

A GROUP OF NON-UNIFORM EXPONENTIAL GROWTH LOCALLY ISOMORPHIC TO $IMG(z^2 + i)$

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ABSTRACT. We prove that a sequence of marked three-generated groups isomorphic to the iterated monodromy group of $z^2 + i$ converges to a group of non-uniform exponential growth, which is an extension of the infinite direct sum of cyclic groups of order 4 by a Grigorchuk group.

1. INTRODUCTION

This paper continues the study of a Cantor set of 3-generated groups defined in the paper [Nek07]. The main result of this paper is showing that it contains a group of non-uniformly exponential growth.

Consider a set of groups \mathcal{D}_w , parametrized by infinite binary sequences $w \in \{0, 1\}^\omega$ and generated by three permutations $\alpha_w, \beta_w, \gamma_w$ of the set $\{0, 1\}^*$ of finite binary words, which act according to the rule

$$\begin{aligned} \alpha_w(0v) &= 1v, & \alpha_w(1v) &= 0v, \\ \beta_w(0v) &= 0\alpha_{s(w)}(v), & \beta_w(1v) &= 1\gamma_{s(w)}(v), \end{aligned}$$

and

$$\gamma_w(0v) = 0\beta_{s(w)}(v), \quad \gamma_w(1v) = 1v$$

if the first letter of w is 0, and

$$\gamma_w(0v) = 0v, \quad \gamma_w(1v) = 1\beta_{s(w)}(v)$$

if the first letter of w is 1. Here $s(w)$ is the shift of w , i.e., $s(x_1x_2\dots) = x_2x_3\dots$

The family of the groups \mathcal{D}_w is related to iterations of the post-critically finite complex 2-dimensional rational mapping $(z, p) \mapsto ((1 - 2z/p)^2, (1 - 2/p)^2)$. This rational map was also studied by J. E. Fornæss and N. Sibony [FS92]. For more details see [Nek07].

This family contains two well-known groups. The group $\mathcal{D}_{000\dots}$ is the iterated monodromy group of $z^2 + i$, while the group $\mathcal{D}_{111\dots}$ is one of Grigorchuk groups, namely the group $\mathcal{G}_{010101\dots}$ from the family studied in [Gri85]. It also appears in the paper [Ers04a] by A. Erschler, where particularly tight estimates

$$\exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right) \preceq \gamma(n) \preceq \exp\left(\frac{n}{\log^{1-\epsilon}(n)}\right)$$

of the growth of $\mathcal{D}_{111\dots}$ were given (here $\epsilon > 0$ is arbitrary).

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Similar to the case of the family of Grigorchuk groups in [Gri85], the family $\{\mathcal{D}_w\}$ contains a countable subset of “exceptional groups” which have more relations than typical groups \mathcal{D}_w . It happens that these exceptional groups are exactly the groups isomorphic to the Grigorchuk group $\mathcal{G}_{010101\dots}$. They are the groups \mathcal{D}_w for which w has finitely many zeros.

The map $w \mapsto (\mathcal{D}_w, \alpha_w, \beta_w, \gamma_w)$ from the set of sequences w with infinitely many zeros to the set of non-exceptional groups is continuous if we consider the product topology on the space of sequences and the natural topology on the space of 3-generated groups (see [Gri85, Cha00]). In the latter topology two groups with marked generating sets are close if their Cayley graphs coincide on a large ball around the identity as marked oriented graphs.

It is hence natural to replace the exceptional groups $\mathcal{D}_{v111\dots}$ by the limits of the groups \mathcal{D}_w as w approaches $v111\dots$ remaining in the set of sequences with infinitely many zeros. Let us denote the redefined groups by G_w (equal to \mathcal{D}_w if w has infinitely many zeros).

The following properties of the families $\{\mathcal{D}_w\}$ and $\{G_w\}$ are proved in [Nek07].

- (1) The map $w \mapsto (G_w, \alpha_w, \beta_w, \gamma_w)$ is continuous one-to-one.
- (2) Two groups G_{w_1} and G_{w_2} are isomorphic if and only if \mathcal{D}_{w_1} and \mathcal{D}_{w_2} are isomorphic, if and only if w_1 and w_2 are cofinal, i.e., if $w_1 = v_1 w$, $w_2 = v_2 w$ for v_1, v_2 finite of equal length.

The main result of this paper (Theorem 4.6) is the following property of the group $G_{111\dots}$.

Theorem 1.1. *The group $G_{111\dots}$ has non-uniform exponential growth.*

Namely, we show that $G_{111\dots}$ is an extension of the infinite restricted direct power of the cyclic group C_4 of order 4 by the group $\mathcal{D}_{111\dots}$, and we conclude from the structure of this extension that $G_{111\dots}$ contains the lamplighter group $C_2^{\mathbb{Z}} \rtimes \mathbb{Z}$, hence is of exponential growth.

The growth is non-uniform, i.e., there is no uniform (not depending on the generating set) lower bound greater than one on the exponent of the growth, since by the properties of the family G_w , the group $G_{111\dots}$ is locally isomorphic to the group $G_{000\dots}$, which is known to have sub-exponential growth (by a result of K.-U. Bux and R. Pérez in [BP06]). Here local isomorphism means that the set of marked 3-generated groups isomorphic to $G_{111\dots}$ has the group $G_{000\dots}$ as an accumulation point. Since the exponent of growth is upper semi-continuous with respect to the topology on the space of groups, we get non-uniformness of the exponential growth of $G_{111\dots}$.

Note that in some sense the first examples of groups of non-uniform exponential growth (see [Wil04b, Wil04a, Bar03]) are based on similar ideas, even though in their case a limit group of intermediate growth is obtained after non-uniformness of the exponential growth is shown. A limit group of intermediate growth is present explicitly in [Bar03].

It is also worthwhile to mention that extensions of abelian groups by groups of intermediate growth have already appeared as interesting examples in different papers. See, for instance, [Ers04b], where non-residually finite groups quasi-isometric to the first Grigorchuk group are constructed, and [Gri87], where an amenable 2-group of exponential growth disproving a conjecture by Rosenblatt appears. Further study of similar extensions might be interesting.

The structure of the paper is as follows. The second section recalls the basic definitions about the space of n -generated groups, exponential growth and non-uniform exponential growth. The third section defines the basic terminology of the theory of self-similar groups, which is the main tool of the paper. All main results, including a review of the results of the paper [Nek07], are given in the last section.

2. SPACE OF GROUPS AND GROUPS OF NON-UNIFORM EXPONENTIAL GROWTH

2.1. Space of groups. Let F_n be the free group and let $\{g_1, g_2, \dots, g_n\}$ be a fixed free generating set of F_n . A *marked n -generated group* $(G, h_1, h_2, \dots, h_n)$ is a quotient of F_n with a fixed epimorphism $\phi : F_n \rightarrow G$ such that $\phi(h_i) = g_i$. In other words, a marked n -generated group is a group with a fixed sequence of n generators.

Every marked n -generated group is uniquely determined by the kernel of the epimorphism ϕ . Hence, we can identify the set of all marked n -generated groups with the set of all normal subgroups of F_n .

The set of all subsets 2^{F_n} of F_n has a natural direct product topology. It is given by the basis consisting of open sets of the form

$$\{A \subset F_n : A \supseteq R, A \cap S = \emptyset\},$$

where $R, S \subset F_n$ are some finite sets.

The set of all marked n -generated groups inherits the topology of 2^{F_n} as a subset. In this topology the basic open set

$$\mathcal{U}_{R,S} = \{A \trianglelefteq F_n : A \supseteq R, A \cap S = \emptyset\}$$

is the set of groups in which all elements of the finite set R are trivial (are *relations*) and all elements of the finite set S are non-trivial (are *inequalities*).

Consequently, another way to define topology on the space \mathfrak{G}_n of marked n -generated groups is to say that two groups are close in this topology if their marked Cayley graphs coincide on a ball of large radius around the identity.

This topology was explicitly introduced by R. Grigorchuk in [Gri85]. See also the end of the paper by M. Gromov on groups of polynomial growth [Gro81a] (end of page 71).

2.2. Groups of non-uniform exponential growth. If G is a group generated by a finite set S , then the corresponding *growth function* $\Gamma(n) = \Gamma_S(n)$ is the number of elements of G which can be represented as products of at most n elements of S and their inverses. We have the obvious inequality

$$\Gamma(n_1 + n_2) \leq \Gamma(n_1)\Gamma(n_2),$$

which implies, by [PS72] Problem 98 of Part I, that

$$\lim_{n \rightarrow \infty} \frac{\log \Gamma(n)}{n} = \inf_{n \geq 1} \frac{\log \Gamma(n)}{n}$$

and that the limit exists.

The group G has *exponential growth* if $e_S(G) = \lim_{n \rightarrow \infty} \frac{\log \Gamma(n)}{n} > 0$. The number $e_S(G)$ is then called the (*algebraic*) *entropy* of G (with respect to the generating set S). The group G has *non-uniform exponential growth* if

$$\inf_S e_S(G) = 0,$$

where infimum is taken over all finite generating sets of G .

The question of the existence of groups of non-uniform exponential growth was asked by M. Gromov in [Gro81b]. See also a survey on questions related to uniform and non-uniform exponential growth in [Har02].

The first example of a group of non-uniform exponential growth was constructed by J. Wilson in [Wil04b].

Proposition 2.1. *Suppose that $(G, g_{1,n}, g_{2,n}, \dots, g_{m,n})$ is a sequence of marked isomorphic groups of exponential growth converging to a group $(H, h_1, h_2, \dots, h_m)$ of sub-exponential growth. Then G has non-uniform exponential growth.*

Proof. For every $R \geq 0$ there exists n_R such that the ball of radius R in the Cayley graph of G with respect to the generating set $S_{n_R} = \{g_{1,n_R}, \dots, g_{m,n_R}\}$ is isomorphic to the ball of radius R in the Cayley graph of H with respect to the generating set $S = \{h_1, \dots, h_m\}$. We have $n_R \rightarrow \infty$ as $R \rightarrow \infty$, since G has exponential growth, while H has sub-exponential growth. In particular,

$$\Gamma_{G, S_{n_R}}(R) = \Gamma_{H, S}(R);$$

hence

$$e_{S_{n_R}}(G) = \inf_{m \geq 1} \frac{\log \Gamma_{G, S_n}(m)}{m} \leq \frac{\log \Gamma_{G, S_{n_R}}(R)}{R} = \frac{\log \Gamma_{H, S}(R)}{R} \rightarrow 0$$

as $R \rightarrow \infty$. Hence G has non-uniform exponential growth. \square

See the paper of L. Bartholdi [Bar03], where the limit of intermediate growth of groups of non-uniform exponential growth is described.

3. SELF-SIMILAR GROUPS

Let X be a finite alphabet. By X^* we denote the set of finite words over X , i.e., the free monoid generated by X . We interpret X^* as a tree with the empty word serving as the root and in which a word v is connected to the words of the form vx for $x \in X$.

The *boundary* ∂X^* of the tree X^* is the set of infinite geodesic rays starting in the root. It is naturally identified with the set X^ω of infinite words $x_1 x_2 \dots$ over the alphabet X (the corresponding geodesic ray consists of the beginnings of the infinite word). The boundary is endowed with the topology of a direct product of finite discrete sets X . If T is a sub-graph of X^* containing the root, then $\partial T \subset \partial X^*$ is the set of geodesic rays of T starting in the root.

The symmetric group $\mathfrak{S}(X)$ acts naturally on the tree X^* by just acting on the first letter of words. Every automorphism of the rooted tree X^* is then written as a product $\sigma \cdot g$ of an element σ of $\mathfrak{S}(X)$ and an element g of the pointwise stabilizer of the first level X^1 of the tree X^* . We write elements of the stabilizer of the first level as sequences $g = (g|_x)_{x \in X}$, where $g|_x$ describes the action of g on the words starting with x :

$$g(xv) = xg|_x(v).$$

In particular, an arbitrary automorphism of the tree X^* is written as a product $\sigma(g|_x)_{x \in X}$. It is easy to see that such products are multiplied by the rule

$$\sigma(g|_x)_{x \in X} \cdot \pi(h|_x)_{x \in X} = \sigma\pi(g|_{\pi(x)}h|_x)$$

(our actions are from the left). In this way the automorphism group $\text{Aut}(X^*)$ of the rooted tree X^* is identified with the wreath product $\mathfrak{S}(X) \wr \text{Aut}(X^*) =$

$\mathfrak{S}(\mathbb{X}) \ltimes \text{Aut}(\mathbb{X}^*)^{\mathbb{X}}$. We call this natural isomorphism $\psi : \text{Aut}(\mathbb{X}^*) \rightarrow \mathfrak{S}(\mathbb{X}) \wr \text{Aut}(\mathbb{X}^*)$ the *wreath recursion*.

More generally, for every automorphism g of the tree \mathbb{X}^* and every word $v \in \mathbb{X}^*$, there exists an automorphism of \mathbb{X}^* , denoted $g|_v$, such that

$$g(vw) = g(v)g|_v(w)$$

for every $w \in \mathbb{X}^*$. The automorphism $g|_v$ is called the *section* of g at v (*restrictions* in [Nek05]).

We have the following properties of sections:

$$g|_{v_1v_2} = g|_{v_1}|_{v_2}, \quad (g_1g_2)|_v = g_1|_{g_2(v)}g_2|_v.$$

Definition 3.1. An automorphism group $G < \text{Aut}(\mathbb{X}^*)$ is called *self-similar* if for every $g \in G$ and for every $x \in \mathbb{X}$ we have $g|_x \in G$.

If G is self-similar, then restriction of the wreath recursion ψ onto G is a homomorphism $G \rightarrow \mathfrak{S}(\mathbb{X}) \wr G$, also called the wreath recursion.

In general, a *wreath recursion* is any homomorphism $\phi : G \rightarrow \mathfrak{S}(\mathbb{X}) \wr G$. If $\phi(g) = \sigma(g_x)_{x \in \mathbb{X}}$ for $\sigma \in \mathfrak{S}(\mathbb{X})$ and $(g_x)_{x \in \mathbb{X}} \in G^{\mathbb{X}}$, then we write

$$g(x) = \sigma(x)$$

and

$$g|_x = g_x$$

for $x \in \mathbb{X}$.

The n th iteration of the wreath recursion $\phi : G \rightarrow \mathfrak{S}(\mathbb{X}) \wr G$ is the wreath recursion $\phi^{\otimes n} : G \rightarrow \mathfrak{S}(\mathbb{X}^n) \wr G$ defined inductively by the conditions

$$g(vx) = g(v)g|_v(x)$$

and

$$g|_{vx} = g|_v|_x,$$

for $v \in \mathbb{X}^{n-1}$, $x \in \mathbb{X}$, where $g(v), g|_v$ are defined by the wreath recursion $\phi^{\otimes(n-1)}$.

Definition 3.2. A self-similar group G (or a wreath recursion $\phi : G \rightarrow \mathfrak{S}(\mathbb{X}) \wr G$) is *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for all words $v \in \mathbb{X}^*$ of length more than n . The smallest set \mathcal{N} satisfying this condition is called the *nucleus* of the group (of the wreath recursion).

For algebraic and dynamical properties of contracting self-similar groups and contracting wreath recursions (permutational bimodules), see [Nek05].

Definition 3.3. A self-similar group G (or a wreath recursion on a group G) is called *self-replicating* if for every $g \in G$ and $x, y \in \mathbb{X}$ there exists $h \in G$ such that $g(x) = y$ and $g|_x = h$.

Self-replicating groups are also sometimes called *recurrent* (see [NS04, Nek05]) or *fractal* (see [BGN03]). It is easy to see that a self-replicating group is always transitive on the levels of the tree \mathbb{X}^* .

4. THE FAMILIES $\{\mathcal{D}_w\}$ AND $\{G_w\}$

In the rest of this paper we denote the binary alphabet $\{0, 1\}$ by X . Consider the automorphism group \mathcal{D} of the 4-tree $\mathbb{T} = (X \times X)^*$ generated by automorphisms α, β, γ , where α is the permutation

$$(0, 0) \leftrightarrow (1, 0), \quad (0, 1) \leftrightarrow (1, 1)$$

and

$$\beta = (\alpha, \gamma, \alpha, \gamma), \quad \gamma = (\beta, 1, 1, \beta),$$

where we order the alphabet $X \times X$ in the sequence

$$(0, 0), (1, 0), (0, 1), (1, 1).$$

It is proved in [Nek07] (Proposition 4.2) that the group \mathcal{D} is contracting with the nucleus

$$\mathcal{N} = \langle \alpha, \beta \rangle \cup \langle \beta, \gamma \rangle \cup \langle \gamma, \alpha \rangle$$

and that the generators satisfy the relations

$$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^8 = (\alpha\gamma)^4 = (\beta\gamma)^8 = 1$$

so that

$$\langle \alpha, \beta \rangle \cong D_8, \quad \langle \beta, \gamma \rangle \cong D_8, \quad \langle \gamma, \alpha \rangle \cong D_4.$$

Here we denote by D_n the dihedral group of order $2n$.

For every sequence $w = y_1 y_2 \dots \in X^\omega$ the set $\mathbb{T}_w \subset \mathbb{T}$ of words of the form $(x_1, y_1)(x_2, y_2) \dots (x_n, y_n)$ is a binary sub-tree of \mathbb{T} , naturally isomorphic to X^* under the isomorphism

$$(x_1, y_1)(x_2, y_2) \dots (x_n, y_n) \mapsto x_1 x_2 \dots x_n.$$

It follows from the recursive definition of the generators of \mathcal{D} that the sub-trees \mathbb{T}_w are \mathcal{D} -invariant.

Definition 4.1. Denote for $w \in X^\omega$ by \mathcal{D}_w the restriction of the action of \mathcal{D} onto \mathbb{T}_w , i.e., the quotient of \mathcal{D} by the subgroup of elements acting trivially on \mathbb{T}_w (equivalently, on $\partial\mathbb{T}_w$).

Denote by G_w the quotient of \mathcal{D} by the subgroup of elements acting trivially on a neighborhood of $\partial\mathbb{T}_w$ in $\partial\mathbb{T}$.

In other words, an element $g \in \mathcal{D}$ belongs to the kernel of the epimorphism $\mathcal{D} \rightarrow G_w$ if and only if there exists n such that g acts trivially on the n th level of the tree \mathbb{T}_w and $g|_v = 1$ for every v from the n th level of \mathbb{T}_w .

It follows from this interpretation that the groups G_w coincide with the groups denoted in the same way in [Nek07] (see the bottom of page 178).

It is also easy to see that the groups \mathcal{D}_w are isomorphic to the groups generated by the automorphisms $\alpha_w, \beta_w, \gamma_w$ of the binary tree X^* , given by the recursions

$$\begin{aligned} \alpha_w &= \sigma, \\ \beta_w &= (\alpha_{s(w)}, \gamma_{s(w)}), \\ \gamma_w &= \begin{cases} (\beta_{s(w)}, 1) & \text{if the first letter of } w \text{ is } 0, \\ (1, \beta_{s(w)}) & \text{if the first letter of } w \text{ is } 1. \end{cases} \end{aligned}$$

Here σ is the transposition $0 \leftrightarrow 1$ and $s(w)$ is the shift of the sequence w , i.e., the sequence obtained by deletion of the first letter.

The following properties of the families $(\mathcal{D}_w, \alpha, \beta, \gamma)$ and $(G_w, \alpha, \beta, \gamma)$ are proved in [Nek07] (see Proposition 5.6, Proposition 6.3 and Theorem 6.5 therein).

Proposition 4.1. *If the sequence $w \in X^\omega$ has infinitely many zeros, then \mathcal{D}_w coincides with G_w (i.e., the kernels of the epimorphisms $\mathcal{D} \rightarrow \mathcal{D}_w$ and $\mathcal{D} \rightarrow G_w$ coincide).*

Proposition 4.2. *The map $w \mapsto (G_w, \alpha, \beta, \gamma)$ from X^ω to the space of three-generated groups is continuous and injective.*

Theorem 4.3. *The following conditions are equivalent:*

- (1) *the groups G_{w_1} and G_{w_2} are isomorphic,*
- (2) *the groups \mathcal{D}_{w_1} and \mathcal{D}_{w_2} are isomorphic,*
- (3) *the sequences w_1 and w_2 are cofinal; i.e., $w_1 = v_1w$ and $w_2 = v_2w$ for some words $v_1, v_2 \in X^*$ of equal length and $w \in X^\omega$.*

4.1. Non-uniform exponential growth of $G_{111\dots}$.

Proposition 4.4. *The group $G_{000\dots}$ is isomorphic to $\text{IMG}(z^2 + i)$ and is of intermediate growth.*

Proof. It is known (see [BGN03, BP06] and [Nek05] Subsection 6.12.5) that the iterated monodromy group of $z^2 + i$ is defined by the recursion

$$\begin{aligned} \alpha &= \sigma, \\ \beta &= (\alpha, \gamma), \\ \gamma &= (\beta, 1). \end{aligned}$$

Consequently, by Proposition 4.1, the group $G_{000\dots} = \mathcal{D}_{000\dots}$ is isomorphic to $\text{IMG}(z^2 + i)$, hence is of intermediate growth, by [BP06]. □

Proposition 4.5. *The group $G_{111\dots}$ is of exponential growth.*

Proof. The group $\mathcal{D}_{111\dots}$ is given by the recursion

$$\alpha = \sigma, \quad \beta = (\alpha, \gamma), \quad \gamma = (1, \beta).$$

This group belongs to a family of groups of intermediate growth, which were studied by Grigorchuk in [Gri85]. Its growth was estimated by Erschler in [Ers04a].

Note, however, that in the group $\mathcal{D}_{111\dots}$ the elements β and γ commute, since

$$\beta\gamma = (\alpha, \gamma\beta), \quad \gamma\beta = (\alpha, \beta\gamma),$$

but $[\beta, \gamma] = (\beta\gamma)^2$ is not trivial in $G_{111\dots}$, since

$$(\beta\gamma)^2 = (1, (\gamma\beta)^2), \quad (\gamma\beta)^2 = (1, (\beta\gamma)^2).$$

Thus $(\beta\gamma)^2$ does not act trivially on any neighborhood of $\partial T_{111\dots}$ (as $(\beta\gamma)^2$ is not trivial in \mathcal{D}).

It follows from the definition of the groups G_w that the group $G_{111\dots}$ is the quotient of the group \mathcal{D} by the union over all $n \geq 1$ of the kernels of the n th iterations $\phi^{\otimes n}$ of the wreath recursion ϕ on \mathcal{D} given by

$$\phi(\alpha) = \sigma, \quad \phi(\beta) = (\alpha, \gamma), \quad \phi(\gamma) = (1, \beta).$$

We will also denote by ϕ the induced (i.e., given by the same formulae) wreath recursion on $G_{111\dots}$.

Note that a similar quotient (for the same wreath recursion), but starting from the free group instead of \mathcal{D} , was studied in [BN06]. Quotients by the unions of kernels of iterations of wreath recursions is a topic of [Ers07].

The nucleus of the wreath recursion ϕ on $G_{111\dots}$ is the set $\{1, \alpha\} \cup \langle \beta, \gamma \rangle$, where $\langle \beta, \gamma \rangle$ is the dihedral group D_8 of order 16. This is easy to check directly.

The set of elements of the nucleus, which are trivial in the quotient $\mathcal{D}_{111\dots}$, is

$$C_4 = \{1, (\beta\gamma)^2, (\beta\gamma)^4, (\beta\gamma)^6\},$$

which is a subgroup isomorphic to the cyclic group of order 4.

Let K be the kernel of the epimorphism $G_{111\dots} \rightarrow \mathcal{D}_{111\dots}$. By contraction of the wreath recursion on $G_{111\dots}$, an element $g \in G_{111\dots}$ belongs to K if and only if there exists n such that for every $v \in X^n$ we have $g(v) = v$ and $g|_v \in C_4$. Consequently, K is equal to the union of the inverse images K_n of $C_4^{X^n}$ under n th iteration of the wreath recursion on $G_{111\dots}$. For every n the group $C_4^{X^n}$ belongs to the range of the n th iteration of the wreath recursion ϕ of $G_{111\dots}$, since ϕ is self-replicating and the image of C_4 under the wreath recursion is trivial on all coordinates except for 1^n , where it is equal to C_4 . We also have

$$(\beta\gamma)^2 = (1, (\gamma\beta)^2);$$

hence K_n embeds into K_{n+1} by the homomorphism $(g_v)_{v \in X^n} \mapsto (h_u)_{u \in X^{n+1}}$, where

$$(4.1) \quad h_{v0} = 1, \quad h_{v1} = g_v^{-1}.$$

Denote by U the set of sequences $w \in X^\omega$ having only a finite number of zeros. For any $g \in K$ there exists n such that the projection of g onto the n th level belongs to $C_4^{X^n}$. Denote by $\Phi(g)$ the function $U \rightarrow C_4$ given by

$$\Phi(g)(v111\dots) = (g|_v)^{(-1)^{|n|}}.$$

It follows from (4.1) that $\Phi(g)$ does not change if we take bigger n and that Φ is an isomorphism between K and the (restricted) direct power C_4^U .

Every element of the nucleus of $\mathcal{D}_{111\dots}$ maps $111\dots \in X^\omega$ to an element of U . Consequently, the set U is $\mathcal{D}_{111\dots}$ -invariant, hence is a $\mathcal{D}_{111\dots}$ -orbit (since $\mathcal{D}_{111\dots}$ is self-replicating).

Looking at the list of the elements of the nucleus, we see that an element $g \in G_{111\dots}$ acts on $K = C_4^U$ by permutation of the coordinates of the direct sum C_4^U according to the original action of g on the $G_{111\dots}$ -orbit U and possibly by inversion of some coordinates (coming from conjugation of C_4 by $\beta(\gamma\beta)^k$ or $\gamma(\beta\gamma)^k$).

This is enough to show that $G_{111\dots}$ has exponential growth. For instance, one can take the subgroup generated by $\tau = \alpha\beta\gamma$ and $(\beta\gamma)^4$. The cyclic group $\langle \tau \rangle$ is level-transitive, since (in $\mathcal{D}_{111\dots}$)

$$\tau^2 = (\sigma(\alpha, \gamma\beta))^2 = (\gamma\beta\alpha, \alpha\gamma\beta),$$

and $\gamma\beta\alpha = \beta\gamma\alpha$ is conjugate to $\alpha\gamma\beta = \alpha\beta\gamma$ in $\mathcal{D}_{111\dots}$. It follows that the $\langle \tau \rangle$ -orbit of $111\dots \in U$ is infinite.

The element $(\beta\gamma)^4 \in C_4$ is of order 2; i.e., it is equal to its inverse, and $\Phi((\beta\gamma)^4)$ is the function $U \rightarrow C_4$ equal to $(\beta\gamma)^4$ at $111\dots \in U$ and trivial on the rest of the coordinates. It is easy to see now that the group $\langle \tau, (\beta\gamma)^4 \rangle$ is isomorphic to the lamplighter group $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes \mathbb{Z}$, hence has exponential growth. \square

Theorem 4.6. *The group $G_{111\dots}$ has non-uniform exponential growth.*

Proof. The sequence $(G_{0^n 111\dots}, \alpha, \beta, \gamma)$ of groups isomorphic to $G_{111\dots}$ (by Theorem 4.3) converges to the group of intermediate growth $(G_{000\dots}, \alpha, \beta, \gamma)$ (by Proposition 4.2), hence $G_{111\dots}$ has non-uniform exponential growth by Proposition 2.1. \square

Corollary 4.7. *The set of growth degrees of the groups G_w contains an uncountable chain and an uncountable anti-chain. For every function of sub-exponential growth $f(n)$ there exists w such that the growth function of G_w is sub-exponential and greater than $f(n)$ for infinitely many values of n .*

Proof. The proof is the same as the proof of similar results for the family of Grigorchuk groups in [Gri85] and [Ers05], since in our family of groups we also have a dense subset of groups of intermediate growth (isomorphic to $\text{IMG}(z^2 + i)$) and a dense subset of groups of exponential growth (isomorphic to $G_{111}\dots$). Therefore, we can “mix” them and find groups which grow for a long while as a group of intermediate growth, and then for a while as a group of exponential growth. \square

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