

QUASIHARMONIC POLYNOMIALS FOR COXETER GROUPS AND REPRESENTATIONS OF CHEREDNIK ALGEBRAS

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ABSTRACT. We introduce and study deformations of finite-dimensional modules over rational Cherednik algebras. Our main tool is a generalization of usual harmonic polynomials for each Coxeter group — the so-called quasiharmonic polynomials. A surprising application of this approach is the construction of canonical elementary symmetric polynomials and their deformations for all Coxeter groups.

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INTRODUCTION

In this paper we introduce and study deformations of finite-dimensional modules over the rational Cherednik algebras $H_c(W)$. Here W is a finite reflection group of the space V , and c is a conjugation-invariant complex-valued function on the set of reflections in W . The algebra $H_c(W)$, first introduced in 1995 by I. Cherednik [4], is a degenerate case of the double affine Hecke algebra and can be thought of as a deformation of the cross product of the group algebra $\mathbb{C}W$ and the symmetric algebra of $V^* \oplus V$; c is the parameter of deformation.

Rational Cherednik algebras have attracted the attention of many authors during the past ten years. Finite-dimensional representations were studied in [2]; see also [6], [7], [8] and a review [24]. Recently I. Gordon ([18], [19], [20]) found remarkable connections between the theory of Cherednik algebras and algebras of diagonal harmonics; the latter were introduced by M. Haiman in [21] and since then have been the subject of intensive research.

This paper is a part of a project having as its goal the construction of canonical bases in the representations of the rational Cherednik algebra. In its turn, this would allow the construction of a canonical basis in the algebra of diagonal harmonics.

For generic c the algebra $H_c(W)$ has no non-trivial finite-dimensional modules. Certain functions c such that $|c| = r \in \mathbb{Z}_{>0}$ (where $|c|$ is, up to a multiple, the sum of all values of c ; see (1.5) for details) are exceptions to this rule: for such c the algebra $H_c(W)$ has a distinguished finite-dimensional module A_r . In particular, this happens for $H_c(S_n)$ when $c = \text{const.} = r/n$ and r and n are coprime ([2]; see also [7] for a simpler proof), for $H_c(I_2(m))$ when m is odd, $c = \text{const.} = r/m \notin \mathbb{Z}$, and when m is even, $c_1 + c_2 = r/m \notin \mathbb{Z}$ ([6]); for other types of Coxeter groups see [2]. The module A_r for all these cases has the form $M(\mathbf{1})/I_r$ where $M(\mathbf{1})$ is the Verma-like $H_c(W)$ -module isomorphic to $S(V)$ as a vector space (first introduced by C. Dunkl in [13]; see Section 1.1 for the exact definition) and $I_r \subset M_r(\mathbf{1})$ is a proper maximal submodule of the form $I_r = V^{(r)} \cdot S(V)$. The subspace $V^{(r)} \subset S^r(V)$ (the space of *singular vectors*) is the common kernel of all *Dunkl operators*.

In the present paper we propose a certain flat c -deformation $V^{(r;c)}$ of the W -module $V^{(r)}$ such that $V^{(r;c)}$ is still a W -module and the quotient algebra $A_{r;c} = S(V)/(V^{(r;c)} \cdot S(V))$ is finite-dimensional and (yet conjecturally) specializes into A_r when c is exceptional as above. The advantage of the deformed algebra $A_{r;c}$ is that it possesses an additional structure of a flat family which, similar to quantum groups, can help to understand canonical bases for A_r .

A principal ingredient of the construction of $V^{(r;c)}$ is the concept of *quasiharmonic elements*, i.e., elements of $S(V)$ killed by almost all W -invariant combinations of Dunkl operators. Namely, $V^{(r;c)}$ is a unique (except for the case when $W = D_{2m}$; see Section 1.3 for details) quasiharmonic W -module in $S^r(V)$ isomorphic to $V^{(r)}$ (and thus we believe that $A_{r;c}$ is a canonical deformation of A_r). A surprising application of this technique is the construction of a remarkable system of canonical invariants $e_k^{(c)} \in S(V)^W$ of any Coxeter group W . For $W = S_n$ these invariants are deformations of the elementary symmetric polynomials e_k ; a more detailed study of these invariants will be the subject of a forthcoming paper.

The paper is organized as follows. In Section 1 we give necessary definitions (Section 1.1), describe the quasiharmonic elements (Section 1.2), define the deformation $V^{(r;c)}$, construct the invariants $e_k^{(c)}$ (Section 1.3) and formulate several conjectures

about properties of $V^{(r;c)}$ and other related objects (Section 1.4). In Section 2 we verify these conjectures for the case $W = I_2(m)$. Section 2.5 deals with a more special question — the algebra $A_{r;c}$ and its Frobenius characteristic polynomial. Section 3 is an appendix summarizing information about standard Frobenius algebras used elsewhere in the paper and containing some remarks we believe necessary for the future development of the subject.

1. QUASIHARMONIC ELEMENTS: GENERAL CASE

1.1. Definitions and notation. For a complex vector space V we denote by $S(V)$ its symmetric algebra, and by $\mathbb{C}[V]$ the algebra of polynomial functions on it. Clearly, $\mathbb{C}[V] = S(V^*)$. Both $S(V)$ and $\mathbb{C}[V]$ are graded algebras; their homogeneous components of degree k are denoted $S^k(V)$ and $\mathbb{C}_k[V]$, respectively.

Let W be a finite reflection group of a Euclidean vector space $V_{\mathbb{R}}$ such that the W -module $V_{\mathbb{R}}$ is irreducible. Denote by $V \stackrel{\text{def}}{=} \mathbb{C} \otimes V_{\mathbb{R}}$ the complexification of $V_{\mathbb{R}}$. Clearly, V is also an irreducible W -module; we will call it the *defining W -module*. The complex space V inherits from $V_{\mathbb{R}}$ a symmetric non-degenerate W -invariant form (\cdot, \cdot) . Denote by $\mathcal{S} \subset W$ the set of all reflections in W . To each $s \in \mathcal{S}$ one associates a positive root $\alpha_s \in V$ and a coroot $\alpha_s^\vee = 2(\alpha_s, \cdot) / (\alpha_s, \alpha_s) \in V^*$.

Fix a function $c : \mathcal{S} \rightarrow \mathbb{C}$ invariant under conjugation: $c(\tau s \tau^{-1}) = c(s)$, and define the *rational Cherednik algebra* $H_c(W)$ as follows. As a vector space, $H_c(W)$ is isomorphic to $S(V) \otimes \mathbb{C}W \otimes S(V^*)$, where $\mathbb{C}W$ is the group algebra. The multiplication is defined by the following requirements:

- (1) The natural inclusions $S(V) \hookrightarrow H_c(W)$, $S(V^*) \hookrightarrow H_c(W)$ and $\mathbb{C}W \hookrightarrow H_c(W)$ are algebra monomorphisms. In particular, $x_1 x_2 = x_2 x_1$ for all $x_1, x_2 \in S(V) \subset H_c(W)$, and the same is true for all $y_1, y_2 \in S(V^*)$.
- (2) The conjugation action of $\mathbb{C}W \subset H_c(W)$ on $V \subset S(V) \subset H_c(W)$ and on $V^* \subset S(V^*) \subset H_c(W)$ is isomorphic to the action of $\mathbb{C}W$ in the defining module V and on its dual V^* , respectively: $wz w^{-1} = w(z)$ for any $w \in W$ and any $z \in V$ or $z \in V^*$.
- (3) For any $x \in V, y \in V^*$ one has

$$(1.1) \quad yx - xy = \langle y, x \rangle - \sum_{s \in \mathcal{S}} c(s) \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle s.$$

Here brackets $\langle \cdot, \cdot \rangle$ mean pairing of V^* and V ; the right-hand side lies in $\mathbb{C}W \subset H_c(W)$.

Evidently, the definition of $H_c(W)$ is sound, i.e. it does not depend on the choice of a positive root α_s for every $s \in \mathcal{S}$.

We can collect all the algebras $H_c(W)$ together. Consider the decomposition $\mathcal{S} = \bigsqcup_{i=1}^k C_i$ of the set of reflections into conjugacy classes (in fact, $k = 1$ or 2 since W is irreducible) and denote $c_i \stackrel{\text{def}}{=} c(s)$ for any $s \in C_i$. Then define a universal algebra $H(W)$ over the ring $\mathbb{C}[c] = \mathbb{C}[c_1, \dots, c_k]$ exactly as in the previous paragraph with the field \mathbb{C} replaced by $\mathbb{C}[c]$ everywhere (including tensor multiplication). For each $c = (c_1, \dots, c_k) \in \mathbb{C}^k$ we have a natural surjective evaluation homomorphism of \mathbb{C} -algebras $\text{ev}_c : H(W) \rightarrow H_c(W)$. The algebra $H(W)$ is a convenient language to study the dependence of various objects inside $H_c(W)$ on the parameter c .

Define the projection $\pi_{12} : H_c(W) \rightarrow S(V) \otimes \mathbb{C}W$ by $\pi_{12}(q \otimes w \otimes p) = p(0) \cdot q \otimes w$ (where $p \mapsto p(0)$ is the evaluation of a polynomial p at 0). In particular, $\pi_{12}(q \otimes w \otimes p) = 0$ for every $p \in S(V^*)_+ = \bigoplus_{m>0} S^m(V^*)$. For every $p \in S(V^*)$ define the *Dunkl operator* $\nabla_p^{(c)} : S(V) \otimes \mathbb{C}W \rightarrow S(V) \otimes \mathbb{C}W$ by the formula

$$\nabla_p^{(c)}(r) = \pi_{12}(pr).$$

(On the right-hand side the product of elements $p \in S(V^*) \subset H_c(W)$ and $r \in S(V) \otimes \mathbb{C}W \subset H_c(W)$ in the algebra $H_c(W)$ is assumed.)

Proposition 1.1 (cf. [12]). *The space $S(V) \otimes \mathbb{C}W$ possesses the following $H_c(W)$ -module structure: elements $q \in S(V)$ act there by multiplication in the first factor, elements $w \in \mathbb{C}W$, as in the module $S(V) \otimes \mathbb{C}W$, and elements $p \in S(V^*)$, by Dunkl operators. Explicitly, the action of the Dunkl operator corresponding to a vector $y \in V^*$ on an element $q \otimes w \in S(V) \otimes \mathbb{C}W$ is given by the following formula:*

$$(1.2) \quad \nabla_y^{(c)}(q \otimes w) = \frac{\partial q}{\partial y} \otimes w - \sum_{s \in \mathcal{S}} c(s) \langle y, \alpha_s \rangle \frac{q - s(q)}{\alpha_s} \otimes sw.$$

In (1.2) the symbol $\frac{\partial q}{\partial y}$ means the directional derivative of a polynomial $q \in S(V) = \mathbb{C}[V^*]$. An element $s \in \mathcal{S} \subset W$ acts on q as in the module $S(V)$; the difference $q - s(q) \in S(V)$ is divisible by $\alpha_s \in V$, so the right-hand side is well-defined.

Proof. Extend the notation ∇ defining $\nabla_u^{(c)}(r) \stackrel{\text{def}}{=} \pi_{12}(ur)$ for every $u \in H_c(W)$ and $r \in S(V) \otimes \mathbb{C}W$. Then show that the correspondence $(u, r) \mapsto \nabla_u^{(c)}(r)$ is an $H_c(W)$ -action on $S(V) \otimes \mathbb{C}W$.

Indeed, for any $x \in H_c(W)$ one has $x - \pi_{12}(x) = \sum_i q_i \otimes w_i \otimes p_i$, where $q_i \in S(V)$, $w_i \in \mathbb{C}W$ and $p_i \in S^{k_i}(V^*)$ with $k_i > 0$. A similar decomposition takes place for the element $u(x - \pi_{12}(x))$ where $u \in H_c(W)$, and therefore $\pi_{12}(u(x - \pi_{12}(x))) = 0$ for all $u, x \in H_c(W)$. This implies the equality $\nabla_{u_1 u_2}^{(c)}(r) = \pi_{12}(u_1 u_2 r) = \pi_{12}(u_1 \pi_{12}(u_2 r)) = \nabla_{u_1}^{(c)}(\nabla_{u_2}^{(c)}(r))$, as required.

Furthermore, the defining relation (1.1) implies, by direct computation, the following relation in $H_c(W)$:

$$(yq - qy)w = \frac{\partial q}{\partial y} \cdot w - \sum_{s \in \mathcal{S}} c(s) \langle y, \alpha_s \rangle \frac{q - s(q)}{\alpha_s} \cdot sw$$

for all $y \in V^*$, $q \in S(V)$, $w \in \mathbb{C}W$. By definition of the projection π_{12} , one also has $\pi_{12}(yqw) = \pi_{12}((yq - qy)w) = (yq - qy)w$. Taking into account the factorization $S(V) \cdot \mathbb{C}W = S(V) \otimes \mathbb{C}W$ (in $H_c(W)$), we obtain (1.2). \square

One can replace $\mathbb{C}W$ in Proposition 1.1 by an arbitrary $\mathbb{C}W$ -module τ . Following [2] we denote this $H_c(W)$ -module by $M(\tau)$; it is isomorphic to $S(V) \otimes \tau$ as a vector space. Modules $M(\tau)$ inherit the grading of $S(V)$: $M(\tau) = \bigoplus_n M_n(\tau)$, where $M_n(\tau) = S^n(V) \otimes \tau$. In particular, setting $\tau = \mathbf{1}$ (the trivial 1-dimensional representation) one obtains a structure of the $H_c(W)$ -module in $M(\mathbf{1}) = S(V)$ where elements of $S(V)$ act by multiplication, elements $w \in \mathbb{C}W$, as in the module

$S(V)$, and elements $p \in V^*$, by the Dunkl operators in the original sense of [12]:

$$(1.3) \quad \nabla_y^{(c)}(q) = \frac{\partial q}{\partial y} - \sum_{s \in \mathcal{S}} c(s) \langle y, \alpha_s \rangle \frac{q - s(q)}{\alpha_s}.$$

Below we will refer to the operators (1.3) as Dunkl operators unless noted otherwise.

We can also consider a “universal” version of the above construction: if $\tilde{\pi}_{12} : H(W) \rightarrow \mathbb{C}[c] \otimes S(V) \otimes CW$ is the natural projection and $p \in S(V^*)$, then the universal Dunkl operator is defined by the formula $\nabla_p(r) \stackrel{\text{def}}{=} \tilde{\pi}_{12}(pr)$. It is clear that the “universal” and “specialized” operators are related by the evaluation morphism $\nabla_p^{(c)} \text{ev}_c = \text{ev}_c \nabla_p$; by a slight abuse of notation we will sometimes be writing ∇_p instead of $\nabla_p^{(c)}$.

Define a linear map $\varepsilon : H_c(W) \rightarrow \mathbb{C}$ by $\varepsilon(q \otimes w \otimes p) = p(0)q(0)$, where $p \mapsto p(0)$ and $q \mapsto q(0)$ are the evaluations of polynomials at 0. In particular, for each $w \in W$, $\varepsilon(S(V^*)_+ \otimes w \otimes S(V)) = 0$ and $\varepsilon(S(V^*) \otimes w \otimes S(V)_+) = 0$. Define a bilinear pairing between the spaces $S(V^*)$ and $S(V)$ by the formula

$$(1.4) \quad \langle p, q \rangle_c = \varepsilon(pq)$$

for $p \in S(V^*)$, $q \in S(V)$; so, $\langle p, q \rangle_c = \delta_{mn} \nabla_p^{(c)}(q)$ for $p \in S^m(V^*)$ and $q \in S^n(V)$. For $c = 0$ equation (1.4) defines the usual pairing between $S(V^*)$ and $S(V)$.

For each function $c : \mathcal{S} \rightarrow \mathbb{C}$ denote

$$(1.5) \quad |c| = \frac{2}{\ell} \sum_{s \in \mathcal{S}} c(s),$$

where $\ell \stackrel{\text{def}}{=} \dim V$. The following result was essentially proved in [2, Proposition 2.1].

Proposition 1.2. *If $q \in V$ and $p \in V^*$, then $\langle p, q \rangle_c = (1 - |c|)\langle p, q \rangle_0$.*

The non-degenerate bilinear form (\cdot, \cdot) on the space V (used in the definition of the reflection group W) defines an isomorphism of vector spaces $\Phi : V \rightarrow V^*$. One can extend this to an isomorphism $\Phi : S(V) \rightarrow S(V^*)$ and further, to the linear involution $\varphi : H_c(W) \rightarrow H_c(W)$ by $\varphi(q \otimes w \otimes p) = \Phi^{-1}(p) \otimes w^{-1} \otimes \Phi(q)$.

Proposition 1.3 (cf. [2]). *The following is true:*

- (1) *The involution φ is an anti-automorphism of $H_c(W)$: $\varphi(ab) = \varphi(b)\varphi(a)$.*
- (2) *The pairing $\langle \cdot, \cdot \rangle_c$ is symmetric with respect to φ : $\langle p, q \rangle_c = \langle \varphi(q), \varphi(p) \rangle_c$.*
- (3) *The Dunkl operators are conjugate to multiplication operators in $S(V^*)$: $\langle rp, q \rangle_c = \langle r, \nabla_p(q) \rangle_c$ for any $p, r \in S^k(V^*)$ and $q \in S^n(V)$.*

Proof. Since $H_c(W)$ is spanned by elements of the form qwp where $q \in S(V)$, $w \in W$ and $p \in S(V^*)$, in order to prove assertion (1) it suffices to verify that

$$(1.6) \quad \varphi(aqwp) = \Phi^{-1}(p)w^{-1}\Phi(q)\varphi(a)$$

for $a \in V \cup W \cup V^*$. For $a \in V$ one has $\varphi(a) = \Phi(a)$, and (1.6) is true because $\Phi : S(V) \rightarrow S(V^*)$ is an algebra homomorphism. For $a \in W$ one has $\varphi(a) = a^{-1}$,

and therefore $\varphi(aqwp) = \varphi(a(q)awp) = \Phi^{-1}(p)w^{-1}a^{-1}\Phi(a(q))$. The group W acts in $S(V)$ by (\cdot, \cdot) -orthogonal operators, so that $\Phi(a(q)) = a(\Phi(q))$ and therefore

$$\varphi(aqwp) = \Phi^{-1}(p)w^{-1}a^{-1}a(\Phi(q)) = \Phi^{-1}(p)w^{-1}\Phi(q)a^{-1}.$$

Now let $a \in V^*$. It follows from (1.1) that for every $b \in V$ one has

$$\begin{aligned} \varphi([a, b]) &= \varphi(\langle a, b \rangle - \sum_{s \in \mathcal{S}} c(s)\langle a, \alpha_s \rangle \langle \alpha_s^\vee, b \rangle s) = \langle a, b \rangle - \sum_{s \in \mathcal{S}} c(s)\langle a, \alpha_s \rangle \langle \alpha_s^\vee, b \rangle s \\ &= \langle \Phi(b), \Phi^{-1}(a) \rangle - \sum_{s \in \mathcal{S}} c(s)\langle \Phi(b), \alpha_s \rangle \langle \alpha_s^\vee, \Phi^{-1}(a) \rangle s = [\Phi(b), \Phi^{-1}(a)]. \end{aligned}$$

Trivial induction shows that $\varphi([a, q]) = [\varphi(q), \varphi(a)]$ for every $q \in S(V)$.

Now one has $aqwp = qawp + [a, q]wp = qw \cdot w^{-1}(a)p + [a, q]wp$. Since

$$\varphi([a, q]wp) = \Phi^{-1}(p)w^{-1}\varphi([a, q]) = \Phi^{-1}(p)w^{-1}[\varphi(q), \varphi(a)]$$

and also

$$\begin{aligned} \varphi(qw \cdot w^{-1}(a)p) &= \Phi^{-1}(w(a)p)w^{-1}\Phi(q) \\ &= \Phi^{-1}(p)w^{-1}(\Phi^{-1}(a))w^{-1}\Phi(q) = \Phi^{-1}(p)w^{-1}\varphi(a)\varphi(q), \end{aligned}$$

one eventually has

$$\varphi(aqwp) = \Phi^{-1}(p)w^{-1}(\varphi(a)\varphi(q) + [\varphi(q), \varphi(a)]) = \varphi(qwp)\varphi(a),$$

so that assertion (1) is proved.

To prove assertion (2) note that the involution φ is graded; hence $\varepsilon(x - \varphi(x)) = 0$ for all $x \in H_c(W)$. Therefore, assertion (1) implies that

$$\langle \varphi(q), \varphi(p) \rangle_c = \varepsilon(\varphi(q)\varphi(p)) = \varepsilon(\varphi(pq)) = \varepsilon(pq) = \langle p, q \rangle_c.$$

Assertion (3) follows from Proposition 1.1:

$$\langle rp, q \rangle_c = \nabla_{rp}^{(c)}(q) = \nabla_r^{(c)}(\nabla_p^{(c)}(q)) = \langle r, \nabla_p^{(c)}(q) \rangle_c.$$

This finishes the proof of the proposition. □

Definition 1.4. An element $x \in S(V)$ is (W, c) -harmonic if $\nabla_p^{(c)}(x) = 0$ for any W -invariant polynomial $p \in S(V^*)_+^W$. An element $x \in \mathbb{C}[c] \otimes S(V)$ is W -harmonic if $\nabla_p(x) = 0$ for any W -invariant polynomial $p \in S(V^*)_+^W$.

A classical Chevalley theorem (see [5]) says that $S(V^*)^W$ is a polynomial algebra of ℓ variables (recall that $\ell = \text{rk}(W) = \dim V$). These variables correspond to some invariant polynomials (*elementary invariants*). The elementary invariants can be chosen in many ways (see, though, Proposition 1.18), but their degrees are uniquely defined and called *exponents* of the group W . Denote the exponents as $d_1 \leq \dots \leq d_\ell = h$ (the largest exponent h is the Coxeter number), and let $e_{d_1}, \dots, e_{d_\ell} = e_h$ be some elementary invariants of W in $S(V^*)$ of degrees d_1, \dots, d_ℓ respectively.

Using this description of invariants one can give a simpler characterization of (W, c) -harmonic elements:

Proposition 1.5. *The following are equivalent for $x \in S(V)$ (respectively, $x \in \mathbb{C}[c] \otimes S(V)$):*

- (1) x is (W, c) -harmonic (respectively, W -harmonic);
- (2) $\nabla_p(x) = 0$ for any W -invariant $p \in S(V^*)_+^W$ such that $\deg(p) \leq h$;
- (3) $\nabla_{e_{d_i}}(x) = 0$ for $i = 1, 2, \dots, \ell$.

The proof is obvious.

Since W is a *real* reflection group (a Coxeter group), one always has the inequality $h = d_\ell > d_{\ell-1}$ ([22]), i.e. the invariant e_h is unique modulo invariants of smaller degrees. By removing e_h from consideration, we obtain the following object:

Definition 1.6. An element $x \in S(V)$ is (W, c) -quasiharmonic if $\nabla_p^{(c)}(x) = 0$ for any W -invariant polynomial $p \in S(V^*)^W$ such that $\deg(p) < h$. An element $x \in \mathbb{C}[c] \otimes S(V)$ is W -quasiharmonic if $\nabla_p(x) = 0$ for any such polynomial. Equivalently, x is (W, c) -quasiharmonic (W -quasiharmonic) if $\nabla_{e_{a_i}}(x) = 0$ for $i = 1, 2, \dots, \ell - 1$.

Denote by $\mathcal{H}^{(c)}$ the space of (W, c) -harmonic elements and by $\mathcal{QH}^{(c)} \supset \mathcal{H}^{(c)}$ the space of (W, c) -quasiharmonic elements. These spaces are graded:

$$\mathcal{H}^{(c)} = \bigoplus_{n \geq 0} \mathcal{H}_n^{(c)}, \quad \mathcal{QH}^{(c)} = \bigoplus_{n \geq 0} \mathcal{QH}_n^{(c)},$$

where $\mathcal{H}_n^{(c)} = \mathcal{H}^{(c)} \cap S^n(V)$ and $\mathcal{QH}_n^{(c)} = \mathcal{QH}^{(c)} \cap S^n(V)$. The same is true for the “universal” versions of these spaces, $\mathcal{H} \subset \mathcal{QH}$. Clearly, $\text{ev}_c \mathcal{H} \subseteq \mathcal{H}^{(c)}$ and $\text{ev}_c \mathcal{QH} \subseteq \mathcal{QH}^{(c)}$.

Remark 1.7. The notions and results of this section are also applicable to *complex* reflection groups. The reason for this is two-fold. First, the algebra of invariants of such a group is free (see [25]), and the exponents $d_1 \leq d_2 \leq \dots \leq d_\ell$ are well-defined. Second, the inequality $d_\ell > d_{\ell-1}$ holds for almost all groups. More precisely, it fails only for the following irreducible complex reflection groups of rank $\ell = 2$ with $(d_1, d_2) = (d, d)$ (see, e.g., [3]):

- (1) The series $G(m, 2, 2)$, $m > 1$, with $d = m$.
- (2) The exceptional groups G_7, G_{11}, G_{19} with $d = 12, 24, 60$, respectively.

1.2. Basic properties of quasiharmonic elements. Following [10], denote by C^{sing} the set of all W -invariant functions $c : \mathcal{S} \rightarrow \mathbb{C}$ such that there exists an element $x \in S(V)$ of positive degree such that $\nabla_y^{(c)}(x) = 0$ for all $y \in V^*$. Elements of C^{sing} are called *singular functions*; other conjugation-invariant functions on \mathcal{S} are called *regular*. Elements $x \in S(V)$ annihilated by all the Dunkl operators are called *singular vectors*. It is easy to see that the group W acts on the set X of singular vectors, and that the space $S(V)X \subset S(V) = M(\mathbf{1})$ is a $H_c(W)$ -submodule. Thus, $c \in C^{sing}$ if and only if the $H_c(W)$ -module $M(\mathbf{1})$ is reducible (see [2] for details). It was proved in [10] that the function $c = const.$ is singular if and only if $c = r/d_i$, where d_i is an exponent of W and r is a positive integer not divisible by d_i . In particular, $c = 0$ is a regular function. Conjugation-invariant functions $c \neq const.$ (taking, actually, two different values) exist for Coxeter groups of types B_n, F_4 , and $I_2(2m)$; see [10] for a description of C^{sing} in these cases.

Proposition 1.8 ([10]). *For all regular c the $S(V^*)$ -actions on $M(\mathbf{1}) = S(V)$ induced by the $H_c(W)$ -action are isomorphic to each other. In particular, they are isomorphic to the natural $S(V^*)$ -action on $S(V)$ by differential operators with constant coefficients (this corresponds to $c = 0$).*

Proposition 1.9. *For each regular $c : \mathcal{S} \rightarrow \mathbb{C}$ the Hilbert series of $\mathcal{QH}^{(c)}$ and of $\mathcal{H}^{(c)}$ are given by the following formula:*

$$\text{hilb}(\mathcal{QH}^{(c)}, t) = \text{hilb}(\mathcal{H}^{(c)}, t) / (1 - t^h) = \frac{(1 - t^{d_1}) \cdots (1 - t^{d_{\ell-1}})}{(1 - t)^\ell}.$$

Proof. Let $I \subset S(V^*)$ be a homogeneous ideal. Since the pairing $\langle \cdot, \cdot \rangle_c$ is non-degenerate, there is an isomorphism of graded vector spaces:

$$S(V^*)/I \cong I^\perp,$$

where $I^\perp = \{x \in S(V) \mid \langle I, x \rangle_c = 0\}$ is the orthogonal complement of I . This implies $\text{hilb}(S(V^*)/I, t) = \text{hilb}(I^\perp, t)$.

Recall that a homogeneous ideal $I \subset S(V^*)$ is called *free* if it admits a free Koszul resolution:

$$(1.7) \quad 0 \rightarrow S(V^*) \otimes \Lambda^k(U) \rightarrow \dots \rightarrow S(V^*) \otimes \Lambda^2(U) \rightarrow S(V^*) \otimes U \rightarrow S(V) \rightarrow S(V^*)/I \rightarrow 0,$$

where $U \subset S(V^*)$ is a k -dimensional space of homogeneous polynomials that generates the ideal I .

One has $\mathcal{H}^{(c)} = I_1^\perp$ and $\mathcal{QH}^{(c)} = I_2^\perp$ where ideals I_1 and I_2 are both free. Their generating sets U_1 and U_2 are linear spans of e_{d_1}, \dots, e_{d_k} , where $k = \ell$ for U_1 and $k = \ell - 1$ for U_2 . Then (1.7) gives

$$\text{hilb}(S(V^*)/I, t) = \frac{(1 - t^{d_1}) \dots (1 - t^{d_k})}{(1 - t)^\ell},$$

and the result follows. □

Corollary 1.10. *For any regular c the following is true:*

- (1) $\mathcal{QH}_r^{(c)} = \mathcal{H}_r^{(c)}$ for all $r < h$.
- (2) $\dim \mathcal{QH}_{h+1}^{(c)} = \dim \mathcal{H}_{h+1}^{(c)} + \ell$.
- (3) For $r \gg 1$ the dimension of $\mathcal{QH}_r^{(c)}$ does not depend on r and is equal to $\frac{|W|}{h} = d_1 d_2 \dots d_{\ell-1}$.

1.3. W -action on the space of quasiharmonics. Proposition 1.9 above admits a representation-theoretic refinement. For any graded W -submodule $U = \bigoplus_{n \geq 0} U_n \subset S(V)$ and an irreducible W -module τ define the graded character $\text{ch}_\tau(U, t)$ by

$$\text{ch}_\tau(U, t) = \sum_{n \geq 0} [\tau : U_n] t^n,$$

where $[\tau : U_n]$ is the multiplicity of τ in U_n .

All the constructions in the proof of Proposition 1.9 are W -equivariant, and each graded component of the ideals involved is a W -module. So we immediately obtain

Proposition 1.11. *For any simple W -module τ and any regular c one has*

$$\text{ch}_\tau(\mathcal{QH}^{(c)}, t) = \frac{\text{ch}_\tau(\mathcal{H}^{(c)}, t)}{1 - t^h}.$$

In particular, if $\tau = \mathbf{1}$ is the trivial 1-dimensional W -module, one has

$$(1.8) \quad \text{ch}_\mathbf{1}(\mathcal{QH}^{(c)}, t) = 1/(1 - t^h) = 1 + t^h + t^{2h} + \dots.$$

1.3.1. Invariants. We obtain the following direct corollary of Proposition 1.11.

Corollary 1.12. *For regular c the degree of each homogeneous quasiharmonic invariant $q \in \mathcal{QH}^W$ is divisible by h . For every $k = 0, 1, \dots$ there is exactly one, up to proportionality, quasiharmonic invariant $e_{kh}^{(c)}$ of degree kh .*

Proof. According to (1.8), for each regular c and each $k \geq 0$,

$$\dim \mathcal{QH}_d^W = \begin{cases} 1 & \text{if } h \text{ divides } d, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for each $k > 0$ there exists a unique (up to a multiple) W -invariant $u_{kh} \in \mathbb{C}(c) \otimes \mathcal{QH}_{kh}$. Multiplying this invariant by an appropriate polynomial in $\mathbb{C}[c]$, we obtain e_{kh} . \square

Polynomials $e_{kh}^{(c)}$ for the case $W = I_2(m)$ (the dihedral group) are computed in Section 2.4. See also Section 1.5 for some explicit formulas in the case $W = S_4$.

The Dunkl operators commute, and therefore they act on the space $\mathcal{QH}^{(c)}$ of quasiharmonics making it a $\mathbb{C}W \ltimes S(V^*)$ -module (where $\mathbb{C}W \ltimes S(V^*)$ denotes the cross-product of $\mathbb{C}W$ and $S(V^*)$ isomorphic to $\mathbb{C}W \otimes S(V^*)$ as a vector space).

Theorem 1.13. *For regular c the $\mathbb{C}W \ltimes S(V^*)$ -module $\mathcal{QH}^{(c)}$ is generated by the elements $e_{kh}^{(c)}$, $k = 0, 1, 2, \dots$. More precisely,*

$$(1.9) \quad \mathcal{QH}^{(c)} = \{ \nabla_p^{(c)}(e_{kh}^{(c)}) \mid p \in \mathcal{H}^*, k = 0, 1, 2, \dots \},$$

where $\mathcal{H}^* \subset S(V^*)$ is the space of harmonic elements of $S(V^*)$.

To prove the theorem we will need some auxiliary results. Let $I^* \subset S(V^*)$ be the ideal generated by all $p \in S(V^*)_+^W$ of degree less than h .

Lemma 1.14. *The ideal I^* is radical.*

Proof. One has a harmonic decomposition $S(V^*) = S(V^*)^W \otimes \mathcal{H}$. Therefore, if we fix generators $e_{d_1}, \dots, e_{d_{\ell-1}}, e_{d_\ell} = e_h$ of $S(V^*)^W$ (so that $S(V^*)^W = \mathbb{C}[e_{d_1}, \dots, e_{d_\ell}]$), we see that the algebra $A \stackrel{\text{def}}{=} S(V^*)/I^*$ is spanned by all the elements $\bar{x} \cdot \bar{e}_h^k$ for $x \in \mathcal{H}$ and $k = 0, 1, \dots$ (here \bar{x} denotes the image of x in A). On the other hand, Proposition 1.9 ensures that the Hilbert series of A and $\mathcal{H}[e_h] = \mathcal{H} \otimes \mathbb{C}[e_h]$ are the same, so that $A \cong \overline{\mathcal{H}}[\bar{e}_h]$ (as a vector space) and $\overline{\mathcal{H}} \cong \mathcal{H}$. Hence, no polynomial of $\bar{e}_h \in A$ with constant coefficients is a zero divisor in A .

The system of equations $e_{d_1} = \dots = e_{d_{\ell-1}} = 0, e_h = 1$ has $|W| = d_1 \dots d_\ell$ distinct solutions. It means that the hypersurfaces defined respectively by $e_i = 0$, $i = 1, \dots, \ell - 1$, and $e_h = 1$ intersect transversally in their common points (the derivatives are linearly independent). Therefore, the algebra $B \stackrel{\text{def}}{=} A/(\bar{e}_h - 1) = S(V^*)/(e_{d_1}, \dots, e_{d_{\ell-1}}, e_h - 1)$ is semisimple of dimension $|W|$ and hence contains no non-zero nilpotents.

Suppose that the ideal I^* is not radical; that is, A contains nilpotent elements. Denote by $z \in A$ a nilpotent element of the smallest degree (as a polynomial in \bar{e}_h). Since the image of z under the natural projection $A \rightarrow B$ is zero, one has $z = (e_h - 1)y$ for some $y \in A$. Then $0 = z^N = (\bar{e}_h - 1)^N y^N$ for some N . Since $(\bar{e}_h - 1)^N$ is not a zero divisor, this implies $y^N = 0$, contrary to the assumption that the degree of z is minimal. The lemma is proved. \square

Consider now the exponential map $\exp(x) \stackrel{\text{def}}{=} \sum_{n \geq 0} \frac{x^n}{n!}$ from V to the completion $\widehat{S(V)}$ of $S(V)$.

- Lemma 1.15.** (1) $\partial_p \exp(v) = p(v) \exp(v)$ for any $v \in V$ and $p \in S(V^*)$, where $\partial_p \stackrel{\text{def}}{=} \nabla_p^{(0)}$ is the differential operator with constant coefficients on $\widehat{S(V)}$ corresponding to the polynomial p .
- (2) If $v_1, \dots, v_N \in V$ are all distinct, then $\exp(v_1), \dots, \exp(v_N) \in \widehat{S(V)}$ are linearly independent.

Proof. The first assertion is obvious. To prove the second one, consider a linear functional $\varphi : V \rightarrow \mathbb{C}$ such that $\lambda_1 \stackrel{\text{def}}{=} \varphi(v_1), \dots, \lambda_N \stackrel{\text{def}}{=} \varphi(v_N)$ are all distinct. Extend φ naturally to $\widehat{S(V)}$; then $\varphi(\exp(v)) = \exp(\varphi(v))$ for all $v \in V$. If $\sum_{i=1}^N \alpha_i \exp(v_i) = 0$ is a linear dependence, then one must have $\sum_{i=1}^N \alpha_i \exp(\lambda_i t) = 0$ for any $t \in \mathbb{C}$. But functions $\exp(t\lambda_i)$ with distinct λ_i are obviously linearly independent. \square

Proposition 1.16. *Let*

$$(1.10) \quad Q \stackrel{\text{def}}{=} \{\partial_p(e_{kh}^{(0)}) \mid p \in S(V^*), k = 0, 1, \dots\},$$

where ∂_p is as in Lemma 1.15. Then $Q^\perp = I^*$, where the annihilator is computed with respect to the standard pairing between $S(V)$ and $S(V^*)$.

Proof. For each $v \in V$ define an element $F(v) \in \widehat{S(V)}$ by the formula

$$F(v) = \sum_{w \in W} \exp(w(v)).$$

Then $\partial_p F(v) = \sum_{w \in W} p(w(v)) \exp(w(v))$ by assertion (1) of Lemma 1.15. Assertion (2) of the same lemma implies that $\partial_p F(v) = 0$ if and only if $p(w(v)) = 0$ for all $w \in W$. Take v_0 such that $p(v_0) = 0$ for any $p \in I^*$; then one has

$$F(tv_0) = \sum_{k \geq 0} e_{kh}^{(0)} t^k$$

for $t \in \mathbb{C}$ (under a certain normalization of coefficients). By Lemma 1.14, $\partial_p F(tv_0) = 0$ if and only if $p \in I^*$. Hence I^* is the annihilator of Q . \square

Proof of Theorem 1.13. By Proposition 1.8 it suffices to prove the theorem for only one regular c . We will do this for $c = 0$. Graded $S(V^*)$ -modules $\mathcal{QH}^{(0)}$ and Q have the same annihilator I^* in $S(V^*)$. Since $Q \subseteq \mathcal{QH}^{(0)}$ we immediately obtain $\mathcal{QH}^{(0)} = Q$. This proves the first assertion of the theorem.

Equation (1.9) holds because $S(V^*) = S(V^*)^W \otimes \mathcal{H}^*$ and, clearly, $\nabla_{e_h}^{(c)}(e_{kh}^{(c)}) \in \mathbb{C} \cdot e_{(k-1)h}^{(c)}$, while $\nabla_p^{(c)}(e_{kh}^{(c)}) = 0$ for $p \in S^r(V^*)^W$, $r < h$, by definition. \square

The theorem is proved. \square

Now denote $\mathcal{QH}^{(c,d)}$ the set of all elements $x \in S(V)$ such that $\langle p, x \rangle_c = 0$ for any W -invariant polynomials $p \in S^k(V^*)$ of degree $k < d$ (so that $\mathcal{QH}^{(c)} = \mathcal{QH}^{(c,h)}$). Clearly, $\mathcal{QH}^{(c,d)}$ is graded: $\mathcal{QH}^{(c,d)} = \bigoplus_{n \geq 0} \mathcal{QH}_n^{(c,d)}$. The “universal” versions of these spaces will be denoted $\mathcal{QH}^{(\cdot,d)}$ and $\mathcal{QH}_n^{(\cdot,d)}$, respectively.

Reasoning as in the proof of Proposition 1.9 and in Proposition 1.11, one obtains

Proposition 1.17. *For each simple W -module τ and any regular c and $i = 1, \dots, \ell$,*

$$(1.11) \quad \text{hilb}_\tau(\mathcal{QH}^{(c,d_i)}, t) = \frac{\text{hilb}_\tau(\mathcal{H}^{(c)}, t)}{(1 - t^{d_1}) \dots (1 - t^{d_\ell})}.$$

In particular, if $\tau = \mathbf{1}$ is the trivial 1-dimensional W -module, one has

$$(1.12) \quad \text{hilb}_1(\mathcal{QH}^{(c,d_i)}, t) = \frac{1}{(1-t^{d_1}) \cdots (1-t^{d_\ell})}.$$

This implies the following important assertion:

Proposition 1.18. *For each $i = 1, 2, \dots, \ell$ and each regular c one has*

$$(1.13) \quad \dim(\mathcal{QH}_{d_i}^{(c,d_i)})^W = \begin{cases} 1, & d_i < d_{i+1}, \\ 2, & d_i = d_{i+1}. \end{cases}$$

Therefore, if all d_i are distinct, then for every $i = 1, 2, \dots, \ell$ there exists a W -invariant $e_{d_i}^{(c)} \in \mathcal{QH}_{d_i}^{(c,d_i)}$ unique up to proportionality.

Proof. Clearly, (1.13) is a direct consequence of (1.12). In its turn, if all d_i are distinct, (1.13) implies that for each $i \leq \ell$ there exists a unique (up to a multiple) W -invariant $u_{d_i} \in \mathbb{C}(c) \otimes \mathcal{QH}_{d_i}^{(c,d_i)}$. Multiplying this invariant by an appropriate polynomial in $\mathbb{C}[c]$, we obtain $e_{d_i}^{(c)}$. The proposition is proved. \square

Remark 1.19. Proportionality in this corollary means the possibility to multiply $e_{d_i}^{(c)}$ by any polynomial in $\mathbb{C}[c]$. We can assume, though, that $e_{d_i}^{(c)} \in \mathbb{C}[c] \otimes S(V)$ is an irreducible polynomial; $e_{d_i}^{(c)}$ is then unique up to the multiplication by a constant. When it does not lead to confusion, we will denote by the same symbol $e_{d_i}^{(c_0)}$ the evaluation $\text{ev}_{c_0}(e_{d_i}^{(c)})$ for a particular function $c_0 : \mathcal{S} \rightarrow \mathbb{C}$.

Remark 1.20. The case $d_i = d_{i+1}$ for irreducible Coxeter groups occurs only if $W = D_\ell$ with even ℓ and $i = \ell/2$. In this case there exists an involutive automorphism σ of V acting non-trivially on the invariants in $\mathcal{QH}_{d_i}^{(c,d_i)}$. We denote $e_{d_i}^{(c)}$ as the invariant of σ , and $e_{d_i,-}^{(c)}$ its skew-invariant.

In complex reflection groups the phenomenon $d_i = d_{i+1}$ is also very rare: in addition to those groups listed in Remark 1.7, it occurs only for $G(kp, p, \ell)$ with $i = \ell/p$, provided p divides ℓ . Therefore, the invariants $e_{d_i}^{(c)}$ make sense for almost all complex reflection groups, too.

The polynomials $e_{d_i}^{(c)}$ should be interesting to study. In particular, $e_h^{(c)}$ is quasi-harmonic and $\mathcal{QH}_h^{(c)} = \mathcal{H}_h^{(c)} \oplus \mathbb{C}e_h^{(c)}$.

Theorem 1.21. *For generic c one has $S(V)^W = \mathbb{C}[e_{d_1}^{(c)}, \dots, e_{d_\ell}^{(c)}]$. In other words, $e_{d_i}^{(c)}$, $i = 1, \dots, \ell$, form a system of elementary W -invariants.*

Proof. First let $c = \text{const.}$, and let $P(c, e_{d_1}^{(c)}, \dots, e_{d_\ell}^{(c)}) = 0$ be an algebraic dependence between $e_{d_1}^{(c)}, \dots, e_{d_\ell}^{(c)}$. Without loss of generality $P(0, x_1, \dots, x_n) \not\equiv 0$; otherwise we could have canceled the extra c^k factor. But $e_{d_i}^{(0)}$ are elementary invariants of the group W in the sense of the classical Chevalley's theorem [5], so they are algebraically independent.

Now let $c \neq \text{const.}$; in this case (possible for Coxeter groups $B_n, I_2(m)$ with m even, and F_4) c assumes two values, c_1 and c_2 . Let $P(c_1, c_2, e_{d_1}^{(c)}, \dots, e_{d_\ell}^{(c)}) = 0$ be an algebraic dependence between $e_{d_1}^{(c)}, \dots, e_{d_\ell}^{(c)}$. Without loss of generality

$P(a, a, x_1, \dots, x_n) \neq 0$; otherwise we could have canceled the extra $(c_1 - c_2)^k$ factor. But $e_{d_i}^{(c)} \Big|_{c_1=c_2=a}$ are algebraically independent for generic $a \in \mathbb{C}$, as we already proved above. \square

Remark 1.22. Apparently, the first construction (recursive) of elementary invariants belongs to Dynkin (see e.g. [23]), who used a similar approach for constructing the elementary Ad-invariant polynomials over semisimple Lie algebras.

1.3.2. *Defining module.* Consider now Proposition 1.11 for $\tau = V$, the defining W -module. It is well-known (see [22]) that

$$(1.14) \quad \text{ch}_V(\mathcal{H}^{(0)}, t) = t^{d_1-1} + t^{d_2-1} + \dots + t^{d_\ell-1}.$$

By Proposition 1.8, the same formula is true for every regular c , so that for $r \geq 1$ one has

$$(1.15) \quad [V : \mathcal{QH}_r^{(c)}] = \begin{cases} 0 & \text{if } r \bmod h \notin \{d_1 - 1, d_2 - 1, \dots, d_\ell - 1\}, \\ 2 & \text{if } r \bmod h = d_{\ell/2} - 1 = d_{\ell/2+1} - 1 = \frac{h}{2} - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, for regular c and for every $r \equiv d_i - 1 \pmod h$, $i = 1, 2, \dots, \ell$, in the case $d_i \neq d_{i\pm 1}$ the space $\mathcal{QH}_r^{(c)}$ contains a unique copy of the defining module V , which we denote by $V^{(r;c)}$. In the case $r \equiv d_{\ell/2} - 1 = d_{\ell/2+1} - 1 = h/2 - 1$ (it occurs only in D_ℓ type with even ℓ) there is an involutive automorphism σ of V acting non-trivially on $\mathcal{QH}_r^{(c)}$. Then we take $V^{(r;c)}$ to be the σ -invariant copy of V in $\mathcal{QH}_r^{(c)}$.

More generally, let τ be an irreducible W -module such that $[\tau : \mathcal{QH}_r^{(0)}] = 1$. By Proposition 1.8, $[\tau : \mathcal{QH}_r^{(c)}] = 1$ for any regular c ; we denote the unique copy of τ in $\mathcal{QH}_r^{(c)}$ by $\tau^{(r;c)}$.

Theorem 1.13 suggests the following procedure for computing $V^{(r;c)}$, where $r \equiv d_i - 1 \pmod h$ for some $i \leq \ell$. For each j denote by V_j^* the unique copy of V^* in the space $\mathcal{H}_{d_j-1}^*$ of the W -harmonic polynomials in $S(V^*)$.

Corollary 1.23 (of Theorem 1.13). *For all regular c and any $r > 0$ of the form $r = kh + d_i - 1$, one has*

$$V^{(r;c)} = \nabla_{V_{\ell+1-i}^*}^{(c)} (e_{(k+1)h}^{(c)})$$

(recall that $d_i + d_{\ell+1-i} = h + 2$ because W is a real reflection group).

More generally, for a submodule $U \subset \mathcal{H}_d^*$, $1 \leq d < h$, one can consider a submodule $U^{(r;c)} \stackrel{\text{def}}{=} \nabla_U(e_{r+d}^{(c)}) \subset \mathcal{QH}_r^{(c)}$, where $r = kh - d$. Theorem 1.13 implies that $U^{(r;c)}$ is isomorphic to U .

1.4. **Flat deformations.** We now formulate some conjectures about the structure of the space of universal quasiharmonics \mathcal{QH} and other related objects. The next section (Section 1.5) contains various computations and other evidence supporting the conjectures.

Conjecture 1.24. *For every singular function c all the singular vectors of $M^{(c)}(\mathbf{1})$ belong to the specialization $\text{ev}_c \mathcal{QH}$ of $\mathcal{QH} \subset H(W)$.*

In other words, we conjecture that the space of singular vectors in $M^{(c)}(\mathbf{1})$ is “deformable” in the class of quasiharmonics. We will prove this conjecture for dihedral groups in Section 2.4 (for singular $c = \text{const.}$ and generic singular $c \neq \text{const.}$). If $W = S_n$ is a symmetric group, then explicit calculations for small n in low degrees of $M(\mathbf{1})$ support the conjecture; see Section 1.5 below.

Suppose now that $r \equiv d_i - 1 \pmod h$ for some $i = 1, \dots, \ell$. Then for regular c the homogeneous component $\mathcal{QH}_r^{(c)} \subset \mathcal{QH}^{(c)}$ contains a copy of the defining module V denoted by $V^{(r;c)}$ above in Section 1.3. Thus, the degree r component \mathcal{QH}_r of the space of universal quasiharmonics \mathcal{QH} contains a unique copy $V^{(r)}$ of $\mathbb{C}[c] \otimes V$ (such that $\text{ev}_c V^{(r)} = V^{(r;c)}$ for all regular c). On the other hand, the degree r component $M_r^{(r/h)}(\mathbf{1})$ of the module $M^{(r/h)}(\mathbf{1})$ is known to contain a subspace V_r of singular vectors isomorphic to V (for the proof, see [10], also [2] and [7] for the case $W = A_n$, [2] for $W = B_n$ and $W = D_n$, and [6] for $W = I_2(m)$). We conjecture that this is a particular case of the following phenomenon:

Conjecture 1.25. *Let $c = \text{const.} = r/h$, where $r \equiv d_i - 1 \pmod h$ for some $i = 1, \dots, \ell$. Then $\mathcal{QH}_r^{(r/h)} = \text{ev}_{r/h} \mathcal{QH}_r$. In particular, $V_r = \text{ev}_{r/h} V^{(r)}$.*

We will prove Conjecture 1.25 for $W = I_2(m)$ in Section 2.4. For the case $W = S_n$ see Section 1.5.

Definition 1.26. We say that a positive integer r is W -good if for each c such that $|c| = r$ the Cherednik algebra $H_c(W)$ admits a finite-dimensional module of the form $S(V)/S(V)V_r$, where $V_r \subset S^r(V)$ is an irreducible W -submodule of dimension $\ell = \dim V$ (consequently, V_r is a space of singular vectors, i.e. is killed by all the Dunkl operators).

Example 1.27. According to [2, Theorem 1.2] (see also [7] for a shorter proof) the number r is S_n -good if $\gcd(r, n) = 1$. By [2, Theorem 1.4], the number r is B_n -good if it is odd and $\gcd(r, n) = 1$, and the number r is D_n -good if it is odd and $\gcd(r, n - 1) = 1$. According to [6], a number r is $I_2(m)$ -good if it is not divisible by m for m odd, and by $m/2$ for m even.

Conjecture 1.28. *Let r be W -good. Then the quotient $S(V)/S(V)V^{(r;c)}$ is finite-dimensional.*

We will prove this conjecture for dihedral groups in Section 2.5.

Remark 1.29. As we noted above (see the beginning of Section 1.2), the ideal $S(V)V_r \subset S(V)$ is also a $H_{r/h}(W)$ -submodule, so that the quotient $S(V)/S(V)V_r$ is a finite-dimensional $H_{r/h}(W)$ -module. It is not true, though, for $S(V)/S(V)V^{(r;c)}$ in general, because the algebra $H_c(W)$ for regular c has no finite-dimensional modules at all. Therefore each module over the $\mathbb{C}[c]$ -algebra $H(W)$ is of infinite length over $\mathbb{C}[c]$ (i.e., after extending coefficients to the field $\mathbb{C}(c)$ it becomes an infinite-dimensional $\mathbb{C}(c)$ -vector space).

1.5. Quasiharmonics and elementary invariants for $W = S_n$. In this section we make various observations concerning Conjectures 1.24, 1.25 and 1.28 in the case when $W = S_n$ is the symmetric group.

The reflections in S_n are transpositions (ij) ; they form a single conjugacy class. Therefore a conjugation-invariant function c must be a constant. The defining module for $W = S_n$ is $V = \mathbb{C}^{n-1}$. It is convenient to assume that $V = \mathbb{C}^n/\mathbb{C}$,

where the additive group \mathbb{C} acts on \mathbb{C}^n by simultaneous translations: $(x_1, \dots, x_n) \mapsto (x_1 + b, \dots, x_n + b)$. The group S_n acts in V by permutation of variables. The symmetric algebra $S(V)$ is isomorphic to the algebra of all translation-invariant polynomials in n variables: $f(x_1 + b, \dots, x_n + b) = f(x_1, \dots, x_n)$ for all $b \in \mathbb{C}$.

Now consider the space $V^{(r;c)} \subset \mathcal{QH}_r^{(c)}(S_n)$ (where n does not divide r), and let $q_{n,r}^{(c)} \in V^{(r;c)}$ be a non-zero invariant of the standard subgroup $S_{n-1} \subset S_n$. Then denote $q_{i,r}^{(c)} \stackrel{\text{def}}{=} w(q_{1,r}^{(c)})$ for any $w \in W$ such that $w(n) = i$. Clearly, the polynomials $q_{i,r}^{(c)}$ are well-defined.

Proposition 1.30. *For each $r > 0$ there exists a constant $\alpha_r(c)$ such that for $i = 1, 2, \dots, n$ one has*

$$(1.16) \quad \nabla_{x_i}^{(c)} q_{i,r}^{(c)} = \alpha_r(c) q_{i,r-1}^{(c)},$$

where we use the convention $q_{i,kn}^{(c)} \stackrel{\text{def}}{=} e_{kn}^{(c)}$ (see Corollary 1.12).

Proof. Let $\varphi_{i,r-1}^{(c)} \stackrel{\text{def}}{=} \nabla_{x_i}^{(c)} q_{i,r}^{(c)}$ for all $r > 0, i = 1, \dots, n$. Clearly, $\varphi_{i,r}^{(c)} \in \mathcal{QH}_{r-1}^{(c)}$ and $w(\varphi_{i,r}^{(c)}) = \varphi_{w(i),r}^{(c)}$ for any $w \in S_n$. The sum $\sum_{i=1}^n \varphi_{i,r}^{(c)}$ is an S_n -invariant element of $\mathcal{QH}_{r-1}^{(c)}$; so, by Corollary 1.12, if $r-1$ is not divisible by n , then the sum is zero. Thus, for $r \not\equiv 0, 1 \pmod n$ the polynomials $\varphi_{i,r}^{(c)}, i = 1, \dots, n$, form a copy of the defining S_n -module inside $\mathcal{QH}_{r-1}^{(c)}$. This copy must be $V^{(r-1,c)}$ by (1.15), so that (1.16) holds for all $r \not\equiv 0, 1 \pmod n$.

Now let $r = kn$ for some $k > 0$. Then each $q_{i,kn}^{(c)}$ is the quasiharmonic invariant $e_{kn}^{(c)}$, and the span of all $\varphi_{i,kn-1}^{(c)}$ is isomorphic to V . Therefore, it is $V^{(kn-1,c)}$. Hence

$$\nabla_{x_i}^{(c)} e_{kn}^{(c)} = \alpha_{kn}(c) q_{i,kn-1}^{(c)},$$

which proves (1.16) in this case.

Now take $r = kn + 1$. The elements $p_{kn,i}^{(c)} \stackrel{\text{def}}{=} \nabla_{x_i}^{(c)} q_{i,kn+1}^{(c)}, i = 1, \dots, n$, span an S_n -submodule U_{kn} in $\mathcal{QH}_{kn}^{(c)}$. This submodule is a quotient of the permutation S_n -module $\mathbf{1} \oplus V$. By (1.15), the S_n -module $\mathcal{QH}_{kn}^{(c)}$ contains no irreducible summands isomorphic to V and, on the other hand, $p_{kn,i}^{(c)} \neq 0$ for c generic. Therefore, the S_n -module U_{kn} must be trivial and $p_{kn,i}^{(c)} = \mathbb{C} \cdot q_{kn}^{(c)}$ for $i = 1, \dots, n$. That is,

$$\nabla_{x_i}^{(c)} q_{i,kn+1}^{(c)} = \alpha_{kn+1}(c) e_{kn}^{(c)}$$

for all $i = 1, \dots, n$.

The proposition is proved. \square

Remark 1.31. The exact value of $\alpha_r(c)$ in (1.16) depends on the normalization of the polynomials $q_{i,r}^{(c)}$ (recall that $q_{i,r}^{(c)}$ are defined up to a multiplicative constant). Explicit calculations for $n, r \leq 5$ show that one can choose the normalization so that $\alpha_r(c) = r - nc$ and $q_{i,r}^{(r/n)} \neq 0$. This supports Conjecture 1.24.

Remark 1.32. In the case $W = S_n$ the singular polynomials of $H_c(W)$ are known. If $c = r/n$, then the space V_r of singular vectors in $S^r(V)$ is spanned by $f_{i,r}^{(r/n)}(x)$,

$i = 1, \dots, n$, where $f_{i,r}^{(c)}(x)$ is the non-symmetric Jack polynomials defined by the formula

$$f_{i,r}^{(c)}(x) = \text{Res}_{z=\infty} \left((1 - x_1/z) \cdots (1 - x_n/z) \right)^c \frac{z^r dz}{z - x_i}$$

(see [13] for proof). Singular vectors for other $c \in C^{sing}$ are also specializations of some other non-symmetric Jack polynomials (see [14], [15] for details).

Note, however, that $q_{i,r}^{(c)}$ is not a multiple of $f_{i,r}^{(c)}$ for regular c . For instance, for $c = 0$ one has $f_{i,r}^{(0)} = x_i^r$, which, clearly, is not quasiharmonic. However, if $c = r/n$, Conjecture 1.25 implies that $q_{i,r}^{(c)}$ and $f_{i,r}^{(c)}$ are proportional.

Conjecture 1.25 claims that the family of vector spaces $\mathcal{QH}_r^{(c)}$ is flat over $c = r/h$. We prove in Section 2.4 (Proposition 2.8) that for the dihedral groups this family is flat everywhere. For $W = S_n$ it is not the case. Consider for example the case $n = 4$ (S_3 is the dihedral group $I_2(3)$). Computations using the MuPad computer algebra system show that the space \mathcal{QH}_3 for the algebra $H(S_4)$ is isomorphic, as an S_4 -module, to $V \oplus (V \otimes \varepsilon)$, where V is the defining module and ε is the sign character. Therefore, $\dim \mathcal{QH}_3^{(c)} = 6$ for generic c , as predicted by Proposition 1.9. Exceptional values are $c = 1/2$ and $c = 1/4$: here $\dim \mathcal{QH}_3^{(c)} = 7$; as a S_4 -module one then has $\mathcal{QH}_3^{(c)} = V \oplus (V \otimes \varepsilon) \oplus \mathbf{1}$. The isotypic component of $V \subset \mathcal{QH}_3^{(c)}$ is $V^{(3;c)}$; the Dunkl operators ∇_{x_i} , $i = 1, \dots, 4$, map it to zero for $c = 3/4$, confirming Conjecture 1.24.

For $r = 4$ similar computations show that $\dim \mathcal{QH}_4(S_4) = 6$. As an S_4 -module $\mathcal{QH}_4^{(c)}$ for generic c is isomorphic to $\mathbf{1} \oplus \tau \oplus (V \otimes \varepsilon)$, where τ is the 2-dimensional irreducible module. Exceptional values are $c = 1/3$, $c = 1/2$ and $c = 3/4$, where $\dim \mathcal{QH}_4^{(c)} = 9$ and $\mathcal{QH}_4^{(c)} = \mathbf{1} \oplus \tau \oplus (V \otimes \varepsilon) \oplus V$.

Here are some formulas for the polynomials $e_k^{(c)}$ in the case $W = S_n$.

Recall that for $W = S_n$ the module $M(\mathbf{1}) = S(V)$ is generated by elements x_1, \dots, x_n with the relation $x_1 + \dots + x_n = 0$. Explicit computations show that

$$\begin{aligned} e_2^{(c)} &= e_2, & e_3^{(c)} &= e_3, \\ e_4^{(c)} &= (n-2)(n-3)(1-nc)e_2^2/2 + (n^2(n-1)c - n(n+1))e_4, \\ e_5^{(c)} &= (n-3)(n-4)(1-nc)e_2e_3 + (n^2(n-1)c - n(n+5))e_5, \end{aligned}$$

where e_s is the s -th elementary symmetric function.

Invariant quasiharmonic polynomials $e_{kh}^{(c)}$ (from Corollary 1.12) are computed by now only for the case $W = S_4$ and small k (plus for all dihedral groups; see Section 2.4, page 250). For example, one has

$$\begin{aligned} e_4^{(c)} &= 4(12c-5)e_4 - (4c-1)e_2^2, \\ e_8^{(c)} &= (16c^2 - 32c + 27)e_2^4 - 24(16c^2 - 40c + 29)e_2^2e_4 - 24(12c - 13)e_2e_3^2 \\ &\quad + 48(12c - 13)(4c - 5)e_4^2. \end{aligned}$$

The authors are planning to write a separate paper dealing with the properties of polynomials $e_k^{(c)}$ and $e_{kh}^{(c)}$.

We finish the section with an easy observation that one cannot replace $\mathcal{QH}^{(c)}$ in Conjectures 1.24 and 1.25 by the space of harmonics $\mathcal{H}^{(c)}$. Namely, take $r = h + 1$. For any irreducible reflection group W the smallest exponent is $d_1 = 2$, so that

$r \equiv d_1 - 1 \pmod h$. Then for regular c one has $[V : \mathcal{QH}_r^{(c)}] = 1$ by (1.15) and $[V : \mathcal{H}_r^{(c)}] = 0$ by (1.14). Thus, the space $V^{(h+1;c)} \subset \mathcal{QH}_{h+1}^{(c)}$ is defined and not annihilated, for regular c , by the operator ∇_{e_h} . For $c = (h + 1)/h \in C^{sing}$ the component $S^{h+1}(V)$ contains a subspace V_{h+1} of singular vectors isomorphic to the defining module V . Naturally, $V_{h+1} \subset \mathcal{H}_{h+1}^{(h+1)/h}$. So $0 = \text{ev}_{(h+1)/h} \mathcal{H}_{h+1} \neq \mathcal{H}_{h+1}^{(h+1)/h}$, i.e. the space of singular vectors cannot be deformed in the class of harmonics. Conjecture 1.24 claims that for quasiharmonics such a situation is impossible — they are, so to say, flexible enough.

2. QUASIHARMONIC ELEMENTS: THE DIHEDRAL GROUP CASE

In this section we study quasiharmonic polynomials in the rational Cherednik algebras for the dihedral group $W = I_2(m)$. In particular, we verify Conjectures 1.24, 1.25 (Corollary 2.13, Corollary 2.16) and 1.28 (Corollary 2.22) for them. We also study the quotient algebra of $M(\mathbf{1})$ by the submodule generated by homogeneous components of $\mathcal{QH}^{(c)}$ (Section 2.5).

2.1. Structure of the dihedral group (a summary). As an abstract group, $I_2(m)$ is generated by two elements s_0, s_1 with the relations $s_0^2 = s_1^2 = (s_0 s_1)^m = 1$. Also, $I_2(m)$ is a finite reflection group acting in the space $V = \mathbb{C}^2$ equipped with the non-degenerate bilinear form (\cdot, \cdot) . The set \mathcal{S} contains m reflections s_0, \dots, s_{m-1} . The action of $I_2(m)$ is a complexification of the action in \mathbb{R}^2 ; to mark this fact we will be using the basis $z \stackrel{\text{def}}{=} x_1 + ix_2, \bar{z} \stackrel{\text{def}}{=} x_1 - ix_2$ in \mathbb{C}^2 instead of the usual x_1, x_2 . Then the reflection s_j acts as follows:

$$(2.1) \quad s_j(z) = -\zeta^j \bar{z}, \quad s_j(\bar{z}) = -\zeta^{-j} z,$$

where $\zeta = e^{2\pi i/m}$ is the m -th primitive root of unity.

If m is odd, then all the reflections $s \in \mathcal{S}$ are conjugate to one another; so, $c = \text{const.}$ in the definition of the Cherednik algebra. If m is even, then the reflections $s \in \mathcal{S}$ split into two conjugacy classes: s_j with j even and s_j with j odd. We will denote $c(s_j) = c_1$ for the even class and $c(s_j) = c_2$ for the odd class.

The $H_c(I_2(m))$ -module $M(\mathbf{1}) = S(V) = \mathbb{C}[z, \bar{z}]$ will be the main object of study throughout Section 2.

The irreducible complex modules over the group $I_2(m)$ have dimensions 1 and 2 and can be described as follows:

- (1) If m is odd, then the group $I_2(m)$ has two 1-dimensional modules: the trivial one $\mathbf{1}$ and the sign one ε where $s_0 = s_1 = -1$.
- (2) If m is even, the group has four 1-dimensional representations: $\mathbf{1}, \varepsilon, \mu_1$ and $\mu_2 = \mu_1 \otimes \varepsilon$, where in μ_1 one has $s_0 = 1$ and $s_1 = -1$, and in μ_2 , vice versa.
- (3) The 2-dimensional irreducible $I_2(m)$ -modules $Z_k, 1 \leq k < m/2$. The element s_0 acts on $Z_k \cong \mathbb{C}^2$ as multiplication by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and the element s_1 by $\begin{pmatrix} 0 & -\zeta^{-k} \\ -\zeta^k & 0 \end{pmatrix}$.

One can define the modules Z_k for all $k \in \mathbb{Z}$ by the same formulas. It is easy to see that $Z_k \cong Z_{-k} \cong Z_{m+k}$; also $Z_0 = \mathbf{1} \oplus \varepsilon$ and, for m even, $Z_{m/2} = \mu_1 \oplus \mu_2$. Any irreducible Z_r is isomorphic to a Z_k with $1 \leq k < m/2$. In particular, Z_1 is the defining $I_2(m)$ -module.

This description allows us to relate the modules $M(\tau)$ over the Cherednik algebra $H_c(W)$ for different 1-dimensional τ . Thus, Dunkl operators for $M(\varepsilon)$ are the same as Dunkl operators for $M(\mathbf{1})$, with the change $c \mapsto -c$. If m is even, then the Dunkl operators for $M(\mu_1)$ are the same as for $M(\mathbf{1})$, with the change $(c_1, c_2) \mapsto (c_1, -c_2)$. By this reason, in the following sections we will describe Dunkl and other related operators mostly for $M(\mathbf{1})$.

2.2. Summary of main results. Here we list, for the reader's convenience, the main results about the dihedral group case to be proved later in this section.

The first group of results are *proofs of some conjectures* mentioned in Section 1:

- Conjecture 1.24 is proved for the dihedral group $W = I_2(m)$ with any m and $c = \text{const.}$ (Corollary 2.14) and for m even and a generic 2-valued c (Corollary 2.17).
- Conjecture 1.25 is proved for all dihedral groups (Corollary 2.13).
- Conjecture 1.28 is proved for all dihedral groups (Corollary 2.22).

The second group of results are various *explicit formulas*:

- In Section 2.3 we derive formulas for polynomials $e_{d_i}^{(c)}$ mentioned in Theorem 1.21. The exponents of the dihedral group $I_2(m)$ are 2 and m ; formulas for $e_2^{(c)}$ and $e_m^{(c)}$ are given in Corollary 2.4 for $c = \text{const.}$ and in Corollary 2.7 for m even and $c \neq \text{const.}$
- In Section 2.4 we give formulas for quasiharmonic polynomials of the dihedral group. According to Definition 1.26 these polynomials are elements of the kernel of a single operator ∇_{e_2} . The dimension of this kernel is 2 for every degree (Proposition 2.8). Explicitly the basic quasiharmonics are given by equations (2.5), (2.6) (see also (2.7) and (2.8)); for the proof, see Theorem 2.10.
- In Section 2.5 we study the quotient of the module $M(\mathbf{1})$ by the ideal generated by quasiharmonics of some fixed degree n . This quotient is a standard Frobenius algebra (see the Appendix for necessary definitions) and thus can be described in terms of its characteristic polynomial. In Theorem 2.28 this polynomial is computed explicitly for $c = \text{const.}$

2.3. Explicit formulas for Dunkl operators. In this section we will describe the Dunkl operators and the invariant operator $\nabla_{e_2} = \nabla_{z\bar{z}}$ of degree 2 in the module $M(\mathbf{1})$ for the dihedral group $I_2(m)$. According to the remark at the end of the previous section, it will allow us to obtain similar formulas for $M(\tau)$ with any 1-dimensional τ . Formulas for $M(Z_k)$ also exist but are cumbersome and do not serve our primary purpose (the study of quasiharmonic elements), so we omit them.

The importance of the operator ∇_{e_2} comes from two reasons. First, the dihedral group $I_2(m)$ has only two exponents: $d_1 = 2$ and $h = d_2 = m$, so the space of quasiharmonics is simply the kernel of ∇_{e_2} . Second, there holds the following proposition:

Proposition 2.1 ([1]). *Let E, F, H be endomorphisms of $\mathbb{C}[z, \bar{z}]$ given respectively by $E = z\bar{z}$, $F = -\nabla_{e_2}$ and $H(P) = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} + (1 - mc)$. Then the operators E , F , and H form a representation of the Lie algebra \mathfrak{sl}_2 , i.e. satisfy the relations*

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

Below we will denote ∇_{e_2} by $-F$.

We consider two separate cases: the case $c = \text{const.}$ (possible for any m) and the general case (possible only for m even) when c takes two different values, c_1 and c_2 , on even-numbered and odd-numbered reflections.

Denote by Y, \bar{Y} the Dunkl operators corresponding, in the notation of (1.3), to the vectors $y = (1/2, -i/2)$ and $y = (1/2, i/2)$, respectively. One can write symbolically $Y = \nabla_z$ and $\bar{Y} = \nabla_{\bar{z}}$ because for $c = 0$ one has $Y = \partial/\partial z$ and $\bar{Y} = \partial/\partial \bar{z}$. Also let $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ denote the operator exchanging z and \bar{z} , so that $(TP)(z, \bar{z}) \stackrel{\text{def}}{=} P(\bar{z}, z)$. Clearly,

$$(2.2) \quad YT = T\bar{Y}, \quad FT = TF.$$

Proposition 2.2. *Let $c = \text{const.}$ Then in the module $M(\mathbf{1})$ one has*

$$Y(z^a \bar{z}^b) = \begin{cases} az^{a-1} \bar{z}^b - mc \sum_{0 \leq k \leq (a-b-1)/m} (-1)^{mk} z^{a-mk-1} \bar{z}^{b+mk} & \text{if } a \geq b \geq 0, \\ az^{a-1} \bar{z}^b + mc \sum_{1 \leq k \leq (b-a)/m} (-1)^{mk} z^{a+mk-1} \bar{z}^{b-mk} & \text{if } b \geq a \geq 0. \end{cases}$$

Proof. Both equations are proved in a similar manner, so we concentrate on the first one. Its first summand is equal to $\frac{\partial z^a \bar{z}^b}{\partial z}$; therefore it is enough to prove that the sum on the right-hand side is equal to $-cD_z(z^a \bar{z}^b)$, where D_z is the ‘‘difference term’’ (the sum over $s \in \mathcal{S}$) of the Dunkl operator; see (1.2).

The polynomial $z\bar{z}$ is $I_2(m)$ -invariant (it is the elementary invariant e_2). So, if $a \geq b$, then $D_z(z^a \bar{z}^b) = (z\bar{z})^b D_z(z^{a-b})$. It is thus enough to consider the case $b = 0$, $a \geq 0$.

Taking into account that $z - s_j(z) = z + \zeta^j \bar{z} = \zeta^{j/2} \alpha_j(x)$ by (2.1), we obtain

$$\begin{aligned} D_z(z^a) &= \sum_{j=0}^{m-1} \frac{z^a - s_j(\zeta^a)}{z - s_j(z)} = \sum_{j=0}^{m-1} \frac{z^a - (-\zeta^j \bar{z})^a}{z - (-\zeta^j \bar{z})} \\ &= \sum_{j=0}^{m-1} \sum_{\ell=0}^{a-1} (-\zeta^j \bar{z})^\ell z^{a-1-\ell} = \sum_{\ell=0}^{a-1} (-1)^\ell z^{a-1-\ell} \bar{z}^\ell \sum_{j=0}^{m-1} \zeta^{j\ell}, \end{aligned}$$

and the identity

$$\sum_{j=0}^{m-1} \zeta^{j\ell} = \begin{cases} m, & \text{if } m \mid \ell, \\ 0, & \text{otherwise} \end{cases}$$

finishes the proof. □

Explicit formulas for \bar{Y} can be obtained from Proposition 2.2 using (2.2).

Proposition 2.3. *Let $c = \text{const.}$ Then in the module $M(\mathbf{1})$ for all $a \geq b \geq 0$ one has*

$$(2.3) \quad F(z^a \bar{z}^b) = (mc-a)bz^{a-1} \bar{z}^{b-1} - mc \sum_{1 \leq k \leq (a-b)/m} (-1)^{mk} (a-b-mk) z^{a-mk-1} \bar{z}^{b+mk-1}.$$

Proof. In the notation of Proposition 2.2 $Y = \frac{\partial}{\partial z} - cD_z$, $\bar{Y} = \frac{\partial}{\partial \bar{z}} - cD_{\bar{z}}$, and therefore

$$(2.4) \quad F = -\frac{\partial^2}{\partial z \partial \bar{z}} + c\left(\frac{\partial}{\partial \bar{z}} D_z + D_{\bar{z}} \frac{\partial}{\partial z}\right) - c^2 D_{\bar{z}} D_z.$$

For $a = b$ the result is evident. If $a > b$, then one has $-\frac{\partial^2}{\partial z \partial \bar{z}}(z^a \bar{z}^b) = -abz^{a-1} \bar{z}^{b-1}$ and also, by Proposition 2.2,

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} D_z(z^a \bar{z}^b) &= m \frac{\partial}{\partial \bar{z}} \sum_{0 \leq k \leq (a-b-1)/m} (-1)^{mk} z^{a-mk-1} \bar{z}^{b+mk} \\ &= \sum_{0 \leq k \leq (a-b-1)/m} (b+mk) (-1)^{mk} z^{a-mk-1} \bar{z}^{b+mk-1} \\ &= mbz^{a-1} \bar{z}^{b-1} + \sum_{1 \leq k \leq (a-b-1)/m} (b+mk) (-1)^{mk} z^{a-mk-1} \bar{z}^{b+mk-1} \end{aligned}$$

and

$$D_{\bar{z}} \frac{\partial}{\partial z}(z^a \bar{z}^b) = a D_{\bar{z}}(z^{a-1} \bar{z}^b) = -ma \sum_{1 \leq k \leq (a-b-1)/m} (-1)^{mk} z^{a-mk-1} \bar{z}^{b+mk-1}.$$

Thus, application to $z^a \bar{z}^b$ of all the terms of (2.4) except the last one already gives the right-hand side of (2.3). As for the last term, write $a = mu + p$, where u and p are integers and $0 \leq p < m$. Then Proposition 2.2 implies that

$$D_{\bar{z}}(z^{mu+p}) = -m \sum_{k=1}^u z^{m(u-k)+p} \bar{z}^{mk-1} = -D_{\bar{z}}(z^p \bar{z}^{mu}).$$

It was noted (see the proof of Proposition 2.2) that $D_{\bar{z}}$ (as well as D_z) commute with the multiplication by $z\bar{z}$. Therefore

$$\begin{aligned} D_{\bar{z}} D_z(z^a) &= D_{\bar{z}} \sum_{k=0}^{[u/2]} (-1)^{mk} (z\bar{z})^{mk} (z^{m(u-2k)+p} + z^p \bar{z}^{m(u-2k)}) \\ &= \sum_{k=0}^{[u/2]} (-1)^{mk} (z\bar{z})^{mk} D_{\bar{z}}(z^{m(u-2k)+p} + z^p \bar{z}^{m(u-2k)}) = 0, \end{aligned}$$

and so $D_{\bar{z}} D_z(z^a \bar{z}^b) = (z\bar{z})^b D_{\bar{z}} D_z(z^{a-b}) = 0$ for all $a > b$. This completes the proof of (2.3). \square

Explicit formulas for $F(z^a \bar{z}^b)$ with $a \leq b$ can be obtained using (2.2).

Corollary 2.4. *For $c = \text{const.}$ the polynomial $z^m + \bar{z}^m$ is quasiharmonic. In terms of Proposition 1.18 one has $\varepsilon_2 = z\bar{z}$ and $\varepsilon_m = z^m + \bar{z}^m$. These polynomials are algebraically independent for all c .*

Now consider the case of general c for m even, so that $c(s_j) = c_1$ for j even and $c(s_j) = c_2$ for j odd.

Proposition 2.5. *In the module $M(\mathbf{1})$ with m even, for any $a \geq b \geq 0$ one has*

$$Y(z^a \bar{z}^b) = az^{a-1} \bar{z}^b - \frac{m}{2} \sum_{0 \leq k \leq 2(a-b-1)/m} (-1)^{mk/2} z^{a-1-mk/2} \bar{z}^{b+mk/2} ((-1)^k c_1 + c_2).$$

For any $b \geq a \geq 0$ one has

$$Y(z^a \bar{z}^b) = az^{a-1} \bar{z}^b + \frac{m}{2} \sum_{1 \leq k \leq 2(b-a)/m} (-1)^{mk/2} z^{a-1+mk/2} \bar{z}^{b-mk/2} ((-1)^k c_1 + c_2).$$

Proposition 2.6. *In the module $M(\mathbf{1})$ with m even, for any $a \geq b \geq 0$ one has*

$$F(z^a \bar{z}^b) = (m(c_1 + c_2)/2 - a)bz^{a-1}\bar{z}^{b-1} - \frac{m}{2} \sum_{1 \leq k \leq 2(a-b)/m} (-1)^{mk/2} z^{a-1-mk/2} \bar{z}^{b-1+mk/2} (a-b-mk/2)((-1)^k c_1 + c_2).$$

Proofs of these propositions are similar to those of Propositions 2.2 and 2.3. Explicit formulas for the operator \bar{Y} and for $F(z^a \bar{z}^b)$ with $b \geq a \geq 0$ can be obtained using (2.2).

Corollary 2.7. *For m even the polynomial*

$$e_m^{(c)} = (c_1 + c_2 - 1)(z^m + \bar{z}^m) + 2(-1)^{m/2}(c_2 - c_1)(z\bar{z})^{m/2}$$

is quasiharmonic. In terms of Proposition 1.18 (and Remark 1.19) one has $e_2^{(c)} = z\bar{z}$. These polynomials are algebraically independent for all the functions c such that $c_1 + c_2 \neq 1$.

2.4. Quasiharmonic polynomials. In this section we describe the space $\mathcal{QH}_n^{(c)} \subset M(\mathbf{1})$, i.e. the kernel of the operator F described above.

Proposition 2.8. *For all c and n the space $\mathcal{QH}_n^{(c)}$ has dimension 2 and is isomorphic, as an $I_2(m)$ -module, to Z_n .*

Proof. Consider the action of \mathfrak{sl}_2 on $M(\mathbf{1}) = \mathbb{C}[z, \bar{z}]$ described in Proposition 2.1. By a classical theorem, $\mathbb{C}[z, \bar{z}]$ splits into a direct sum of spaces spanned by the bases a, Ea, E^2a, \dots , where $\deg a = r$ for some r and therefore $\deg E^k a = r + 2k$. On such a basis the operator F acts as $FE^k a = 2kE^{k-1}a$. The operator E has a trivial kernel; therefore the operator $F : \mathbb{C}_n[z, \bar{z}] \rightarrow \mathbb{C}_{n-2}[z, \bar{z}]$ has a trivial co-kernel. This implies the equality $\dim \mathcal{QH}_n^{(c)} = \dim \mathbb{C}_n[z, \bar{z}] - \dim \mathbb{C}_{n-2}[z, \bar{z}] = 2$ for all c and n .

The space $M_n(\mathbf{1}) = \mathbb{C}_n[z, \bar{z}]$ splits into a sum of $I_2(m)$ -isotypic components corresponding to irreducible representations described in Section 2.1. Now note that the space $\mathcal{QH}_n^{(c)} \subset M_n(\mathbf{1})$ for any n and c lies in the kernel of the operator F and therefore cannot lie in the image of E .

Thus, $\mathcal{QH}_n^{(c)}$ contains polynomials $P_1 = z^n + \dots$ and $P_2 = \bar{z}^n + \dots$, where the dots mean a linear combination of $z^{n-k} \bar{z}^k$ with $k = 1, \dots, n-1$. Since $\dim \mathcal{QH}_n^{(c)} = 2$, it is isomorphic to the $I_2(m)$ -module generated by z^n and \bar{z}^n , that is, to Z_n . \square

In particular, if $n \equiv \pm 1 \pmod m$, then $\mathcal{QH}_n^{(c)}$ is isomorphic to the defining W -module.

Suppose now that the function c is constant. Let $n = mq + r$, where q, r are non-negative integers and $0 \leq r \leq m-1$. For a non-negative integer k denote $\lambda_k(c) = c(c-1) \cdots (c-k)$ (with the convention $\lambda_{-1}(c) = 1$). Consider a polynomial

$$(2.5) \quad R_{n,c}(z, \bar{z}) = \sum_{p=0}^q (-1)^{mp} \binom{q}{p} \lambda_{q-p}(c) \lambda_{p-1}(c) z^{n-mp} \bar{z}^{mp},$$

and take

$$(2.6) \quad \varrho_{n,c}(z, \bar{z}) \stackrel{\text{def}}{=} R_{n,c} / \lambda_{\lfloor q/2 \rfloor}(c).$$

Equation (2.5) can be rewritten in another form:

$$(2.7) \quad R_{n,c}(z, \bar{z}) = cq!z^r \operatorname{Res}_{w=0} (1 + (-\bar{z})^m/w)^c (1 + (-z)^m/w)^{c-1} w^{q-1} dw.$$

For $c = n/m$ this becomes a formula for singular vectors from [10].

One more form of (2.5) is recursive:

$$(2.8) \quad R_{n,c} = \begin{cases} zR_{n-1,c}, & r \neq 0, \\ (c - q)zR_{n-1,c} + (-1)^{mq}c\bar{z}\bar{R}_{n-1,c}, & r = 0. \end{cases}$$

Proofs of (2.7) and (2.8) are immediate.

Proposition 2.9 ([11], p. 181). *Suppose that $c = \text{const}$. Then the polynomial $\varrho_{n,c}$ is quasiharmonic for all n and c . One has*

$$Y(\varrho_{n,c}) = \begin{cases} (n - mc)\varrho_{n-1,c} & \text{if } r \neq 0, \\ mq(2c - q)\varrho_{n-1,c} & \text{if } r = 0 \text{ and } q \text{ is odd,} \\ 2mq\varrho_{n-1,c} & \text{if } r = 0 \text{ and } q \text{ is even,} \end{cases}$$

and

$$\bar{Y}(\varrho_{n,c}) = 0$$

for all n and c .

Proposition 2.9 implies the following description of the space of quasiharmonics with $c = \text{const}$.

Theorem 2.10. *Suppose that $c = \text{const}$. If $r \neq 0$, then the polynomials $\varrho_{n,c}$ and $\bar{\varrho}_{n,c} \stackrel{\text{def}}{=} T\varrho_{n,c}$ form a basis for the space $\mathcal{QH}_n^{(c)}$. If $r = 0$ (that is, $n = mq$), then the basis consists of two polynomials, $\varrho_{mq,c} + \bar{\varrho}_{mq,c}$ and $(\varrho_{mq,c} - \bar{\varrho}_{mq,c})/(2c - q)$.*

Remark 2.11. In fact, $\varrho_{mq,c}$ and $\bar{\varrho}_{mq,c}$ form a basis in $\mathcal{QH}_{mq}^{(c)}$, too, unless q is odd and $c = q/2$; see the proof below.

Proof. In view of Proposition 2.8 and Proposition 2.9 it suffices to prove that $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ are linearly independent unless $r = 0$, q is odd, and $c = q/2$. The element $\gamma = s_0s_1 \in W \subset H_c(W)$ (where s_0, s_1 are the defining reflections) acts on $M(\mathbf{1}) = \mathbb{C}[z, \bar{z}]$ as follows: $\gamma P(z, \bar{z}) = P(\zeta z, \bar{\zeta} \bar{z})$, where $\zeta = \exp(2\pi i/m)$ is a primitive m -th root of unity. The polynomials $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ are eigenvectors of the operator γ with eigenvalues ζ^n and $\bar{\zeta}^n$, respectively. If $r \neq 0$, then the eigenvalues are different, and $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ can be linearly dependent only if they are identically zero, which never happens.

Now let $r = 0$; that is, $n = mq$. By (2.5) and (2.6) the coefficients at $z^{mk}\bar{z}^{m(q-k)}$ and $z^{ml}\bar{z}^{m(q-l)}$ of the polynomials $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ form a matrix

$$A_{kl} = \begin{pmatrix} \frac{\lambda_{k-1}(c)\lambda_{q-k}(c)}{\lambda_{[q/2]}(c)} & \frac{\lambda_{l-1}(c)\lambda_{q-l}(c)}{\lambda_{[q/2]}(c)} \\ \frac{\lambda_{q-k-1}(c)\lambda_k(c)}{\lambda_{[q/2]}(c)} & \frac{\lambda_{q-l-1}(c)\lambda_l(c)}{\lambda_{[q/2]}(c)} \end{pmatrix}.$$

Apparently, $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ are linearly dependent only if $\det A_{k,l}(c) = 0$ for all k and l .

One has

$$(2.9) \quad \begin{aligned} \det A_{kl} &= \frac{\lambda_{k-1}(c)\lambda_{l-1}(c)\lambda_{q-k-1}(c)\lambda_{q-l-1}(c)}{\lambda_{[q/2]}^2(c)} \det \begin{pmatrix} c-q+k & c-q+l \\ c-k & c-l \end{pmatrix} \\ &= (k-l)(2c-q) \frac{\lambda_{k-1}(c)\lambda_{l-1}(c)\lambda_{q-k-1}(c)\lambda_{q-l-1}(c)}{\lambda_{[q/2]}^2(c)}. \end{aligned}$$

Suppose first that q is even, so that $[q/2] = q/2$. Then $\det A_{kl}(c)$ can be zero only if c is an integer between 0 and q . Now one has $\det A_{0q} = q(2c-q) \frac{c\lambda_{q-1}(c)}{\lambda_{q/2}(c)^2}$, so that $\det A_{0q}(0) \neq 0$. Also $\det A_{01} = (2c-q) \frac{c\lambda_{q-1}(c)\lambda_{q-2}(c)}{\lambda_{q/2}(c)^2}$, and therefore $\det A_{01}(c) \neq 0$ for $c = 1, \dots, q/2 - 1$. Similarly, $\det A_{q/2, q/2+1}(c) \neq 0$ for $c = q/2, \dots, q$.

Thus, for every c there exist k, l such that $\det A_{kl}(c) \neq 0$, and the polynomials $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ are linearly independent.

If q is odd, then the reasoning is the same except for the case $c = q/2$. One gets $\det A_{kl}(q/2) = 0$ for all k, l , so that $\varrho_{n,c}$ and $\bar{\varrho}_{n,c}$ for $c = q/2$ are linearly dependent (actually, equal). A similar computation shows that in this case $\varrho_{mq,c} + \bar{\varrho}_{mq,c}$ and $(\varrho_{mq,c} - \bar{\varrho}_{mq,c})/(2c-q)$ form a basis in $\mathcal{QH}_{mq}^{(c)}$. \square

Corollary 2.12 ([10]). *A function $c = \text{const.}$ is singular for the module $M(\mathbf{1})$ of the dihedral group $W = I_2(m)$ if $c = n/m$, where n is a positive integer not divisible by m , or $c = \ell + 1/2$, where ℓ is a non-negative integer.*

Corollary 2.13 (of Proposition 2.9, Theorem 2.10 and Proposition 2.8). *Conjectures 1.24 and 1.25 are valid for the dihedral group $W = I_2(m)$ and $c = \text{const.}$*

Proof. For $c = n/m$ the deformation in question is $V_r(c) = \mathcal{QH}_r^{(c)}$. For $c = \ell + 1/2$ the deformation is $\varrho_{m(2\ell+1),c}$. \square

Note that the polynomials $e_{km}^{(c)} = \varrho_{km,c} + \bar{\varrho}_{km,c}$, $k = 1, 2, \dots$, are exactly the quasiharmonic invariants mentioned in Corollary 1.12. So, Theorem 2.10 and Proposition 2.9 imply

Corollary 2.14 (of Theorem 2.10 and Corollary 2.12). *Conjecture 1.24 is valid for dihedral groups with $c = \text{const.}$*

Later we are going to prove (see Corollary 2.17) a similar statement for $I_2(m)$ with m even and generic function c assuming 2 values.

Recall that the space $\mathcal{QH}_n^{(c)}$ is isomorphic to Z_n by Proposition 2.8. Denote $S_{n,c} \in \mathcal{QH}_n^{(c)}$ an eigenvector with the eigenvalue ζ^n of the operator $\gamma = s_0 s_1$. If n is not divisible by $m/2$, then the two eigenvalues of γ are different and $S_{n,c}$ is defined uniquely up to proportionality; we normalize it by the condition $S_{n,c} = \lambda_{[2(n-1)/m]}(c_1 + c_2)z^n + \dots$ (recall that we denote $\lambda_k(c) = c(c-1) \cdots (c-k)$; square brackets $[\alpha]$ mean the biggest integer not exceeding α). If $2n/m = q \in \mathbb{Z}$, then $\gamma = (-1)^q \text{Id}$. In this case we define $S_{n,c} = \sum_{k=0}^{2q} r_k z^{n-mk/2} \bar{z}^{mk/2} \in \mathcal{QH}_n^{(c)}$ by the conditions $r_0 = \lambda_{[2(n-1)/m]}(c_1 + c_2)$, $r_{2q} = 0$. Note that if $2n/m \notin \mathbb{Z}$, then it follows from Proposition 2.6 that $S_{n,c} = \sum_{0 \leq k \leq 2n/m} r_k z^{n-mk/2} \bar{z}^{mk/2}$ does not contain the term \bar{z}^n either. So the coefficient at \bar{z}^n in $S_{n,c}$ vanishes for all n , and the coefficient at z^n is always equal to $\lambda_{[2(n-1)/m]}(c_1 + c_2)$. Also, denote $\bar{S}_{n,c} \stackrel{\text{def}}{=} TS_{n,c}$; then $\bar{S}_{n,c} = \lambda_{[2(n-1)/m]}(c_1 + c_2)\bar{z}^n + \dots$ and does not contain the term z^n .

Clearly, for c generic the polynomials $S_{n,c}$ and $\bar{S}_{n,c}$ are linearly independent and therefore form a basis in $\mathcal{QH}_n^{(c)}$.

Polynomials $S_{n,c}$ (with a different normalization) were first considered in [10]. The action of Dunkl operators on $\mathcal{QH}_n^{(c)}$ is described as follows:

Proposition 2.15 ([10]). *Let m be even, $n = mq/2 + r$, $q, r \in \mathbb{Z}$, $0 \leq r \leq m/2 - 1$. Then for generic 2-valued function c one has*

$$(2.10) \quad Y(S_{n,c}) = \begin{cases} \left(n - \frac{m}{2}(c_1 + c_2)\right) S_{n-1,c}, & r \neq 1, \\ \left(n - \frac{m}{2}(c_1 + c_2)\right) \left((c_1 + c_2 - q) S_{n-1,c} \right. \\ \quad \left. + (-1)^{mq/2} ((-1)^q c_1 + c_2) \bar{S}_{n-1,c} \right) & r = 1 \end{cases}$$

and

$$(2.11) \quad \bar{Y}(S_{n,c}) = \begin{cases} 0, & r \neq 0, \\ (-1)^{mq/2} \frac{m}{2} ((-1)^q c_1 + c_2) \bar{S}_{n-1,c}, & r = 0. \end{cases}$$

Corollary 2.16. *Conjecture 1.24 is valid for dihedral groups $I_2(m)$ with m even and generic c .*

The polynomial $e_{mk}^{(c)} = S_{mk,c} + \bar{S}_{mk,c}$ is the quasiharmonic invariant from Corollary 1.12.

Corollary 2.17. *Conjecture 1.24 is valid for dihedral groups with even m and generic 2-valued function c .*

Remark 2.18. In view of Corollary 2.17, Conjecture 1.24 for dihedral groups is reduced to the following statement: the kernel of Dunkl operators applied to quasiharmonic invariants is larger than usual exactly when the Harish-Chandra pairing has a non-trivial kernel. We proved this for $c = const.$ (see Corollary 2.14 above).

Explicit (though rather complicated) formulas for the polynomials $S_{n,c}$ can be found in [10]. In particular, Proposition 2.15 implies the following recursive formula for $S_{n,c}$, similar to (2.8):

$$(2.12) \quad S_{n+1,c} = \begin{cases} z S_{n,c}, & r \neq 0, \\ (c_1 + c_2 - q) z S_{n,c} + (-1)^{mq/2} ((-1)^q c_1 + c_2) z \bar{S}_{n,c}, & r = 0. \end{cases}$$

2.5. Multiplication modulo quasiharmonics. The module $M(\mathbf{1})$ is isomorphic to $\mathbb{C}[z, \bar{z}]$, so, it possess a natural algebra structure. Denote by $J_{n,c} \subset \mathbb{C}[z, \bar{z}]$ the ideal generated by the space $\mathcal{QH}_n^{(c)}$ of quasiharmonics. We are going to investigate the multiplication in the algebra $\mathcal{P}_{n,c} = M(\mathbf{1})/J_{n+1,c}$.

We will be using some known facts from the theory of standard Frobenius algebras (the algebra $\mathcal{P}_{n,c}$ is standard Frobenius for c generic).

The necessary statements, as well as notation, are summarized in the Appendix (Section 3).

Lemma 2.19. *For every c and for every $P \in J_{n,c}$ one has $z\bar{z}P \in J_{n+1,c}$. If $c = const.$ is regular and $z\bar{z}P \in J_{n+1,c}$, then $P \in J_{n,c}$.*

Proof. The first statement follows obviously from (2.8) with $c = \text{const.}$ and from (2.12) for m even and general c .

Now apply the map T (exchanging z and \bar{z}) to (2.8):

$$\bar{R}_{n,c} = \begin{cases} \bar{z}\bar{R}_{n-1,c}, & r \neq 0, \\ (c-q)\bar{z}\bar{R}_{n-1,c} + (-1)^{mq}czR_{n-1,c}, & r = 0. \end{cases}$$

Using this together with (2.8) one obtains the relation

$$z\bar{z}R_{n-1,c} = \begin{cases} \bar{z}R_{n,c}, & r \neq 0, \\ \frac{1}{q(q-2c)}(\bar{z}R_{n,c} - (-1)^n cz\bar{R}_{n,c}), & r = 0, \end{cases}$$

which implies the second statement of the lemma. \square

Remark 2.20. It is natural to expect that for m even and regular $c \neq \text{const.}$ the converse of Lemma 2.19 is also true. Numerical examples support this expectation, but at the time of this writing the proof was not known.

Proposition 2.21. *The algebra $\mathcal{P}_{n,c}$ is finite-dimensional for regular $c = \text{const.}$ and for generic $c \neq \text{const.}$ if m is even.*

Corollary 2.22. *Conjecture 1.28 holds for the dihedral group $I_2(m)$.*

Proof of Proposition 2.21. We are going to prove that $M_{2n+1}(\mathbf{1}) \subset J_{n+1,c}$ using induction by n . The base $n = 1$ can be checked immediately. Suppose that for $n-1$ the assertion is proved. It then follows from Lemma 2.19 that $z\bar{z}M_{2n-1}(\mathbf{1}) \subset J_{n+1,c}$. For $c = 0$ the ideal $J_{n+1,0}$ is generated by the polynomials z^{n+1} and \bar{z}^{n+1} ; therefore the generators $\varrho_{n+1,c}$ (for $c = \text{const.}$) and $S_{n+1,c}$ (for generic c and m even) have non-zero coefficient $b(c)$ at z^{n+1} for generic c .

If $c = \text{const.}$, then Theorem 2.10 and Corollary 2.12 imply a stronger statement: the coefficient $b(c)$ is non-zero for every regular c .

Now let $n+1 = mq + r$. If $r \neq 0$, then for regular c one has $z^n \varrho_{n+1,c} = b(c)z^{2n+1} + z\bar{z}\varphi(c)$ with $b(c) \neq 0$, and therefore $z^{2n+1} \in J_{n+1,c}$.

Similarly, $\bar{z}^{2n+1} \in J_{n+1,c}$, so that $M_{2n+1}(\mathbf{1}) \in J_{n+1,c}$ in this case.

If $r = 0$ (that is, $n+1 = mq$), consider the polynomial $\det A_{0q}(c)$ from the proof of Theorem 2.10. It is non-zero for $c = 0$ and therefore non-zero for generic c ($c \neq \text{const.}$ included). Explicit formulas (2.9) also imply that $A_{0q}(c) \neq 0$ for any regular $c = \text{const.}$ So, in both cases there exists a linear combination $\mu \stackrel{\text{def}}{=} \alpha z^n \varrho_{n,c} + \beta \bar{z}^n \bar{\varrho}_{n,c}$ such that $\mu = b(c)z^{2n+1} + z\bar{z}\varphi$ with $b(c) \neq 0$. So, $z^{2n+1} \in J_{n+1,c}$ for $r = 0$, too. Similarly, $\bar{z}^{2n+1} \in J_{n+1,c}$, and the induction is finished. \square

It now follows from Proposition 3.13 that the algebra $\mathcal{P}_{n,c}$ is standard Frobenius. Its degree is $2n$; denote $p_{2n,c}$ as its characteristic polynomial. It is defined up to a constant multiple; we do not fix the normalization unless otherwise stated.

Proposition 2.23. *Let $c = \text{const.}$ For every n there exists a rational function $a_n(c)$ such that $\Delta p_{2n,c} = a_n(c)p_{2(n-1),c}$ (here $\Delta = -\frac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplace operator).*

Proof. Let $u(z, \bar{z})$ be a polynomial of degree $2n-2$. By the definition of the characteristic polynomial, $u \in J_{n,c}$ if and only if $\langle p_{2(n-1),c}, u \rangle = 0$. By Lemma 2.19 for generic $c = \text{const.}$ this is equivalent to $z\bar{z}u \in J_{n+1,c}$, and therefore $0 = \langle p_{2n,c}, z\bar{z}u \rangle = \langle \Delta p_{2n,c}, u \rangle$. So, the annihilators of the polynomials $p_{2(n-1),c}$ and

$\Delta p_{2n,c}$ are the same, and therefore these polynomials are proportional: $\Delta p_{2n,c} = a_n(c)p_{2(n-1),c}$. Here a_n is a rational function of c ; if we choose the normalization of $p_{2n,c}$ for all n as in (3.2), then $a_n(c)$ will actually be a polynomial. \square

For $c = \text{const.}$ we will compute $p_{2n,c}$. Denote $Q_1 = (c-1)z^{2m} + (-1)^m c z^m \bar{z}^m$, and $Q_k = \prod_{i=0}^k (c-i) \cdot z^{2km} + (-1)^{mk+m+k} \prod_{i=-1}^{k-1} (c+i) \cdot z^{m(k+1)} \bar{z}^{m(k-1)}$ for $k \geq 2$.

Lemma 2.24. *Let $n = mq + r$. Then in the notation of (2.5) one has*

$$(2.13) \quad z^{mq} \bar{z}^r R_{n,c} = \sum_{k=0}^{q-2} (-1)^{mk} \binom{q}{k} \prod_{i=0}^{k-1} (c-i) \cdot (z\bar{z})^{mk+r} Q_{q-k} \\ + (-1)^{mq} \prod_{i=0}^{q-1} (c-i) \cdot (z\bar{z})^{m(q-1)+r} Q_1$$

(an empty product for $k = 0$ is assumed to be 1).

Proof. By (2.5) it suffices to prove (2.13) for $r = 0$. We will be using the notation of (2.5), where $\lambda_k(c) \stackrel{\text{def}}{=} c(c-1) \cdots (c-k)$.

The term containing Q_r on the right-hand side is actually equal to

$$(-1)^{mk} \lambda_{k-1}(c) (z\bar{z})^{mk} (\lambda_{q-k}(c) z^{2m(q-k)} \bar{z}^{mk} \\ + (-1)^{m(q-k)+m+q-k} \lambda_{q-k}(c+q-k-1) z^{m(q-k+1)} \bar{z}^{m(q-k-1)}) \\ = (-1)^{mk} \lambda_{k-1}(c) \lambda_{q-k}(c) z^{m(2q-k)} \bar{z}^{mk} \\ + (-1)^{m(q+1)+q-k} \lambda_{k-1}(c) \lambda_{q-k}(c+q-k-1) z^{m(q+1)} \bar{z}^{m(q-1)}$$

(where we abbreviated $r = q - k$), and the Q_1 term is

$$(-1)^{m(q+1)} \lambda_0(c) \lambda_{q-1}(c) z^{mq} \bar{z}^{mq} + (-1)^{mq} \lambda_0(c-1) \lambda_{q-1}(c) z^{m(q+1)} \bar{z}^{m(q-1)}.$$

So, every monomial $z^{m(2q-p)} \bar{z}^{mp}$ with $0 \leq p \leq q$, $p \neq q - 1$ enters the right-hand side of (2.13) exactly once, and the coefficient is the same as on the left-hand side (cf. (2.5)).

The coefficient at $z^{m(q+1)} \bar{z}^{m(q-1)}$ on the left-hand side is equal, by (2.5), to $(-1)^{m(q-1)} \binom{q}{q-1} \lambda_1(c) \lambda_{q-2}(c) = (-1)^{m(q+1)} qc(c-1) \lambda_{q-2}(c)$, while the corresponding coefficient on the right-hand side is

$$\sum_{k=0}^{q-2} (-1)^{m(q+1)+q-k} \binom{q}{k} \lambda_{k-1}(c) \lambda_{q-k}(c+q-k-1) + (-1)^{m(q+1)} (c-1) \lambda_{q-1}(c) \\ = (-1)^{m(q+1)} c(c-1) \sum_{0 \leq k \leq q, k \neq q-1} (-1)^{q-k} \binom{q}{k} \lambda_{q-2}(c+q-k-1).$$

$\lambda_{q-2}(x)$ is a polynomial of degree $q - 1$, so its q -th iterated difference is identically zero:

$$\sum_{k=0}^q (-1)^k \binom{q}{k} \lambda_{q-2}(x - k + 1) = 0.$$

Taking $x = c+q$, we obtain the desired identity for the coefficients of $z^{m(q+1)} \bar{z}^{m(q-1)}$. \square

Proposition 2.25. *For any c one has $z^r \bar{z}^r Q_q \in J_{n+1,c}$.*

Proof. Proposition 3.2 implies that the corollary is true when c belongs to some closed subset of \mathbb{C} ; therefore it is enough to prove it for generic c .

Use induction in q . For $q = 1$ the inclusion can be checked immediately. For bigger q , it follows from the induction hypothesis and Lemma 2.19 that all the terms in (2.13), except $z^r \bar{z}^r Q_q$, are in $J_{n,c}$; therefore, $z^r \bar{z}^r Q_q \in J_{n,c}$, too. \square

Corollary 2.26. *If $c = -1, -2, \dots, -q + 1$, then $z^{2n-r} \bar{z}^r \in J_{n+1,c}$.*

Proposition 2.27. *If $c = 1, \dots, q$, then $z^{n+m+r} \bar{z}^{n-m-r} \in J_{n+1,c}$.*

Proof. Similar to Proposition 2.25 one proves that $z^{2r} Q_q \in J_{n+1,c}$ for any c . \square

Theorem 2.28. *Let $c = \text{const.}$, $n = mq + r$, $q, r \in \mathbb{Z}$, $0 \leq r \leq m - 1$. Then the characteristic polynomial of the ideal $J_{n+1,c}$ is equal to*

$$(2.14) \quad p_{2n,c} = z^{(n)} \bar{z}^{(n)} + \sum_{k=1}^q (-1)^{mk} \frac{c(c+1) \cdots (c+k-1)}{(1-c)(2-c) \cdots (k-c)} z^{(n-mk; n+mk)},$$

where $z^{(p;q)} \stackrel{\text{def}}{=} z^{(p)} \bar{z}^{(q)} + z^{(q)} \bar{z}^{(p)}$.

Proof. Use induction in q ; base $q = 0$. The ideal $J_{n+1,c}$ is $I_2(m)$ -invariant, and therefore $p_{2n,c}$ is $I_2(m)$ -invariant, too. So, $p_{2n,c}$ can be expressed as a polynomial of elementary invariants $e_2 = z\bar{z}$ and $e_m = z^m + \bar{z}^m$. If $q = 0$, then the degree $2n < 2m$, and e_m cannot enter the expression. Therefore $p_{2n,c} = e_2^n$, up to a multiple.

Induction step: now let $q > 0$. Normalize the polynomial $p_{2n,c}$ as in (3.2). (Actually, this means multiplying all the terms of (2.14) by $(1-c) \cdots (q-c)$.) Then it is easy to see that if $r \neq 0$, then the function $a_n(c)$ of Proposition 2.23 is equal to 1, while for $r = 0$ it is a polynomial of degree 1 with leading coefficient 1; that is, $a_n(c) = c - \alpha$. So, for $c = \alpha$ one has $\Delta p_{2n,c} = 0$, and therefore $z\bar{z}u \in J_{n+1,c}$ for any polynomial $u \in \mathbb{C}_{2(n-1)}[z, \bar{z}]$. On the other hand, if $c = \alpha$ is a regular value, then Lemma 2.19 implies that $u \in J_{n,c}$. Therefore, $\mathbb{C}_{2n-2}[z, \bar{z}] \subset J_{n,c}$, which is impossible because $J_{n,c}$ is generated by two polynomials of degree n . So, α is a singular value. Induction hypothesis and Proposition 2.27 show that $\alpha = q$ is the only possibility.

Thus, $a_n(c) = c - q$. This proves (2.14) by induction up to a term P annihilated by the operator Δ . If $r \neq 0$, then this term is zero because of $I_2(m)$ -invariance. If $r = 0$, then it looks like $P = \mu_n(c)(z^{2n} + \bar{z}^{2n})$, where μ_n is a polynomial of degree q . Corollary 2.26 shows that $a_n(c)$ is divisible by $(c+1) \cdots (c+q-1)$.

Now let $c = 0$. In this case $F = \Delta$ by Proposition 2.3, and therefore $J_{n+1,c}$ is invariant under the action of the orthogonal group $\text{SO}(2) = \{\xi \mid |\xi| = 1\} \subset \mathbb{C}^*$. So, $p_{2n,0}$ is $\text{SO}(2)$ -invariant, too, and therefore $p_{2n,0} = z^n \bar{z}^n$ (as usual, up to a multiplicative constant). So, $\mu_n(0) = 0$ and $\mu_n(c) = M_n c(c+1) \cdots (c+q-1)$. Equation (3.2) allows us to determine the normalizing coefficient: $M_n = 1$. \square

For $c \neq \text{const.}$ (and m even) an explicit formula for $p_{2n,c}$ is not yet known.

3. APPENDIX: STANDARD FROBENIUS ALGEBRAS

In this section we collect definitions and theorems about standard Frobenius algebras used elsewhere in the paper. Our main sources are the book [16] and the review [9].

3.1. Main properties and operations.

Definition 3.1. A graded commutative finite-dimensional \mathbb{C} -algebra $A = \bigoplus_{k=0}^N A_k$ with a unit $\mathbf{1} \in A_0$ is called *standard Frobenius* if

- (1) $\dim A_0 = 1$.
- (2) A is generated by the component A_1 .
- (3) The operation $(a, b) \mapsto \text{pr}_N(ab)$ defines a non-degenerate bilinear form on A (here pr_N means taking the homogeneous component of degree N).

A graded commutative algebra possessing properties (1) and (3) only is called Frobenius; the bilinear form mentioned in (3) is called a Frobenius form. Algebras possessing property (2) are called standard. The number N is called a degree of the algebra A ; the number $r = \dim A_1$ is called a rank of A .

Non-degeneracy of the Frobenius form implies symmetry of the Hilbert series of a standard Frobenius algebra: $\dim A_k = \dim A_{N-k}$ for any $k = 0, 1, \dots, N$; in particular, $\dim A_N = 1$.

For each standard Frobenius algebra $A = \bigoplus_{k=0}^N A_k$ one can define its *characteristic polynomial* $p_A \in \mathbb{C}[A_1]$, which appears to encode all the information about the algebra. It follows from property (2) that $A \cong S(A_1)/I$, where $I = \bigoplus_{k \geq 0} I_k$ is a graded ideal. Since $\dim A_N = 1$, the component I_N is a co-dimension 1 subspace of $S^N(A_1)$. Therefore it is a kernel of a linear function $p_A \in (S^N(A_1))^*$. A natural identification $(S^N(A_1))^* = S^N(A_1^*) = \mathbb{C}_N[A_1]$ allows us to consider p_A a polynomial function of degree N on A_1 .

Clearly, p_A is defined up to a scalar multiple; so, strictly speaking, p_A is an element of the projective space $\mathbb{P}(\mathbb{C}_N[A_1])$.

Proposition 3.2. For every $k = 0, \dots, N$ the homogeneous component A_k of the algebra A is equal to $S^N(A_1)/V_k$, where the subspace $V_k \subset S^k(A_1)$ is the common kernel of all the partial derivatives of order $N - k$ of the polynomial p_A .

Proof. Let $P \in S^k(A_1)$. Since the Frobenius form on A is non-degenerate, one has $P \in J_k$ if and only if $PQ \in J_N$ for all $Q \in S^{N-k}(A_1)$. By the definition of $p_A \in (S^k(A_1))^*$, this is equivalent to $\langle p_A, PQ \rangle = 0$. Fix a basis $x_1, \dots, x_r \in A_1$. Then $P \in J_k$ if and only if for all $\alpha_1, \dots, \alpha_r \in \mathbb{Z}_{\geq 0}$ such that $\alpha_1 + \dots + \alpha_r = N - k$ one has $0 = \langle p_A, x_1^{\alpha_1} \dots x_r^{\alpha_r} P \rangle = \langle \frac{\partial^{N-k} p_A}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}}, P \rangle$. □

This result can be reformulated in a more elegant coordinate-free manner. To do this, consider the action of the symmetric algebra $S(V)$ on the algebra of polynomial functions $\mathbb{C}[V]$ by the differential operators with constant coefficients. This action is graded: $S^m(V) \times \mathbb{C}_n[V] \rightarrow \mathbb{C}_{n-m}[V]$ for any $n \geq m$.

Proposition 3.3. Let $A = \bigoplus_{k=0}^N A_k$ be a standard Frobenius algebra with the characteristic polynomial $p = p_A$. Then the dual space $A^* \subset \mathbb{C}[A_1]$ is a cyclic $S(A_1)$ -module $S(A_1)(p)$. More precisely, $A_k^* = S^k(V)(p)$ for all $k = 0, \dots, N$.

Corollary 3.4. For any $v \in A_1$ the directional derivative $\frac{\partial p_A}{\partial v}$ does not vanish identically. In other words, the partial derivatives $\frac{\partial p_A}{\partial x_1}, \dots, \frac{\partial p_A}{\partial x_r}$ are linearly independent.

This follows from the equality $\dim A_1 = r$. In other words, the polynomial p_A depends essentially on r variables and cannot be written as a polynomial of a

smaller number of variables. We denote by $\mathbb{C}_N^0[V]$ the set of all $p \in \mathbb{C}_N[V]$ having this property. Clearly, $\mathbb{C}_N^0[V]$ is an open dense $GL(V)$ -invariant subset of $\mathbb{C}_N[V]$.

Corollary 3.5. *As a graded vector space the algebra A is isomorphic to the set of all partial derivatives of the polynomial p_A .*

Theorem 3.6. *A graded isomorphism $f : A \rightarrow B$ of standard Frobenius algebras carries p_A to p_B . For any polynomial $p \in \mathbb{C}_N^0[V]$ there exists a unique standard Frobenius algebra A of degree N such that $A_1 = V$ and $p_A = p$.*

Proof. The first assertion follows immediately from the definition of p_A . The uniqueness in the second assertion is a consequence of Proposition 3.2.

We now prove the existence of a standard Frobenius algebra with a given $p_A = p$. Call an algebra *standard almost Frobenius* if it possesses properties (1) and (2) from Definition 3.1. Note that for each standard quasi-Frobenius algebra \tilde{A} the radical $\text{rad}(\tilde{A}) = \{v \in \tilde{A} \mid (v, w) = 0 \forall w \in \tilde{A}\}$ of its Frobenius form (\cdot, \cdot) is a graded ideal. One can easily show that $\text{rad}(\tilde{A})$ is the maximal graded ideal in \tilde{A} not intersecting the highest degree component \tilde{A}_N .

Now let $p \in \mathbb{C}_N^0[V]$. Let $I_N \subset S^N(V)$ be the kernel of the corresponding linear function $S^N(V) \rightarrow \mathbb{C}$. Denote $J \stackrel{\text{def}}{=} I_N \oplus_{k>N} S^k(V)$. Clearly, this is an ideal in $S(V)$. Denote $\tilde{A} \stackrel{\text{def}}{=} S(V)/J$ and define a bilinear form on this algebra by the formula $(v, w)_{\tilde{A}} = \langle p, uv \rangle$.

Clearly, \tilde{A} is a standard quasi-Frobenius algebra. We define the algebra A as the quotient: $A \stackrel{\text{def}}{=} \tilde{A}/\text{rad}(\tilde{A})$. An easy check shows that A is standard Frobenius and $p_A = p$. Since $p \in \mathbb{C}_N^0[V]$, the ideal $\text{rad}(\tilde{A})$ does not intersect A_1 , and the equality $A_1 = V$ follows. □

Remark 3.7. This result implies that standard Frobenius algebras are in 1-1 correspondence with their characteristic polynomials modulo linear changes of variables. In other words, the isomorphism classes of degree N standard Frobenius algebras generated by $A_1 = V$ are parametrized by the points of the orbit space $\mathbb{P}\mathbb{C}_N^0[V]/GL(V)$, where $\mathbb{P}\mathbb{C}_N^0[V]$ is a dense subset of $\mathbb{P}(\mathbb{C}_N[V])$ corresponding to $\mathbb{C}_N^0[V]$.

One can define several operations over the standard Frobenius algebras. One of them is the tensor product:

Proposition 3.8. *The tensor product of standard Frobenius algebras A and B has a natural structure of a standard Frobenius algebra with the characteristic polynomial $p_{A \otimes B} = p_A \otimes p_B$.*

Proof. If $A = \bigoplus_{k=0}^N A_k$, $B = \bigoplus_{l=0}^M B_l$, then the tensor product $C = A \otimes B$ is a graded algebra: $C = \bigoplus_{r=0}^{M+N} C_r$, where

$$C_r = \bigoplus A_k \otimes B_{r-k}.$$

The algebra C is generated $C_1 = A_1 \otimes B_0 \oplus A_0 \otimes B_1$. Also,

$$C_{M+N} = (C_1)^{M+N} = (A_1 \otimes B_0 \oplus A_0 \otimes B_1)^{M+N} = A_N \otimes B_M.$$

It remains to prove that the bilinear form $(\cdot, \cdot)_C$ is non-degenerate. Indeed, for any $a, a' \in A, b, b' \in B$,

$$\begin{aligned} (a \otimes b, a' \otimes b') &= \text{pr}_{C_{M+N}}(a \otimes b)(a' \otimes b') = \text{pr}_{C_{M+N}}(aa' \otimes bb') \\ &= \text{pr}_{A_N \otimes B_M}(aa' \otimes bb') = \text{pr}_{A_N}(aa') \text{pr}_{B_M}(bb') = (a, a')(b, b'); \end{aligned}$$

that is, $(\cdot, \cdot)_{A \otimes B} = (\cdot, \cdot)_A \otimes (\cdot, \cdot)_B$, which implies non-degeneracy and the equality for characteristic polynomials. \square

Another important operation is the internal product:

Definition 3.9. Let $A = \bigoplus_{k=0}^N A_k$ be a standard Frobenius algebra and $U \subset A_1$ be a subspace such that $U^N = A_N$. Define the *restriction* $A|_U$ of A to U to be the quotient $\langle U \rangle / \mathcal{R}$, where $\langle U \rangle \subset A$ is the subalgebra generated by U and \mathcal{R} is the radical of the restriction of the Frobenius form (\cdot, \cdot) to $\langle U \rangle$.

Definition 3.10. Let A and B be standard Frobenius algebras such that $A_1 = B_1 = V$. Define the *internal product* $A * B$ of the algebras as the restriction $A \otimes B|_{\Delta(V)}$, where $\Delta(V) = \{v \otimes 1 + 1 \otimes v \mid v \in V\} \subset (A \otimes B)_1$ is the diagonal.

The following proposition is obvious:

Proposition 3.11. *Characteristic polynomials of standard Frobenius algebras satisfy the identities $p_{A|_U} = p_A|_U$ and $p_{A*B} = p_A p_B$.*

Corollary 3.12. *The internal product is a monoidal operation on the category of standard Frobenius algebras generated by a given space V .*

In other words, $A * (B * C) = (A * B) * C$ and $A * E = E * A = A$, where E is a trivial one-dimensional algebra.

3.2. Free standard Frobenius algebras. Recall from [16] that a graded zero-dimensional complete intersection cone in \mathbb{C}^ℓ is a finite-dimensional quotient of $\mathbb{C}[x_1, \dots, x_\ell]$ by an ideal J generated by ℓ homogeneous polynomials P_1, \dots, P_ℓ .

Proposition 3.13 ([16]). *Each graded zero-dimensional complete intersection cone in \mathbb{C}^ℓ is a standard Frobenius algebra of degree $\deg A = \sum_{i=1}^\ell (\deg P_i - 1)$ and total dimension $\dim_{\mathbb{C}} A = \prod_{i=1}^\ell \deg P_i$.*

For the sake of simplicity, based on Proposition 3.13, we will refer to graded zero-dimensional complete intersection cones as *free standard Frobenius algebras*. Borrowing a term from the Invariant Theory, we call the numbers $m_i = \deg(P_i) - 1$ *exponents* of the algebra A . It is easy to see that for each free standard Frobenius algebra A the elements P_1, \dots, P_ℓ are algebraically independent. The Hilbert polynomial $h_A(t) = \sum_{k=0}^N \dim A_k t^k$ of A is

$$(3.1) \quad h_A(t) = \prod_{i=1}^{\ell} \frac{1 - t^{m_i+1}}{1 - t}.$$

Here are some examples of free standard Frobenius algebras.

Example 3.14. We say that a standard Frobenius algebra A is monomial if there exists a basis e_1, \dots, e_ℓ in A_1 such that $p_A = e_1^{m_1} e_2^{m_2} \cdots e_\ell^{m_\ell}$ for some integers $1 \leq m_1 \leq m_2 \leq \cdots \leq m_\ell$ such that $m_1 + m_2 + \cdots + m_\ell = N$.

Each monomial standard Frobenius algebra A with the characteristic polynomial p_A of this form is free with the exponents m_1, m_2, \dots, m_ℓ . Moreover, $A \cong \bigotimes_{i=1}^\ell A(m_i)$, where $A(m) \stackrel{\text{def}}{=} \mathbb{C}[x]/(x^{m+1})$.

Example 3.15. Another example of a free standard Frobenius algebra is the algebra of coinvariants of a reflection group. Let W be a finite reflection group of a space V . Denote by I_W the ideal in $S(V)$ generated by all homogeneous W -invariant elements of positive degrees. It is well-known (see [9]) that the algebra $A_W = S(V)/I_W$ of coinvariants is standard Frobenius.

Let $\Delta = \Delta_W \in \mathbb{C}[V]$ be the polynomial skew-invariant of W of the smallest degree. Clearly, Δ is unique up to a constant multiple and factors into the product of all positive roots of W . If W is irreducible, then the degree of Δ is equal to $|\mathcal{S}| = h\ell/2$, where h is the Coxeter number of W and \mathcal{S} is the set of all reflections in W (see Section 1.1).

Proposition 3.16 ([9]). *The coinvariant algebra A_W is free standard Frobenius with the characteristic polynomial Δ .*

Example 3.17. Finite-dimensional quotients of the module $M(\mathbf{1})$ over a rational Cherednik algebra $H_c(W)$ (see Section 1.1 above for notation) are free standard Frobenius algebras. According to [2], if one identifies $M(\mathbf{1}) = S(V)$ and considers the module in question as a quotient algebra, then this algebra is free standard Frobenius. See Section 2.5 for a complete description of this algebra for $W = I_2(m)$.

3.3. Rank 2 case. We refine Proposition 3.13 when $\dim A_1 = 2$ as follows.

Proposition 3.18 ([16]). *Let A be a quotient algebra of $\mathbb{C}[x_1, x_2]$ by a homogeneous ideal J . Then the following are equivalent:*

- (1) A is standard Frobenius.
- (2) A is free standard Frobenius.
- (3) The ideal J is generated by two coprime homogeneous polynomials R_1, R_2 .

For such algebras it is possible to write an explicit expression for the characteristic polynomial provided R_1, R_2 are known.

Let $A = \mathbb{C}[x_1, x_2]/J$ be a standard Frobenius algebra and the ideal J be generated by coprime polynomials $R_1, R_2 \in \mathbb{C}_{n+1}[x_1, x_2]$. Then by Proposition 3.13 the degree of A is $N = 2n$, and the component J_N of the ideal J is spanned by all the polynomials $x_1^\alpha x_2^\beta R_i$, where $\alpha + \beta = n - 1$ and $i = 1, 2$. If $R_1 = \sum_{i=0}^{n+1} a_i x_1^i x_2^{n+1-i}$ and $R_2 = \sum_{i=0}^{n+1} b_i x_1^i x_2^{n+1-i}$, then the functional $p_A : \mathbb{C}_{2n}[x_1, x_2] \rightarrow \mathbb{C}$ should vanish on all the polynomials $\sum_{i=0}^{n+1} a_i x_1^{i+\alpha} x_2^{2n-i-\alpha}$ and $\sum_{i=0}^{n+1} b_i x_1^{i+\alpha} x_2^{2n-i-\alpha}$, where $\alpha = 0, \dots, n - 1$. It means that

$$(3.2) \quad p_A = \sum_{i=0}^{2n} u_i y_1^{(i)} y_2^{(2n-i)},$$

where $a^{(b)} \stackrel{\text{def}}{=} a^b/b!$ and u_i is the minor of the $(2n+1) \times (2n)$ matrix U composed of the coefficients of these polynomials:

$$U = \begin{pmatrix} a_0 & a_1 & \dots & a_n & a_{n+1} & 0 & \dots & 0 \\ 0 & a_0 & \dots & a_{n-1} & a_n & a_{n+1} & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_{n+1} \\ b_0 & b_1 & \dots & b_n & b_{n+1} & 0 & \dots & 0 \\ 0 & b_0 & \dots & b_{n-1} & b_n & b_{n+1} & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & b_0 & b_1 & b_2 & \dots & b_{n+1} \end{pmatrix}$$

obtained by the deletion of its $(i+1)$ -th column.

Let $p = \sum_{i=0}^n c_i x_1^{(i)} x_2^{(n-i)}$ be a homogeneous polynomial (a binary form) of degree N . Define the matrix $H_k(p)$ by $(H_{k,n-k}(p))_{ij} = c_{i+j}$ for $i = 0, 1, \dots, k$, $j = 0, 1, \dots, n-k$ (see e.g. [17]). Clearly, the transposed of $H_{n-k,k}(p)$ is $H_{k,n-k}(p)$ for all $k \leq n$.

Proposition 3.19. *Let A be a rank 2 standard Frobenius algebra with the characteristic polynomial p , and let $k \leq \frac{n}{2}$. Then $\dim A_k = k+1$ if and only if the matrix $H_{k,n-k}(p)$ has the maximal rank $k+1$.*

The proof follows from (3.2) and Proposition 3.2.

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