

REFLECTION SUBGROUPS OF COXETER GROUPS

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ABSTRACT. We use the geometry of the Davis complex of a Coxeter group to investigate finite index reflection subgroups of Coxeter groups. The main result is the following: if G is an infinite indecomposable Coxeter group and $H \subset G$ is a finite index reflection subgroup, then the rank of H is not less than the rank of G . This generalizes earlier results of the authors (2004). We also describe the relationship between the nerves of the group and the subgroup in the case of equal rank.

1. INTRODUCTION

In [3], M. Davis constructed for any Coxeter system (G, S) a contractible piecewise Euclidean complex, on which G acts properly and cocompactly by reflections. In this paper, we use this complex to study finite index reflection subgroups of infinite indecomposable Coxeter groups from a geometrical point of view. We define convex polytopes in the complex to prove the following result:

Theorem 1.1. *Let (G, S) be a Coxeter system, where G is infinite and indecomposable, and S is finite. If P is a compact polytope in Σ , then the number of facets of P is not less than $|S|$.*

This generalizes results of [7], where a similar result was proved for fundamental polytopes of finite index subgroups of cocompact (or finite covolume) groups generated by reflections in hyperbolic and Euclidean spaces. M. Dyer [5] proved that any reflection subgroup of a Coxeter group is also a Coxeter group. Using Theorem 1.1 and a criterion for finiteness of a Coxeter group provided by V. Deodhar [4], we obtain the main result of this paper:

Theorem 1.2. *Let (G, S) be a Coxeter system, where G is infinite and indecomposable and S is finite. Let $H \subset G$ be a finite index reflection subgroup. Then any set of reflections generating H contains at least $|S|$ elements.*

Further, we consider the geometry of the Davis complex itself. E. M. Andreev [1] obtained the following result for polytopes in hyperbolic and Euclidean spaces.

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Theorem 1.3 (Andreev [1]). *Let P be an acute-angled polytope in \mathbb{E}^n or \mathbb{H}^n , and let a and b be two faces of P . If $a \cap b = \emptyset$, then the minimal planes containing, respectively, a and b are also disjoint.*

In Section 4, we define acute-angled polytopes in the Davis complex and prove a counterpart of Andreev's theorem for non-intersecting facets of acute-angled polytopes (see Lemma 4.3).

In Section 5, we focus on the case of a Coxeter group G having a finite index reflection subgroup H of equal rank; i.e., this is the minimum possible rank for such an H . We show that the nerve of H can be obtained from the nerve of G by deleting some simplices (Lemma 5.1). We also give conditions on the combinatorics of (G, S) which are necessary in order for G to contain such an H (Lemma 5.4).

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2. PRELIMINARIES

2.1. Coxeter systems. A *Coxeter group* is a group with presentation

$$\langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

for all $s_i, s_j \in S$, where $m_{ii} = 1$ and $m_{ij} \in \{2, 3, \dots, \infty\}$ for $i \neq j$. We further require S to be finite. S is called a *standard generating set*. A pair (G, S) is called a *Coxeter system*.

If S is fixed, G is called *decomposable* if $S = S_1 \cup S_2$, where S_1 and S_2 are non-empty subsets such that $s_i s_j = s_j s_i$ for all $s_i \in S_1, s_j \in S_2$. If G is not decomposable, it is called *indecomposable*.

An element $s \in G$ is called a *reflection* if it is conjugate to some of the $s_i \in S$ (in particular, any $s_i \in S$ is a reflection). A subgroup $H \subset G$ is called a *reflection subgroup* of G if H is generated by reflections.

For any $T \subset S$ a reflection subgroup G_T generated by all $s_i \in T$ is called a *standard* subgroup of G .

The *rank* of a Coxeter system (G, S) is the number of reflections in S . We denote it by $|S|$.

The proof of the following lemma is suggested by R. B. Howlett.

Lemma 2.1. *Let (G, S) be a Coxeter system of rank n . Then G cannot be generated by less than n reflections.*

Proof. Let $S = \{s_1, \dots, s_n\}$. Consider the Tits representation of (G, S) on a real vector n -space V (see e.g. [2]). Suppose that G is generated by $k < n$ reflections r_1, r_2, \dots, r_k along vectors v_1, v_2, \dots, v_k , i.e.

$$r_i(x) = x - 2 \frac{(v_i, x)}{(v_i, v_i)} v_i.$$

Let L be the linear subspace spanned by v_1, \dots, v_k . Let $g = r_{i_1} r_{i_2} \dots r_{i_l}$ be any element of G . We prove by induction on l that for any $v \in V$, $g(v) \in v + L$. For $l = 0$ the statement is evident (since $v \in v + L$). Suppose the statement holds for

all elements $g = r_{i_1} r_{i_2} \dots r_{i_{l-1}}$. Let $g = r_{i_1} g_1$, where $g_1 = r_{i_2} \dots r_{i_l}$. Then

$$g(v) = r_{i_1} g_1(v) = g_1(v) - 2 \frac{(v_{i_1}, g_1(v))}{(v_{i_1}, v_{i_1})} v_{i_1} \in v + L + \lambda v_{i_1} \subset v + L$$

(where $\lambda \in \mathbb{R}$). Hence, $g(v) \in v + L$ for any $g \in G$.

Suppose that $g \in G$ is a reflection along some vector v . Then $-v = g(v) \in v + L$. Therefore, $v \in L$. Hence, by the construction of the Tits representation of (G, S) , L should coincide with V , which is impossible for a space spanned by $k < n$ vectors. \square

2.2. Davis complex. For any Coxeter system (G, S) there exists a contractible piecewise Euclidean cell complex $\Sigma(G, S)$ (called the *Davis complex*) on which G acts discretely, properly and cocompactly. The construction was introduced by Davis [3]. In [9] Moussong proved that this complex yields a natural complete piecewise Euclidean metric which is *CAT*(0). We give a brief description of this complex following [10].

For a finite group G the complex $\Sigma(G, S)$ is just one cell, which is obtained as a convex hull C of G -orbit of a suitable point p in the standard linear representation of G as a group generated by reflections. The point p is chosen in such a way that its stabilizer in G is trivial and all the edges of C are of length 1. The faces of C are naturally identified with Davis complexes of the subgroups of G conjugate to standard subgroups.

If G is infinite, the complex $\Sigma(G, S)$ is built up of the Davis complexes of maximal finite subgroups of G gluing together along their faces corresponding to common finite subgroups. The 1-skeleton of $\Sigma(G, S)$ considered as a combinatorial graph is isomorphic to the Cayley graph of G with respect to the generating set S .

The action of G on $\Sigma(G, S)$ is generated by reflections. The *walls* in $\Sigma(G, S)$ are the fixed point sets of reflections in G . The intersection of a wall α with cells of $\Sigma(G, S)$ supplies α with a structure of a piecewise Euclidean cell complex with finitely many isometry types of cells. Walls are totally geodesic: any geodesic joining two points of α lies entirely in α . Since Σ is *CAT*(0), any two points of Σ may be joined by a unique geodesic.

Any wall divides $\Sigma(G, S)$ into two connected components. The complement of the union of all walls in $\Sigma(G, S)$ is a disjoint collection of connected components, and the closure of each of these components is compact; these closures are called *chambers*. Any chamber is a fundamental domain of G -action on $\Sigma(G, S)$. The set of all chambers with appropriate adjacency relation is isomorphic to the Cayley graph of G with respect to S .

A *nerve* of a Coxeter system (G, S) is a simplicial complex with vertex set S . A collection of vertices spans a simplex if and only if the corresponding reflections generate a finite group. It is easy to see that the combinatorics of the chamber of $\Sigma(G, S)$ is completely determined by the nerve: a collection of walls of a chamber has a non-empty intersection if and only if the corresponding reflections generate a finite reflection group.

For any subgroup $H \subset G$ we say that a wall α is a *mirror* of H if H contains the reflection with respect to α .

In what follows, if G and S are fixed, we write Σ instead of $\Sigma(G, S)$.

2.3. Convex polytopes in Σ . For any wall α of Σ we denote by α^+ and α^- the closures of the connected components of $\Sigma \setminus \alpha$; we call these components *halfspaces*.

A *convex polytope* $P \subset \Sigma$ is an intersection of finitely many halfspaces $P = \bigcap_{i=1}^n \alpha_i^+$, such that P is not contained in any wall. Clearly, any convex polytope $P \subset \Sigma$ is a union of closed chambers. P is compact if and only if it contains finitely many chambers. Since the walls are totally geodesic, any convex polytope is convex in the usual sense: for any two points $p_1, p_2 \in P$ the geodesic segment connecting p_1 with p_2 belongs to P .

In what follows, by writing $P = \bigcap_{i=1}^n \alpha_i^+$ we assume that the collection of walls α_i is minimal: for any $j = 1, \dots, n$ we have $P \neq \bigcap_{i \neq j} \alpha_i^+$. A *facet* of P is an intersection $P \cap \alpha_i$ for some $i \leq n$. For any $I \subset \{1, \dots, n\}$ a set $\bigcap_{i \in I} \alpha_i$ is called a *face* of P if it is not empty.

We can easily define a dihedral angle formed by two walls: if $\alpha_i \cap \alpha_j \neq \emptyset$ there exists a maximal cell C of Σ intersecting $\alpha_i \cap \alpha_j$. We define the angle $\angle(\alpha_i, \alpha_j)$ to be equal to the corresponding Euclidean angle formed by $\alpha_i \cap C$ and $\alpha_j \cap C$. Clearly, any dihedral angle formed by two intersecting walls in Σ is equal to $k\pi/m$ for some positive integers k and m . A convex polytope P is called *acute-angled* if each of the dihedral angles of P does not exceed $\pi/2$.

A convex polytope P is called a *Coxeter polytope* if all its dihedral angles are integer submultiples of π . Clearly, a fundamental domain of any reflection subgroup of G is a Coxeter polytope in Σ . Conversely, Theorem 4.4 of [5] implies that any Coxeter polytope in Σ is a fundamental chamber for the subgroup of G generated by reflections in its walls.

3. PROOFS OF THE THEOREMS

In what follows we write $|P|$ for the number of facets of a convex polytope $P \subset \Sigma$.

Let a_i and a_j be intersecting facets of a convex polytope $P \subset \Sigma$. We say that a dihedral angle of P formed by a_i and a_j is *decomposed* if there exists a wall γ containing $a_i \cap a_j$ and intersecting $\text{int}(P)$.

The following lemma is proved by V. Deodhar.

Lemma 3.1 (Deodhar [4], Proposition 4.2). *Let (G, S) be a Coxeter system, where G is an infinite indecomposable Coxeter group and $S = \{s_0, s_1, \dots, s_k\}$. Let $H = \langle s_{i_1}, \dots, s_{i_l} \rangle$ be a proper standard subgroup of G . Then $[G : H] = \infty$.*

For the proof of Theorem 1.1 we need to consider decomposable groups also. More precisely, we use the following corollary of Lemma 3.1.

Corollary 3.2. *Let (G, S) be a Coxeter system, where G is an infinite Coxeter group and $S = \{s_0, s_1, \dots, s_k\}$. Suppose that for some $l \leq k$ the group $G_l = \langle s_0, \dots, s_l \rangle$ is infinite and indecomposable, and let m be a positive integer not exceeding k . Then a subgroup $H = \langle s_m, \dots, s_k \rangle$ has infinite index in G .*

Proof. If $l = k$, then G is indecomposable, and Lemma 3.1 applies. So, we may assume that G is decomposable. Permuting elements of S , we may assume that there exists p such that a group $G_p = \langle s_0, \dots, s_p \rangle$ is indecomposable and commutes with all s_i for $i > p$. Then the index $[G : H]$ is equal to the index $[G_p : H \cap G_p]$. Since $m \geq 1$, the subgroup G_p is not contained in H , so the index $[G_p : H \cap G_p]$ is infinite due to Lemma 3.1. \square

Theorem 1.1. *Let (G, S) be a Coxeter system, where G is infinite and indecomposable, and S is finite. If P is a compact polytope in Σ , then $|P| \geq |S|$.*

Proof. Suppose the conclusion of the theorem does not hold. Then there exist a Coxeter system (G, S) and a polytope P in Σ such that $|P| < |S|$. The proof is by induction on $|P|$, i.e. for any $k < |S|$ we assume that there is no compact polytope $P' \subset \Sigma$ such that $|P'| < k$ and prove that there is no compact polytope $P \subset \Sigma$ such that $|P| = k$. (Since G is infinite, this trivially holds for the cases $|P'| = 0$ and 1 .)

Suppose there exists a compact polytope $P \subset \Sigma$ such that $|P| = k < |S|$. Recall that \mathcal{P} is the set of all compact convex polytopes in Σ . Define

$$\mathcal{P}_1 = \{P_1 \in \mathcal{P} \mid P_1 \subset P, |P_1| = k\}.$$

Since P is a compact polytope, P is intersected by finitely many walls of Σ , so \mathcal{P}_1 is a finite set. \mathcal{P}_1 is not empty as it contains P . Let $P_{\min} \in \mathcal{P}_1$ be a polytope minimal with respect to inclusion.

Claim 1. P_{\min} has no decomposed dihedral angle.

Indeed, let a and b be facets of P_{\min} , and let μ be a wall decomposing a dihedral angle formed by a and b . Then $\mu^+ \cap P_{\min} \in \mathcal{P}$, $\mu^+ \cap P_{\min} \subset P$, and $|\mu^+ \cap P_{\min}| \leq k$ (since μ^+ contains only one of the facets a and b). By the induction assumption, the case $|\mu^+ \cap P_{\min}| < k$ is impossible. Hence, $|\mu^+ \cap P_{\min}| = k$, which contradicts the assumption that P_{\min} is a minimal by inclusion polytope in \mathcal{P}_1 .

Define the set of polytopes \mathcal{P}'_1 in the following way:

$$\mathcal{P}'_1 = \{P'_1 \in \mathcal{P} \mid P'_1 \subset P_{\min}, |P'_1| = k + 1,$$

and all but one facet of P'_1 are facets of $P_{\min}\}$.

Clearly, \mathcal{P}'_1 is a finite set. To show that \mathcal{P}'_1 is not empty, notice that P_{\min} is not a chamber of Σ since $|P_{\min}| = k < |S|$, while a chamber has $|S|$ facets. Therefore, there exists a wall μ decomposing P_{\min} into two polytopes, namely $P_{\min} \cap \mu^+$ and $P_{\min} \cap \mu^-$. It is clear that $|P_{\min} \cap \mu^+| \leq k + 1$. The case $|P_{\min} \cap \mu^+| < k$ is impossible by the induction assumption, and the case $|P_{\min} \cap \mu^+| = k$ is impossible since P_{\min} is a minimal by inclusion element of \mathcal{P}_1 . Hence, $|P_{\min} \cap \mu^+| = k + 1$, so $P_{\min} \cap \mu^+ \in \mathcal{P}'_1$.

Let $\mathcal{P}'_{1\min} \subset \mathcal{P}'_1$ be the set of polytopes of \mathcal{P}'_1 which are minimal by inclusion, and let $P'_{\min} \in \mathcal{P}'_{1\min}$.

Claim 2. P'_{\min} has no decomposed dihedral angle.

Again, let a and b be facets of P'_{\min} , and let μ be a wall decomposing the dihedral angle formed by a and b . Then μ decomposes P'_{\min} into two polytopes, namely $P'_{\min} \cap \mu^+$ and $P'_{\min} \cap \mu^-$. The polytope $\mu^+ \cap P'_{\min} \in \mathcal{P}$, $\mu^+ \cap P'_{\min} \subset P_{\min}$, and $|\mu^+ \cap P'_{\min}| \leq k + 1$ (since μ^+ does not contain either a or b). By the induction assumption, the case $|\mu^+ \cap P'_{\min}| < k$ is impossible. The case $|\mu^+ \cap P'_{\min}| = k$ is not possible either, since P_{\min} is a minimal by inclusion polytope in $\mathcal{P}'_{1\min}$. Hence, $|\mu^+ \cap P'_{\min}| = k + 1$, which contradicts the assumption that P'_{\min} is a minimal by inclusion polytope in $\mathcal{P}'_{1\min}$.

In particular, Claims 1 and 2 imply that all dihedral angles of P_{\min} and P'_{\min} are equal to some dihedral angles of a chamber of Σ , so P_{\min} and P'_{\min} are Coxeter polytopes. Therefore, $(\Gamma_{P'_{\min}}, S_{P'_{\min}})$ is a Coxeter system for the group $\Gamma_{P'_{\min}}$, where $S_{P'_{\min}}$ is the complete collection of the reflections with respect to the facets of P'_{\min} ,

and $\Gamma_{P'_{\min}}$ is the group generated by all these reflections. Similarly, $(\Gamma_{P_{\min}}, S_{P_{\min}})$ is a Coxeter system for the group $\Gamma_{P_{\min}}$, and $\Gamma_{P_{\min}}$ is a standard subgroup of $\Gamma_{P'_{\min}}$.

The group $\Gamma_{P_{\min}}$ may be decomposable. However, any maximal indecomposable component of $\Gamma_{P_{\min}}$ is infinite. Indeed, suppose that $G_0 = \langle s_1, \dots, s_l \rangle$ is a maximal indecomposable standard subgroup of $\Gamma_{P_{\min}}$, and G_0 is finite. Consider the union of all polytopes gP_{\min} , $g \in G_0$. This union is a compact Coxeter polytope with $k-l$ facets consisting of $|G_0|$ copies of P_{\min} . This contradicts the induction assumption that we have no compact polytopes in Σ with less than k facets.

Now consider the facet μ of P'_{\min} which is not a facet of P_{\min} , and let us prove the following statement.

Claim 3. There exists $\tilde{P}'_{\min} \in \mathcal{P}'_{1\min}$ such that μ is not orthogonal to all facets of P_{\min} .

Suppose the contrary. Take any $P'_{\min} \in \mathcal{P}'_{1\min}$, and denote by s the reflection in μ . Then $P'_{\min} \cup s(P'_{\min}) = P_{\min}$; i.e., μ divides P_{\min} into two congruent parts. Since s commutes with all the other standard generators of $\Gamma_{P'_{\min}}$, the group $\Gamma_{P'_{\min}}$ is decomposable, which implies that it is a proper subgroup of G . In particular, P'_{\min} is not a chamber of Σ , so there is a wall μ_1 dividing P'_{\min} .

Suppose that μ_1 is not orthogonal to all the facets of P_{\min} . Then consider the subset of \mathcal{P}'_1 consisting of polytopes lying inside $P'_{\min} \cap \mu_1^+$. This set is not empty since it contains $P'_{\min} \cap \mu_1^+$. Take a minimal by inclusion polytope P'^+_{\min} . It is cut off from P_{\min} by a wall μ'_1 . By assumption, μ'_1 is orthogonal to all the facets of P_{\min} . Therefore, $\mu'_1 \neq \mu_1$, and P_{\min} consists of two copies of P'^+_{\min} . Now consider a subset of \mathcal{P}'_1 consisting of polytopes lying inside $P'_{\min} \cap \mu_1^-$, and take a minimal by inclusion polytope P'^-_{\min} . Clearly, P_{\min} consists of two copies of P'^-_{\min} as well. However, P'^-_{\min} does not intersect P'^+_{\min} , so a copy of P'^-_{\min} should contain P'^+_{\min} , and a copy of P'^+_{\min} should contain P'^-_{\min} . The contradiction shows that any wall dividing P_{\min} is orthogonal to all facets of P_{\min} .

Now consider a chamber F of Σ contained in P_{\min} . We may assume that at least one facet of F belongs to some face of P_{\min} . Facets of F are of two types: some of them belong to facets of P_{\min} ; the others belong to walls dividing P_{\min} . Both sets are non-empty. However, any facet from one of these sets is orthogonal to any facet from another one. This implies that the group $G = \Gamma_F$ is not indecomposable, which contradicts the assumption of the theorem.

By Claim 3, we may assume that the reflection s in the facet μ of \tilde{P}'_{\min} does not commute with all the elements of $S_{P_{\min}}$. Consider the maximal indecomposable component Γ of $\Gamma_{\tilde{P}'_{\min}}$ containing s . Since all maximal indecomposable components of $\Gamma_{P_{\min}}$ are infinite, Γ is also infinite. So, we are in the assumptions of Corollary 3.2, and $[\Gamma_{\tilde{P}'_{\min}} : \Gamma_{P_{\min}}] = \infty$. This implies that P_{\min} contains infinitely many copies of \tilde{P}'_{\min} ; thus, $P_{\min} \subset P$ contains infinitely many chambers of Σ . This contradicts the assumption that P is a compact polytope in Σ . \square

Theorem 1.2. *Let (G, S) be a Coxeter system, where G is infinite and indecomposable, and S is finite. Let $H \subset G$ be a finite index reflection subgroup. Then any set of reflections generating H contains at least $|S|$ elements.*

Proof. Let (H, S') be a Coxeter system, where S' consists of some reflections of (G, S) . By Lemma 2.1, any set of generating reflections of H contains at least $|S'|$ reflections. So, we are left to show that $|S'| \geq |S|$.

Consider a fundamental chamber P of the subgroup H acting on the complex $\Sigma = \Sigma(G, S)$. Since H is a finite index subgroup, P is a compact polytope in Σ . By Theorem 1.1, this implies that $|P| \geq |S|$. Since, $|P| = |S'|$, we obtain the required inequality. \square

Remark 3.3. The conditions for G to be infinite and indecomposable are essential in both Theorems 1.1 and 1.2. Clearly, any finite group contains a trivial subgroup of finite index. Concerning decomposable groups, the group

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_3)^2 = (s_2 s_3)^2 = 1 \rangle$$

contains a subgroup $\langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$ of index two.

4. ANDREEV'S THEOREM

In this section we prove a counterpart of Andreev's theorem for facets of an acute-angled polytope in the complex Σ .

Remark 4.1. In the special case where P is a single chamber, then the statement is tautological even for arbitrary faces: by construction of Σ a collection of walls (respectively, facets of a chamber) has a non-empty intersection if and only if the corresponding reflections generate a finite group. It is easy to see that the same is true if P is any Coxeter polytope in Σ : for that it is sufficient to consider P as a chamber of the corresponding subgroup Γ_P of G .

Lemma 4.2. *Let P_1 and P_2 be convex polytopes in Σ and let c be a common facet of P_1 and P_2 . If all the dihedral angles of P_1 and P_2 formed by c with other facets of P_1 and P_2 are acute, then $P = P_1 \cup P_2$ is a convex polytope.*

Proof. Let $P_1 = \gamma^+ \cap \left(\bigcap_{i=1}^k \alpha_i^+ \right)$ and $P_2 = \gamma^- \cap \left(\bigcap_{j=1}^l \beta_j^+ \right)$, where γ is a wall containing c , α_i is a wall containing a facet a_i of P_1 , and β_j is a wall containing a facet b_j of P_2 . We will prove that $P = P_1 \cup P_2 = \left(\bigcap_{i=1}^k \alpha_i^+ \right) \cap \left(\bigcap_{j=1}^l \beta_j^+ \right)$. To prove this, it is sufficient to show that $P_2 \subset \alpha_i^+$ for all $i = 1, \dots, k$, and $P_1 \subset \beta_j^+$ for all $j = 1, \dots, l$. We prove the former of these statements; the latter may be shown in the same way. Given a facet $a_i = \alpha_i \cap P_1$, we consider two cases: either $\alpha_i \cap \gamma$ contains a face of P or it does not.

At first, consider a facet $a = a_i$ such that $\alpha \cap \gamma$ contains a face of P (where α is the wall containing a). Then there exists a facet $b = b_j$ of P_2 such that $(a \cap c) \cap P = (b \cap c) \cap P$. Denote by β the wall containing b . To prove that $P_2 \subset \alpha^+$, it is enough to prove that $\gamma^- \cap \beta^+ \subset \alpha^+$. For this, it is sufficient to show that $\text{int}(\gamma^- \cap \beta^+) \cap \alpha = \emptyset$ (since $c \subset \alpha^+$). Since γ divides Σ into two connected components, γ divides α into $\alpha \cap \gamma^+$ and $\alpha \cap \gamma^-$. The component $\alpha \cap \gamma^+$ does not intersect $\text{int}(\gamma^- \cap \beta^+)$ since these two sets belong to different halfspaces with respect to γ . The component $\alpha \cap \gamma^-$ does not intersect $\text{int}(\gamma^- \cap \beta^+)$ since $\alpha \cap \gamma^-$ belongs to β^- (the latter statement follows immediately while considering any maximal cell of Σ intersecting $c \cap b \cap a$).

Thus, we have proved that $P \subset \alpha_i^+$ for all $i \in I$, where $i \in I$ if and only if $(\gamma \cap \alpha_i) \cap P \neq \emptyset$. Consider $Q = \bigcap_{i \in I} \alpha_i^+$. Notice that $P_2 \subset Q \cap \gamma^-$, and $\gamma \cap Q = \gamma \cap P_1 = c$.

Now, consider a facet a_j such that $j \notin I$. Suppose that $\alpha_j \cap \text{int}(P_2) \neq \emptyset$. Then $\alpha_j \cap (Q \cap \gamma^-)$ is not empty either. Consider two points $p_1 \in \text{rel int}(P_1 \cap \alpha_j)$ and $p_2 \in \text{int}(P_2) \cap \alpha_j$. Here we require p_1 to belong to the relative interior of $P_1 \cap \alpha_j$ to be sure that p_1 lies in the interior of Q . Clearly, $p_2 \in \text{int}(Q)$, too. Notice that there is one special case when we cannot take $p_1 \in \text{rel int}(P_1 \cap \alpha_j)$: this happens if a_j is a point. However, in this case α_j is also a point, so it cannot intersect P_2 .

Consider a geodesic segment ξ joining p_1 with p_2 . Since α_j is a wall, and any wall is totally geodesic, $\xi \subset \alpha_j$. On the other hand, since Q is a convex polytope and $p_1, p_2 \in \text{int}(Q)$, we see that $\xi \subset \text{int}(Q)$. Moreover, $p_1 \in \gamma^+$ and $p_2 \in \gamma^-$, so, there exists a point $p \in \xi$ such that $p \in \gamma$. Since $\xi \subset \text{int}(Q)$, we obtain that $p \in \text{int}(Q) \cap \gamma = \text{int}(c)$. In particular, $\alpha_j \cap \text{int}(c) \neq \emptyset$, which is impossible by the definition of the convex polytope P_1 . Therefore, $\alpha_j \cap \text{int}(P_2) = \emptyset$, and the lemma is proved. \square

Lemma 4.3. *Let P be an acute-angled polytope in Σ . Let a and b be facets of P and α and β be the walls containing a and b respectively. If $a \cap b = \emptyset$, then $\alpha \cap \beta = \emptyset$.*

Proof. Let r_α and r_β be the reflections with respect to α and β . Suppose that $\alpha \cap \beta \neq \emptyset$. Then, by construction of Σ , r_α and r_β generate a finite dihedral group.

Consider a sequence of polytopes $P_0 = P$, $P_1 = r_\alpha P_0$, $P_2 = r_\alpha r_\beta r_\alpha P_1 = r_\alpha r_\beta P_0$, $P_3 = (r_\alpha r_\beta) r_\alpha (r_\alpha r_\beta)^{-1} P_2 = r_\alpha r_\beta r_\alpha P_0$, and so on, i.e. $P_i = r_\alpha r_\beta P_{i-2}$. By construction, P_i and P_{i+1} have a common facet, and P_i is symmetric to P_{i+1} with respect to this facet. Notice that each of the P_i is an acute-angled polytope (as an image of an acute-angled polytope). Let $Q_0 = P_0$ and $Q_i = Q_{i-1} \cup P_i$.

We claim that Q_i is a convex polytope and any dihedral angle of Q_i formed by $c_i = P_i \cap P_{i+1}$ and any other facet is acute. The proof is by induction on i . Indeed, $Q_0 = P$ is a convex acute-angled polytope. Suppose that the statement is true for Q_{i-1} . Then Lemma 4.2 implies that Q_i is a convex polytope. Any dihedral angle of Q_i formed by c_i and any other facet is a dihedral angle of an acute-angled polytope P_i , and hence, is acute. Therefore, Q_i is a convex polytope for any i .

Notice that the wall γ_{i-1} (containing the facet c_{i-1} of Q_{i-1}) decomposes Q_i into two convex polytopes $Q_i \cap \gamma_{i-1}^+ = Q_{i-1}$ and $Q_i \cap \gamma_{i-1}^- = P_i$, and hence, $\text{int}(P_i)$ does not intersect $\text{int}(Q_{i-1})$. On the other hand, $r_\alpha r_\beta$ has finite order, and hence, there exists $j \in \mathbb{N}$ such that $P_j = P_0$. The contradiction completes the proof of the lemma. \square

5. THE CASE OF EQUAL RANKS

Throughout this section we suppose that G is an infinite indecomposable Coxeter group with a finite set S of standard generators, $H \subset G$ is a finite index reflection subgroup with a set S' of standard generators, and the ranks of G and H are equal (i.e. $|S'| = |S|$). Let $|S'| = k$.

Lemma 5.1. *The nerve of a Coxeter system (H, S') can be obtained from the nerve of (G, S) by deleting some simplices.*

Proof. Let P be a fundamental chamber of H . Define the set of polytopes $\mathcal{P}(P)$ in the following way:

$$\mathcal{P}(P) = \{P' \in \mathcal{P} \mid P' \subset P, \text{ all but one facet of } P' \text{ are facets of } P\}.$$

$\mathcal{P}(P)$ is finite and non-empty: since P is not a chamber of Σ , at least one wall of Σ decomposes P into two smaller polytopes, each of them belonging to $\mathcal{P}(P)$. Thus, the set $\mathcal{P}_{\min}(P)$ of polytopes minimal in $\mathcal{P}(P)$ by inclusion is not empty.

Claim. There exists $P_1 \in \mathcal{P}_{\min}(P)$, such that P_1 is a Coxeter polytope, and $|P_1| = k$.

The first statement is evident: any polytope $P'_1 \in \mathcal{P}_{\min}(P)$ is a Coxeter polytope. Indeed, let μ_1 be the facet of P'_1 which is not a facet of P . If P'_1 is not a Coxeter polytope, a dihedral angle formed by μ_1 and some facet of P is decomposed by some wall μ'_1 . Then μ'_1 cuts out of P a polytope which is contained in P'_1 , which contradicts the minimality of P'_1 .

By Theorem 1.1, any $P'_1 \in \mathcal{P}_{\min}(P)$ has at least k facets. If it has more than k facets (i.e. $k+1$ facets), then all the facets of P are facets of P'_1 , so H is a standard subgroup of the group $\Gamma_{P'_1}$ which is generated by reflections in the facets of P'_1 .

The rest of the proof of this claim is similar to the proof of Claim 3 from the proof of Theorem 1.1.

Suppose that any $P'_1 \in \mathcal{P}_{\min}(P)$ has exactly $k+1$ facets. It follows from Theorem 1.1 that any indecomposable component of H is infinite. So, if for some $P'_1 \in \mathcal{P}_{\min}(P)$ the corresponding facet μ_1 is not orthogonal to all the facets of P , then we are in the assumptions of Cor. 3.2 for the groups $\Gamma_P \subset \Gamma_{P'_1}$. Therefore, we may assume that for any $P'_1 \in \mathcal{P}_{\min}(P)$ the corresponding facet μ_1 is orthogonal to all the facets of P , and P consists of two copies of P'_1 . In particular, $\Gamma_{P'_1}$ is decomposable, so P'_1 is not a chamber of Σ .

Now take any wall μ_2 dividing P'_1 , and suppose that μ_2 is not orthogonal to all the facets of P . Then consider two polytopes $P^+, P^- \in \mathcal{P}_{\min}(P)$ lying inside $P \cap \mu_2^+$ and $P \cap \mu_2^-$ respectively. P consists of two copies of each of these polytopes; however, they do not intersect, so a copy of P^- should contain P^+ , and a copy of P^+ should contain P^- . The contradiction implies that any wall dividing P is orthogonal to all the facets of P . Then, as in Claim 3 from the proof of Theorem 1.1, take a chamber F of Σ contained in P and show that the group $G = \Gamma_F$ is not indecomposable. The contradiction completes the proof of the claim.

The claim above implies that we may take $P_1 \in \mathcal{P}_{\min}(P)$ such that the corresponding facet μ_1 does not intersect exactly one facet of P , say f_1 . All the remaining facets of P are facets of P_1 . Therefore, we have a one-to-one correspondence between facets of P and P_1 : facet μ_1 corresponds to f_1 , and any other facet corresponds to itself. This implies a correspondence of vertices of the nerves of H and Γ_{P_1} . We want to prove that the nerve of H can be obtained from the nerve of Γ_{P_1} by deleting some simplices. For this we show that if a collection of facets of P has a non-empty intersection in P , then the corresponding collection of facets of P_1 (i.e. substituting μ_1 by f_1) has a non-empty intersection in P_1 .

Suppose that a collection $J = \{f_{i_1}, f_{i_2}, \dots, f_{i_n}\}$ of facets of P defines a face of P ; i.e., the intersection of all the facets contained in J is not empty. If $f_1 \notin J$, then the facets contained in J have non-empty intersection in P_1 (see Remark 4.1). If $f_1 \in J$, then consider the face f of P defined by the collection $J \setminus f_1$. By our assumption, f intersects f_1 . On the other hand, it was shown above that f contains

a face of P_1 . Since P_1 and f_1 are contained in distinct halfspaces with respect to μ_1 , this implies that f intersects μ_1 .

Therefore, the nerve of H can be obtained from the nerve of Γ_{P_1} by deleting some simplices. Now, substituting P by P_1 in the construction above, we may choose a polytope P_2 with k facets which is minimal in $\mathcal{P}(P_1)$. Since $P_2 \neq P_1$, P_2 contains a smaller number of chambers of Σ than P_1 does. Following this procedure (i.e. P_3 is minimal in $\mathcal{P}(P_2)$ and so on), we see that for some m the polytope P_m is a chamber of Σ , so $\Gamma_{P_m} = G$. The same proof as above shows that for any i the nerve of Γ_{P_i} can be obtained from the nerve of $\Gamma_{P_{i+1}}$ by deleting some simplices. Hence, the nerve of H also can be obtained from the nerve of G by deleting some simplices. \square

Example 5.2. As an example of the situation described in Lemma 5.1, consider a group $\Gamma = \Gamma(2, 3, \infty)$ generated by reflections in the sides of a hyperbolic triangle with angles $\pi/2, \pi/3$ and 0 . The nerve N of the group consists of 3 vertices joined by two edges, corresponding to the dihedral groups of orders 4 and 6. This group has exactly three reflection subgroups of rank 3. The subgroup $\Gamma_1 = \Gamma(3, 3, \infty)$ of index two is generated by reflections in the sides of a triangle with angles $\pi/3, \pi/3$ and 0 . Its nerve N_1 is isomorphic to N . The subgroup $\Gamma_2 = \Gamma(2, \infty, \infty)$ of index three is generated by reflections in the sides of a triangle with angles $\pi/3, 0$ and 0 . Its nerve N_2 consists of 3 vertices, only two of which are joined by an edge (corresponding to a dihedral group of order 4). The subgroup $\Gamma_3 = \Gamma(\infty, \infty, \infty)$ of index six is generated by reflections in the sides of an ideal triangle. Its nerve N_3 consists of 3 vertices and no edges.

Remark 5.3. Given an arbitrary Coxeter system (G, S) , a deleting of simplices usually does not lead to any finite index reflection subgroup. For example, it is easy to see that a group generated by reflections in the sides of an equilateral hyperbolic triangle with angles $(\pi/5, \pi/5, \pi/5)$ has no finite index reflection subgroups at all. Moreover, it is not even clear when a deleting of a simplex from a nerve of (G, S) corresponds to a reflection subgroup of G , probably of infinite index.

Finally, we provide a necessary condition for (G, S) to have a finite index reflection subgroup of the same rank.

Lemma 5.4. *Let (G, S) be a Coxeter system, where G is infinite indecomposable, and S is finite. Suppose that there exists a finite index reflection subgroup H of G of rank $|S|$. Then there exists $s_0 \in S$ such that at least one of the following holds:*

- (1) s_0 commutes with all but one element of S ;
- (2) the order of $s_0 s$ is finite for all $s \in S$.

Proof. Let \mathcal{P}_k be the set of polytopes in Σ with exactly k facets. Let $P_0 \in \mathcal{P}_k$ be any polytope containing no elements of \mathcal{P}_k except chambers of Σ , and P_0 itself is not a chamber of Σ . Since the rank of H is equal to the rank of G , such a P_0 does exist (for example, one can take as P_0 a fundamental domain of H in Σ). Clearly, there is at least one wall of Σ which divides P_0 .

Suppose that P_0 contains a decomposed dihedral angle formed by facets f_1 and f_2 of P_0 . Let μ be a wall of Σ which decomposes that angle. Then μ decomposes P_0 into two polytopes P_0^+ , and P_0^- and each of them has at most k facets (since one of f_1 and f_2 , say f_2 , is not a facet of P_0^+ , and the other is not a facet of P_0^-). By Theorem 1.1, each of P_0^+ and P_0^- has exactly k facets. Thus, P_0^+ and P_0^- are

chambers of Σ , and P_0^+ can be obtained from P_0^- by reflecting in μ . Since $k - 1$ of k facets of P_0^+ are facets of P_0^- , a facet $\mu \cap P_0$ of P_0^+ is orthogonal to all but one facet of P_0^+ , so condition (1) holds. A unique facet of P_0^+ which is not orthogonal to $\mu \cap P_0$ is f_1 . By assumption, μ intersects f_1 , so in this case condition (2) also holds.

Now suppose that P_0 contains no decomposed dihedral angles. In particular, P_0 is a Coxeter polytope. As in the proof of Lemma 5.1, consider the set $\mathcal{P}(P_0)$, and take a minimal (by inclusion) element P_1 . Again, P_1 is a Coxeter polytope, and $|P_1| = k$. By the choice of P_0 , this implies that P_1 is a chamber of Σ . Let μ be the wall of Σ which contains a facet of P_1 but contains no facets of P_0 . Let $P_2 = P_0 \setminus P_1$. By Theorem 1.1, P_2 has at least k facets. It is also clear that P_2 has at most $k + 1$ facets.

If P_2 has $k + 1$ facets, then any facet of P_0 contains a facet of P_2 . Therefore, any wall containing a facet of P_1 contains a facet of P_2 . Since P_1 and P_2 are contained in distinct halfspaces with respect to μ , this implies that μ intersects all facets of P_1 , so condition (2) holds.

If P_2 has k facets, then it is also a chamber of Σ , and P_2 can be obtained from P_1 by reflecting in μ . Thus, the number of facets of P_0 is the number of facets of P_1 that are orthogonal to μ plus twice the number of remaining facets of P_1 except $\mu \cap P_0$. Solving this linear equation, we see that $k - 2$ facets of P_1 are orthogonal to μ , so condition (1) holds. \square

Remark 5.5. Let (G, S) be a Coxeter system satisfying condition (1) of Lemma 5.4. If the Coxeter relation between s_0 and s' (which is a unique generator not commuting with s_0) has an even exponent (or if there is no relation), then the first condition of Lemma 5.4 is also sufficient: the set $\{S \setminus s_0, s_0 s' s_0\}$ is a set of standard generators for a subgroup of index 2. However, in the case of an odd exponent the first condition is not sufficient: a group generated by reflections in the sides of a hyperbolic triangle with angles $(\pi/2, \pi/5, \pi/5)$ has no finite index reflection subgroups of rank 3 (see, for example, [8] or [6]). The same example shows that the second condition is not sufficient either: any two standard generators of the group above generate a finite dihedral group.

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