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ON TUTTE'S CHROMATIC INVARIANT

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Abstract. Consider a simple connected graph G embedded in the plane together with a contractible circuit J. For a partition ϕ of the vertex set of J we denote by $P_{(\mathsf{G},\phi)}(t)$ the number of ways of assigning one of t given colours to each vertex of G so that vertices in the same block of ϕ have the same colour. Tutte showed that this polynomial may be expressed uniquely as a linear combination of $P_{(\mathsf{G},\pi)}(t)$ over all planar partitions π of J, with scalars $\vartheta_{\phi,\pi}(t)$ that are independent of G. We show that the (chromatic) invariants $\vartheta_{\phi,\pi}$ have a natural algebraic setting in terms of the orthogonal projection from the partition algebra $\mathbb{P}_r(t)$ to the Temperley-Lieb subalgebra $\mathbb{TL}_r(t,1)$. We define the genus of a partition and give an extension of the invariants to arbitrary genus g. Finally, we summarise the rôle of the genus 0 invariants in the algebraic approach of Birkhoff and Lewis to the Four Colour Theorem.

Contents

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1. Introduction

The result of Tutte we discuss contains objects we have termed Tutte's (chromatic) invariants. These invariants, which are defined in Section 2, encode relations between (free) chromatic polynomials. Our aim is to provide a richer context within which to study these Tutte invariants. We do this in Section 3 by realizing them as coefficients of the orthogonal projection from the partition algebra $\mathbb{P}_r$ to the Temperley-Lieb algebra $\mathbb{TL}_r \subset \mathbb{P}_r$. This suggests a higher genus generalization of Tutte invariants. In Section 4 we summarize the relationship between the genus zero invariants and the Four Colour Theorem.

It is worth mentioning that there are other relationships between the Temperley-Lieb algebra, chromatic polynomials and the Four Colour Theorem. One such relationship directly relates the chromatic polynomial of a graph G to the Jones polynomial of its associated alternating link (see [K]). This relation in effect translates the delete-and-contract equation for the chromatic polynomials of graphs into the skein relation for the Jones polynomial. See also [M] (section 1.5 onward) for a discussion of the traditional physicist's approach to the relationship between Temperley-Lieb algebras and (di)chromatic polynomials.

Another relationship translates the statement of the Four Colour Theorem in terms of the vanishing of a particular element in the Temperley-Lieb algebra (see [KT]). In our case the relationship is between Tutte's chromatic invariants, which are in turn related to the Four Colour Theorem, and a projection from the partition algebra to the Temperley-Lieb algebra. These three approaches are different.

2. Tutte's invariants

2.1. **Planar and non-planar partitions of J.** Let J be a circuit of length r of a graph G. We shall refer to J as an r-ring. A *partition* ϕ of J is a partition of the vertex set of J into subsets called *blocks*. Let $\ell(\pi)$ denote the number of blocks of π. Denote by $\Phi(\mathsf{J})$ the set of all partitions of the vertex set of J.

A partition π of J is said to be *planar* if lines in the interior of J can be drawn between each pair of vertices which are part of the same block of π such that they do not meet. We represent each block by a region in the interior of J as illustrated in Figure 1. We denote the set of planar partitions of J by $\Pi(\mathsf{J})$. For example, the number of partitions of J_4 is 15 but $|\Pi(\mathsf{J}_4)| = 14$, so there is a unique non-planar partition in $\Phi(\mathsf{J}_4)$, namely $\big\{[1,3],[2,4]\big\}$.

Define a partial order $\preccurlyeq$ on $\Phi(\mathsf{J})$ by $\phi' \preccurlyeq \phi$ if ϕ' is obtained from ϕ by refinement (see, *e.g.*, [S, §3.1.1]). Clearly, $\ell(\phi') \geq \ell(\phi)$ for all $\phi' \preccurlyeq \phi$. The poset $(\Phi(\mathsf{J}), \preccurlyeq)$ is a *lattice*. For every pair of partitions $\phi, \phi' \in \Phi(\mathsf{J})$, the *join* $\phi \vee \phi'$ is defined by $\phi \vee \phi' = \min_{\sigma\in\Phi(\mathsf{J})}\{\sigma \mid \sigma \succcurlyeq \phi, \phi'\}$. For example, if $\phi = \big\{[1,3],[2],[4]\big\}$ and $\phi' = \big\{[1],[2,4],[3]\big\}$, then $\ell(\phi) = 3 = \ell(\phi')$ with $\phi \vee \phi' = \big\{[1,3],[2,4]\big\}$. Notice that even though ϕ and ϕ' are planar partitions their join $\phi \vee \phi'$ is not (so the set of planar partitions is not closed under join and hence lattice-theoretic techniques cannot be used).

2.2. **Free chromials.** We consider throughout only undirected graphs G. A circuit J of length r in the embedded graph G is *contractible* if its interior contains no vertices or edges. We will consider pairs (G, ϕ) where $\phi \in \Phi(\mathsf{J})$ is a partition of the vertex set of the circuit J.

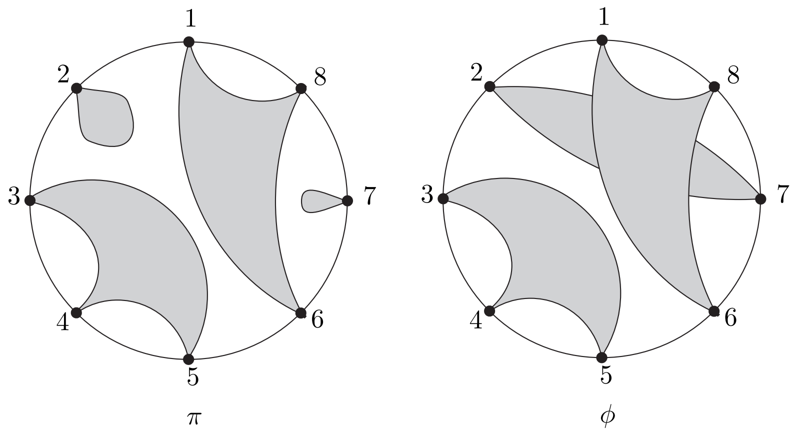

FIGURE 1. A planar partition $\pi = \{[1,6,8],[3,4,5],[2],[7]\}$ and a non-planar partition $\phi = \{[1,6,8],[3,4,5],[2,7]\}$.

A *(vertex) colouring* of a graph G with t colours is a colouring of vertices of G with colours $\{1,\ldots,t\}$ such that no edge joins vertices of the same colour. We denote by $P_{\mathsf{G}}(t)$ the number of such colourings. It is called the *chromial* of G. The *free chromial* $P_{(\mathsf{G},\phi)}(t)$ of (G,ϕ) is the number of ways of colouring (G,ϕ) with at most t colours such that the vertices of G in the same block of ϕ receive the same colour. Clearly, the free chromial $P_{(\mathsf{G},\phi)}(t)$ specialises to the chromial $P_{\mathsf{G}}(t)$ when $\phi=\varepsilon$, the partition of $V(\mathsf{J})$ into singleton blocks.

2.3. **Tutte's invariants.** The starting point is the following theorem due to Tutte. It provides relations between free chromials. Remarkably, these relations are independent of the graph G. Hence the coefficients $\vartheta_{\phi,\pi}$ in the relations are fundamental invariants which we call *Tutte's Chromatic Invariants.*

Theorem 2.1 (Tutte's Invariants [T1])**.** *Let* G *be a graph embedded in the plane and* J *a contractible circuit of* G *of length* $r<\infty$*. Let* $\phi\in\Phi(\mathsf{J})$ *(any partition of the vertices in* J*). Then there exist invariants* $\vartheta_{\phi,\pi}(t)\in\mathbb{Q}((t))$ *independent of* G*, where* $(\phi,\pi)\in\Phi(\mathsf{J})\times\Pi(\mathsf{J})$*, such that*

$$P_{(\mathsf{G},\phi)}(t) \quad = \sum_{\pi\in\Pi(\mathsf{J})}\vartheta_{\phi,\pi}(t)P_{(\mathsf{G},\pi)}(t). \tag{1}$$

Moreover, $\vartheta_{\phi,\pi}(t)$ *can be determined from the equation*

$$\left[t^{\ell(\phi\vee\sigma)}\right]_{B_r\times C_r} = \left[\vartheta_{\phi,\pi}\right]_{B_r\times C_r}\left[t^{\ell(\pi\vee\sigma)}\right]_{C_r\times C_r},$$

where $B_r=|\Phi(\mathsf{J})|$ *and* $C_r=|\Pi(\mathsf{J})|$*.*

Proof. We use induction on the number of edges of G. Suppose e is an edge of G without both ends on the ring J. By deletion and contraction, $P_{(\mathsf{G},\phi)}(t) = P_{(\mathsf{G}-e,\phi)}(t) - P_{(\mathsf{G}/e,\phi)}(t)$. By the induction hypothesis, Tutte's relation holds for both $\mathsf{G}-e$ and G/e while $\vartheta_{\phi,\pi}$ is independent of $\mathsf{G}-e$ and G/e, so

$$\begin{aligned} P_{(\mathsf{G},\phi)}(t) &= P_{(\mathsf{G}-e,\phi)}(t) - P_{(\mathsf{G}/e,\phi)}(t) \\ &= \sum_{\pi\in\Pi(\mathsf{J})} \vartheta_{\phi,\pi}(t)\left(P_{(\mathsf{G}-e,\pi)}(t) - P_{(\mathsf{G}/e,\pi)}(t)\right) = \sum_{\pi\in\Pi(\mathsf{J})} \vartheta_{\phi,\pi}(t) P_{(\mathsf{G},\pi)}(t). \end{aligned}$$

Tutte's relation also stays true if we remove isolated vertices introduced by the delete-and-contract procedure since an isolated vertex contributes a factor of t to both sides of (1). Thus we are reduced to proving the theorem for the case where G contains vertices only on J (this is the base case of the induction). In this base case the graph G is embedded in the plane and the edges join vertices on J.

Consider an arbitrary method T of deleting and contracting all the remaining edges of G. This identifies vertices on J into blocks which form the blocks of a partition σ_T of the vertices of J. This partition is planar since delete-and-contract operations preserve planarity. Now $P_{(\mathsf{G},\phi)}$ can be expressed as a sum over all such T. The contribution of one such T to $P_{(\mathsf{G},\phi)}$ is $(-1)^{c(T)} t^{\ell(\phi\vee\sigma_T)}$, where $c(T)$ is the number of edges contracted by T. Thus $P_{(\mathsf{G},\phi)} = \sum_T (-1)^{c(T)} t^{\ell(\phi\vee\sigma_T)}$. Similarly we get $P_{(\mathsf{G},\pi)} = \sum_T (-1)^{c(T)} t^{\ell(\pi\vee\sigma_T)}$. So equation (1) becomes

$$\sum_T (-1)^{c(T)} t^{\ell(\phi\vee\sigma_T)} = \sum_{\pi\in\Pi(\mathsf{J})} \vartheta_{\phi,\pi} \sum_T (-1)^{c(T)} t^{\ell(\pi\vee\sigma_T)} = \sum_T (-1)^{c(T)} \sum_{\pi\in\Pi(\mathsf{J})} \vartheta_{\phi,\pi} t^{\ell(\pi\vee\sigma_T)}.$$

Hence it suffices to show that there exist $\vartheta_{\phi,\pi}$ such that for any $(\phi,\pi) \in (\Phi(\mathsf{J}), \Pi(\mathsf{J}))$ we have

$$t^{\ell(\phi\vee\sigma)} = \sum_{\pi\in\Pi(\mathsf{J})} \vartheta_{\phi,\pi} t^{\ell(\pi\vee\sigma)}.$$

Let $\mathbf{B}_r = \left[t^{\ell(\phi\vee\sigma)}\right]_{B_r\times C_r}$, $\mathbf{A}_r = [\vartheta_{\phi,\pi}]_{B_r\times C_r}$, and $\mathbf{M}_r = \left[t^{\ell(\pi\vee\sigma)}\right]_{C_r\times C_r}$. Then the relation above can be rewritten as $\mathbf{B}_r = \mathbf{A}_r\mathbf{M}_r$. Hence such $\vartheta_{\phi,\pi}$ exist if and only if $\mathbf{M}_r$ is invertible.

We now prove the invertibility of $\mathbf{M}_r$. Denote the planar partitions by $\pi_1, \ldots, \pi_m$ (where $m = C_r$). A general term in the expansion of $\det \mathbf{M}_r$ is $\pm t^{\ell(\pi_1\vee\pi_{i_1})} \cdots t^{\ell(\pi_m\vee\pi_{i_m})}$. Note that $\ell(\pi_1 \vee \pi_{i_1}) \le \ell(\pi_1 \vee \pi_1)$ with equality if and only if $\pi_{i_1} \preccurlyeq \pi_1$. Consequently

$$\ell(\pi_1 \vee \pi_{i_1}) + \cdots + \ell(\pi_m \vee \pi_{i_m}) \le \ell(\pi_1 \vee \pi_1) + \cdots + \ell(\pi_m \vee \pi_m)$$

with equality if and only if $\pi_{i_1} \preccurlyeq \pi_1, \ldots, \pi_{i_m} \preccurlyeq \pi_m$. By Lemma 2.2, which follows below, this means that equality holds if and only if $i_1 = 1, \ldots, i_m = m$. Thus there is a unique term of highest degree in the expansion of $\det \mathbf{M}_r$ (namely the product of the diagonal elements of $\mathbf{M}_r$). This element occurs with non-zero coefficient, whence $\mathbf{M}_r$ is invertible. □

Lemma 2.2. *Let $(\mathcal{P}, \preccurlyeq)$ be a finite poset with elements $p_1, \ldots, p_m$. If $i_1, \ldots, i_m$ is a permutation of $1, \ldots, m$ such that $p_{i_1} \preccurlyeq p_1, \ldots p_{i_m} \preccurlyeq p_m$, then $i_1 = 1, \ldots, i_m = m$.*

Proof. Without loss of generality assume p_1 is a minimal element. Then $p_{i_1} \preccurlyeq p_1$ implies that $i_1 = 1$. Now delete p_1 from the poset and repeat the argument. This process terminates since $(\mathcal{P}, \preccurlyeq)$ is finite. □

Since $\vartheta_{\phi,\pi}(t)$ is independent of G we shall call it Tutte's *chromatic invariant*. If ϕ happens to be planar, then $\vartheta_{\phi,\pi} = \delta_{\phi,\pi}$ and Tutte's relation gives no information. The matrix $\mathbf{M}_r = \left[t^{\ell(\pi\vee\sigma)}\right]_{C_r\times C_r}$ is usually called the *matrix of chromatic joins*.

Remark. Similarly, it can be shown that for any graph G embedded in the plane with circuit J and arbitrary exterior as well as interior we have

$$P_{(\mathsf{G},\phi)}(t) = \sum_{\rho} \vartheta'_{\rho,\phi} P_{(\mathsf{G},\rho)}(t)$$

for some $\vartheta'_{\rho,\phi}$, where the sum is over all partitions ρ that can be expressed as $\rho = \sigma_1 \vee \sigma_2$ for some planar partitions σ_1 and σ_2. This follows because the base case in the proof of this involves a circuit J with $\mathrm{ext}_{\mathsf{J}}\mathsf{G}$ and $\mathrm{int}_{\mathsf{J}}\mathsf{G}$ encoded by planar partitions σ_1 and σ_2. Thus $P_{(\mathsf{G},\phi)} = t^{\ell(\sigma_1 \vee \sigma_2 \vee \phi)}$ and the σ in the proof of Theorem 2.1, which was planar, is now replaced by $\sigma_1 \vee \sigma_2$. The matrix which is analogous to $\mathbf{M}_r$ is $\left[t^{\ell(\rho_1 \vee \rho_2)}\right]$, where both ρ_1 and ρ_2 can be expressed as the join of two planar partitions. All that remains to prove such a result is to show that the determinant of this matrix is not identically zero. This can be proved using the above ideas to show that $\mathbf{M}_r$ is invertible.

2.4. **Completeness of Tutte's relations.** Next we show that Tutte's relations (1) account for all the linear relations between free chromials. By such a linear relation we mean an equation of the form

$$\sum_{\phi \in \Phi(\mathsf{J})} f_\phi(t) P_{(\mathsf{G},\phi)}(t) = 0,$$

where the $f_\phi(t)$ are independent of G.

Theorem 2.3. *Tutte's relations* (1) *form a basis over* $\mathbb{R}(t)$ *for the set of all linear relations between free chromials.*

Proof. Suppose we have such a linear relation $\sum_{\phi \in \Phi(\mathsf{J})} f_\phi(t) P_{(\mathsf{G},\phi)}(t) = 0$. Using equation (1) we can replace each occurrence of $P_{(\mathsf{G},\phi)}(t)$ where ϕ is non-planar by a linear combination involving terms $P_{(\mathsf{G},\pi)}(t)$ where the π are all planar. Thus we get a relation $\sum_{\pi \in \Pi(\mathsf{J})} f_\pi(t) P_{(\mathsf{G},\pi)}(t) = 0$, where the sum is over *planar* partitions. It is enough to show that in such a relation we have $f_\pi(t) = 0$ for all $\pi \in \Pi(\mathsf{J})$.

Take G to have no vertices outside of J so that it is given by some planar partition $\sigma \in \Pi(\mathsf{J})$. Then $P_{(\mathsf{G},\pi)}(t) = t^{\ell(\pi \vee \sigma)}$ and the relation reads $\sum_{\pi \in \Pi(\mathsf{J})} f_\pi(t) t^{\ell(\pi \vee \sigma)} = 0$. Since this holds for any planar partition σ and the chromatic join matrix $\left[t^{\ell(\pi \vee \sigma)}\right]$ is invertible it follows that $f_\pi(t) = 0$ for all planar partitions π. □

3. The relation between Tutte's invariants, the Temperley-Lieb algebra and the partition algebra

The bivariate Temperley-Lieb algebra $\mathbb{TL}_r(x, y)$ will enable us to gain further insight into Tutte's invariants. We introduce this algebra and then use it to determine the determinant of the matrix $\mathbf{M}_r$ of chromatic joins. Part of the motivation for doing this is to find further properties of $\mathbf{M}_r$. The Temperley-Lieb algebra is used to obtain a natural decomposition of $\mathbf{M}_r^{-1}$ into the product of an upper triangular matrix $\mathbf{P}_r$ and its transpose. Further details regarding the structure of $\mathbb{TL}_r(x, y)$ and its relationship to $\mathbf{M}_r$ are given in [CJ].

3.1. **The Temperley-Lieb algebra.** The *bivariate Temperley-Lieb algebra* $\mathbb{TL}_r(x, y)$ is the free additive algebra over $\mathbb{R}(x, y)$ with multiplicative generators

$1, \mathsf{e}_1, \ldots, \mathsf{e}_{r-1}$ subject to the relations

S1 $\mathsf{e}_i^2 = x_i \mathsf{e}_i$ for $i = 1, \ldots, r-1$, where $x_i = x$ if i odd, and $x_i = y$ if i even,

S2 $\mathsf{e}_i \mathsf{e}_j = \mathsf{e}_j \mathsf{e}_i$ if $|i-j| > 1$,

S3 $\mathsf{e}_i \mathsf{e}_{i\pm 1} \mathsf{e}_i = \mathsf{e}_i$ for $i = 1, \ldots, r-1$,

where x and y commute with all elements, and 1 is the multiplicative identity. Thus

$$\mathbb{TL}_r(x,y) = \mathbb{R}(x,y)\langle 1, \mathsf{e}_1, \ldots, \mathsf{e}_{r-1}\rangle/(S1, S2, S3).$$

$TL_r(x,y)$ can be constructed from the standard Temperley-Lieb algebra $\mathbb{TL}_r(q)$ by the change of variables $\mathsf{e}_i' := t\mathsf{e}_i$ if i is odd and $\mathsf{e}_i' := \frac{1}{t}\mathsf{e}_i$ if i is even, where $x = qt$ and $y = q/t$. Consequently, representation theoretically, $\mathbb{TL}_r(x,y)$ is no more interesting than $\mathbb{TL}_r(q)$. Our interest in $\mathbb{TL}_r(x,y)$ lies in its connection to 2-coloured planar strand diagrams. For a more detailed study of $\mathbb{TL}_r(x,y)$, see [CJ].

3.1.1. *Planar partitions and strand diagrams.* There is an elementary bijection between the set Π_r of all planar partitions of J_r and the set $\mathcal{S}_r$ of rooted, 2-coloured, planar strand diagrams (see, for example, Section 6.2.1 of [M]). This bijection is best described by an example. Consider the planar partition $\pi_0 = \{[1,2,6],[3,4],[5]\} \in \Pi_6$. Figure 2 shows how the planar diagram for π_0 is transformed via vertex splitting into a strand diagram representing an element in the Temperley-Lieb algebra. This construction is clearly reversible and extends naturally to all planar partitions.

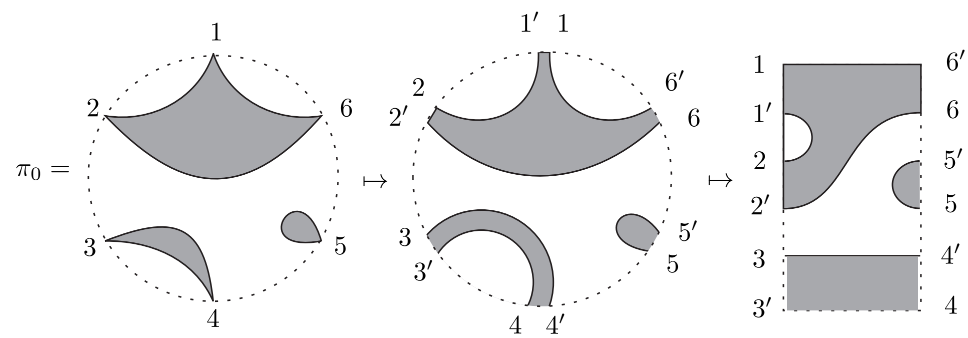

FIGURE 2. Construction of the strand diagram for the planar partition $\pi_0 = \{[1,2,6],[3,4],[5]\}$ via vertex splitting.

The connection of $\mathbb{TL}_r(x,y)$ to planar strand diagrams is through the combinatorial presentation of the generators $1, \mathsf{e}_1, \ldots, \mathsf{e}_{n-1}$ as properly 2-coloured *planar strand diagrams* as illustrated in Figure 3. This connection is known (see example 2.8 in [J]).

The product in $\mathbb{TL}_r(x,y)$ corresponds to the concatenation of strand diagrams with the convention that each black loop so formed is marked by an x and deleted, and each white loop is marked by a y and deleted. It is a good exercise to check the three relations $(S1, S2, S3)$ diagrammatically. Consequently, there is a bijection between monomials in $\mathbb{TL}_r(x,y)$ and planar strand diagrams.

We summarize the correspondences as follows. To every planar partition $\pi \in \Pi_r$ there corresponds a planar 2-coloured strand diagram, henceforth denoted by p_π.

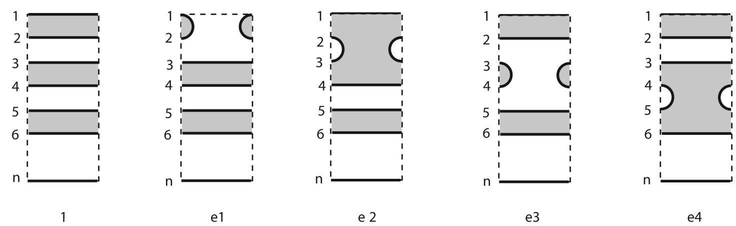

FIGURE 3. Strand diagrams corresponding to generators of $\mathsf{1}, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4 \in \mathbb{TL}_n$.

Moreover, each such strand diagram corresponds to a monomial $\mathsf{p}_\pi \in \mathbb{TL}_r(x,y)$. With a minor abuse of notation, we denote a monomial $\mathsf{p}_\pi \in \mathbb{TL}_r(x,y)$ and the corresponding strand diagram by the same symbol.

3.1.2. *A bilinear form on* $\mathbb{TL}_r(x,y)$. For $\pi \in \Pi_r$ the closure $\overline{\mathsf{p}_\pi}$ of p_π is obtained by joining the ends of the strands of p_π by arcs. Figure 4 illustrates the closure of p_{π_0}.

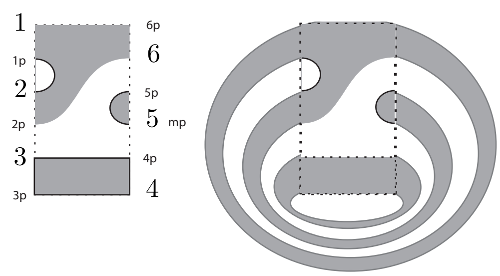

FIGURE 4. Loop diagram for the closure of p_{π_0}.

If $\pi \in \Pi_r$, let $\mathrm{sh}(\pi)$ and $\mathrm{ush}(\pi)$ be, respectively, the number of (finite) shaded and unshaded regions in the closure $\overline{\mathsf{p}_\pi}$ of p_π. Let

$$\mathrm{tr}\colon \mathbb{TL}_r(x,y) \to \mathbb{R}(x,y)\colon \mathsf{p}_\pi \mapsto x^{\mathrm{sh}(\pi)} y^{\mathrm{ush}(\pi)},$$

extended linearly to $\mathbb{TL}_r(x,y)$. For example, $\mathrm{tr}(\pi_0) = x^2y^2$. Notice that $\mathrm{tr}(\mathsf{ab}) = \mathrm{tr}(\mathsf{ba})$ for all $\mathsf{a}, \mathsf{b} \in \mathbb{TL}_r(x,y)$, so tr is a trace function in $\mathbb{TL}_r(x,y)$. It is worthwhile noting that the obvious generalization of this trace is no longer well-defined if the bivariate Temperley-Lieb algebra is replaced by the obvious multivariate generalization (where $\mathsf{e}_i^2 = x_i \mathsf{e}_i$ for some indeterminates x_i).

The *transpose* p_π^t of a strand diagram p_π is obtained by flipping the diagram about a vertical axis. If $\mathsf{a} = \mathsf{p}_\pi \in \mathbb{TL}_r(x,y)$ is a monomial, then we define the transpose of a by $\mathsf{a}^t = \mathsf{p}_\pi^t$, extending linearly to all elements of $\mathbb{TL}_r(x,y)$. Hence we have:

- $\mathsf{e}_i^t = \mathsf{e}_i$ for $i = 1, \ldots, r-1$,
- $(\mathsf{ab})^t = \mathsf{b}^t \mathsf{a}^t$ for all $\mathsf{a}, \mathsf{b} \in \mathbb{TL}_r(x,y)$ and
- $(c_1\mathsf{a} + c_2\mathsf{b})^t = c_1\mathsf{a}^t + c_2\mathsf{b}^t$ for all $\mathsf{a}, \mathsf{b} \in \mathbb{TL}_r(x,y)$ and all $c_1, c_2 \in \mathbb{R}(x,y)$.

We now define a symmetric bilinear form on $\mathbb{TL}_r(x,y)$ as follows:

$$\langle\cdot,\cdot\rangle\colon \mathbb{TL}_r(x,y)\times\mathbb{TL}_r(x,y)\to\mathbb{R}(x,y)\colon (\mathsf{a},\mathsf{b})\mapsto \operatorname{tr}(\mathsf{b}^t\mathsf{a}). \tag{2}$$

The Gram matrix of $\langle\cdot,\cdot\rangle$ is denoted by $\mathbf{M}_r(x,y)$. In other words, the rows and columns of $\mathbf{M}_r(x,y)$ are indexed by planar partitions with the entry in position (π_1,π_2) being $\langle \mathsf{p}_{\pi_1},\mathsf{p}_{\pi_2}\rangle$. The key point is:

Proposition 3.1. *The Gram matrix* $\mathbf{M}_r(t,1)$ *is the same as the matrix of chromatic joins.*

Proof. Recall that $\langle \mathsf{p}_{\pi_1},\mathsf{p}_{\pi_2}\rangle = x^a y^b$, where a is the number of shaded regions in the closure of $\mathsf{p}^t_{\pi_2}\pi_{\pi_1}$ and b is the number of unshaded regions. Thus, if we set $x=t$ and $y=1$, it is not hard to see that by construction $\langle \mathsf{p}_{\pi_1},\mathsf{p}_{\pi_2}\rangle = t^{\ell(\pi_1\vee\pi_2)}$. Hence the Gram matrix is identified with the chromatic join matrix. □

This provides a link between the chromatic join matrix (or Tutte invariants) and the Temperley-Lieb algebra $\mathbb{TL}_r(t,1)$. For instance, in [CJ] we computed the determinant of the chromatic join matrix, or more generally the determinant of the Gram matrix $\mathbf{M}_r(x,y)$, by using the structure of $\mathbb{TL}_r(x,y)$.

Remark. In [CJ] we computed $\det \mathbf{M}_r(x,y)$ by working in $\mathbb{TL}_r(x,y)$. That argument can be simplified by working instead in the left ideal $\mathbb{K}_{2r}\subset\mathbb{TL}_{2r}(x,y)$ generated by the monomials corresponding to the set $\widehat{\Pi}_r$ of partitions in Π_r that are augmented by singletons; *i.e.*, $\widehat{\Pi}_r=\{\pi\cup\{[r+1],\dots,[2r]\}\colon \pi\in\Pi_r\}$. If $\pi\in\Pi_r$, then the corresponding augmented element $\pi\cup\{[r+1],\dots,[2r]\}$ is denoted by $\widehat{\pi}$. Then if $\pi,\gamma\in\Pi_r$, it is easy to see $\langle\widehat{\mathsf{a}},\widehat{\mathsf{b}}\rangle = t^{r+\ell(\pi\vee\gamma)}$ and hence $\mathbf{M}_r(t,1)=t^{-r}[\langle\widehat{\mathsf{a}},\widehat{\mathsf{b}}\rangle]_{\widehat{\mathsf{a}},\widehat{\mathsf{b}}\in\mathbb{K}_{2r}}$. Then one can use the orthonormal basis from [CJ] to diagonalize $\mathbf{M}_r(t,1)$ and compute its determinant. The advantage is that to construct this orthonormal basis for $\mathbb{K}_{2r}$ is easier than to construct it for $\mathbb{TL}_r(x,y)$ since one can avoid using Jones-Wenzl projectors.

3.2. **The partition algebra.** In this section we show that Tutte's chromatic invariant $\vartheta_{\phi,\pi}(t)$ can be understood in terms of an orthogonal projection from the partition algebra $\mathbb{P}_r$ to the Temperley-Lieb algebra $\mathbb{TL}_r\hookrightarrow\mathbb{P}_r$. A detailed account of the partition algebra is given in [HR]. Also, chapter 8 of [M] contains relevant constructions of representations of the partition algebra.

We have seen that $\mathbb{TL}_r(x,y)$ is an algebra whose standard basis is in bijection with the set Π_r of planar partitions of $\{1,\dots,r\}$. This idea can be extended naturally to include all partitions of $\{1,\dots,r\}$ and gives us the partition algebra $\mathbb{P}_r(t)$. The standard basis of $\mathbb{P}_r(t)$ is in bijection with all partitions of $\{1,\dots,r\}$. As before, we draw partitions as 2-coloured strand diagrams except that now the strands can cross (see, for example, Figure 5). Multiplication corresponds to concatenation of strand diagrams. The variable t is used to mark shaded components. Unfortunately it is not possible to have a bivariate version of $\mathbb{P}_r(t)$, so we shall have only an inclusion $\mathbb{TL}_r(t,1)\hookrightarrow\mathbb{P}_r(t)$. Given a partition $\phi\in\Phi_r$ of $\{1,\dots,r\}$, we shall denote by p_ϕ the corresponding monomial in $\mathbb{P}_r(t)$.

The trace tr and the bilinear form $\langle\cdot,\cdot\rangle$ can be extended from $\mathbb{TL}_r(t,1)$ to $\mathbb{P}_r(t)$ by extending our old definitions. That is to say, the trace is obtained by closing up a diagram and counting the number of shaded regions which are marked by t. The inner product is $\langle\mathsf{a},\mathsf{b}\rangle=\operatorname{tr}(\mathsf{a}\mathsf{b}^t)=\operatorname{tr}(\mathsf{b}^t\mathsf{a})$, where b^t denotes the transpose of b (the diagram obtained from b by flipping).

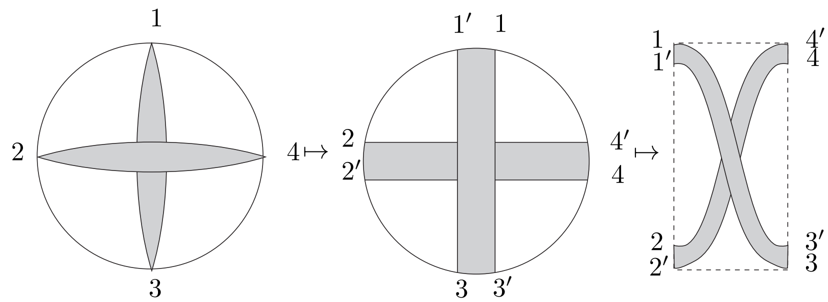

FIGURE 5. The non-planar partition $\phi = \{[1,3],[2,4]\}$ and the corresponding element $\mathsf{p}_\phi \in \mathbb{P}_4$.

3.2.1. *An orthogonal projection.*

Lemma 3.2. *The restriction of the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{P}_r(t)$ to $\mathbb{TL}_r(t,1)$ is non-degenerate.*

Proof. The inner product $\langle \cdot, \cdot \rangle$ is non-degenerate if the determinant of its Gram matrix is non-zero. But the Gram matrix for $\mathbb{TL}_r(t,1)$ is $\mathbf{M}_r(t,1)$ whose determinant is calculated in [CJ, Theorem 5.6] to be non-zero. $\square$

The following theorem recovers Tutte's chromatic invariants $\vartheta_{\phi,\pi}$ from $\mathbb{P}_r(t)$ and $\mathbb{TL}_r(t,1)$.

Theorem 3.3. *Let $\phi \in \Phi_r$ be any partition and denote by* $\mathrm{proj}^\perp \colon \mathbb{P}_r(t) \to \mathbb{TL}_r(t,1)$ *the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$. Then*

$$\mathrm{proj}^\perp(\mathsf{p}_\phi) = \sum_{\pi \in \Pi_r} \vartheta_{\phi,\pi} \mathsf{p}_\pi.$$

Proof. First notice that $\mathrm{proj}^\perp$ is well-defined by Lemma 3.2. From Theorem 2.1,

$$P_{(\mathrm{ext}_\mathsf{J}\mathsf{G},\phi)}(t) = \sum_{\pi \in \Pi_r} \vartheta_{\phi,\pi}(t) P_{(\mathrm{ext}_\mathsf{J}\mathsf{G},\pi)}(t),$$

where $\vartheta_{\phi,\pi}(t)$ is independent of G. Select $\mathrm{ext}_\mathsf{J}\mathsf{G}$ to be a graph embedded in the plane with no vertices in the exterior of J. This graph induces a planar partition π' on $V(\mathsf{J})$. Moreover, $P_{\mathrm{ext}_\mathsf{J}(\mathsf{G},\phi)}(t) = t^{\ell(\pi' \vee \phi)}$ and $P_{(\mathrm{ext}_\mathsf{J}(\mathsf{G}),\pi)}(t) = t^{\ell(\pi' \vee \pi)}$. Thus

$$t^{\ell(\pi' \vee \phi)} = \sum_{\pi \in \Pi_r} \vartheta_{\phi,\pi}(t) t^{\ell(\pi' \vee \pi)} \tag{3}$$

for any planar partition π' and any partition ϕ. Now $\mathrm{proj}^\perp(\mathsf{p}_\phi) \in \mathbb{TL}_r$, so $\mathrm{proj}^\perp(\mathsf{p}_\phi) = \sum_{\pi \in \Pi_r} b_{\phi,\pi} \mathsf{p}_\pi$ for some scalars $b_{\phi,\pi}$. Then $\langle \mathsf{p}_\phi, \mathsf{p}_{\pi'} \rangle = \langle \mathrm{proj}^\perp(\mathsf{p}_\phi), \mathsf{p}_{\pi'} \rangle = \sum_{\pi \in \Pi_r} b_{\phi,\pi} \langle \mathsf{p}_\pi, \mathsf{p}_{\pi'} \rangle$ since $\mathrm{proj}^\perp$ is an orthogonal projection. So $t^r t^{\ell(\phi \vee \pi')} = \sum_{\pi \in \Pi_r} b_{\phi,\pi} t^r t^{\ell(\pi \vee \pi')}$, whence $t^{\ell(\pi' \vee \phi)} = \sum_{\pi \in \Pi_r} b_{\phi,\pi} t^{\ell(\pi' \vee \pi)}$. It follows from (3) that $b_{\phi,\pi} = \vartheta_{\phi,\pi}(t)$. $\square$

Note that if $\phi = \pi'$ is planar, then from Theorem 3.3, $\vartheta_{\pi,\pi'}(t) = \delta_{\pi,\pi'}$ as expected.

3.3. **Higher genus Tutte invariants.** The *genus* $g(\phi)$ of a partition ϕ is defined to be the smallest integer $g(\phi)$ so that there exists an embedding of the partition diagram of ϕ in a surface of genus $g(\phi)$. For example, $g(\phi) = 0$ if and only if ϕ is planar. We denote by Φ_r^g and $\Phi_r^{\leq g}$ the partitions of $\{1, \ldots, r\}$ of genus g and genus less than or equal to g, respectively. For example, $\Phi_r^0 = \Pi_r$ while for $g \gg 0$ we have $\Phi_r^g = \Phi_r^{\leq g} = \Phi_r$ because any partition of $\{1, \ldots, r\}$ has genus at most r.

On the other hand, a graph G has genus g if it can be embedded without crossing edges on a surface of genus g while it cannot be embedded on a surface of genus less than g. For example, graphs embeddable in the plane have genus 0.

The following theorem shows that there exist higher genus Tutte chromatic invariants.

Theorem 3.4. *Let G be a graph embedded in a surface of genus g and let J be a contractible circuit of G of length $r < \infty$. Then for every partition $\phi \in \Phi_r$ there exist invariants $\vartheta^g_{\phi,\pi}(t) \in \mathbb{Q}((t))$, where $\pi \in \Phi^{\leq g}$, which are independent of G and satisfy*

$$P_{(\mathsf{G},\phi)}(t) = \sum_{\pi \in \Phi_r^{\leq g}} \vartheta^g_{\phi,\pi}(t) P_{(\mathsf{G},\pi)}(t).$$

Moreover, $\vartheta^g_{\phi,\pi}(t)$ can be determined from the equation

$$\left[t^{\ell(\phi\vee\sigma)}\right]_{B_r \times C_r^g} = \left[\vartheta_{\phi,\pi}\right]_{B_r \times C_r^g} \left[t^{\ell(\pi\vee\sigma)}\right]_{C_r^g \times C_r^g},$$

where C_r^g is the number of partitions of $\{1, \ldots, r\}$ of genus less than or equal to g.

Proof. The proof is the same as that of Theorem 2.1 except for the base case of the induction in which the circuit J is now on a surface of genus g. The same argument as before goes through except that now we can express the free chromials $P_{(\mathsf{G},\phi)}(t)$ as a sum over partitions of genus less than or equal to g instead of a sum over partitions of genus 0.

□

Let $\mathbb{TL}_r^g(t)$ be the span of p_ϕ for all $\phi \in \Phi^{\leq g}$. Note that $\mathbb{TL}_r^g(t) \subset \mathbb{P}_r(t)$ is not a subalgebra (unless $g = 0$ or $g \gg 0$). Nevertheless, they do form a graded filtration $\mathbb{TL}_r^0 \subset \mathbb{TL}_r^1 \subset \cdots \subset \mathbb{TL}_r^{g \gg 0} = \mathbb{P}_r$ since $\mathbb{TL}_r^{g_1} \cdot \mathbb{TL}_r^{g_2} \subseteq \mathbb{TL}_r^{g_1+g_2}$.

Lemma 3.5. *The restriction of the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{P}_r(t)$ to $\mathbb{TL}_r^g(t)$ is non-degenerate.*

Proof. To show that the determinant of the Gram matrix of $\mathbb{TL}_r^g$ is non-zero, one proceeds as in the proof of Theorem 2.1 (which deals with the case of $\mathbb{TL}_r^0$). □

The following generalizes Theorem 3.3.

Theorem 3.6. *Let $\phi \in \Phi_r$ be any partition and denote by $\mathrm{proj}^\perp_{\mathbb{TL}_r^g} : \mathbb{P}_r(t) \to \mathbb{TL}_r(t)$ the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$. Then*

$$\mathrm{proj}^\perp_{\mathbb{TL}_r^g}(\mathsf{p}_\phi) = \sum_{\pi \in \Phi_r^{\leq g}} \vartheta^g_{\phi,\pi} \mathsf{p}_\pi.$$

Proof. The proof is analogous to that of Theorem 3.3. By Lemma 3.5, $\mathrm{proj}^\perp$ is well-defined. From Theorem 3.4,

$$P_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t) = \sum_{\pi \in \Phi_r^{\leq g}} \vartheta^g_{\phi,\pi}(t) P_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\pi)}(t),$$

where $\vartheta^g_{\phi,\pi}(t)$ is independent of G. Select $\mathrm{ext}_{\mathsf{J}}\mathsf{G}$ to be a graph embedded in a surface of genus g with no vertices in the exterior of J. This graph induces a partition π' of genus $\leq g$ on $V(\mathsf{J})$. Moreover, $P_{\mathrm{ext}_{\mathsf{J}}(\mathsf{G},\phi)}(t) = t^{\ell(\pi'\vee\phi)}$ and $P_{(\mathrm{ext}_{\mathsf{J}}(\mathsf{G}),\pi)}(t) = t^{\ell(\pi'\vee\pi)}$. Thus

$$t^{\ell(\pi'\vee\phi)} = \sum_{\pi\in\Phi_r^{\leq g}} \vartheta^g_{\phi,\pi}(t) t^{\ell(\pi'\vee\pi)} \tag{4}$$

for any partition π' of genus $\leq g$ and any partition ϕ. Now $\mathrm{proj}^{\perp}(\mathsf{p}_\phi) \in \mathbb{TL}^g_r$, so $\mathrm{proj}^{\perp}(\mathsf{p}_\phi) = \sum_{\pi\in\Phi_r^{\leq g}} b_{\phi,\pi}\mathsf{p}_\pi$ for some scalars $b_{\phi,\pi}$. Then $\langle \mathsf{p}_\phi, \mathsf{p}_{\pi'}\rangle = \langle \mathrm{proj}^{\perp}(\mathsf{p}_\phi), \mathsf{p}_{\pi'}\rangle$ $= \sum_{\pi\in\Phi_r^{\leq g}} b_{\phi,\pi}\langle \mathsf{p}_\pi, \mathsf{p}_{\pi'}\rangle$ since $\mathrm{proj}^{\perp}$ is an orthogonal projection. So $t^r t^{\ell(\phi\vee\pi')} = \sum_{\pi\in\Phi_r^{\leq g}} b_{\phi,\pi} t^r t^{\ell(\pi\vee\pi')}$, whence $t^{\ell(\pi'\vee\phi)} = \sum_{\pi\in\Phi_r^{\leq g}} b_{\phi,\pi} t^{\ell(\pi'\vee\pi)}$. It follows from (4) that $b_{\phi,\pi} = \vartheta^g_{\phi,\pi}(t)$. □

4. The connection between the Tutte invariants, the Birkhoff-Lewis equations and the Four Colour Theorem

The familiar approach to proving the Four Colour Theorem involves first using a discharging algorithm to find a complete set of unavoidable graphs, that is, a set such that any minimal counterexample to the theorem would have to contain one of these graphs. Kempe chains are then used to obtain relations between (constrained) chromatic chromials of these graphs. If the graphs are large enough, then these relations can be used to obtain a contradiction (thus concluding the graphs cannot in fact occur as a subgraph of a minimal counterexample to the Four Colour Theorem). However, as the graphs grow, so does their number (this forces the use of a computer). Recent approaches such as those of Appel and Haken ([AH1],[AHK]) or Robertson *et al.* [RSST] involve reducing the size of the set of unavoidable graphs. The relations obtained from Kempe chains may only comprise a fraction of all the existing relations. Consequently, one characteristic of their approach is that many graphs are required but only some of the relations are used.

In contrast one can try to generate more relations in the hope of being able to use a smaller set of graphs. The Tutte invariants provide a way to obtain all linear relations (see Theorem 2.3) in a systematic manner rather than by using Kempe chains. The tradeoff is having to generate more equations instead of having to check many graphs, so we are somewhat pessimistic that this approach will lead to an algebraic proof of the Four Colour Theorem which is less dependent on computer assistance than the current one.

We now describe the rôle played by the Tutte invariants in the construction of the Birkhoff-Lewis equations [BL] and summarise how these relate to the Four Colour Theorem. Let $\mathrm{int}_{\mathsf{J}}\mathsf{G}$ and $\mathrm{ext}_{\mathsf{J}}\mathsf{G}$ be, respectively, the subgraphs of G in the interior and exterior of J that, by convention, include J.

4.1. **Constrained chromials.** Let G be a graph and let J be a circuit of G of length r with a non-trivial interior and exterior. The *constrained chromial* $\dot{P}_{(\mathsf{G},\phi)}(t)$ of (G,ϕ) is the number of ways of colouring G with t colours such that the vertices of G in the same block of ϕ receive the same colour and vertices in different blocks receive different colours. The chromial $P_{\mathsf{G}}(t)$ may be expressed immediately in terms of constrained chromials of graphs with fewer vertices as follows.

Lemma 4.1. *Let* G *be a graph with a circuit* J*. Then*

$$P_{\mathsf{G}}(t) = \sum_{\phi \in \Phi(\mathsf{J})} \frac{1}{(t)_{l(\phi)}} \dot{P}_{(\mathrm{int}_{\mathsf{J}}\mathsf{G},\phi)}(t) \dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t),$$

where $(t)_k = t(t-1)\cdots(t-k+1)$*. (Note that* $\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)/(t)_{l(\phi)}$ *is a polynomial in* t*.)*

Proof. Any colouring of G induces a colouring of J and thence a unique partition $\phi \in \Phi(\mathsf{J})$ whose blocks comprise vertices of J with the same colour. The number of colourings of $(\mathrm{int}_{\mathsf{J}}\mathsf{G}, \phi)$ with at most t colours is $\dot{P}_{(\mathrm{int}_{\mathsf{J}}\mathsf{G},\phi)}(t)$. □

Lemma 4.2. *Let* G *be a graph and let* J *be a circuit of* G*. Then, for* $\phi \in \Phi(\mathsf{J})$*,*

$$P_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t) = \sum_{\substack{\phi' \in \Phi(\mathsf{J}),\\ \phi' \succeq \phi}} \dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi')}(t). \tag{5}$$

Proof. Trivial. □

4.2. **Examples.** Substituting equation (5) into Tutte's relations (1) yields what are commonly known as the *Birkhoff-Lewis equations*. Birkhoff and Lewis [BL] gave such equations explicitly for $r = 4, \dots, 7$, and proposed a somewhat arbitrary method for constructing them in general. Although the function $\vartheta_{\phi,\pi}$ was implicit in their work, there was not an explicit algebraic characterisation of it until the appearance of Tutte's Theorem.

4.2.1. *The 4-ring equation and an example.* The following example illustrates the use of constrained chromials and Tutte's equations to show that the four-wheel W_4 cannot occur as a subgraph of a minimal counterexample to the Four Colour Theorem. The example serves as the model for the method of excluding a particular graph as a subgraph of a minimal counterexample.

Table 1. Notation for the constrained chromials for the 4-ring.

$\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)$	$\phi_i \in \Phi(\mathsf{J})$
$\dot{A}_i$	$\{[i,i+2],[i+1,i+3]\}$
$\dot{B}_i$	$\{[i,i+2],[i+1],[i+3]\}$
$\dot{C}_i$	$\{[i],[i+1],[i+2],[i+3],[i+4]\}$

Lemma 4.3. *If* G *is a minimal counterexample to the Four Colour Theorem, then* W_4 *is not a subgraph of* G*.*

Proof. Suppose G is a minimal counterexample containing W_4 as a subgraph. Let J be such that $\mathrm{int}_{\mathsf{J}}\mathsf{G} = \mathsf{W}_4$. Let $V(\mathsf{J}) = \{1,2,3,4\}$. The constrained chromials of $\mathrm{ext}_{\mathsf{J}}\mathsf{G}$ are $\dot{A}, \dot{B}_1, \dot{B}_2$ and $\dot{C}$, as indicated in Table 1. For example, $\dot{A}$ is the constrained chromial corresponding to the only non-planar partition $\{[1,3],[2,4]\}$. Let A, B_1, B_2 and C denote the corresponding free chromials.

There are two sources of relations between these constrained chromials, namely Tutte's relations (Theorem 2.1) and Lemma 4.1, and we consider these in turn.

Theorem 2.1 gives one relation between the free chromials for each non-planar partition. Since the 4-ring has only one such partition, there is precisely one Tutte relation which, by direct calculation, is

$$(t^2 - 5t + 5)A - (t-1)(t-3)(B_1 + B_2) + (t-2)(t-3)C = 0.$$

From Lemma 4.2, which expresses the free chromials in terms of the constrained chromials, we obtain

$$\begin{array}{rcl} A & = & \dot{A}, \\ B_1 & = & \dot{B}_1 + \dot{A}, \\ B_2 & = & \dot{B}_2 + \dot{A}, \\ C & = & \dot{C} + \dot{B}_1 + \dot{B}_2 + \dot{A}, \end{array}$$

so, substituting these into Tutte's relation gives

$$\dot{A} - (t-3)(\dot{B}_1 + \dot{B}_2) + (t-2)(t-3)\dot{C} = 0 \tag{6}$$

and there are no other relations between free chromials (and hence constrained chromials) by Theorem 2.3.

We now use Lemma 4.1 to obtain further relations. Suppose G is not 4-colourable. Then $P_{\mathsf{G}}(4) = 0$, so, from Lemma 4.1, $\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)} = 0$ whenever $\dot{P}_{(\mathrm{int}_{\mathsf{J}}\mathsf{G},\phi)} \neq 0$. Thus $\dot{P}_{(\mathsf{W}_4,\phi)} \neq 0$ if $\phi = \{[1,3],[2,4]\}$ (since the vertices of W_4 on the 4-ring may be coloured 1212 in cyclic order) and hence $\dot{A}(4) = 0$. Similarly, $\dot{B}_1(4) = \dot{B}_2(4) = 0$. This accounts for all of the relations that can be obtained in this way from Lemma 4.1.

Combining these with (6) gives $\dot{C}(4) = 0$. But, from Lemma 4.2, we see that $\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G})}(4) = \dot{A}(4) + \dot{B}_1(4) + \dot{B}_2(4) + \dot{C}(4) = 0$. Thus $\mathrm{ext}_{\mathsf{J}}\mathsf{G}$ is a subgraph of G that is *not* 4-colourable, so G is not a minimum counterexample, forcing the contradiction. □

4.2.2. *The* 5*-ring equations and an example.* As a further example, we use the Tutte relations (Theorem 2.1) to rederive the Birkhoff-Lewis equations for the 5-ring.

We shall index the vertices of the 5-ring J by $1, \ldots, 5$ in cyclic order. In Table 2,

TABLE 2. Notation for the constrained chromials for the 5-ring.

$\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)$	$\phi_i \in \Phi(\mathsf{J})$
$\dot{A}_i$	$\{[i],[i-1,i+2],[i+1],[i+3]\}$
$\dot{B}_i$	$\{[i],[i-1,i+2],[i+1,i+3]\}$
$\dot{G}$	$\{[1],[2],[3],[4],[5]\}$

it is understood that the elements in partitions are taken modulo 5, and that 0 is replaced by 5. Suppressing the (routine) details of the matrix computation, we have, from Theorem 2.1, that

$$(t^2 - 3t + 1)B_{i+4} = G + (t-2)(A_{i+1} + A_{i+2}) + A_{i+4}.$$

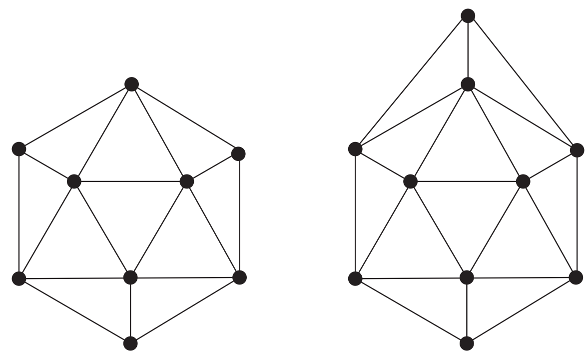
(a) (b)

FIGURE 6. Candidates for an unavoidable set.

These can be expressed wholly in terms of constrained chromials through the relations

$$\begin{aligned} A_2 &= \dot{A}_2 + \dot{B}_5 + \dot{B}_4, \\ A_3 &= \dot{A}_3 + \dot{B}_1 + \dot{B}_5, \\ A_5 &= \dot{A}_5 + \dot{B}_2 + \dot{B}_3, \\ B_5 &= \dot{B}_5, \\ G &= \dot{G} + (\dot{A}_1 + \cdots + \dot{A}_5) + (\dot{B}_1 + \cdots + \dot{B}_5), \end{aligned}$$

obtained from Lemma 4.2. Substituting these into the above relation, we have

$$\dot{G} + (t-2)(t-3)\dot{B}_{i+4} + \dot{A}_i + \dot{A}_{i+3} - (t-3)(\dot{A}_{i+1} + \dot{A}_{i+2} + \dot{B}_1 + \dot{B}_{i+3}) = 0.$$

Let $R_1, \ldots, R_5$ denote these five relations.

These relations differ from the Birkhoff-Lewis equations (eq. (8.4) in [BL]), which were derived by the method of Kempe chains. We can recover the Birkhoff-Lewis equations from $R_1, \ldots, R_5$ by taking the linear combination $R_1 + (t-3)R_2 + (t^2-5t+5)R_3 + (t-3)R_4 + R_5$, giving the relation $(t^2-3t+1)(\dot{G} + (t-3)(t-4)\dot{B}_2) = (t^2-3t+1)(t-4)(\dot{A}_4 + \dot{A}_5)$, whence, cancelling the factor t^2-3t+1, which has no integer roots, $\dot{G} + (t-3)(t-4)\dot{B}_2 = (t-4)(\dot{A}_4 + \dot{A}_5)$, again with the cyclic permutation of indices modulo 5. This is indeed equation (8.4) of [BL]. The fact that we needed to take linear combinations suggests that this approach may differ from the Kempe chain approach employed by Birkhoff and Lewis.

4.2.3. *The 6-ring equations and an example.* As a final example, we examine the case $\mathrm{int}_{\mathsf{J}}\mathsf{G} = \mathsf{T}$, where T is given in Figure 6(a). It is considered by Thomas [To] as a candidate for an unavoidable set of configurations, with the hope that this may lead to a set that is smaller than the set used in [RSST]. Its boundary is a circuit of length 6, so the Birkhoff-Lewis equations, which are known for the 6-ring, are required. We shall see that the Birkhoff-Lewis equations for the 6-ring are not enough to prove that T cannot occur as a subgraph of a minimum counterexample (if they were, the present proof of the Four Colour Theorem could be substantially shortened). The names of each of the constrained chromials $\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)$ are given in Table 3 for each partition ϕ of the vertices of J. The index i takes values in

$\{1,\ldots,6\}$, and elements of blocks are to be evaluated modulo 6 (with 0 to be replaced with 6). The unknown constrained chromials are

$$\dot{B}_i, \dot{F}_i, \dot{G}_i, \dot{J}_i, i=1,\ldots,6; \quad \dot{C}_i, \dot{H}_i, \dot{I}_i, \dot{K}_i, i=1,\ldots,3; \quad \dot{E}_i, i=1,2; \quad \dot{A}, \dot{D}, \dot{L}, \tag{7}$$

since $\dot{C}_{i+3}=\dot{C}_i, \dot{E}_{i+2}=\dot{E}_i, \dot{H}_{i+3}=\dot{H}_i, \dot{I}_{i+3}=\dot{I}_i$, and $\dot{K}_{i+3}=\dot{K}_i$.

Table 3. Notation for the constrained polynomial for the 6-ring.

$\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)$	$\phi_i \in \Phi(\mathsf{J})$
$\dot{A}$	$\{[i,i+2,i+4],[i+1,i+3,i+5]\}$
$\dot{B}_i$	$\{[i],[i+1,i+3,i+5],[i+2,i+4]\}$
$\dot{C}_i$	$\{[i,i+3],[i+1,i+5],[i+2,i+4]\}$
$\dot{D}$	$\{[i,i+3],[i+1,i+4],[i+2,i+5]\}$
$\dot{E}_i$	$\{[i],[i+1,i+3,i+5],[i+2],[i+4]\}$
$\dot{F}_i$	$\{[i,i+3],[i+1,i+5],[i+2],[i+4]\}$
$\dot{G}_i$	$\{[i],[i+1,i+3],[i+2,i+4],[i+5]\}$
$\dot{H}_i$	$\{[i],[i+1,i+5],[i+2,i+4],[i+3]\}$
$\dot{I}_i$	$\{[i],[i+1,i+4],[i+2,i+5],[i+3]\}$
$\dot{J}_i$	$\{[i],[i+1,i+5],[i+2],[i+3],[i+4]\}$
$\dot{K}_i$	$\{[i],[i+3],[i+1],[i+2],[i+4,i+5]\}$
$\dot{L}$	$\{[i],[i+1],[i+2],[i+3],[i+4],[i+5]\}$

The following is the complete set of Birkhoff-Lewis equations for the 6-ring and is taken from [BL], p. 442.

$$(8)\quad \left\{\begin{aligned} (t-3)\Big((t-2)\dot{A}-\dot{B}_i-\dot{B}_{i+1}+\dot{B}_{i+3}-\dot{B}_{i+5}\Big)+\dot{G}_i+\dot{G}_{i+1}-\dot{H}_i&=0, && (.1)\\ (t-3)(\dot{B}_i-\dot{C}_i)-\dot{E}_i+\dot{F}_i&=0, && (.2)\\ \dot{F}_{i+1}+\dot{F}_{i+2}-\dot{G}_{i+5}-\dot{I}_i&=0, && (.3)\\ (t-3)(\dot{D}-\dot{C}_i)+\dot{H}_i-\dot{I}_i&=0, && (.4)\\ (t-3)(t-4)\dot{B}_i-(t-4)(\dot{E}_i+\dot{H}_i)+\dot{J}_i&=0, && (.5)\\ (t-4)(-\dot{F}_i+\dot{G}_{i+3})-\dot{J}_{i+5}+\dot{K}_i&=0, && (.6)\\ (t-3)(t-4)(t-5)\dot{C}_i-(t-4)(t-5)\dot{H}_i-(t-5)\dot{K}_i+\dot{L}&=0, && (.7) \end{aligned}\right.$$

for $i=1,\ldots,6$, and for all $t\geq 1$.

We now attempt to show that T cannot occur in a minimum counterexample. Suppose that G is a minimum counterexample with ($\mathrm{int}_{\mathsf{J}}\mathsf{G}=\mathsf{T}$). It is readily found that the only partitions ϕ that allow a 4-colouring of (T,ϕ) are associated with the constrained chromials $\dot{B}_1,\dot{B}_3,\dot{B}_5,\dot{E}_1,\dot{E}_2,\dot{F}_2,\dot{F}_4,\ \dot{F}_6,\dot{G}_1,\ldots,\dot{G}_6$. Thus

$$\dot{b}_1=\dot{b}_3=\dot{b}_5=\dot{e}_1=\dot{e}_2=\dot{f}_2=\dot{f}_4=\dot{f}_6=\dot{g}_1=\cdots=\dot{g}_6=0, \tag{9}$$

where, for example, $\dot{b}_i$ denotes $\dot{B}_i(4)$.

If $\ell(\phi)>4$, then $\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(4)=0$, so we set $\dot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)=(t-4)\ddot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)$, where $\ddot{P}_{(\mathrm{ext}_{\mathsf{J}}\mathsf{G},\phi)}(t)$ is a polynomial in t. Let $\ddot{J}_i,\ddot{K}_i,\ddot{L}_i$ be, respectively, the corresponding polynomials for $\dot{J}_i,\dot{K}_i$ and $\dot{L}_i$ where these are the constrained polynomials associated with partitions ϕ having $\ell(\phi)>4$. Now, from (8.5), we have

$(t-4)((t-3)\dot{B}_i - (\dot{E}_i + \dot{H}_i) + \ddot{J}_i) = 0$, so $(t-3)\dot{B}_i - (\dot{E}_i + \dot{H}_i) + \ddot{J}_i = 0$ for all $t \geq 1$. Doing the same with (8.6) and (8.7), we get the equations

$$\text{(10)} \qquad \left\{ \begin{aligned} (t-3)\dot{B}_i - \dot{E}_i - \dot{H}_i + \ddot{J}_i &= 0, \quad (.5') \\ -\dot{F}_i + \dot{G}_{i+3} - \dot{J}_{i+5} + \ddot{K}_i &= 0, \quad (.6') \\ (t-3)(t-5)\dot{C}_i - (t-5)\dot{H}_i - (t-5)\ddot{K}_i + \ddot{L} &= 0, \quad (.7') \end{aligned} \right.$$

for all t, to replace equations (8.5), (8.6) and (8.7), respectively. Using (9), the Birkhoff-Lewis equations therefore become

$$\text{(11)} \qquad \left\{ \begin{aligned} 2\dot{a} - \dot{b}_i - \dot{b}_{i+1} + \dot{b}_{i+3} - \eth_i &= 0, \quad (.1) \\ \dot{b}_i - \dot{c}_i + \dot{f}_i &= 0, \quad (.2) \\ \dot{f}_{i+1} + \dot{f}_{i+2} - \dot{\imath}_i &= 0, \quad (.3) \\ \dot{d} - \dot{c}_i + \eth_i - \dot{\imath}_i &= 0, \quad (.4) \\ \dot{b}_i - \eth_i + \ddot{\jmath}_i &= 0, \quad (.5') \\ -\dot{f}_i - \ddot{\jmath}_{i+5} + \ddot{k}_i &= 0, \quad (.6') \\ -\dot{c}_i + \eth_i + \ddot{k}_i + \ddot{\ell} &= 0, \quad (.7') \end{aligned} \right.$$

for $i = 1, \dots, 6$.

From (11.1) with $i = 2$, and (9), we have $\dot{b}_2 = 2\dot{a} - \eth_2$. Again from (11.1), with $i = 1$, and (9), we have $2\dot{a} - \dot{b}_2 + \dot{b}_4 - \dot{b}_6 - \eth_1 = 0$, so elimination of $\dot{b}_2$ between these gives $\eth_2 + \eth_3 = 2\eth_1$. Similarly, $\eth_1 + \eth_2 = 2\eth_1$, so, on eliminating $\eth_2$ between the latter, $\eth_1 = \eth_3$, and similarly for $\eth_2$. Thus

$$\text{(12)} \qquad \eth_1 = \eth_2 = \eth_3.$$

Then, using the above expression for $\dot{b}_2$ we have $\dot{b}_2 = \dot{b}_4 = \dot{b}_6$ and $2\dot{a} = \eth_2 + \dot{b}_2$.

From (11.2), for i odd, we have $\dot{c}_1 = \dot{f}_1, \dot{c}_2 = \dot{f}_3, \dot{c}_3 = \dot{f}_5$ and, again, for i even, $\dot{c}_1 = \dot{b}_4, \dot{c}_2 = \dot{b}_2, \dot{c}_3 = \dot{b}_6$, so combining these with the above observations of $\dot{b}_i$, we have $\dot{b}_2 = \dot{b}_4 = \dot{b}_6 = \dot{c}_1 = \dot{c}_2 = \dot{c}_3 = \dot{f}_1 = \dot{f}_3 = \dot{f}_5$. But, from (11.3), $\dot{\imath}_1 = \dot{\imath}_2 = \dot{\imath}_3 = \dot{f}_1 = \dot{f}_3 = \dot{f}_5$, so from the previous equation we have

$$\text{(13)} \qquad \dot{b}_2 = \dot{b}_4 = \dot{b}_6 = \dot{c}_1 = \dot{c}_2 = \dot{c}_3 = \dot{f}_1 = \dot{f}_3 = \dot{f}_5 = \dot{\imath}_1 = \dot{\imath}_2 = \dot{\imath}_3.$$

From (11.4), (12) and (13), $\dot{d} = -\eth_2 + 2\dot{b}_2$. Also, from (11.5′), (12) and the relation among the $\dot{b}_i$ we have $\ddot{\jmath}_1 = \ddot{\jmath}_3 = \ddot{\jmath}_5 = \eth_1 = \eth_2 = \eth_3$ and $\ddot{\jmath}_2 = \ddot{\jmath}_4 = \ddot{\jmath}_6 = \eth_2 - \dot{b}_2$. But, from (8.6′), $\ddot{\jmath}_1 = \ddot{k}_2, \ddot{\jmath}_3 = \ddot{k}_1, \ddot{\jmath}_5 = \ddot{k}_2$, so, combining these results,

$$\text{(14)} \qquad \eth_1 = \eth_2 = \eth_3 = \ddot{\jmath}_1 = \ddot{\jmath}_3 = \ddot{\jmath}_5 = \ddot{k}_1 = \ddot{k}_2 = \ddot{k}_3,$$

and $\ddot{\jmath}_2 = \eth_2 - \dot{b}_2$. The only new relation that (11.7′) provides is $-\ddot{\ell} = 2\eth_2 - \dot{b}_2$.

We would have liked to have shown that enough of the constrained polynomials are zero at $t = 4$ to be able to obtain a contradiction, as was possible in the case of W_4. Notice that we are in fact quite close in the sense that we have equations (9), (13) and (14), and we have expressed the remaining unknowns $(\dot{a}, \dot{d}, \ddot{\jmath}_2, \ddot{\ell})$ only in terms of $\dot{b}_2$ and $\eth_2$. Nevertheless, the Birkhoff-Lewis equations do not show that G does not contain T as a subgraph.

We remark in passing that Birkhoff and Lewis [BL] proved (using the Birkhoff-Lewis equations) that the graph in Figure 6(b) cannot occur as a subgraph of a minimum counterexample to the Four Colour Theorem. Interestingly, this is the first graph in the unavoidable set given by [RSST].

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References

[AH1] K. Appel and W. Haken, Every planar map is four colorable. Part I. Discharging, *Illinois J. Math.* **21** (1977), 429-490. MR0543792 (58:27598a)

[AHK] K. Appel, W. Haken, and J. Koch, Every planar map is four colorable. Part II. Reducibility, *Illinois J. Math.* **21** (1977), 491-567. MR0543793 (58:27598b)

[BL] G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355–451. MR0018401 (8:284f)

[CJ] S. Cautis and D. M. Jackson, The matrix of chromatic joins and the Temperley-Lieb algebra, *J. Combin. Theory (Ser. B)* **89** (2003), 109–155. MR1999738 (2004g:05046)

[K] L. Kauffman, Statistical mechanics and the Jones polynomial, *Contemp. Math.* **78** (1988), 263–297. MR975085 (89j:57002)

[KT] L. Kauffman and R. Thomas, *Temperley-Lieb algebras and the Four Colour Theorem.* Available at http://math.uic.edu/∼kauffman/TLFCT.pdf.

[HR] T. Halverson and A. Ram, Partition algebras, *European J. Combinatorics* **26** (2005), 869–921. MR2143201 (2006g:05228)

[J] V. F. R. Jones, Planar Algebras, I. http://xxx.lanl.gov/abs/math/9909027.

[M] P. Martin, *Potts Models and Related Problems in Statistical Mechanics*, World Scientific, 1991. MR1103994 (92m:82030)

[RSST] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four colour theorem, *J. Combin. Theory* (Ser. B) **70** (1997), 2–44. MR1441258 (98c:05065)

[S] R. P. Stanley, *Enumerative Combinatorics*, vols. 1, 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1997, 1999. MR1442260 (98a:05001)

[To] Private communication *per* J. Geelen, December 2004.

[T1] W. T. Tutte, On the Birkhoff-Lewis equations, *Discrete Math.* **92** (1991), 417–425. MR1140602 (92k:05052)

[T2] W. T. Tutte, The matrix of chromatic joins, *J. Combin. Theory (Ser. B)* **57** (1993), 269–288. MR1207492 (94a:05144)

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