i are (proper) epimorphisms. Then, by the assumption that C is Ω-perfect, for all $n \geq 0$, $\tau^n f_i$ and $\tau^n g_i$ are epimorphisms. This yields a sequence of proper epimorphisms:

$$\cdots \longrightarrow \tau^3 C \xrightarrow{\tau^2 f_i \tau^2 g_i} \tau^2 C \xrightarrow{\tau f_i \tau g_i} \tau C \xrightarrow{f_i g_i} C.$$

This contradicts the assumption that the complexity of C is 1, and hence there is a bound on the lengths of the $\tau^n C$.

If both f_i and g_i are (proper) monomorphisms, for some i, then a similar argument works, since C has finite length. $\square$

We end this section with a result about the number of indecomposable modules appearing in the middle of an Auslander-Reiten sequence ending at Ω-perfect modules. Let $0 \to \tau C \to E \to C \to 0$ be the Auslander-Reiten sequence ending at C. Recall that $\alpha(C)$ is the usual notation for the number of indecomposable summands of E.

Proposition 2.12. *Let R be a selfinjective Artin algebra and C an indecomposable Ω-perfect R-module such that $\beta_n(C) = 1$ for an infinite number of $n \geq 0$. Then $\alpha(C) = 1$.*

Proof. Let

$$0 \to \tau C \xrightarrow{(f_1, f_2, \dots, f_t)^T} E_1 \oplus \cdots \oplus E_t \xrightarrow{(g_1, \dots, g_t)} C \to 0$$

be the Auslander-Reiten sequence ending at C, with each E_i an indecomposable R-module, and assume that $t > 1$. If g_1 is a monomorphism, then we claim that f_1 cannot be a monomorphism too. If f_1 was a monomorphism, then $g_1 f_1 : \tau C \to C$ would be a (proper) monomorphism. Since C is Ω-perfect, it follows that $\tau^n g_1 \tau^n f_1 : \tau^{n+1} C \to \tau^n C$ is a (proper) monomorphism, for all $n \geq 0$. But this is impossible since C has finite length. Thus, if g_1 is a monomorphism, f_1 must be an epimorphism.

If f_1 is an epimorphism, then we obtain a short exact sequence

$$0 \to \mathrm{Ker} f_1 \longrightarrow \tau C \longrightarrow E_1 \to 0.$$

But, since the map $f_1 \colon \tau C \to E_1$ is Ω-perfect, we have, for each $i \geq 0$, irreducible epimorphisms $\Omega^i \tau C \to \Omega^i E_1$. Hence $\beta_i(\mathrm{Ker} f_1) < \beta_i(\tau C)$, for all $i \geq 0$. Our assumption on Betti numbers implies that, for some i, $\beta_i(\tau C) = 1$ and this contradicts the preceding inequality.

Assume now that g_1 is an epimorphism. But then, so is the map

$$E_1 \oplus \cdots \oplus E_{t-1} \xrightarrow{(g_1, \ldots, g_{t-1})} C.$$

This immediately implies that the map $f_t \colon \tau C \to E_t$ is also an irreducible epimorphism, and we repeat the preceding argument. $\square$

3. Modules with bounded Betti numbers

Let C be a module with $\mathrm{cx}(C) = 1$. We need an important invariant of C. Let $\beta(C) = \max_{k \geq 0}\{\beta_k(C)\}$. It is very easy to see that we have $\beta(\Omega^i(C)) \leq \beta(C)$, for every $i \geq 0$. Therefore, $\beta(C) \geq \beta(\tau C)$, and the length of C is bounded by $\beta(C)\ell(R)$, where $\ell(R)$ denotes the length of R. Of course, this bound is not sharp.

Now let C be an indecomposable module lying in a stable tube, or in a regular $\mathbb{Z}A_\infty$-component containing no simple R-modules. If g is an irreducible epimorphism in this component, its kernel lies on the boundary of the component and, hence, is not simple. Since each induced irreducible map τg, and hence $\Omega^2 g$, is also an epimorphism, we see that g is Ω-perfect. In fact, one can show that every module in such a component is Ω-perfect. We have the following proposition:

Proposition 3.1. *Let R be a selfinjective Artin algebra, and let C be an indecomposable module with complexity* 1*, lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver and such that $\beta(C)$ is minimal among the modules in $\mathcal{C}$. Assume further that $\mathcal{C}$ is either not a stable tube, or, that if a stable tube, then $\mathcal{C}$ contains no simple modules. Then $\alpha(C) = 1$.*

Proof. Without loss of generality we may assume by Proposition 2.10, and by the preceding remarks, that C is an Ω-perfect module. Since $\beta(\tau^n C) \leq \beta(C)$, and since $\beta(C)$ is minimal, we have $\beta(\tau^n C) = \beta(C)$, for all $n \geq 0$. Let

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \cdots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be the Auslander-Reiten sequence ending at C, with each E_i an indecomposable R-module, and assume that $t > 1$. We now proceed precisely as in the proof of Proposition 2.12 to obtain $\beta_i(E_t) < \beta_i(\tau C)$, for all $i \geq 0$. By the minimality of $\beta(C)$, we have that $\beta(\tau C) = \beta(C)$. Thus we obtain a contradiction to the minimality of $\beta(C)$ in its component, and we conclude that $t = 1$. $\square$

As an immediate application we have:

Proposition 3.2. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver containing no simple modules and containing a module of complexity* 1*. Then, for each irreducible epimorphism $g\colon B \to C$ in $\mathcal{C}$, and for each $i \geq 0$, we have*

$$\beta_i(\operatorname{Ker} g) + \beta_i(C) = \beta_i(B). \qquad \square$$

The following result is part of the folklore, and we include its proof for completeness.

Lemma 3.3. *Let R be an Artin algebra, and let $\mathcal{C}$ be a connected component of the Auslander-Reiten quiver of R. Let $M \in \mathcal{C}$ and assume that all the predecessors of M in $\mathcal{C}$ have lengths bounded by some positive integer b. Then, if X is an indecomposable R-module such that $\operatorname{Hom}_R(X, M) \neq 0$, then $X \in \mathcal{C}$ and X is a predecessor of M.*

Proof. Let X be an indecomposable R-module, not isomorphic to M, such that $\operatorname{Hom}_R(X, M) \neq 0$. By [14], we either have a chain of irreducible maps with nonzero composition from X to M passing through indecomposable modules, or else there exist arbitrarily long chains of irreducible maps through indecomposable modules with nonzero composition ending at M. The second case cannot occur by [2], Corollary VI,1.3, since each predecessor of M has length bounded by b. Hence, we obtain a finite chain of irreducible maps starting at X and ending at M whose composition is nonzero. $\square$

Lemma 3.4. *Let R be a selfinjective Artin algebra and let*

$$0 \to \tau C \to E_1 \oplus \cdots \oplus E_t \to C \to 0$$

be an Auslander-Reiten sequence ending at C, where C is an indecomposable Ω-perfect module of complexity 1*. Suppose that $\beta_i(E_1) > \beta_i(C)$, for all $i \geq 0$. Then, for all $i \geq 0$ and $j \geq 2$, $\beta_i(\tau C) > \beta_i(E_j)$. In particular, we have that $\beta(\tau C) > \beta(E_j)$, for all $j \geq 2$.*

Proof. By Proposition 2.8(3), we have that

$$\beta_i(\tau C) + \beta_i(C) = \beta_i(E_1) + \sum_{j=2}^{t} \beta_i(E_j).$$

It follows that $\beta_i(\tau C) > \sum_{j=2}^{t} \beta_i(E_j) \geq \beta_i(E_j)$, for all $j \geq 2$. $\square$

Using the previous result, we also have the following property of Auslander-Reiten sequences ending in an Ω-perfect module of complexity 1, and lying in a regular component:

Proposition 3.5. *Let R be a selfinjective Artin algebra and let C be an Ω-perfect module of complexity* 1*, belonging to a regular component of the Auslander-Reiten quiver. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \cdots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be the Auslander-Reiten sequence ending at C. Then precisely one of the maps g_i is an epimorphism and all the other ones are monomorphisms.

Proof. Assume that two of the g_i's are epimorphisms, say g_1 and g_2. Then so is the map $(g_2, \ldots, g_t)$, and therefore the map f_1 is an epimorphism too, contradicting Lemma 2.11. Now assume that all the g_i are monomorphisms. Then, by invoking again Lemma 2.11, all the maps f_i are irreducible epimorphisms.

We will show that all the predecessors of C in its Auslander-Reiten component have length bounded by $\beta(C)\ell(R)$. By Lemma 3.3, this would imply that our component contains a projective R-module, for instance the projective cover of any composition factor of C, yielding a contradiction, and thus proving our proposition. It suffices to show that if X is a predecessor of C, then $\beta(X) \leq \beta(C)$. We also know that if M is an indecomposable R-module of complexity 1, then $\beta(\tau^k M) \leq \beta(M)$ for all nonnegative integers k. Using this fact, it is enough to prove that if we have a path

$$E^k \to E^{k-1} \to \cdots \to E^1 \to C,$$

where each module lies on a different τ-orbit, then $\beta(E^k) \leq \beta(C)$.

Assume $E^k \to E^{k-1} \to \cdots \to E^1 \to C$ is such a path, where $k \geq 1$, and let $E^0 = C$. We know that $E^1 = E_j$ for some $j = 1, \ldots, t$. Consider the Auslander-Reiten sequences ending at E^j,

$$0 \to \tau E^j \to E^{j+1} \oplus \tau E^{j-1} \oplus L_j \to E^j \to 0,$$

where L_j may decompose. Since f_j is an Ω-perfect irreducible epimorphism, we have that, for each i, $\beta_i(\tau C) > \beta_i(E^1)$ (by Proposition 2.8(3)). Hence, we have the inequalities

$$\beta(E^1) < \beta(\tau C) \leq \beta(C).$$

Thus, if $k = 1$, we are done. But the fact that f_j is an Ω-perfect irreducible epimorphism also implies that so is the irreducible epimorphism $\tau E^1 \to E^2$. By induction, it follows that each map $\tau E^j \to \tau E^{j+1}$ is an Ω-perfect irreducible epimorphism; hence we obtain by 2.8(3) that $\beta_i(E^{j+1}) < \beta_i(\tau E^j)$, for each i. We obtain by induction

$$\beta(E^k) < \beta(\tau E^{k-1}) \leq \beta(E^{k-1}) < \cdots < \beta(E^1) < \beta(M),$$

and the proof is now complete. □

We are now in a position to find a strong bound on the number of summands of the middle term of an Auslander-Reiten sequence ending at a module with bounded Betti numbers lying in a regular component.

Theorem 3.6. *Let R be a selfinjective Artin algebra and let M be an indecomposable module of complexity* 1 *belonging to a regular component. Then $\alpha(M) \leq 2$.*

Proof. If the module M is τ-periodic, then the result follows by [11], so without loss of generality we may assume that the component containing M is not a tube. By taking enough powers of the Auslander-Reiten translate τ, we may assume that the module M is Ω-perfect. Assume that the Auslander-Reiten sequence ending at M is

$$0 \to \tau M \to E_1 \oplus E_2 \oplus E_3 \oplus L \to M \to 0,$$

where each E_i is indecomposable and L is possibly zero. By applying the previous proposition, we may assume that the irreducible map $g_1 \colon E_1 \to M$ is an epimorphism.

Applying τ to the above sequence, we obtain another Auslander-Reiten sequence:

$$0 \to \tau^2 M \to \tau E_1 \oplus \tau E_2 \oplus \tau E_3 \oplus \tau L \to \tau M \to 0.$$

Since M is Ω-perfect, $\tau E_1 \to \tau M$ is again an irreducible epimorphism. Next we consider the Auslander-Reiten sequences ending at E_2 and at E_3:

$$0 \to \tau E_2 \to \tau M \oplus C \to E_2 \to 0 \quad \text{and} \quad 0 \to \tau E_3 \to \tau M \oplus D \to E_3 \to 0,$$

where C and D are not necessarily indecomposable R-modules. Computing lengths on the first two Auslander-Reiten sequences, we obtain the following equality:

$$\sum_{i=1}^{3}(\ell(E_i) + \ell(\tau E_i)) + \ell(L) + \ell(\tau L) = \ell(\tau^2 M) + 2\ell(\tau M) + \ell(M).$$

Computing lengths on the remaining two Auslander-Reiten sequences, we obtain the following equality:

$$2\ell(\tau M) + \ell(C) + \ell(D) = \ell(E_2) + \ell(E_3) + \ell(\tau E_2) + \ell(\tau E_3).$$

Adding these two equalities and cancelling terms where possible yields

$$\ell(E_1) + \ell(\tau E_1) + \ell(L) + \ell(\tau L) + \ell(C) + \ell(D) = \ell(\tau^2 M) + \ell(M).$$

From this equality, we conclude that

$$\ell(E_1) + \ell(\tau E_1) \le \ell(\tau^2 M) + \ell(M).$$

Recalling that $E_1 \to M$ and $\tau E_1 \to \tau M$ are Ω-perfect irreducible epimorphisms, we see that $\ell(M) + \ell(\tau M) < \ell(\tau^2 M) + \ell(M)$. Hence,

$$\ell(\tau M) < \ell(\tau^2 M).$$

Replacing M by $\tau^i M$, the above argument shows that we obtain a sequence of strict inequalities

$$\ell(\tau M) < \ell(\tau^2 M) < \ell(\tau^3 M) < \cdots .$$

But, $\ell(\tau^i M) \le \beta(M)\ell(R)$, for each i, and we have arrived at a contradiction, proving the result. □

The structure of a regular Auslander-Reiten component that contains a module of complexity 1 can be completely determined now, yielding the main result of this section.

Theorem 3.7. *Let R be a selfinjective Artin algebra and $\mathcal{C}$ a regular component of the Auslander-Reiten quiver containing a module of complexity* 1. *Then $\mathcal{C}$ is a stable tube or a component of type $\mathbb{Z}A_\infty$.*

Proof. If the component contains a τ-periodic module, we are done by [11]. Assume that there are no τ-periodic modules in $\mathcal{C}$. By Theorem 3.6, we know that if $C \in \mathcal{C}$, then $\alpha(C) \le 2$. Moreover, we infer by Proposition 3.5, that if the middle term of the Auslander-Reiten sequence ending at C has two indecomposable summands, then one of the irreducible maps ending at C is an epimorphism, and the other one is a monomorphism. By Proposition 3.1, there is some $M \in \mathcal{C}$ with $\alpha(M) = 1$, and such a module must lie on the boundary of $\mathcal{C}$. The result follows from these observations. □

As an immediate corollary, we obtain the following consequence, which will prove quite useful.

Corollary 3.8. *Let R be a selfinjective Artin algebra and let C be an indecomposable R-module whose Betti numbers are eventually equal to* 1 *lying in a regular component. Then $\alpha(C) = 1$.*

Proof. We may assume that C is Ω-perfect and that $\beta(C) = 1$. Therefore $\beta(C)$ is minimal in the class of modules having bounded Betti numbers. The result now follows immediately. $\square$

4. Modules with eventually constant Betti numbers

Let R be a selfinjective Artin algebra. We study components of the Auslander-Reiten quiver that contain modules whose Betti numbers eventually are constant. In this section, we use the notion of the quasi-length; see [16]. We have shown that such a component must be a stable tube or of type $\mathbb{Z}A_\infty$. Our first result studies the case where there is a module on the boundary of the component whose Betti numbers are eventually constant.

Proposition 4.1. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R containing a module M whose Betti numbers are eventually equal to b and with $\alpha(M) = 1$. Assume also that if $\mathcal{C}$ is a stable tube, then it contains no simple R-modules. Then every module B in $\mathcal{C}$ has eventually constant Betti numbers equal to $ql(B)b$, where $ql(B)$ denotes the quasi-length of B.*

Proof. Since the Betti numbers of M are eventually constant, we see that $\mathrm{cx}(M) = 1$, and hence $\mathcal{C}$ is either a stable tube or of type $\mathbb{Z}A_\infty$ by Theorem 3.7. Without loss of generality, we may also assume that M is Ω-perfect. We will use induction on the quasi-length n of B. Consider the following portion of $\mathcal{C}$.

$$\begin{tikzcd}
 & \cdots & B_4 \arrow[dr] & & \tau^{-1}B_4 \arrow[dr] & & & \\
\cdots & \tau B_3 \arrow[ur] \arrow[dr] & & B_3 \arrow[ur] \arrow[dr] & & \tau^{-1}B_3 \arrow[dr] & & \\
\tau^2 B_2 \arrow[ur] \arrow[dr] & & \tau B_2 \arrow[ur] \arrow[dr] & & B_2 \arrow[ur] \arrow[dr] & & \tau^{-1}B_2 & \\
\cdots & \tau^2 M \arrow[ur] \arrow[rr, dash, dashed] & & \tau M \arrow[ur] \arrow[rr, dash, dashed] & & M \arrow[ur] & & \cdots
\end{tikzcd}$$

We begin by considering $n = 2$. Since $0 \to \tau M \to B_2 \to M \to 0$ is the Auslander-Reiten sequence ending at M, we see that $\beta_i(B_2) = \beta_i(M) + \beta_i(\tau M)$, for all i, by Proposition 2.8. It follows that the Betti numbers of B_2 are eventually equal to $2b$.

Let $n \geq 3$ and assume that we have shown that B_i has Betti numbers eventually equal to ib, for $i = 2, \ldots, n-1$. We show that B_n has Betti numbers eventually equal to nb. Since $\mathcal{C}$ is a component of the form $\mathbb{Z}A_\infty$ or a stable tube, we have that the Auslander-Reiten sequence ending at B_{n-1} is

$$0 \to \tau B_{n-1} \to B_n \oplus \tau B_{n-2} \to B_{n-1} \to 0.$$

(If $n = 3$, then take $B_{n-2} = M$.) Since B_{n-1} and τB_{n-1} have Betti numbers eventually equal to $(n-1)b$ and τB_{n-2} has Betti numbers eventually equal to $(n-2)b$, by the usual application of Proposition 2.8, we see that B_n has Betti numbers eventually equal to nb and the proof is complete. $\square$

Keeping our usual hypothesis that R is a selfinjective Artin algebra, let M be an R-module with $\mathrm{cx}(M) = 1$ lying in a regular component of the Auslander-Reiten quiver of R that, if a stable tube, contains no simple modules. The above theorem shows that if $\beta(M)$ is minimal in its Auslander-Reiten component, and if M has eventually constant Betti numbers, then every module in the component of M has eventually constant Betti numbers. Note that the minimality of $\beta(M)$ is equivalent to M lying on the boundary of the component, which is equivalent to $\alpha(M) = 1$. Example 4.7 below shows that the existence of just one module M having eventually constant Betti numbers and $\alpha(M) = 2$ need not imply that every module in the component of M has eventually constant Betti numbers. On the other hand, it is easy to see that if M is a module having eventually constant Betti numbers equal to 2, then every module in the component of M has eventually constant Betti numbers.

Along these lines, we have the following result.

Proposition 4.2. *Let R be a selfinjective Artin algebra and M and N indecomposable R-modules whose Betti numbers are eventually constant. Assume that M lies in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver of R that, if a tube, contains no simple modules. If there is an irreducible homomorphism $M \to N$, then every module in $\mathcal{C}$ has eventually constant Betti numbers.*

Proof. By our assumption on Betti numbers, both M and N have complexity 1. Hence, by applying τ a sufficient number of times, Proposition 2.10 lets us assume that both M and N are Ω-perfect. We also observe that if

$$0 \to \tau N \to M \oplus M' \to N \to 0$$

is the Auslander-Reiten sequence ending at N, then M' has eventually constant Betti numbers. This follows immediately from Proposition 2.8(3), since we have the equality $\beta_n(M') = \beta_n(N) + \beta_n(\tau N) - \beta_n(M)$. Now let X be in the component. Then $\tau^i X$ lies on a sectional path ending at N, or at M for some $i \in \mathbb{Z}$. The result now follows by a repeated application of Proposition 2.8(3). □

Eisenbud [5] has proved that if R is a local complete intersection, then every indecomposable module with bounded Betti numbers is eventually Ω-periodic with period 2 and has eventually constant Betti numbers. He also showed that for $R = KG$ for some finite group G with $\mathrm{char}(K)$ dividing the order of G, then every indecomposable module with bounded Betti numbers is Ω-periodic. If R is a local commutative ring, Eisenbud also asked: are all modules with bounded Betti numbers eventually Ω-periodic? The answer to this question is negative (see [9]). For the noncommutative local case, nonperiodic modules with constant Betti numbers have been found.

Let R be a local selfinjective Artin algebra and M an indecomposable R-module with eventually constant Betti numbers lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver of R. We do not know how to prove (or even if it is true) that every module in $\mathcal{C}$ has eventually constant Betti numbers except in some special cases. We do know that this need not be true in the nonlocal case; see the example later in this section. We say that an R-module C has *periodic Betti numbers* if there is some positive integer n such that $\beta_i(C) = \beta_{i+n}(C)$, for all $i \geq 0$ and we say that C has *eventually periodic Betti numbers* if there are some positive integers n and k such that $\beta_i(C) = \beta_{i+n}(C)$, for all $i \geq k$. In this direction, we can prove the following:

Proposition 4.3. *Let R be a selfinjective Artin algebra, and let M be an indecomposable R-module with eventually constant Betti numbers lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver, that, if a tube, contains no simple modules.*

Then every module in $\mathcal{C}$ has eventually periodic Betti numbers. Furthermore, the eventual period of the Betti numbers of a module in $\mathcal{C}$ divides $2ql(M)$.

Proof. Without loss of generality we may assume that M is Ω-perfect, and that it has constant Betti numbers. If M lies on the boundary of $\mathcal{C}$, every module has eventually constant Betti numbers by Proposition 4.1 and we are done. Now assume that n is the quasi-length of M. Then there is a sectional path of n irreducible epimorphisms

$$M \twoheadrightarrow M_1 \twoheadrightarrow M_2 \twoheadrightarrow \cdots \twoheadrightarrow M_{n-1} \twoheadrightarrow M_n,$$

where M_n lies on the boundary of $\mathcal{C}$, and a sectional path of n irreducible monomorphisms

$$\tau^n M_n \hookrightarrow \tau^{n-1} M_{n-1} \hookrightarrow \tau^{n-2} M_{n-2} \hookrightarrow \cdots \hookrightarrow \tau M_1 \hookrightarrow M.$$

It is easy to show that the composition of the monomorphisms above is the kernel of the irreducible epimorphism $M \twoheadrightarrow M_1$. In particular, we have a short exact sequence $0 \to \tau^n M_n \to M \to M_1 \to 0$. As usual, we have $\beta_i(\tau^n M_n) + \beta_i(M_1) = \beta_i(M)$, for all $i \geq 0$. Replacing M by M_1, we obtain $\beta_i(\tau^{n-1} M_n) + \beta_i(M_2) = \beta_i(M_1)$, for all $i \geq 0$. Continuing, we get $\beta_i(\tau^{n-j} M_n) + \beta_i(M_{j+1}) = \beta_i(M_j)$, for all $0 \leq i \leq n$. From these formulas, we obtain the following equality:

$$\beta_i(M) = \sum_{j=0}^{n} \beta_i(\tau^j M_n),$$

for all $i \geq 0$. In a similar fashion, we obtain

$$\beta_i(\tau M) = \sum_{j=1}^{n+1} \beta_i(\tau^j M_n),$$

for all $i \geq 0$. By our assumption that M has constant Betti numbers, we have $\beta_i(M) = \beta_i(\tau M)$, for all $i \geq 0$ and it follows that

$$\sum_{j=0}^{n} \beta_i(\tau^j M_n) = \sum_{j=1}^{n+1} \beta_i(\tau^j M_n).$$

Canceling like terms, we see that

$$\beta_i(\tau^{n+1} M_n) = \beta_i(M_n),$$

for each $i \geq 0$. Therefore M_n, and hence every module on the boundary, has eventually periodic Betti numbers with period dividing $2(n+1)$. The result now follows by noting that if $0 \to \tau C \to B \to C \to 0$ is an Auslander-Reiten sequence with modules in the component $\mathcal{C}$, then $\beta_i(B) = \beta_i(\tau C) + \beta_i(C)$, for all $i \geq 0$ and by doing induction on the quasi-length. $\square$

Our next result is an easy combinatorial exercise whose proof we leave to the reader.

Lemma 4.4. *Suppose that $\mathcal{C}$ is a $\mathbb{Z}A_\infty$-component of the Auslander-Reiten quiver of a selfinjective Artin algebra R and that C is an Ω-perfect R-module on the boundary of $\mathcal{C}$. If*

$$M_n \twoheadrightarrow M_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow C$$

is a sectional path of irreducible epimorphisms in $\mathcal{C}$, then $\beta_i(M_n) = \sum_{j=0}^{n} \beta_{2j+i}(C)$, for all $i \geq 0$ and $n \geq 1$. $\square$

The next result is consequence of Proposition 4.3 and the above lemma.

Proposition 4.5. *Let R be a selfinjective Artin algebra and let M an indecomposable R-module with $\mathrm{cx}(M) = 1$ lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver, that, if a tube, contains no simple modules. Assume the length of a sectional path from M to a module C on the boundary of the component is n; that is, the quasi-length of M is n. Then M has eventually constant Betti numbers if and only if C has eventually periodic Betti numbers with period dividing $2ql(M)$, and, for sufficiently large m, $\sum_{j=0}^{n} \beta_{2j}(\tau^m C) = \sum_{j=0}^{n} \beta_{2j+1}(\tau^m C)$.*

Proof. Without loss of generality, by applying τ sufficiently many times, it is enough to show that M has constant Betti numbers if and only if C has periodic Betti numbers with period dividing $2\mathrm{ql}(M)$, and

$$\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C).$$

It is also clear that we may assume that both M and C are Ω-perfect. First suppose that C has periodic Betti numbers, and that we also have the equality $\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$. By Lemma 4.4, $\beta_i(M) = \sum_{j=0}^{n} \beta_{2j+i}(C)$. By the periodicity assumption, $\sum_{j=0}^{n} \beta_{2j+i}(C) = \sum_{j=0}^{n} \beta_{2j+s}(C)$, where $s = 0$ if i is even, and $s = 1$ if i is odd. Since $\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$, we see that M has constant Betti numbers.

Now assume that M has constant Betti numbers. By the proof of Proposition 4.3, we see that the Betti numbers of C are periodic with period dividing $2(n+1)$. It remains to prove the equality $\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$. But, by Lemma 4.4, $\beta_0(M) = \sum_{j=0}^{n} \beta_{2j}(C)$ and $\beta_1(M) = \sum_{j=0}^{n} \beta_{2j+1}(C)$. Since $\beta_0(M) = \beta_1(M)$, the proof is complete. $\square$

As a consequence of the above proposition, we have the following result.

Theorem 4.6. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R containing a module having eventually constant Betti numbers. Assume also that, if a tube, $\mathcal{C}$ contains no simple R-modules. Then*

(1) *There is an infinite family $\{M^n\}_{n=1}^{\infty}$ of modules in $\mathcal{C}$ having constant Betti numbers $\{b_n\}_{n=1}^{\infty}$; that is, $\beta_i(M^n) = b_n$, for all $i \geq 0$.*
(2) *The sequence $\{b_n\}$ is strictly increasing.*
(3) *The R-modules M^n lie on distinct τ-orbits.*
(4) *There is a positive integer d, such that, for each $t \geq 1$, M^t can be chosen having constant Betti numbers $b_t = td$.*

Proof. Clearly, it suffices to prove part (4). If there is a module with eventually constant Betti numbers lying on the boundary of $\mathcal{C}$, we are done by Proposition 4.1. Now assume that $\mathcal{C}$ contains a module with eventually constant Betti numbers which does not lie on the boundary of $\mathcal{C}$. The proof of Proposition 4.5 shows that there is a module C lying on the boundary having periodic Betti numbers, with period dividing $2(n+1)$, for some positive integer n. Furthermore, $\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$. Again from the proof of Proposition 4.5, we obtain a module with constant Betti numbers equal to $\sum_{j=0}^{n} \beta_{2j}(C)$. But the Betti numbers of C are also periodic with period dividing $2t(n+1)$, for all $t \geq 1$. From this we obtain the existence of a module having constant Betti numbers equal to $t(\sum_{j=0}^{n}\beta_{2j}(C))$ since $\sum_{j=0}^{tn}\beta_{2j}(C) = \sum_{j=0}^{tn}\beta_{2j+1}(C)$, and from the periodicity of the Betti numbers of C. Setting $d = \sum_{j=0}^{n}\beta_{2j}(C)$, the proof is complete. $\square$

The following example shows that there are examples of regular Auslander-Reiten components containing a module M with periodic Betti numbers, on the boundary of the component, satisfying $\sum_{j=0}^{n}\beta_{2j}(M) = \sum_{j=0}^{n}\beta_{2j+1}(M)$, and yet not all modules in the component have eventually constant Betti numbers. By Proposition 4.5, such a component must contain some modules with constant Betti numbers.

Example 4.7. Let K be a field and $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} \overset{3}{\circ} \arrow[r, bend left, "d"] & \overset{1}{\circ} \arrow[l, bend left, "a"] \arrow[r, bend left, "b"] & \overset{2}{\circ} \arrow[l, bend left, "c"] \end{tikzcd}$$

Let $R = K\mathcal{Q}/I$, where I is generated by $bc - ad, ca, db$. Note that R is a selfinjective algebra of finite representation type. It is not hard to check that the sequence of Betti numbers for the simple R-module at vertex 1 is

$$1, 2, 1, 1, 2, 1, 1, 2, \ldots.$$

Let $\Gamma = R \ltimes D(R)$ be the trivial extension of R by the dual of R. Then Γ is a symmetric finite dimensional K-algebra of infinite representation type, and we have also that $\mathrm{cx}(S) = 2$, for all simple Γ-modules S.

Since R is a symmetric algebra, Γ is a projective left R-module. Furthermore, since $vR \otimes_R \Gamma$ is isomorphic to $v\Gamma$, for every vertex $v \in \mathcal{Q}$, we see that if M is a right R-module, then $\beta_n(M) = \beta_n(M \otimes_R \Gamma)$, where the left hand side is the nth-Betti number of M, as an R-module, and the right hand side is the n-th Betti number of $M \otimes_R \Gamma$, as a Γ-module.

Let M denote the simple right R-module at vertex 1. Thus, $M \otimes_R \Gamma$ has $1, 2, 1, 1, 2, 1, 1, 2, \ldots$ as its sequence of Betti numbers and hence $M \otimes_R \Gamma$ has periodic Betti numbers. We know that $M \otimes_R \Gamma$ is an indecomposable Γ-module since $\beta_0(M \otimes_R \Gamma) = 1$. Let $\mathcal{C}$ be the component containing $M \otimes_R \Gamma$. Since each simple Γ-module has complexity 2 and M is an Ω-periodic R-module, we infer that $\mathcal{C}$ is a stable tube and that $M \otimes_R \Gamma$ is Ω-perfect. Since $\beta_n(M \otimes_R \Gamma) = 1$ for infinitely many n, $M \otimes_R \Gamma$ lies on the boundary by Proposition 2.12. Finally, we note that we have the equality $\sum_{j=0}^{3}\beta_{2j}(M \otimes_R \Gamma) = \sum_{j=0}^{3}\beta_{2j+1}(M \otimes_R \Gamma)$; the period of $\{\beta_n(M \otimes_R \Gamma)\}$ divides 6, and hence, by Proposition 4.5, the component that contains $M \otimes_R \Gamma$ also constains modules with constant Betti numbers.

We note that we do not know if such an example exists in the local case. We use the following well-known result in the next proposition and whose proof we include for completeness.

Lemma 4.8. *Let*

$$\cdots \to V^n \xrightarrow{f^n} V^{n-1} \xrightarrow{f^{n-1}} V^{n-2} \to \cdots \to V^1 \xrightarrow{f^1} V^0$$

be an infinite exact sequence of modules of finite length over a ring R. Assume that, for each $i \geq 0$, the lengths $\ell(V^{2i}) = a$ and $\ell(V^{2i+1}) = b$, for some positive integers a and b. Then $a = b$.

Proof. For each $k > 0$, we have an exact sequence

$$0 \to \mathrm{Im}(f^{2k+1}) \to V^{2k} \to V^{2k-1} \to \cdots \to V^1 \to \mathrm{Im}(f^1) \to 0.$$

Taking the Euler characteristic of this sequence, we obtain

$$\ell(\mathrm{Im}(f^{2k+1})) + k(b-a) - \ell(\mathrm{Im}(f^1)) = 0.$$

Hence $k(b-a) = \ell(\mathrm{Im}(f^1)) - \ell(\mathrm{Im}(f^{2k+1}))$. But $-a \leq \ell(\mathrm{Im}(f^1)) - \ell(\mathrm{Im}(f^{2k+1})) \leq a$, and k is arbitrary, and we conclude that $a = b$. □

We are now ready to prove the next theorem.

Theorem 4.9. *Let $(R, \mathbf{m})$ be a local selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R, which, if a tube, does not contain $R/\mathbf{m}$. Then $\mathcal{C}$ contains a module with eventually constant Betti numbers if and only if there is a module on the boundary of C with eventually periodic Betti numbers.*

Proof. It follows from our assumptions that every module in the component is eventually Ω-perfect. By Proposition 4.5, we only have to prove that, if there is a module on the boundary of C with eventually periodic Betti numbers, then $\mathcal{C}$ contains a module with eventually constant Betti numbers. Assume that C is an Ω-perfect module lying on the boundary with periodic Betti numbers, and period dividing $2(n+1)$ for some $n \geq 0$. By the proof of Proposition 4.5, we need only show the equality

$$\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C).$$

Suppose not. Let M be a module in $\mathcal{C}$ with a sectional path of length n of epimorphisms ending at C. By Lemma 4.4, we have that $\beta_i(M) = \sum_{j=0}^{n} \beta_{2j+i}(C)$, for all $i \geq 0$ and $n \geq 1$. But then we see that $\beta_i(M) = \sum_{j=0}^{n} \beta_{2j}(C)$ if i is even, and $\beta_i(M) = \sum_{j=0}^{n} \beta_{2j+1}(C)$ if i is odd, by the periodicity of the Betti numbers of C. Thus the sequence of Betti numbers of M is $a, b, a, b, a, b, \ldots$, where $a = \beta_i(M) = \sum_{j=0}^{n} \beta_{2j}(C)$ and $b = \beta_i(M) = \sum_{j=0}^{n} \beta_{2j+1}(C)$. We let

$$\mathcal{V}: \quad \cdots \to V^n \to V^{n-1} \to \cdots \to V^2 \to V^1 \to M \to 0$$

be a minimal free R-resolution of M. Then, for each i, $\ell(V^i) = \ell(R)\beta_i(M)$. The result follows by applying Lemma 4.8. □

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