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ALGEBRAIZATION OF BUNDLES ON NON-PROPER SCHEMES

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Abstract. We study the algebraization problem for principal bundles with reductive structure groups on a non-proper formal scheme. When the formal scheme can be compactified by adding a closed subset of codimension at least 3, we show that any such bundle admits an algebraization. For codimension 2 we provide a necessary and sufficient condition.

1. Introduction

This work is a contribution toward an algebraic understanding of the Uhlenbeck compactification. Recall, cf. [DK], that for a complex projective surface S the moduli space M_n of semistable vector bundles with fixed rank, determinant and $c_2 = n$ is non-compact, but the union $Uhl_n = \coprod_{s \geq 0} M_{n-s} \times Sym^s S$ can be given a topology of a compact space (since one deals with semistable bundles for $s \gg 0$ the space M_{n-s} will be empty). We will call Uhl_n the Uhlenbeck moduli space, although sometimes this name is reserved for the closure of M_n in Uhl_n.

Some time ago, see e.g. [Li], [BFG], [FGK], the Uhlenbeck moduli space started to appear in algebraic geometry and the higher dimensional Langlands Program. For instance, it is a convenient tool for the study of higher versions of Hecke correspondences which modify a vector bundle on S (more generally, a principal bundle) along a divisor, obtaining a new bundle. For several reasons, we would like to have the definition of Uhl_n as a "functor"; i.e. we want to be able to describe in geometric terms the set of maps $F(T)$ (actually, a category of maps) from any test scheme $T = Spec(A)$ to Uhl_n. First, that would allow us to define Uhl_n over any field k and not to require stability. Second, in the study of the cohomology of Uhl_n and the action of Hecke correspondences on it, one needs to deal with the phenomenon of unexpected dimension of Uhl_n. A possible approach involves defining a "derived moduli space" $DUhl_n$ in the sense of [Lu] which would amount to considering more general "spaces" T. Thus, defining Uhl_n as a functor is a necessary preliminary step to constructing $DUhl_n$.

Very roughly, it is expected that a map $T \to Uhl_n$ should be described by a vector bundle F on an open subset $U \subset T \times S$ such that its complement Z is finite over T, a family ξ of effective zero cycles on S parameterized by T plus an agreement condition between ξ and F. Such a definition gives a "reasonable space" Uhl_n if it satisfies a criterion due to Artin, cf. [Ar], or its "derived" generalization proved in [Lu]. The most difficult part of Artin's criterion is the effectiveness condition: if A is a complete noetherian local k-algebra with maximal ideal $\mathfrak{m}$ and $A_p = A/\mathfrak{m}^{p+1}$,

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one needs to show that $F(Spec(A)) = \varprojlim F(Spec(A_p))$. Ignoring the family of zero cycles ξ (as will be done in this paper), if $X = Spec(A) \times_k S$ and $\widehat{X}$ is its formal completion along the fiber over the closed point of $Spec(A)$, we try to find whether a bundle $\mathcal{F}$ on an open subset $\widehat{U} \subset \widehat{X}$ comes from a bundle F on an open subset $U \subset X$. Such an F is called an *algebraization* of $\mathcal{F}$.

In this paper we prove that, when S has arbitrary dimension and $\widehat{U}$ has a complement of codimension ≥ 3, algebraization always exists (for vector bundles and principal bundles over reductive groups). If $\widehat{U}$ has a complement of codimension ≥ 2, then algebraization exists only under an additional condition (which, in the Uhlenbeck functor case, is guaranteed by the presence of ξ). For the codimension ≥ 3 case the algebraization result is used in [Ba] to show that the corresponding functor of bundles "away from codimension 3" is an algebraic stack.

Earlier similar questions were studied for coherent sheaves on proper schemes by Grothendieck, see [EGAIII], and in the case of Lefschetz type theorems by Grothendieck and Raynaud in [SGA2] and [R]. Although these results do not apply to our case directly, our proof is based on the tools developed in [EGAIII] and [SGA2].

In Section 2 we fix the notation, give examples illustrating some issues to be encountered, and prove algebraization results for vector bundles, summarized in Corollary 2.12. In Section 3 we formulate an algebraization criterion for principal bundles over reductive groups; see Theorem 3.1. Finally, Section 4 provides a categorical restatement of our results; see Theorem 4.2.

2. Algebraization for vector bundles

2.1. **Setup.** We refer the reader to Expose III in [SGA2] regarding basic properties of depth and its relation to local cohomology. Let S be an irreducible noetherian scheme of finite type over a field k. We will assume that S is proper and satisfies Serre's S_2 condition: for any $s \in S$, $depth_s \mathcal{O}_S \geq \min(\dim \mathcal{O}_{S,s}, 2)$. Let $V \subset S$ be an open subset with closed complement of codimension ≥ 2 in S and A a complete noetherian local k-algebra with residue field $K = A/\mathfrak{m}$ and associated graded K-algebra $gr(A) = \bigoplus_{p \geq 0} gr_p(A) = \bigoplus_{p \geq 0} \mathfrak{m}^p/\mathfrak{m}^{p+1}$. Define $X = S \times_k Spec(A)$ and

$$X_p = S \times_k Spec(A/\mathfrak{m}^{p+1}), \quad U_p = V \times_k Spec(A/\mathfrak{m}^{p+1}), \qquad p \geq 0.$$

Let $i_p : U_p \to X_p$ be the natural open embeddings. The completion $\widehat{X}$ of X along X_0 may be viewed at the limit of $\{X_p\}_{p \geq 0}$; cf. Section 10.6 in [EGAI]. The limit of i_p gives an open formal subscheme $\widehat{i} : \widehat{U} \to \widehat{X}$. The ideal sheaf of X_0 in X will be denoted by $\mathcal{J}_X$ and the closed subset $X_0 \setminus U_0$ by Z_0. Finally, $f : X \to Spec(A)$ is the natural proper projection and, for any $s \in Spec(A)$, X_s stands for the fiber $f^{-1}(s)$.

Observe that X may no longer satisfy the S_2 condition (since we made no depth assumptions on A). However, a particular case of Lemma 2 in [Ba] gives

Lemma 2.1. *For any $x \in X$ with $f(x) = s$, depth $\mathcal{O}_{X,x} \geq \min(\dim \mathcal{O}_{X_s,x}, 2)$.* □

Consider a vector bundle $\mathcal{F}$ on $\widehat{U}$, i.e. a sequence of vector bundles F_p on U_p with isomorphisms

$$F_p|_{U_{p-1}} \simeq F_{p-1}, \qquad p \geq 1. \tag{2.1}$$

Definition 2.2. We will say that a vector bundle $\mathcal{F}$ on $\widehat{U}$ *admits an algebraization* (U, F) if there exists an open subset $U \subset X$ with $U \cap X_0 = U_0$ and a vector bundle F on U such that $\mathcal{F}$ is isomorphic to the completion of F; i.e. for $\mathcal{J}_U = \mathcal{J}_X|_U$ there exist isomorphisms $F_p \simeq F/\mathcal{J}_U^{p+1}F$ compatible with (2.1). In Section 3 we apply similar terminology to principal bundles.

Let Z be the closed subset $X \setminus U$ and $i : U \hookrightarrow X$ the open embedding.

Lemma 2.3. *Assume that* $codim_{X_0} Z_0 \geq 2$ *and an open subset* $U \subset X$ *satisfies* $U \cap X_0 = U_0$. *For any* $s \in Spec(A)$, *define* $Z_s = Z \cap X_s$. *Then* $codim_{X_s} Z_s \geq 2$ *for all* $s \in SpecA$ *and* $codim_X Z \geq 2$.

Proof. Since f is proper, the image $f(\overline{Z}_s)$ contains the unique closed point $s_0 \in Spec(A)$. Therefore $\overline{Z}_s \cap X_0 \subset Z_0$ is not empty. By semicontinuity of dimensions in the fibers we have $codim_{X_s} Z_s \geq codim_{X_0}(\overline{Z}_s \cap X_0) \geq codim_{X_0} Z_0 = 2$. The second assertion of the lemma follows from the first. $\square$

Below we repeatedly use the following results.

Proposition 2.4. *In the notation introduced above*

(i) *Completion along* X_0 *induces an equivalence between the category of coherent sheaves on* X *and the category of coherent sheaves on the formal scheme* $\widehat{X}$.

(ii) *For any locally free sheaf* F *(resp.* F_0*) on* U *(resp.* U_0*) its direct image* i_*F *(resp* $(i_0)_*F_0$*) is coherent. If* $codim_{X_0} Z_0 \geq 3$, *then* $R^1(i_0)_*F_0$ *is also coherent.*

(iii) *Let* E *be a coherent sheaf on* X *and* $\psi : E \to i_*i^*E$ *the canonical morphism. Then* ψ *is an isomorphism if and only if* $depth_x E \geq 2$ *for any point* $x \in Z = X \setminus U$.

Proof. Part (i) follows from Corollary 5.1.6 in [EGAIII]. To check the coherence of i_*F, by Corollary VIII.2.3 in [SGA2] it suffices to check that $depth_x F \geq 1$ for any point $x \in U$ such that $\overline{\{x\}} \cap Z$ has codimension 1 in $\overline{\{x\}}$. But Lemma 2.1 and local freeness of F imply that any x with $depth_x F = 0$ must be generic in its fiber, and Lemma 2.3 implies that $\overline{\{x\}} \cap Z$ would in fact have codimension 2 in $\overline{\{x\}}$. The same proof applies to $(i_0)_*F_0$. If $codim_{X_0} Z_0 \geq 3$ then the above argument can also by applied to $R^1(i_0)_*F_0$ once we show that $depth_x F_0 \geq 2$ for any $x \in U_0$ such that $\overline{\{x\}} \cap Z_0$ has codimension 1 in $\overline{\{x\}}$. But by S_2 condition $depth_x F_0 \leq 1$ can only hold for points x of codimension ≤ 1 in U_0, which would imply that $\overline{\{x\}} \cap Z_0$ has codimension ≥ 2 in $\overline{\{x\}}$. This proves *(ii)*. Part *(iii)* is a particular case of Corollary II.3.5 in *loc.cit.* $\square$

2.2. **Examples.** The first example with $codim_{X_0} Z_0 = 3$ and $K = k$ shows that one may not be able to take $U = U_0 \times_k Spec(A)$.

Example 2.5. Take $S = X_0 = \mathbb{P}^3$ with homogeneous coordinates $[x : y : z : w]$ and set $V = U_0 = S \setminus [0 : 0 : 0 : 1]$, $A = k[[t]]$ (formal power series in t). Define vector bundles F_p as kernels of

$$\varphi_p : \mathcal{O}_{U_p}^{\oplus 3} \to \mathcal{O}(1)_{U_p}; \quad (s_1 \oplus s_2 \oplus s_3) \mapsto s_1 x + s_2 y + s_3(z - tw).$$

Observe that φ_p is surjective since t is nilpotent on U_p and $[0 : 0 : 0 : 1] \notin U_p$.

Lemma 2.6. *The bundle $\mathcal{F}$ admits no algebraization (U, F) with $U = U_0 \times_k Spec(A)$.*

Proof. Set F to be the kernel of morphism $\varphi : \mathcal{O}^{\oplus 3} \to \mathcal{O}(1)$of vector bundles on U, given by the same formula as for φ_p. By definition, φ is not surjective only at $P = [0 : 0 : t : 1] \in U$, which projects to the generic point $\xi = Spec(k[t^{-1}, t]) \in Spec(A)$. The specialization at $t = 0$ is not in U_0, hence P is closed in U and $U \setminus P$ is an open subset containing U_0. Since on $U \setminus P$ we have the short exact sequence of locally free sheaves

$$0 \to F \to \mathcal{O}^{\oplus 3} \to \mathcal{O}(1) \to 0,$$

the restriction of F to each U_p is given by F_p, i.e. $\mathcal{F}$ is indeed the completion of F. On the other hand, F is not locally free at P: from $0 \to F \to \mathcal{O}_U^{\oplus 3} \to \mathcal{O}_U \to k_P \to 0$ we immediately get $\mathcal{E}xt^1(F, \mathcal{O}_U) \simeq \mathcal{E}xt^3(k_P, \mathcal{O}) \simeq k_P$ since the middle two terms are projective.

Suppose that E is a locally free sheaf on U with completion isomorphic to $\mathcal{F}$. We will see later in Proposition 2.11(ii) that in such a situation we must have $\widehat{i_* E} \simeq \widehat{i}_* \mathcal{F} \simeq \widehat{i_* F}$; hence by Proposition 2.4(i), $i_* F \simeq i_* E$, which contradicts $\mathcal{E}xt^1(F, \mathcal{O}_U) \neq 0$. □

The second example illustrates that for $codim_{X_0}\, Z_0 = 2$, a pair (U, F) may not exist at all.

Example 2.7. Consider $A = k[[t]]$ and $S = X_0 = \mathbb{P}^2$ with homogeneous coordinates $(x : y : z)$. Let $V = U_0 = X_0 \setminus P$, where $P = (0 : 0 : 1)$, and define a rank 2 bundle F_p on $U_p = U_0 \times_k Spec(k[t]/t^{p+1})$ as follows. The affine open subsets $U_p^{(x)}$, $U_p^{(y)}$ given by non-vanishing of x, resp. y, form a covering of U_p, and we can glue trivial rank 2 bundles on these open sets, using the transition function

$$\begin{pmatrix} 1 & \sum_{m=0}^{p} \left(\frac{tz^2}{xy}\right)^m \\ 0 & 1 \end{pmatrix}$$

on $U_p^{(x)} \cap U_p^{(y)}$. Clearly $F_p|_{U_{p-1}} \simeq F_{p-1}$ in a natural way, and we obtain a vector bundle $\mathcal{F}$ on $\widehat{U}$.

Lemma 2.8. *There exists no vector bundle F on $U = X \setminus Z$ with $\widehat{F} \simeq \mathcal{F}|_{\widehat{U} \setminus (Z \cap U_0)}$ for any closed subset $Z \subset X$ such that $Z_0 \subset (Z \cap X_0)$ and $codim_{X_0}(Z \cap X_0) \geq 2$.*

Proof. Suppose otherwise, and take the direct image of F with respect to the open embedding $i : U \to X$. By Proposition 2.4, $i_* F$ is coherent and has *depth* ≥ 2 at all codimension 2 points of X. Since modules of depth 2 over two-dimensional regular local rings are free by the Auslander-Buchsbaum formula, $i_* F$ will be locally free in codimension two. Therefore by shrinking Z we can assume that Z has codimension 3 in X, which in our case means that Z is a finite set of points in X_0. Then the short exact sequence of sheaves on $X \setminus Z$

$$0 \to F \xrightarrow{t^{p+1}} F \to F_p \to 0$$

(we identify F_p with its direct image on $X \setminus Z$ abusing notation) gives a long exact sequence on X:

$$0 \to i_* F \xrightarrow{t^{p+1}} i_* F \to i_* F_p \to R^1 i_* F \xrightarrow{t^{p+1}} R^1 i_* F,$$

where $R^1 i_* F$ is coherent for the same reason as in Proposition 2.4(ii). Since $R^1 i_* F$ is supported at the finite set Z of closed points, it has finite length at each of them and the last arrow is zero for $p \geq p_0$. For such p we can write $i_* F \to i_* F_p \to R^1 i_* F \to 0$ which gives

$$i_* F \otimes_{\mathcal{O}_X} k(P) \to i_* F_p \otimes_{\mathcal{O}_X} k(P) \to R^1 i_* F \otimes_{\mathcal{O}_X} k(P) \to 0.$$

To prove the lemma it suffices to show that $\dim_k i_* F_p \otimes_{\mathcal{O}_X} k(P)$ is unbounded as $p \to \infty$.

To that end, replace X_0 with the affine open subset $\widetilde{X}_0 \simeq \mathbb{A}^2$ given by the non-vanishing of z, with affine coordinates $u = \frac{x}{z}$, $v = \frac{y}{z}$. Set $W_0 = U_0 \cap \widetilde{X}_0$ and similarly for $\widetilde{X}_p$, W_p, $W_p^{(x)}$ and $W_p^{(y)}$. Then $i_* F_p|_{\widetilde{X}_p}$ is the sheaf associated to $H^0(W_p, F_p|_{W_p})$ viewed as a module over $A(\widetilde{X}_p) = k[u,v,t]/t^{p+1}$. By its definition, F_p is an extension of $\mathcal{O}_{U_p}$ with $\mathcal{O}_{U_p}$ which leads to the long exact sequence

$$0 \to H^0(W_p, \mathcal{O}_{W_p}) \to H^0(W_p, F_p|_{W_p}) \to H^0(W_p, \mathcal{O}_{W_p}) \to H^1(W_p, \mathcal{O}_{W_p}),$$

where the last arrow sends the constant function 1 to the class of the extension. Let M_p be the kernel of the last arrow. It suffices to show that $dim_k(M_p/\langle u,v,t\rangle M_p)$ is unbounded. Computing M_p via the affine covering $\{W_p^{(x)}, W_p^{(y)}\}$, we identify it with the kernel of

$$k[u,v,t]/t^{p+1} \xrightarrow{\pi_p \circ \psi_p} \frac{1}{uv} k[u^{-1}, v^{-1}, t]/t^{p+1},$$

where ψ_p is multiplication by $\sum_{l=0}^{p} \left(\frac{t}{uv}\right)^l$ (i.e. the upper right corner of the transition matrix in the definition of F_p) and π_p is the natural projection

$$k[u, u^{-1}, v, v^{-1}, t]/t^{p+1} \to \frac{1}{uv} k[u^{-1}, v^{-1}, t]/t^{p+1}.$$

It follows that M_p is generated by the monomials $t^p, t^i u^{p-i}, t^i v^{p-i}$ for $i = 0, \ldots, p-1$; thus

$$\dim_k(M_p/\langle u,v,t\rangle M_p) = 2p+1 \to \infty \qquad \text{as } p \to \infty.$$

$\square$

Example 2.9 (Suggested to the author by V. Drinfeld)**.** The bundle in the previous example has trivial determinant, but if we don't insist on that, there is a rank one example: glue two trivial line bundles on $U_p^{(x)}, U_p^{(y)}$ using the transition function $\sum_{m=0}^{p} \left(\frac{tz^2}{xy}\right)^m$. The resulting line bundle admits no algebraization since again $\dim_k (i_p)_* F_p \otimes_{\mathcal{O}_X} k(P)$ is not bounded as $p \to \infty$.

2.3. **Algebraization of vector bundles.**

Theorem 2.10. *In the notation of Section 2.1,*

- (i) *If $codim_{X_0} Z_0 \geq 3$, then $\mathcal{F}$ admits an algebraization.*
- (ii) *If $codim_{X_0} Z_0 \geq 2$ and the cokernel of the natural morphism $(i_p)_* F_p|_{X_{p-1}} \to (i_{p-1})_* F_{p-1}$ is supported in codimension ≥ 3 for all p large enough, then $\mathcal{F}$ admits an algebraization.*
- (iii) *In either of the two situations (codimension ≥ 3 or codimension ≥ 2 with the additional support assumption) the projective system $\{(i_p)_* F_p\}_{p \geq 0}$ satisfies the Mittag-Leffler condition, and the direct image $\widehat{i}_* \mathcal{F}$ is coherent and isomorphic to $\varprojlim (i_p)_* F_p$.*

Proof. We split the proof of (i) and (ii) in a number of steps. Part (iii) will follow from Step 2.

Step 1. Suppose that $\widehat{i}_*\mathcal{F}$ is coherent. By Proposition 2.4(i) there exists a unique coherent sheaf E on X such that $\widehat{E} \simeq \widehat{i}_*\mathcal{F}$. The subset $U \subset X$ of points where E is locally free is open and contains U_0 (e.g. by Nakayama's Lemma). Shrinking U if necessary we can achieve $U \cap X_0 = U_0$. Now set $F = E|_U$.

Step 2. Therefore (i) and (ii) are reduced to showing that, under the conditions stated, $\widehat{i}_*\mathcal{F}$ is coherent. To that end we modify the argument of 0.13.7.7 in [EGAIII] which will also prove (iii). First, as in 0.13.7.2 of *loc. cit.*, we choose injective resolutions $F_k \to L_k^\bullet$ such that $L_{k+1}^\bullet/\mathcal{J}_U^{k+1}L_{k+1}^\bullet \simeq L_k^\bullet$ and the natural filtrations by $\mathcal{J}_U^n(\ldots)$ agree with those on F_k. Each $\widehat{i}_*(L_k^\bullet)$ is a filtered complex and has a spectral sequence with the E_1 term given by

$$E_1^{pq} = R^{p+q}\widehat{i}_*(\mathcal{J}_U^pF_k/\mathcal{J}_U^{p+1}F_k).$$

As in 0.13.7.3 of *loc.cit.* we pass to the limit as $k \to \infty$ and get a spectral sequence with

$$E_1^{p,q} = R^{p+q}\widehat{i}_*(F_p/F_{p+1}) \simeq R^{p+q}\widehat{i}_*(F_0) \otimes_K (\mathfrak{m}^p/\mathfrak{m}^{p+1}) = R^{p+q}\widehat{i}_*(F_0) \otimes_K gr_p(A).$$

We are interested in the components

$$E_1^0 = \bigoplus_{p+q=0} E_1^{p,q} = \widehat{i}_*(F_0) \otimes_K gr(A); \quad E_1^1 = \bigoplus_{p+q=1} E_1^{p,q} = R^1\widehat{i}_*(F_0) \otimes_K gr(A).$$

We would like to show that the spectral sequence converges at the $E^0 = \bigoplus E^{p,-p}$ terms. Note that each $E_{k+1}^1 = \bigoplus E_{k+1}^{p,1-p}$ is a quotient of a subsheaf in E_k^1, while each E_{k+1}^0 is a subsheaf E_k^0 (since $E^{p,-1-p}$ terms are zero). Taking successive preimages of the boundaries in $E_{r-1}, E_{r-2}, \ldots, E_1$ we get a sequence of boundary subsheaves $B_1 \subset B_2 \subset B_3 \subset \ldots \subset E_1^1$, and taking preimages of cycles in E_k we get a sequence of cycle subsheaves $E_1^0 \supset Z_1 \supset Z_2 \supset Z_3 \supset \ldots$. By 0.13.7.6 in *loc.cit.* these are actually $\mathcal{O}_{X_0} \otimes_K gr(A)$-submodules.

Suppose that the sequence of cycles stabilizes; i.e. for some r_0 one has $Z_r = Z_{r_0}$ whenever $r \geq r_0$. Then by 0.13.7.4 in [EGAIII] the projective system $\{\widehat{i}_*(F_k)\}_{k\geq 0}$ satisfies the Mittag-Leffler condition and the associated graded of $\widehat{i}_*(\mathcal{F})$ is precisely $Z_{r_0} \subset \widehat{i}_*(F_0) \otimes_K gr(A)$. But $\widehat{i}_*(F_0)$ is a coherent by Proposition 2.4(ii), hence the subsheaf $gr(\widehat{i}_*\mathcal{F}) \subset \widehat{i}_*(F_0) \otimes_K gr(A)$ is a coherent $\mathcal{O}_{X_0} \otimes_K gr(A)$-module by the noetherian property of X_0 and A. By *loc.cit.* 13.7.7.2, $\widehat{i}_*\mathcal{F}$ is itself coherent on $\widehat{X}$. Also, $\widehat{i}_*\mathcal{F} \simeq \varprojlim (i_p)_*F_p$ by 0.13.7.5.1 in *loc.cit.*

Step 3. Now the assertion of the theorem is reduced to showing that the sequence of cycles $Z_1 \supset Z_2 \supset \ldots$ stabilizes. By definition of Z_i this is equivalent to saying that the higher differentials of the spectral seqence $d_r : E_r^0 \to E_r^1$ become zero for $r \geq r_0$. That in turn is equivalent to saying that the sequence of boundaries $B_1 \subset B_2 \subset B_3 \subset \ldots$ also stabilizes.

If $codim_{X_0}Z_0 \geq 3$ by Proposition 2.4(ii), $R^1(i_0)_*F_0$ is also coherent and $\{B_r\}_{r\geq 1}$ stabilizes by the noetherian property of $R^1(i_0)_*F_0 \otimes_K gr(A)$, which proves (i). If $codim_{X_0}Z_0 \geq 2$ we need to find a coherent subsheaf of $R^1(i_0)_*F_0 \otimes_K gr(A)$ containing B_r for all $r \geq 1$.

Step 4. At this point we reduced (ii) to showing that, under the assumptions stated, there exists a coherent subsheaf $G \subset R^1(i_0)_*F_0$ such that $B_r \subset G \otimes_K gr(A)$ for all r. By 0.11.2.2 in [EGAIII] for $r \geq p$, the term $B_r^{p,1-p}$ is the image of the connecting homomorphism

$$\widehat{i}_*F_p \to \widehat{i}_*F_{p-1} \overset{\rho_p}{\to} R^1\widehat{i}_*F_0 \otimes_K (\mathfrak{m}^p/\mathfrak{m}^{p+1})$$

in the long exact sequence obtained by applying $R\widehat{i_*}$ to the short exact sequence on $\widehat{U}$:

$$0 \to F_0 \otimes_K (\mathfrak{m}^p/\mathfrak{m}^{p+1}) \to F_p \to F_{p-1} \to 0.$$

Observe that by our assumptions each $Im(\rho_p)$ is coherent and supported in codimension ≥ 3 for $p \gg 0$. Thefore we are done once we show that the subsheaf of $R^1(i_0)_*F_0$ formed by all sections with support in codimension ≥ 3 is coherent whenever $codim_{X_0} Z_0 \geq 2$ and F_0 is locally free on U_0.

Step 5. Set $Q = (i_0)_*F_0$, a coherent sheaf on X_0 by Step 2. By the standard exact sequence we have $\mathcal{H}^2_{Z_0}Q = R^1(i_0)_*Q|_{U_0} = R^1(i_0)_*F_0$, so it suffices to show that $\mathcal{H}^0_{\geq 3}\mathcal{H}^2_{Z_0}Q$ is coherent where $\mathcal{H}^0_{\geq 3}$ is the functor of sections supported in codimension ≥ 3. Let $\mathcal{H}^i_{\geq 3}$ be the higher derived functors.

First, the standard spectral sequence for the composition of functors $R\mathcal{H}^0_{\geq 3}$, $R\mathcal{H}^0_{Z_0}$ has $E_2^{p,q} = \mathcal{H}^p_{\geq 3}\mathcal{H}^q_{Z_0}Q$. But $\mathcal{H}^i_{Z_0}Q = 0$ for $i = 0, 1$ by Proposition 2.4(iii), so

$$\mathcal{H}^0_{\geq 3}\mathcal{H}^2_{Z_0}Q \simeq \mathcal{H}^2_{\Phi}Q,$$

where the local cohomology $\mathcal{H}^2_{\Phi}$ has a family of supports

$$\Phi = \{\text{all codim} \geq 3 \text{ closed subsets in } Z_0\}.$$

Step 6. To show that $\mathcal{H}^2_{\Phi}Q$ is coherent note that by [Ha2] the scheme X_0 has a dualizing complex ω of the form

$$0 \to \mathcal{K}^0 \to \ldots \to \mathcal{K}^{\dim_K X_0} \to 0$$

with $\mathcal{K}^i = \bigoplus_{dim \mathcal{O}_{X_0,x}=i} J(x)$ and each $J(x)$ is the direct image of the injective envelope of the residue field $k(x)$ with respect to the natural morphism $i^x : Spec(\mathcal{O}_{X_0,x}) \to X_0$. By definition of a dualizing complex, the double complex $\mathcal{K}^{p,q} = \mathcal{H}om(\mathcal{H}om(Q, \mathcal{K}^{-q}), \mathcal{K}^p)$ has a total complex quasi-isomorphic to Q. Moreover, by Proposition IV.2.1 and the remark on page 123 in [Ha2], the total complex is a flasque resolution of Q and hence can be used to compute $\mathcal{H}^{\bullet}_{\Phi}(Q)$. This leads to a spectral sequence:

$$E_2^{p,q} = \mathcal{E}xt^p_{\Phi}(\mathcal{E}xt^{-q}(Q, \omega), \omega) \Rightarrow \mathcal{H}^{p+q}_{\Phi}(Q),$$

where $\mathcal{E}xt^p_{\Phi} = R^p(\Gamma_{\Phi} \circ \mathcal{H}om)$ and the $\mathcal{E}xt$ sheaves are understood in the sense of hypercohomology.

Only finitely many terms $E_2^{p,q}$ with $p + q = 2$ will be non-trivial: since $\mathcal{K}^q$ are injective, the non-vanishing implies $0 \leq (-q) \leq \dim_K X_0$. Thus it suffices to show that $E_2^{p,2-p} = \mathcal{E}xt^p_{\Phi}(\mathcal{E}xt^{p-2}(Q, \omega), \omega)$ is coherent for $p \geq 2$.

An important observation which we use below is that $\mathcal{K}^p$ has no sections supported in codimension $\geq p + 1$.

Step 7. First observe that $\mathcal{E}xt^2_{\Phi}(G, \omega) = 0$ for any quasi-coherent sheaf G since $\mathcal{K}^2$ has no sections supported in codimension ≥ 3 and hence no sections with support in Φ. Hence we can assume that $p \geq 3$.

We first claim that $codim_{X_0} Supp(\mathcal{E}xt^{p-2}(Q,\omega)) = d \geq p \geq 3$. In fact, let $x \in Supp(\mathcal{E}xt^{p-2}(Q,\omega))$ be a point with $\dim \mathcal{O}_{X_0,x} = d$. By local duality, cf. V.6 in [Ha2], the non-vanishing of the stalk $\mathcal{E}xt^{p-2}(Q,\omega)_x$ is equivalent to the non-vanishing of local cohomology $\mathcal{H}_x^{d+2-p}(Q)$, which implies $d+2-p \geq 0$ and $d \geq p-2 \geq 1$. If $d = 1$, then $p = 3$ and also $x \notin Z_0$; hence the stalk Q_x is free. Thus $\mathcal{H}_x^0(\mathcal{O}) \neq 0$, contradicting the S_2 assumption. If $d \geq 2$, then applying the S_2 condition when $x \notin Z_0$ and Proposition 2.4(iii) when $x \in Z_0$, we actually have $d+2-p \geq 2$, so $d \geq p$ as required.

By primary decomposition, the coherent sheaf $\mathcal{E}xt^{p-2}(Q,\omega)$ admits a finite filtration by coherent subsheaves such that all successive quotients have irreducible supports of codimension $\geq p$. By the standard long exact sequence for $\mathcal{E}xt^{\bullet}_{\Phi}(\cdot,\omega)$ is suffices to show that $\mathcal{E}xt^p_{\Phi}(G,\omega)$ is coherent whenever $p \geq 3$ and G is a coherent sheaf with irredicuble support Y of codimension $\geq p$.

If $Y \nsubseteq Z_0$ for any W in the family Φ, the intersection $Y \cap W$ is not equal to Y and therefore has codimension $\geq p+1$. But then $\mathcal{E}xt^p_{\Phi}(G,\omega) = 0$, because any section ρ of $\mathcal{H}om(G,\mathcal{K}^p)$ representing a class in $\mathcal{E}xt^p_{\Phi}(G,\omega)$ has zero values since $\mathcal{K}^p$ has no sections supported in codimension $\geq p+1$. If $Y \subseteq Z_0$, then Y is an element of Φ and $\mathcal{E}xt^p_{\Phi}(G,\omega) \simeq \mathcal{E}xt^p(G,\omega)$ since all sections of $\mathcal{H}om(G,\mathcal{K}^t)$ have support in Φ. But $\mathcal{E}xt^p(G,\omega)$ is coherent, which finishes the proof. □

The converse to Theorem 2.10 can be formulated as follows.

Proposition 2.11. *In the setting of Section 2.1, assume that $\mathcal{F}$ admits an algebraization (U,F) and view each F_p as a sheaf on U. Then*

(i) *The cokernel of $i_*F_p \to i_*F_{p-1}$ is supported in codimension ≥ 3 for $p \gg 0$.*

(ii) *The isomorphism $\widehat{F} \simeq \mathcal{F}$ extends to direct images: $\widehat{i_*F} \simeq \widehat{i}_*\mathcal{F}$. In particular, $\widehat{i}_*\mathcal{F}$ is coherent.*

Proof. To prove (i) observe that the cokernel of $i_*F_p \to i_*F_{p-1}$ is annihilated by $\mathcal{J}_X$, being a subsheaf of $R^1 i_*F_0 \otimes_K gr_p(A)$, and is therefore isomorphic to the cokernel of $i_*F_p|_{X_0} \to i_*F_{p-1}|_{X_0}$.

We will first show that the natural map $i_*F_p|_{X_0} \to i_*F_0$ is an embedding of sheaves for all p. Considering the exact sequence

$$0 \to \mathcal{J}_X(i_*F_p) \to i_*F_p \to i_*F_p|_{X_0} \to 0$$

and its map to the first terms of the sequence

$$0 \to i_*(\mathcal{J}_U F_p) \to i_*F_p \to i_*F_0 \to R^1 i_*(\mathcal{J}_U F_p) \to \dots,$$

we see that $i_*F_p|_{X_0} \to i_*F_0$ is an embedding precisely when the natural map $\mathcal{J}_X(i_*F_p) \to i_*(\mathcal{J}_U F_p)$ is an isomorphism. Observe that $i_*\mathcal{O}_U = \mathcal{O}_X$; hence $i_*\mathcal{J}_U$ is a sheaf of ideals in $\mathcal{O}_X$.

Using Lemma 2.1 and the Cohen-Macaulay assumption on X_0, we see that $\mathcal{H}^t_Z \mathcal{O}_X = \mathcal{H}^t_{Z_0}\mathcal{O}_{X_0} = 0$ for $t = 0,1$. By the short exact sequence $0 \to \mathcal{J}_X \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0$ we derive $\mathcal{H}^t_Z \mathcal{J}_X = 0$ for $t = 0,1$, and hence $\mathcal{J}_X = i_*\mathcal{J}_U$ by Proposition 2.4(iii). Then

$$i_*(\mathcal{J}_U F_p) = (i_*\mathcal{J}_U)(i_*F_p) = \mathcal{J}_X i_*F_p,$$

as required. Similarly, $i_*F|_{X_0} \to i_*F_0$ is an embedding. So for any $p \geq 1$ we have the embeddings

$$i_*F|_{X_0} \hookrightarrow i_*F_p|_{X_0} \hookrightarrow i_*F_{p-1}|_{X_0} \hookrightarrow i_*F_0.$$

Consequently, the coherent sheaf $\mathcal{K} = Coker(i_*(F)|_{X_0} \to i_*F_0)$ has a decreasing filtration by images of $i_*F_p|_{X_0}$, and each $Coker(i_*F_p|_{X_0} \to i_*F_{p-1}|_{X_0})$ is its successive quotient. But $\mathcal{K}$ is a coherent sheaf with $Supp(\mathcal{K}) \subset Z_0$ and Z_0 has at most finitely many points of codimension 2. Since for each point $x \in X_0$ of codimension 2 the localization $\mathcal{K}_x$ is a module of finite length, only finitely many successive quotients of the filtration of $\mathcal{K}$ can be non-trivial in codimension 2, which proves (i).

To prove (ii) first observe that $\widehat{i}_*\mathcal{F}$ and $E = i_*F$ are coherent by Theorem 2.10(iii) and Proposition 2.4(ii), respectively. By Proposition 2.4(i) we can find a sheaf E' such that $\widehat{E'} \simeq \widehat{i}_*\mathcal{F}$. The isomorphism $\widehat{E}|_{\widehat{U}} \simeq \mathcal{F} = \widehat{i}^*\widehat{i}_*\mathcal{F}$ extends uniquely to a morphism of sheaves $\widehat{\phi} : \widehat{E} \to \widehat{i}_*\mathcal{F} = \widehat{E'}$. By Proposition 2.4(i), $\widehat{\phi}$ is the completion of a unique morphism $\phi : E \to E'$ which by Corollary 10.8.14 in [EGAI] should be an isomorphism on an open subset W containing U_0. Shrinking W if necessary we can assume $W \subset U$. By Lemma 2.3, each point $x \in U \setminus W$ has codimension ≥ 2 in its fiber; hence $depth_xE \geq 2$ by Lemma 2.1. For $x \in X \setminus U$ we still have $depth_xE \geq 2$ by Proposition 2.4(iii). Applying the same result to $j : W \hookrightarrow X$ instead of U we see that $E = j_*j^*E$. By adjunction of j^* and j_* the isomorphism $(\phi|_W)^{-1} : j^*E' \to j^*E$ extends uniquely to a morphism $\psi : E' \to j_*j^*E = E$.

By construction, the composition $\psi\phi : E \to E$ restricts to identity on W; hence $\psi\phi = Id_E$, by the same adjunction. Similarly, the composition $\widehat{\phi}\widehat{\psi} = \widehat{E'} \to \widehat{E'}$ restricts to identity on $\widehat{U}$, and since $\widehat{E'} \simeq \widehat{i}_*\mathcal{F}$, we must have $\widehat{\phi}\widehat{\psi} = Id_{\widehat{E'}}$, so $\phi\psi = Id_{E'}$ by Proposition 2.4(i). We have proved that $E = i_*F \simeq E'$. Since $\widehat{E'} = \widehat{i}_*\mathcal{F}$ we conclude that $\widehat{i_*F} = \widehat{i}_*\mathcal{F}$. ☐

Corollary 2.12. *The following conditions are equivalent:*

(i) *The cokernel of* $(i_p)_*F_p|_{X_{p-1}} \to (i_{p-1})_*F_{p-1}$ *is supported in codimension* ≥ 3 *for* $p \gg 0$.

(ii) *The projective system* $\{\widehat{i}_*F_p\}_{p\geq 1}$ *satisfies the Mittag-Leffler condition.*

(iii) *The direct image* $\widehat{i}_*\mathcal{F}$ *is coherent.*

(iv) *The bundle* $\mathcal{F}$ *admits an algebraization.*

Proof. The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are established in the proof of Theorem 2.10. The implication (iv) ⇒ (i) is proved in Proposition 2.11. If the projective system $\{\widehat{i}_*F_p\}_{p\geq 1}$ satisfies the Mittag-Leffler condition, by 0.13.3.1 in [EGAIII] the natural map $\widehat{i}_*\mathcal{F} \to \varprojlim \widehat{i}_*F_p$ is an isomorphism. By the Mittag-Leffler condition we can replace $\widehat{i}_*F_p$ by a system of subsheaves $G_p \subset \widehat{i}_*F_p$ so that the property $\widehat{i}_*\mathcal{F} \simeq \varprojlim G_p$ still holds and $G_p|_{X_{p-1}} \to G_{p-1}$ is surjective. Since each G_p is coherent by the noetherian property of X_p, Proposition 10.11.3 in [EGAI] tells us that $\varprojlim G_p$ is also coherent. Therefore, (ii) ⇒ (iii). ☐

Remark 2.13. Suppose that X_0 is a smooth projective surface over K, $\xi = k_1P_1 + \ldots + k_lP_l$ an effective zero cycle and that F_0 is a rank n vector bundle on $U_0 = X_0 \setminus \{P_1, \ldots, P_l\}$. The pair (F_0, ξ_0) should define a point $Spec(K) \to Uhl_n$ of the Uhlenbeck functor. Assume that $(F, \xi) : Spec(A) \to Uhl_n$ extends (F_0, ξ_0). Then it is expected that $Coker(i_*F \to i_*F_0)$ can be supported only at the points $P_1, \ldots, P_l$, with multiplicities bounded by $k_1, \ldots, k_l$, respectively. (In the differential geometry picture, cf. [DK], ξ_0 represents the singular part of a connection which may be smoothed out by F but may not acquire any negative coefficients. Since the multiplicities of $Coker(i_*F \to i_*F_0)$ measure the local change of c_2, one obtains the bound mentioned.) But the proof of Proposition 2.11 shows that the multiplicities of $Coker(i_*F \to i_*F_0)$ give an upper bound for the total sum, over all p, of similar multiplicities for $Coker((i_p)_*F_p|_{X_{p-1}} \to (i_{p-1})_*F_{p-1})$. Hence the condition of Corollary 2.12(i) is rather natural from the point of view of Uhlenbeck spaces.

3. Algebraization of principal bundles

Let G be an affine algebraic group over k. We keep the notation of Section 2.1 and consider left principal G-bundles which are locally trivial in fppf topology. For such a G-bundle P (over $\widehat{U}$ or an open subset $U \subset X$) and any scheme Y over k with left G-action, denote by $P_Y = G \setminus (Y \times_k P)$ the associated fiber bundle, i.e. the quotient by the left diagonal action of G. For instance, when $\rho : G \to H$ is a homomorphism of linear algebraic groups over k we can consider a left G-action on H given by $g \cdot h = h\rho(g)^{-1}$, and then P_H is simply the principal H-bundle induced via ρ.

Theorem 3.1. *Assume that the identity component G° is reductive. Then a principal G-bundle $\mathcal{P}$ over the formal scheme $\widehat{U}$ admits an algebraization if and only if for a fixed exact representation $G \hookrightarrow GL(V)$ the associated vector bundle $\mathcal{P}_V$ admits an algebraization, i.e. satisfies the conditions of Corollary* 2.12.

The "only if" part is obvious. Since by a result of Haboush, cf. Theorem 3.3 in [Ha1], the quotient $GL(V)/G$ is affine, the "if" part follows from the following general statement.

Proposition 3.2. *Let H be an affine algebraic group over k and let G be its closed subgroup such that H/G is affine. Suppose that $\mathcal{P}$ is a principal G-bundle over $\widehat{U}$ such that the associated principal H-bundle $\mathcal{Q} = \mathcal{P}_H$ admits an algebraization. Then $\mathcal{P}$ admits an algebraization.*

First we establish a preparatory result. As before, $U \subset X$ is an open subset satisfying $U \cap X_0 = U_0$.

Lemma 3.3. *Let H be a linear algebraic group overe k, Q be a principal H-bundle on U and $\widehat{Q}$ be its completion. Also let Y be an affine H-variety. Then for any section $\widehat{s} : \widehat{U} \to \widehat{Q}_Y$ there exists a section $s : W \to Q_Y$ on an open subset $W \subset U$ containing U_0, with completion equal to $\widehat{s}$. If (W, s) and (W', s') are two such algebraizations, then $s = s'$ on $W \cap W'$.*

Proof. One can find an H-invariant linear subspace $V^\vee \subset k[Y]$ containing a set of generators of $k[Y]$ as a k-algebra. Then the surjection $Sym_k^*(V^\vee) \to k[Y]$ gives an H-equivariant closed embedding $Y \hookrightarrow V$ into the dual space V. This induces closed embeddings $Q_Y \hookrightarrow Q_V$ and $\widehat{Q}_Y \hookrightarrow \widehat{Q}_V$.

Therefore $\widehat{s}$ becomes a section of the vector bundle $\widehat{Q}_V$. By Proposition 2.11(ii) the completion of the coherent sheaf i_*Q_V is isomorphic to $\widehat{i}_*\widehat{Q}_V$, and therefore by Proposition 2.4(i) there exists a unique section $\tilde{s}$ of i_*Q_V with completion given by $\widehat{i}_*\widehat{s}$. Set $s = \tilde{s}|_U$.

It remains to show that $s(W) \subset Q_V$ on some W as above. Let $\mathcal{A} = Sym^*(Q_V^\vee)$ be the sheaf of symmetric algebras on U corresponding to Q_V and $\mathcal{I} \subset \mathcal{A}$ be the ideal sheaf of Q_Y. The section s gives the evaluation morphism $\rho : \mathcal{A} \to \mathcal{O}_U$. The sheaf $G = \rho(\mathcal{I})$ is coherent, being a subsheaf of $\mathcal{O}_U$. Since $\widehat{s}$ takes values in $\widehat{Q}_Y$, the completion $\widehat{G}$ is zero. By Corollary 10.8.12 in [EGAI] this implies $Supp(G) \cap U_0 = \emptyset$; hence $W = U \setminus Supp(G)$ satisfies the conditions of the lemma. The uniqueness of s follows from the uniqueness of $\widetilde{s}$. □

Proof of Proposition 3.2. Let (U, Q) be an algebraization of $\mathcal{Q}$. In general, giving a principal G-bundle is equivalent to giving a principal H-bundle $\mathcal{R}$ together with a reduction to G, i.e. a section of the associated bundle $\mathcal{R}_{H/G}$ with the fiber H/G. Since $\mathcal{Q}$ is induced from $\mathcal{P}$, we get a section $\widehat{s} : \widehat{U} \to \mathcal{Q}_{H/G}$, and by the above lemma there exists $s : W \to Q_{H/G}$ such that $\widehat{s}$ is equal to its completion. Then $\mathcal{P}$ admits an algebraization (W, P) where P is the pullback of the principal G-bundle $Q \to Q_{H/G}$ via $s : W \to Q_{H/G}$. □

4. Categorical formulations

Proposition 4.1. *The functor* $F \mapsto \widehat{F}|_{\widehat{U}}$ *induces an eqivalence between the full subcategory of all coherent sheaves* E *on* X*, which are locally free at the points of* $U_0 \subset X$ *and have* $depth_x E \geq 2$ *at the points where* E *is not locally free, and the full subcategory of locally free sheaves on* $\widehat{U}$ *admitting algebraization.*

Proof. Let (U, F) be an algebraization of $\mathcal{F}$. Then the sheaf $E = i_* F$ satisfies $E \simeq i_* i^* E$; hence by Proposition 2.4(iii) $depth_x E \geq 2$ for all $x \in Z = X \setminus U$. We also observe that E is uniquely determined by $\mathcal{F}$, since by Propositions 2.4(i) and 2.11(ii) it is the unique coherent sheaf on X such that $\widehat{E} \simeq \widehat{i}_* \mathcal{F}$. Thus the functor described is essentially surjective on objects. For the morphisms, let $\mathcal{F}_1, \mathcal{F}_2$ be a pair of vector bundles on $\widehat{U}$ with algebraizations (U, F_1) and (U, F_2), respectively, which we may assume to be defined on the same U. Denote by $E_1 = i_* F_1, E_2 = i_* F_2$ the corresponding coherent sheaves on X. Then $Hom_{\widehat{U}}(\mathcal{F}_1, \mathcal{F}_2) = Hom_{\widehat{X}}(\widehat{i}_* \mathcal{F}_1, \widehat{i}_* \mathcal{F}_2) = Hom_X(E_1, E_2)$, where the first equality is by adjunction of i^* and i_* and the second by Propositions 2.4(i) and 2.11(ii). □

To formulate a result for principal bundles, let $\mathcal{B}(G, U_0)$ be the groupoid category in which the objects are given by pairs (U, P), where $U \subset X$ is an open subset with $U \cap X_0 = U_0$, and P is a principal G-bundle on U. Morphisms from (U, P) to (U', P') are given by the set of equivalence classes of pairs (W, ψ), where $W \subset U \cap U'$ is an open subset with $W \cap X_0 = U_0$ and $\psi : P|_W \to P'|_W$ an isomorphism of G-bundles. Two such pairs (W, ψ) and (W, ψ') are equivalent if $\psi = \psi'$ on $W \cap W'$. Also denote by $Bun(G, \widehat{U})$ the groupoid category of G-bundles on the formal scheme $\widehat{U}$. Completion along U_0 defines a functor $\Psi : \mathcal{B}(G, U_0) \to Bun(G, \widehat{U})$. The following statement summarizes our results on algebraization of principal bundles.

Theorem 4.2. *With the notation of Section 2.1,*

(i) *For any affine algebraic group* G *over* k, $\Psi : \mathcal{B}(G, U_0) \to Bun(G, \widehat{U})$ *is full and strict.*

(ii) *For* $G = GL_n(k)$ *the essential image of* Ψ *is the full subcategory of rank* n *vector bundles* $\mathcal{F} = \varprojlim F_p$ *on* $\widehat{U}$ *which satisfy the equivalent conditions* (i)-(iii) *of Corollary* 2.12.

(iii) *Let* $G \hookrightarrow H$ *be a closed embedding of affine algebraic groups over* k *such that* H/G *is affine. Then the natural functor from* G*-bundles to* H*-bundles induces an equivalence of categories*

$$\mathcal{B}(G, U_0) \simeq Bun(G, \widehat{U}) \times_{Bun(H,\widehat{U})} \mathcal{B}(H, U_0).$$

Proof. To prove (i) suppose that $\mathcal{P}, \mathcal{P}'$ are two principal bundles on $\widehat{U}$ admitting algebraizations P, P', respectively, which we may assume to be defined on the same $U \subset X$. Let $\widehat{\psi} : \mathcal{P} \to \mathcal{P}'$ be an isomorphism. We need to prove that there exists (perhaps after shrinking U) a unique isomorphism $\psi : P \to P'$ with completion given by $\widehat{\psi}$. Let $Isom(P, P')$ be the bundle of isomorphisms $P \to P'$. Considering graphs of isomorphisms, we can identify $Isom(P, P') \simeq G \backslash (P \times_U P')$. On the other hand, $P \times_U P'$ is a principal bundle over $G \times_k G$. Define a left action of $G \times_k G$ on G by $(g, h) \cdot f = gfh^{-1}$; then $G \setminus (P \times_U P') \simeq (P \times_U P')_G$. Since $\widehat{\psi}$ gives a section $\widehat{s}$ of $Isom(\mathcal{P}, \mathcal{P}')$, applying Lemma 3.3 to $H = G \times_k G$ and $Y = G$, we get a unique algebraization $s : W \to (P \times_U P')_G \simeq Isom(P, P')|_W$, which corresponds to the required isomorphism ψ. This proves (i).

The statement of (ii) for objects holds by Corollary 2.12 and for morphisms by (i).

For (iii) first observe that the compositions $\mathcal{B}(G, U_0) \to \mathcal{B}(H, U_0) \to Bun(H, \widehat{U})$ and $\mathcal{B}(G, U_0) \to Bun(G, \widehat{U}) \to Bun(H, \widehat{U})$ are canonically isomorphic; therefore one does get a functor

$$\mathcal{B}(G, U_0) \to Bun(G, \widehat{U}) \times_{Bun(H,\widehat{U})} \mathcal{B}(H, U_0).$$

On objects, this functor is an equivalence if for a G-bundle $\mathcal{P}$ on $\widehat{U}$, an H-bundle Q on $U \subset X$ and an isomorphism $\phi : \mathcal{P}_H \simeq \widehat{Q}$, there exists an open subset $W \subset U$ with $W \cap X_0 = U_0$, a G-bundle P on W and isomorphisms $\widehat{P} \simeq \mathcal{P}$ and $P_H \simeq Q|_W$ which induce ϕ in a natural way. This is equivalent to finding an algebraization of the section $\widehat{s} : \widehat{U} \to \widehat{Q}_{H/G}$ induced by ϕ, which was done in the proof of Proposition 3.2. On morphisms, without loss of generality it suffices to consider two G-bundles P, P' defined on the same open set U, and isomorphisms $\psi : P_H \simeq P'_H$, $\widehat{\phi} : \widehat{P} \to \widehat{P}'$ which have the same image in $Bun(H, \widehat{U})$. We need to show that there exists a unique isomorphism $\phi : P \to P'$ inducing $\widehat{\phi}$ and ψ in the natural sense. But by (i) there exists a unique ϕ with completion equal to $\widehat{\phi}$. Since by assumption the isomorphisms $\psi' = \phi_H$ and ψ are equal after completion, $\psi' = \psi$ by part (i). This finishes the proof. □

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