

STRONG UNIQUE CONTINUATION FOR PRODUCTS OF ELLIPTIC OPERATORS OF SECOND ORDER

FERRUCCIO COLOMBINI AND HERBERT KOCH

ABSTRACT. We study strong unique continuation for products of elliptic operators. The main tools are Carleman inequalities for second order elliptic operators. We obtain strong unique continuation assuming either Gevrey regularity of the coefficients or some pointwise conditions on the coefficients.

1. INTRODUCTION

Solutions to the scalar second order differential inequalities

$$(1) \quad \left| \sum_{i,j=1}^n a^{ij}(x) \partial_{ij}^2 u \right| \leq c(|u| + |\nabla u|)$$

have the *Strong Unique Continuation Property* if the coefficients a^{ij} are real, uniformly positive definite, and Lipschitz continuous: If u satisfies (1), if it is flat at the origin in the sense that

$$(2) \quad \sup_{r \leq \delta} r^{-N} \|u\|_{L^2(B(0,r))} < \infty$$

for all N , then u vanishes near the origin.

Building on the seminal work by Carleman [5], N. Aronszajn, A. Krzywicki, and J. Szarski [3] have obtained this basic result using Carleman inequalities. By now the understanding of the scalar second order case is fairly complete, and we refer to [10] for a more detailed exposition of its history.

In this note we study conditions on the differential equations

$$Pu = \sum_{|\beta| \leq 2m} c_\beta(x) \partial^\beta u$$

which ensure strong unique continuation. For simplicity we consider products of second order operators $P = A_M \dots A_1$, where (using the Einstein convention)

$$A_j u = a_{kl}^j(x) \partial_{kl}^2 u.$$

We assume that the coefficients a_{kl}^j are in the Gevrey class G^σ (see Section 2) and that there exists $\alpha > 0$ such that the eigenvalues $\nu_1 \leq \nu_2 \leq \dots \leq \nu_n$ of $a_{kl}^j(x)$ satisfy

$$(3) \quad \alpha > \frac{\nu_n - \nu_1}{\nu_1}$$

uniformly in x and j .

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Theorem 1. *Suppose that the operators A_k and α satisfy (3) at $x = 0$ (where we replace A by A_k). Let $\sigma < 1 + 1/\alpha$. Then the Strong Unique Continuation Property holds for the differential equation*

$$Pu = \sum_{|\beta| \leq [3M/2]} a_\beta \partial^\beta u$$

provided the coefficients are in the Gevrey class G^σ .

The case of higher order operators is very different from the situation in (1). In general strong unique continuation does not hold (see Alinhac [1]); however a strengthened flatness assumption like (4) below sometimes implies vanishing.

The proof of the theorem consists of two parts. First Gevrey regularity of the elliptic equation implies Gevrey regularity of the solution (see Lerner [12] and Section 2). Together with the vanishing of infinite order this implies

$$(4) \quad |u| \lesssim e^{-|x|^{-s}} \text{ near } x = 0 \text{ and for some } s > 0.$$

In a second step Carleman inequalities show that (4) implies vanishing near the origin.

The situation improves for powers of the Laplacian, where one can prove Carleman inequalities by applying more standard Carleman inequalities to each factor. This line of approach has been taken in [6] and [2] for the differential inequalities

$$|\Delta^m u| \leq \sum_{j=0}^m c_j |x|^{j-2m} |D^j u|$$

for c_m sufficiently small. It has been improved by [11] for $m = 2$ to the same conclusion for the inequality

$$(5) \quad |\Delta^2 u| \leq c_0 |x|^{-4} |u| + c_1 |x|^{-3} |Du| + c_2 |x|^{-2} |D^2 u| + c_3 |x|^{-1+\varepsilon} |D^3 u|,$$

where again c_2 is small.

In this note we generalize the result of [8] to operators of all orders, and we allow derivatives on the right hand side. We also improve upon the ε in (5) and generalize the result of [11] to products of elliptic operators of second order for which $a_{kl}^j(0) = \delta_{kl}$. At the same time we provide a unified proof for Theorem 1 and for the generalization of [11].

Let u satisfy a differential equality or inequality. Weak unique continuation is the implication: Vanishing in an open set implies vanishing. The book of Zuily [14] gives an interesting and detailed survey of this theory; it is instructive but beyond the scope of this paper to compare the results on strong and weak unique continuation. In view of Watanabe [13] one might conjecture that solutions to

$$|\Delta^3 u + \sum_{|\alpha|=5} c_\alpha \partial^\alpha u| \leq \sum_{j \leq 4} c_j |D^j u|$$

with Lipschitz continuous functions c_α have the strong unique continuation property.

The paper at hand implements the procedure above in a fairly straightforward manner. All constructions rely on the following observations: The weight $|x|^\tau$ is only degenerate pseudoconvex. We replace it by $|x|^{h_\tau(-\ln|x|)}$ with convex functions h_τ . Linear functions h allow us to control one derivative for the second order operator and M derivatives for a product of M second order operators—provided

the coefficients are δ^{ij} at $x = 0$. Convexity of h allows us to control perturbations from the Laplacian and $M/2$ more derivatives. The amount of convexity of h however is limited by the at most linear growth translating to polynomial vanishing for u . We may use more convexity at the expense of requiring more decay as in (4).

2. GEVREY REGULARITY FOR ELLIPTIC OPERATORS

Definition 2.1. Let U be an open set, $f \in C^\infty(U)$, and $\sigma > 1$. We say that f lies in the Gevrey class \mathcal{G}^σ if there exist constants c and A so that

$$|\partial^\alpha f| \leq cA^{|\alpha|} |\alpha|!^\sigma \quad \text{in } U.$$

It is not hard to see that we get the same space if we replace $|\alpha|!^\sigma$ by $(\alpha!)^\sigma$ or by $|\alpha|^{\sigma|\alpha|}$.

Lemma 2.2. Suppose that $0 \in U$, $\sigma \geq 1$, and that $f \in \mathcal{G}^\sigma(U)$ satisfies

$$\partial^\alpha f(0) = 0 \quad \text{for all multiindices } \alpha.$$

Let $\sigma - 1 < \rho$. Then

$$|f(x)| \leq e^{-|x|^{-1/\rho}}$$

near $x = 0$.

This estimate follows from Taylor’s formula by optimizing the order. See [12] for a complete proof. This decay result is fairly sharp because the following example can be checked by a straightforward calculation, which again can be found in [12].

Lemma 2.3. We have

$$e^{-|x|^{-1/\rho}} \in \mathcal{G}^\sigma(\mathbb{R})$$

provided $1 + \rho = \sigma$.

Gevrey classes are good classes for elliptic regularity; see [4], Proposition 2.13:

Lemma 2.4. Let

$$P(x, D)u = f \quad \text{in } U$$

be an elliptic differential equation with coefficients and right hand side in the Gevrey class $\mathcal{G}^\sigma(U)$. Then $u \in \mathcal{G}^\sigma(V)$ for all bounded $V \subset\subset U$.

3. CARLEMAN INEQUALITIES FOR THE WEIGHT $e^{\tau|x|^{-\alpha}}$
FOR ELLIPTIC OPERATORS OF SECOND ORDER

Let

$$A = \sum_{i,j=1}^n a^{ij}(x) \partial_i \partial_j$$

be an elliptic operator with real Lipschitz continuous coefficients. The following Carleman inequality has been proven in [8]:

Proposition 3.1. Suppose that the coefficients of A are constant and positive definite, and denote the eigenvalues by $\nu_1 \leq \nu_2 \cdots \leq \nu_n$. We assume that $\alpha > 0$ in (3). There exist $\tau_0 > 0$ and $r_0 > 0$ such that for $\tau > \tau_0$ the following inequality holds for all smooth v supported in $B_{r_0}(0) \setminus \{0\}$:

$$\tau^{-1/2} \|D^2(|x|^{\alpha/2} e^{\tau|x|^{-\alpha}} v)\|_{L^2} + \tau^{3/2} \| |x|^{-2-3\alpha/2} e^{\tau|x|^{-\alpha}} v \|_{L^2} \lesssim \|e^{\tau|x|^{-\alpha}} Av\|_{L^2}.$$

Its most interesting feature is the sharp inequality (3). Checking the proof, it is not hard to see that the inequality remains true for operators

$$Av = \sum_{i,j} \partial_i a^{ij}(x) \partial_j$$

with uniformly elliptic Lipschitz coefficients which satisfy (3) near $x = 0$ and for functions supported in a small punctured disc.

We easily obtain a slightly stronger consequence: Under the same assumptions the estimate

$$(6) \quad \sum_{j=0}^2 \tau^{3/2-j} \| |x|^{\alpha/2} |x|^{(j-2)(1+\alpha)} D^j e^{\tau|x|^{-\alpha}} v \|_{L^2} \lesssim \| e^{\tau|x|^{-\alpha}} Av \|_{L^2}$$

holds. To see this, we set $w = e^{\tau|x|^{-\alpha}} v$ and write

$$|x|^{\alpha/2} D^2 w = D^2(|x|^{\alpha/2} w) + O(|x|^{\alpha/2-1}) Dw + O(|x|^{\alpha/2-2}) w,$$

thus

$$\| |x|^{\alpha/2} D^2 w \|_{L^2} \leq \| D^2(|x|^{\alpha/2} w) \|_{L^2} + c \| |x|^{\alpha/2-2} w \|_{L^2} + c \| |x|^{\alpha/2-1} Dw \|_{L^2},$$

and by integration by parts for the first order derivatives

$$\| |x|^{\alpha/2} D^2 w \|_{L^2} \leq \| D^2(|x|^{\alpha/2} w) \|_{L^2} + c \| |x|^{\alpha/2-2} w \|_{L^2}.$$

We combine this with the estimate of Proposition 3.1 to get

$$\tau^{-1/2} \| |x|^{\alpha/2} D^2 w \|_{L^2} + \tau^{3/2} \| |x|^{-2-3\alpha/2} w \|_{L^2} \lesssim \| e^{\tau|x|^{-\alpha}} Av \|_{L^2}.$$

We can sharpen the estimate from the integration by parts to

$$(7) \quad \tau^{1/2} \| |x|^{\alpha/2} |x|^{-1-\alpha} Dw \|_{L^2} \leq C \left(\tau^{3/2} \| |x|^{\alpha/2} |x|^{-2(1+\alpha)} w \|_{L^2} + \tau^{-1/2} \| |x|^{\alpha/2} D^2 w \|_{L^2} \right),$$

which completes the proof of inequality (6).

It will be useful in the sequel to commute the derivatives and the exponential. We compute

$$|x|^{\alpha/2} |x|^{-1-\alpha} e^{\tau|x|^{-\alpha}} Dv = |x|^{\alpha/2} |x|^{-1-\alpha} Dw + \alpha \tau |x|^{\alpha/2} |x|^{-2-2\alpha} |x|^{-1} w x$$

and hence

$$\tau^{1/2} \| |x|^{\alpha} |x|^{-1-\alpha} e^{\tau|x|^{-\alpha}} Dv \|_{L^2} \lesssim \| e^{\tau|x|^{-\alpha}} Av \|_{L^2}.$$

Similarly we can deal with the second order derivative. We obtain the Carleman inequality in the equivalent form

$$(8) \quad \sum_{j=0}^2 \tau^{3/2-j} \| e^{\tau|x|^{-\alpha}} |x|^{\alpha/2} |x|^{(j-2)(1+\alpha)} D^j v \|_{L^2} \lesssim \| e^{\tau|x|^{-\alpha}} Av \|_{L^2}.$$

In the sequel we also need higher order derivatives and different powers in the estimate.

Corollary 3.2. *Let*

$$Au = \sum_{i,j} \partial_i a^{ij}(x) \partial_j u$$

with a^{jj} positive definite and continuous. Suppose that (3) holds at $x = 0$. Let $\sigma > 0$, and assume that the coefficients are of class C^σ . Then there exists τ_0 such that for all $|s|, k \leq \sigma$, and $\tau \geq \tau_0$

$$(9) \quad \begin{aligned} & \sum_{j=0}^{k+2} \tau^{3/2-j} \| |x|^{\alpha/2+j(1+\alpha)} |x|^s e^{\tau|x|^{-\alpha}} D^j v \|_{L^2} \\ & \lesssim \sum_{j=0}^k \tau^{-j} \| |x|^{(2+j)(1+\alpha)} |x|^s e^{\tau|x|^{-\alpha}} D^j A v \|_{L^2}. \end{aligned}$$

Proof. We apply estimate (8) to the functions $u = |x|^{s+(2+k)(1+\alpha)} \partial^\beta v$ with $|\beta| = k$, and we obtain

$$\begin{aligned} & \sum_{j=0}^2 \tau^{3/2-j-k} \| |x|^{\alpha/2} |x|^{(j-2)(1+\alpha)} e^{\tau|x|^{-\alpha}} D^j (|x|^{s+(2+k)(1+\alpha)} \partial^\beta v) \|_{L^2} \\ & \lesssim \tau^{-k} \| e^{\tau|x|^{-\alpha}} A |x|^{s+(2+k)(1+\alpha)} \partial^\beta v \|_{L^2}. \end{aligned}$$

We sum over β with $|\beta| = k$. On the right hand side we commute A and $|x|^{s+(2+k)(1+\alpha)} D^k$, which yields commutator terms which are controlled by the left hand side. On the left hand side we commute D^j and $|x|^{s+(2+k)(1+\alpha)}$, where the commutator terms are controlled by induction on k . \square

4. PRODUCTS OF ELLIPTIC OPERATORS

We study

$$P = A_M \dots A_1.$$

Let $(\nu_{jk})_{1 \leq j \leq n}$ be the eigenvalues of the coefficients of the k th operator at $x = 0$. We assume that (3) is satisfied for all j . Corollary 3.2 is applied iteratively. Let $v_l = A_l \dots A_1 v$. Then

$$(10) \quad \begin{aligned} & \sum_{j=0}^{2M} \tau^{3M/2-j} \| |x|^{M(-2-3\alpha/2)} |x|^{j(1+\alpha)} e^{\tau|x|^{-\alpha}} D^j v \|_{L^2} \\ & \lesssim \sum_{j=0}^{2M-2} \tau^{3(M-1)/2-j} \| |x|^{(1-M)(2+3\alpha/2)} |x|^{j(1+\alpha)} e^{\tau|x|^{-\alpha}} D^j v_1 \|_{L^2} \\ & \text{by Corollary 3.2 with } k = 2(M-1) \text{ and } s = -M(2 + \frac{3}{2}\alpha) - \alpha/2 \\ & \lesssim \sum_{j=0}^4 \tau^{3-j} \| |x|^{-4-3\alpha} |x|^{j(1+\alpha)} e^{\tau|x|^{-\alpha}} D^j v_{M-2} \| \end{aligned}$$

after $M-1$ applications of Corollary 3.2

$$\lesssim \sum_{j=0}^2 \tau^{3/2-j} \| |x|^{-2-3\alpha/2} |x|^{j(1+\alpha)} e^{\tau|x|^{-\alpha}} D^j v_{M-1} \|$$

by Corollary 3.2 with $k = 2$ and $s = -4 - 7\alpha/2$

$$\lesssim \| e^{\tau|x|^{-\alpha}} v_M \|$$

by Corollary 3.2 with $k = 0$ and $s = -2(1 + \alpha)$.

Proof of Theorem 1: The operator is hypoelliptic in the Gevrey class σ . Thus vanishing of infinite order implies that

$$|u| \lesssim e^{-|x|^{-\beta}}$$

for some $\beta > \alpha$ and small arguments. Now the Carleman inequality (10) implies vanishing by standard arguments. \square

A related result about weak unique continuation can be found in Zuily [14], Theorem 2.5.1.

5. PRODUCTS OF LAPLACIANS

Here we study products of operators

$$A_l = a_l^{ij} \partial_{ij}^2$$

with

$$(11) \quad a_l^{ij}(0) = \delta^{ij}$$

and

$$(12) \quad a_l^{ij} \in C^{2(M-l)}$$

for $1 \leq l \leq M$. We study again the Strong Unique Continuation Property, but this time we want to assume only vanishing of infinite order and not the stronger exponential bound.

Theorem 2. *Suppose that the operators A_l satisfy (11). Let $C > 0$. Then there exists $\delta > 0$ such that the Strong Unique Continuation Property holds for the differential inequality*

$$\begin{aligned} |A_M A_{M-1} \dots A_1 u| &\leq \sum_{j=0}^{M-1} C |x|^{j-2m} |D^j u| + \delta |x|^{-M} |D^M u| \\ &\quad + C \sum_{j=M+1}^{[3M/2]} |\ln |x||^{M-j-\varepsilon} |x|^{j-2m} |D^j u| \end{aligned}$$

if $\varepsilon > 0$.

Given $1 \ll \tau$, we use weight functions $e^{h(-\ln(|x|))}$ with

$$(13) \quad \tau \sim h',$$

i.e., there exists $C > 1$ with $C^{-1}\tau \leq h' \leq C\tau$,

$$(14) \quad \frac{1}{4} \leq h'' + d(2h', \mathbb{Z}),$$

$$(15) \quad |h'''| \lesssim (h'' + 1) \lesssim \tau.$$

Such functions are used to dominate error terms in the Carleman inequalities. They can be constructed as follows. Let $(c_j)_{j \in \mathbb{Z}}$ be a sequence of nonnegative numbers with

$$\sum c_j < \delta$$

for a small number δ which will be independent of τ . Let (a_j) be the discrete convolution of (c_j) and $2^{-\mu|j|}$ for some small $\mu > 0$. Then $c_j \leq a_j$ and $2^{-\mu} a_{j+1} \leq a_j \leq 2^\mu a_{j+1}$ for all j , a property which we call slowly varying. We follow the construction of Lemma 6.1 of [10] which deals with a slightly more complicated

situation. We define a function h with $h(0) = 0$, $h'(-\infty) = [\tau] + \frac{5}{4}$ ($[\cdot]$ the largest integer smaller than τ),

$$h'' = \sum b_j \chi_{[j, j+1]}$$

where the b_j are integers satisfying

$$\begin{aligned} b_j &= 0 && \text{if } a_j \leq \mu\tau^{-1}, \\ b_j &\in [\tau a_j, 2\tau a_j] && \text{if } a_j \geq \mu\tau^{-1}. \end{aligned}$$

This function h has almost all the desired properties if δ and μ are sufficiently small, and we obtain the missing statement (15) by a standard mollification.

Proposition 5.1. *Suppose that A is a second order elliptic operator as above with Lipschitz continuous coefficients, and let $1 \ll \tau$. Then there exists h which satisfies (13), (14), (15) and*

$$(16) \quad \sum_{j=0}^2 \tau^{1-j} \|(1 + h'')^{\frac{1}{2}} |x|^{-2+j} e^h D^j v\| \lesssim \|e^h A v\|$$

for all v supported in $B(0, 1) \setminus \{0\}$ with a constant in inequality (16) independent of τ and h .

Proof. The result is contained in [10], and it is a small modification of classical arguments. Since the argument is elementary, we include it for completeness, while explicitly disclaiming originality. We may assume that $|x|^{-1}(1 + h''(-\ln(|x|))^{\frac{1}{2}})$ is large. Then first order terms can be controlled, and it suffices to prove the estimate for operators in divergence form, $Au = \partial_i(a^{ij}(x)\partial_j u)$.

We prove the estimate parallel for the Laplace operator and in the general case. Let $t = -\ln|x|$, $y = x/|x| \in \mathbb{S}^{n-1}$. Then

$$|x|^{\frac{n+2}{2}} \Delta |x|^{\frac{2-n}{2}} = \partial_t^2 + \Delta_{\mathbb{S}^{n-1}} - \left(\frac{n-2}{2}\right)^2 = L_0,$$

and, in symbolic notation,

$$L = |x|^{\frac{n+2}{2}} A |x|^{\frac{2-n}{2}} = L_0 + DbD + Dc_1 + c_2D + d,$$

where the operator on the right hand side is expressed in the $\mathbb{S}^{n-1} \times \mathbb{R}$ coordinates. The coefficients b, c_1, c_2 , and d are bounded by $C_0 e^{-t}$.

We recall that by the chain rule $\sum_{j=0}^l |x|^j |D_x^j u| \sim \sum_{j=0}^l |D_{t,y}^j u|$, and the Jacobian determinant leads to

$$\int f dx = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} e^{nt} f dt dy.$$

Thus the desired inequality is equivalent (modulo adding the half of an integer to h') to

$$(17) \quad \sum_{j=0}^2 \tau^{1-j} \|(1 + h'')^{1/2} e^{h(t)} D_{t,y}^j v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|e^{h(t)} L v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

and, repeating previous arguments and using $h^{(j)} \lesssim \tau$ for $j = 1, 2$, to

$$(18) \quad \sum_{j=0}^2 \tau^{1-j} \|(1 + h'')^{1/2} D_{t,y}^j v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|L_h v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

where L_h is the conjugated operator

$$L_h = e^h L e^{-h}.$$

For simplicity we do the calculation first for L_0 :

$$L_{0,h} = e^h L_0 e^{-h} = L_0 + (h')^2 - (\partial_t h' + h' \partial_t).$$

We denote $L_{0,h}^s = L_0 + (h')^2$ and $L_{0,h}^u = -(\partial_t h' + h' \partial_t)$. Then

$$\|L_{0,h} v\|_{L^2}^2 = \|L_{0,h}^s v\|_2^2 + \|(\partial_t h' + h' \partial_t) v\|^2 - ([L_{0,h}^u, L_{0,h}^s] u, u)$$

where

$$([L_{0,h}^u, L_{0,h}^s] u, u) = -4 \int h' h'' h' u^2 + h'' (\partial_t u)^2 - h^{(4)} u^2 dy dt.$$

The last term is negligible. We obtain the desired estimate for u in the case of the Laplacian. First and second order terms are controlled by elliptic estimates on annuli using the bounds for u and $\|L_{0,h}^s u\|_{L^2}^2$.

The calculation for L is similar. We again decompose L_h into a symmetric and an antisymmetric part. The commutator contains additional error terms compared to $L_{0,h}$, which can be controlled by

$$C \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} e^{-t} (|Du|^2 + \tau^2 |u|^2) dt d\sigma.$$

This term can be controlled by the left hand side of (16) if

$$h'' + \frac{1}{4} \geq 2C\tau e^{-t}$$

as long as the coefficients b, c_1, c_2, d and their derivatives are small. We choose h with

$$\begin{aligned} h'(t) &= [\tau] + \frac{1}{4} && \text{if } t \geq 2C \ln \tau, \\ h''(t) &= 2C\tau e^{-t} && \text{if } 0 \leq t \leq C \ln \tau. \end{aligned}$$

With this argument we obtain the estimate provided u is supported in a small punctured disc with size independent of τ . The general estimate follows using arguments as in Hörmander, [9] Chapter 17.2. \square

Corollary 5.2. *Let $\sigma > 0$ and suppose that the coefficients are $\sigma + 1$ times continuously differentiable. Then there exists τ_0 such that for all $|s|, k \leq \sigma$, and $\tau \geq \tau_0$ the estimate*

$$\sum_{j=0}^{k+2} \tau^{1-j} \|(1 + h'')^{\frac{1}{2}} |x|^{j+s} e^h D^j v\| \lesssim \sum_{j=0}^k \tau^{-j} \||x|^{2+j+s} e^h D^j A v\|$$

holds.

Proof. We argue as in the previous section. There is no difference in the argument. \square

Again we apply this iteratively.

Proposition 5.3. *Suppose that $A_l, 1 \leq l \leq M$, are elliptic operators of second order with coefficients in $C^{1+2(M-l)}$ with $a_l^{ij}(0) = \delta^{ij}$. Let*

$$P = A_M \dots A_1.$$

Then the following Carleman inequality holds for τ sufficiently large and $v \in C_0^\infty(B_1(0) \setminus \{0\})$:

$$(19) \quad \sum_{j=0}^{2M} \tau^{M-j} \|(1+h'')^{M/2} |x|^{-2M+j} e^h D^j v\|_{L^2} \lesssim \|e^h P v\|.$$

Proof. Again we define for $1 \leq j \leq M$

$$v_j = A_j A_{j-1} \dots A_1 v.$$

We apply Corollary 5.2 iteratively:

$$(20) \quad \begin{aligned} & \sum_{j=0}^{2M} \tau^{M-j} \|(1+h'')^{M/2} |x|^{-2M+j} e^h D^j v\|_{L^2} \\ & \lesssim \sum_{j=0}^{2M-2} \tau^{M-1-j} \|(1+h'')^{(M-1)/2} |x|^{-2(M-1)+j} e^h D^j v_1\|_{L^2} \\ & \lesssim \sum_{j=0}^4 \tau^{2-j} \|(1+|h''|) |x|^{-4+j} e^h D^j v_{M-2}\| \\ & \lesssim \sum_{j=0}^2 \tau^{1-j} \|(1+h'')^{1/2} |x|^{-2+j} e^h D^j v_{M-1}\| \\ & \lesssim \|e^h v_M\|. \end{aligned} \quad \square$$

We study strong unique continuation for solutions to the inequality

$$|Pv| \leq \sum_{j=0}^{3M/2} b_j(x) |x|^{-2M+j} |D^j v|.$$

We have to choose h such that

$$b_j \lesssim \tau^{M-j} (1+h'')^{M/2}.$$

This reduces to boundedness for $j < M$ and for $j = M$ to

$$\limsup_{x \rightarrow 0} b_M(x) \ll 1$$

with an implicit constant C depending only on h and n . For all j we need

$$h'' \geq \max \left\{ \left(\sum_{j=M+1}^{\lfloor 3M/2 \rfloor} b_j \tau^{j-M} \right)^{2/M} - 1, 0 \right\}$$

and

$$\int_1^\infty h''(s) ds \lesssim \tau.$$

This is satisfied under the assumptions of Theorem 2. The Strong Unique Continuation Property follows again by standard arguments.

Let $c_{j,k} = \max\{b_j(x) : 2^{-k} \leq |x| \leq 2^{-k+1}\}$. Thus we need

$$\sum_k c_{j,k}^{2/M} \lesssim \tau^{3-2j/M},$$

where we sum only over those k for which

$$c_{j,k} \geq \tau^{M-j}.$$

We work out these condition for $M = 2, 3, 4$.

The case $M = 2$. In this case we consider a fourth order operator. It suffices to assume that b_0 and b_1 are large constants, b_2 is a small constant, and the coefficients $c_{3,k}$ are summable.

The case $M = 3$. Here it suffices to require that b_0, b_1 , and b_2 be large constants. The coefficient b_3 of the third order terms has to be small. Since $5 > 3 * 3/2$, we require $b_5 = 0$, and for b_4 we obtain the condition

$$(21) \quad \limsup_{\lambda \rightarrow 0} \lambda^{1/3} \sum_{k: c_{4,k}^{2/3} \geq \lambda} c_{4,k}^{2/3} \ll 1.$$

Lemma 5.4. *Let $(a_j) \in l^p$. Then*

$$\sup_{\lambda > 0} \lambda^{p-1} \sum_{|a_j| \geq \lambda} |a_j| \lesssim \|(a_j)\|_{l^p}^p.$$

Proof. By Hölder's and Tschebysheff's inequalities

$$\sum_{|a_j| \geq \lambda} |a_j| \leq \|(a_j)\|_{l^p} (\#\{j : |a_j| \geq \lambda\})^{1-1/p} \leq \lambda^{1-p} \|(a_j)\|_{l^p}^p. \quad \square$$

Thus (21) is satisfied if

$$(c_{4,k}) \in l^1.$$

The case $M = 4$. Here we need that b_0, b_1, b_2 , and b_3 are arbitrary constants, b_4 is a small constant, $b_7 = 0$,

$$\sum_k c_{6,k}^{1/2} < \infty,$$

and

$$\sum_k c_{5,k} < \infty.$$

The case M even. We need b_j bounded for $j < M$, $b_M \ll 1$, $b_j = 0$ for $j > 3M/2$,

$$\sum_k c_{3M/2,k}^{2/M} < \infty,$$

and, if $M \geq 6$,

$$\sum_{k: c_{3M/2-1,k} \geq \tau^{1-M/2}} c_{3M/2-1,k}^{2/M} \lesssim \tau^{2/M}$$

which is satisfied if $(c_{3M/2-1,k}) \in l^{\frac{2}{M-2}}$.

In general we obtain the sufficient condition

$$(c_{M+j,k}) \in l^{\frac{1}{j}}$$

for $0 < j \leq \frac{M}{2}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PISA, VIA F BUONARROTI 2, 56127 PISA, ITALY

MATHEMATISCHES INSTITUT, UNIVERSITY OF BONN, BERINGSTRASSE 1, 53115 BONN, GERMANY