

DIVERGENCE IN LATTICES IN SEMISIMPLE LIE GROUPS AND GRAPHS OF GROUPS

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ABSTRACT. Divergence functions of a metric space estimate the length of a path connecting two points A, B at distance $\leq n$ avoiding a large enough ball around a third point C . We characterize groups with non-linear divergence functions as groups having cut-points in their asymptotic cones. That property is weaker than the property of having Morse (rank 1) quasi-geodesics. Using our characterization of Morse quasi-geodesics, we give a new proof of the theorem of Farb-Kaimanovich-Masur that states that mapping class groups cannot contain copies of irreducible lattices in semi-simple Lie groups of higher ranks. It also gives a generalization of the result of Birman-Lubotzky-McCarthy about solvable subgroups of mapping class groups not covered by the Tits alternative of Ivanov and McCarthy.

We show that any group acting acylindrically on a simplicial tree or a locally compact hyperbolic graph always has “many” periodic Morse quasi-geodesics (i.e. Morse elements), so its divergence functions are never linear. We also show that the same result holds in many cases when the hyperbolic graph satisfies Bowditch’s properties that are weaker than local compactness. This gives a new proof of Behrstock’s result that every pseudo-Anosov element in a mapping class group is Morse.

On the other hand, we conjecture that lattices in semi-simple Lie groups of higher rank always have linear divergence. We prove it in the case when the \mathbb{Q} -rank is 1 and when the lattice is $SL_n(\mathcal{O}_S)$ where $n \geq 3$, S is a finite set of valuations of a number field K including all infinite valuations, and \mathcal{O}_S is the corresponding ring of S -integers.

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1. INTRODUCTION

1.1. The divergence. Roughly speaking, the divergence of a pair of points (a, b) in a metric space X relative to a point $c \notin \{a, b\}$ is the length of the shortest path from a to b avoiding a ball around c of radius δ times the distance $\mathrm{dist}(c, \{a, b\})$ for some $\delta \in (0, 1)$ fixed in advance. If no such path exists, then we define the divergence to be infinity. The divergence of a pair (a, b) is the supremum of the divergences of a, b relative to all $c \in X$. The divergence function $\mathrm{Div}(n; \delta)$ is the maximum of all divergencies of pairs (a, b) with $\mathrm{dist}(a, b) \leq n$. As usual with asymptotic invariants of metric spaces and groups, we consider the divergence function up to the natural equivalence:

$$f \equiv g \quad \text{if} \quad \frac{1}{C}g\left(\frac{n}{C}\right) - Cn - C \leq f(n) \leq Cg(Cn) + Cn + C$$

for some $C > 1$ and all n (a similar equivalence is used for functions in more than one variable). Then the divergence function is obviously a quasi-isometry invariant of a metric space.

For example, the divergence function of the plane \mathbb{R}^2 or the Cayley graph of \mathbb{Z}^2 is linear (for every δ), the divergence function of a tree or of a group with infinitely many ends is infinity for all $n > 0$, the divergence function of any hyperbolic group is at least exponential [Gro87, Al91], and the divergence function of any mapping class group is at least quadratic [Beh06]. There are in fact several other definitions of divergence in the literature: one can restrict the choice of c in a different way. For example, one can only consider the case when c is on a geodesic $[a, b]$. The choice of c can be further restricted by assuming that it is the midpoint

of a geodesic $[a, b]$. The divergence of pairs of rays can also be defined [Al91], and so on. S. Gersten ([Ger94b], [Ger94a]) used a version of the divergence function to study Haken manifolds. It is proved in [Ger94b], [Ger94a] and [KL98] that for every fundamental group of a Haken 3-manifold the divergence is linear, quadratic or exponential, with the quadratic divergence occurring precisely for graph manifolds and exponential divergence for manifolds with at least one hyperbolic geometric component. It is also proved in [Ger94b] that the divergence function of a semidirect product $H \rtimes \mathbb{Z}$ is at most the distortion function of H in $H \rtimes \mathbb{Z}$. The equality can be strict, as for instance in the Heisenberg group $\mathbb{Z}^2 \rtimes \mathbb{Z}$ which has linear divergence while \mathbb{Z}^2 is quadratically distorted.

In Section 3, we show that different definitions of divergence give equivalent functions, so we can speak of the divergence of a metric space.

1.2. Super-linear divergence and cut-points in asymptotic cones. For applications (see below), it is important to distinguish cases when the divergence is/is not linear. Since the (equivalence class of) divergence function is a quasi-isometry invariant, we can say that, for example, a group has linear or superlinear divergence without specifying a generating set. Note that there are finitely generated groups whose divergence is not linear but is arbitrarily close to being linear (and in fact is bounded by a linear function on arbitrary long intervals) [OOS05].

In Section 3, we characterize groups with superlinear divergence as groups whose asymptotic cones have global cut-points. In particular we prove

Proposition 1.1 (See Lemma 3.17). *All asymptotic cones of a finitely generated group G have no cut-points if and only if the divergence function $\text{Div}(n, \frac{1}{4})$ is linear.*

We also characterize groups, one of whose asymptotic cones has cut-points. Note that there are groups with some asymptotic cones having cut-points and some asymptotic cones having no cut-points.

The importance of having cut-points in asymptotic cones has been shown in [DS05] and [DS06]. In particular it is proved in [DS05] that if a non-virtually cyclic finitely generated group G has cut-points in one of its asymptotic cones, then a direct power of G contains a free non-Abelian subgroup. Hence G does not satisfy a non-trivial law (cannot be bounded torsion or solvable, etc.). Also if one asymptotic cone of G has cut-points, then G cannot have an infinite cyclic subgroup in its center, unless G is virtually cyclic [DS05]. Note that G can still have infinite locally finite center [OOS05].

Subgroups of groups with cut-points in all their asymptotic cones display some form of rigidity: if a subgroup H has infinitely many homomorphisms into G that are pairwise non-conjugate in G , then H acts non-trivially on an asymptotic cone of G which in many cases implies that H acts non-trivially on a simplicial tree [DS06], and so H splits non-trivially into an amalgamated product or an HNN extension.

1.3. Morse quasi-geodesics. A formally stronger property than superlinear divergence is the existence of the so called *Morse quasi-geodesics* (also called *stable quasi-geodesics* or *rank 1 quasi-geodesic*). A bi-infinite quasi-geodesic \mathfrak{q} in X is called *Morse* if every (L, C) -quasi-geodesic with endpoints on \mathfrak{q} is at bounded distance from \mathfrak{q} (the bound depends only on L, C). Proposition 3.24 below provides several other equivalent definitions of Morse quasi-geodesics. It turns out that a

quasi-geodesic \mathfrak{q} is Morse if and only if every point x on the limit of \mathfrak{q} in every asymptotic cone \mathcal{C} separates the two halves (before the point and after the point) of the limit, i.e. the two halves are in different connected components of $\mathcal{C} \setminus \{x\}$. This implies in particular that every point on the limit of \mathfrak{q} is a cut-point of \mathcal{C} . The converse implication (i.e. existence of cut-points in every asymptotic cone implies existence of a Morse quasi-geodesic) is not known in general (but it does hold in particular cases, for instance for non-positively curved manifolds; see the following paragraph). We show (Proposition 3.20) that if an asymptotic cone \mathcal{C} of G has cut-points, then some asymptotic cone \mathcal{C}' of G has a non-trivial geodesic with the above property (we call it a *transverse geodesic*). By Remark 3.21, if the Continuum Hypothesis is true, one can take $\mathcal{C}' = \mathcal{C}$.

In a Gromov hyperbolic space, every bi-infinite quasi-geodesic is Morse, a property which is crucial in the proof of Mostow rigidity in the rank 1 case. A similar property is true for relatively hyperbolic spaces [DS05].

Suppose that a finitely generated group G acts on a space X by isometries. Recall that a quasi-geodesic \mathfrak{q} in X is called *periodic* if $h \cdot \mathfrak{q} = \mathfrak{q}$ for some $h \in G$, and $\mathfrak{q}/\langle h \rangle$ is bounded. The most common sources of such quasi-geodesics are just orbits $\{h^n \cdot x, n \in \mathbb{Z}\}$ of $h \in G$ in X . Note that if an orbit of h in X is quasi-geodesic, then the sequence $\{h^n, n \in \mathbb{Z}\}$ is also a periodic bi-infinite quasi-geodesic in (any Cayley graph of) G . W. Ballmann [Bal95] (see also Kapovich-Kleiner-Leeb [KKL98, Proposition 4.5]) proved that in a locally compact complete CAT(0)-space X with a co-compact group action, a periodic quasi-geodesic is Morse if and only if it does not bound a half-plane. Moreover [KKL98] if X is a non-flat de Rham irreducible manifold on which a discrete group acts cocompactly by isometries, then X is either a symmetric space of non-compact type and rank at least two or X contains a periodic Morse geodesic. Consequently, in this particular case the existence of cut-points in asymptotic cones implies the existence of Morse quasi-geodesics.

The existence of Morse geodesics in finite dimensional locally compact CAT(0)-spaces with co-compact group action implies the existence of free non-cyclic subgroups in the group [Bal95]. One cannot drop the CAT(0) assumption in this statement because by Olshanskii-Osin-Sapir [OOS05], there exist Tarski monsters (non-virtually cyclic groups where all proper subgroups are cyclic), where every non-trivial cyclic subgroup is a Morse quasi-geodesic.

1.4. Morse elements. We shall call elements $g \in G$ such that $\{g^n, n \in \mathbb{Z}\}$ is a Morse quasi-geodesic *Morse elements* (these elements are also sometimes called *elements of rank 1*). For example every element of infinite order in a hyperbolic group is Morse [Gro87]. In relatively hyperbolic groups, every element of infinite order that is not in a parabolic subgroup is Morse [DS05]. In the mapping class group $\text{MCG}(S_g)$, every pseudo-Anosov element is Morse [Beh06]. The fundamental group of a compact irreducible manifold of non-positive sectional curvature is either a lattice in a higher rank semi-simple Lie group or it contains a Morse element ([Bal95], [KKL98]). This dichotomy has been extended to fundamental groups of locally CAT(0) spaces (complexes), with lattices of isometries of Euclidean buildings added to the list of possibilities ([BBr95], [BBr99], [BBu06]).

Existence of Morse elements immediately implies some algebraic properties of the group G . In particular, suppose that a finitely generated subgroup $H \leq G$ contains a Morse element g . Then, by Lemma 3.25, in the word metric of H , $\{g^n, n \in \mathbb{Z}\}$ is also a Morse quasi-geodesic. In particular, all the asymptotic cones of the group

H (considered as a metric space with its own word metric) must have cut-points. Thus we have the following statement.

Proposition 1.2 (See Proposition 3.26 below). *If $H < G$ does not have its own Morse elements (say, H is torsion or satisfies a non-trivial law or, more generally, does not have cut-points in its asymptotic cones), then H cannot contain any Morse elements of G .*

This proposition has applications to subgroups of relatively hyperbolic groups and mapping class groups (see below).

1.5. The main results of the paper. Proposition 1.2 shows that it is useful to study both the class of groups with Morse elements and the class of groups without Morse elements. In Section 4, we study the former, and in Sections 5, 6, we study the latter.

The existence of Morse elements in non-trivial free products of groups follows (in particular) from the fact that free products are hyperbolic relative to their free factors and from [DS05]. We generalize this fact by proving the following theorem (the terms used in the theorem are explained after the formulation).

Theorem 1.3 (See Theorems 4.1, 4.4). *Let X be a simplicial tree or a uniformly locally finite hyperbolic graph, and let G be any finitely generated group acting on X acylindrically. Then any loxodromic element of G is Morse in G .*

Recall that an action of G on X is *acylindrical* if for some $l > 0$ the stabilizers in G of pairs of points in X at distance $\geq l$ are finite of uniformly bounded sizes (in this case we say that the action is l -acylindrical).

It is known [Bow] that if X is a tree or a locally finite hyperbolic graph, then for every isometry α of X some power α^k of α either fixes a point in X (i.e. α is *elliptic*) or stabilizes a bi-infinite geodesic \mathfrak{p} and $\langle \alpha^k \rangle$ acts on \mathfrak{p} co-compactly (i.e. α is *loxodromic*). If a group G acts on X by isometries and every element of G is elliptic, then G has a bounded orbit.

In Section 4.3, we generalize Theorem 1.3 to groups acting on the so called *Bowditch graphs*; our result implies the theorem of Behrstock, stating that every pseudo-Anosov element in a mapping class group of a surface is Morse. The notion of Bowditch graph (see a formal definition in Section 4.3) is in a sense an abstract version of the curve complex of a surface. We postulate the existence of a set of *tight geodesics* which is invariant under G . The set should be large enough so that every two vertices in X are connected by a tight geodesic. On the other hand the set of tight geodesics should be small enough so that, for example, for every pair of points $a, b \in X$ and every point c on a tight geodesic $[a, b]$, far enough from a, b , every ball $B(c, r)$ contains only finitely many points from tight geodesics connecting a and b . Note that every simplicial tree is a Bowditch graph where every finite geodesic is considered tight.

In Sections 5 and 6, we study lattices in higher rank semi-simple Lie groups. We conjecture that every such lattice has linear divergence. The conjecture is true for uniform lattices because an asymptotic cone of a uniform lattice Γ in a Lie group L is bi-Lipschitz equivalent to an asymptotic cone of L . If L is semi-simple of higher rank and non-compact type, then every asymptotic cone of L is a Euclidean building of rank ≥ 2 [KL97] and every two points in the asymptotic cone belong to

a 2 dimensional flat. Hence there are no cut-points in the asymptotic cones of L . For non-uniform lattices the question is still open. We prove

Theorem 1.4 (See Corollary 5.13 and Theorem 6.1). *Let Γ be an irreducible lattice in a semi-simple Lie group of \mathbb{R} -rank ≥ 2 . Suppose that Γ is either of \mathbb{Q} -rank 1 or is of the form $\mathrm{SL}_n(\mathcal{O}_S)$ where $n \geq 3$, S is a finite set of valuations of a number field K including all infinite valuations, and \mathcal{O}_S is the corresponding ring of S -integers. Then Γ has linear divergence.*

In the proof of Theorem 1.4, we heavily use the theorem of Lubotzky-Mozes-Raghunathan [LMR93, LMR00] which says that the word metric on an irreducible lattice in a semi-simple Lie group of higher rank is quasi-isometric to the restriction of any left-invariant Riemannian metric of the Lie group itself. In the case of \mathbb{Q} -rank 1 we use the structure of asymptotic cones. In that case every asymptotic cone is a product of Euclidean buildings [KL97]. We use results from Druțu [Dru97, Dru98, Dru04] and results about buildings from Kleiner-Leeb [KL97]. In the case of $\mathrm{SL}_n(\mathcal{O}_S)$ we explicitly construct a path connecting two given matrices and avoiding a ball centered at a third matrix.

1.6. Applications to subgroups of mapping class groups. One of the most important results about subgroups of the mapping class groups is the Tits alternative proved by McCarthy [McC85] and Ivanov [Iva84]: *every subgroup of a mapping class group $\mathcal{MCG}(S)$ either contains a free non-Abelian subgroup or it contains a free Abelian subgroup of rank at most $\xi(S)$ and of index at most $N = N(S)$.* Thus every subgroup of a mapping class group not containing a free non-Abelian subgroup is virtually Abelian. This improved a previous result of Birman-Lubotzky-McCarthy [BLM83] that a solvable subgroup of a mapping class group must be virtually Abelian.

Proposition 1.2 and Theorem 1.3 immediately imply the following new version of the Tits alternative.

Theorem 1.5. *If a group H does not have Morse elements and is a subgroup of the mapping class group $\mathcal{MCG}(S)$ (where S is a surface with possible punctures), H then stabilizes a (finite) collection of pairwise disjoint simple closed curves on S .*

Proof. Indeed, the fact that the curve complex of a surface S is a Bowditch graph is proved in [Bow]. By Proposition 1.2 and Theorem 1.3, H cannot contain any loxodromic elements for the action of $\mathcal{MCG}(S)$ on the curve complex of S . Thus some power of every element in H must fix a curve on S . By Ivanov and McCarthy [Iva92] H then stabilizes a collection of pairwise disjoint simple closed curves on S . \square

Note that this theorem can be used to give a proof of the Birman-Lubotzky-McCarthy theorem [BLM83] cited above. Indeed, non-virtually cyclic solvable groups do not have cut-points in their asymptotic cones [DS05], so if H is a solvable non-virtually cyclic subgroup of $\mathcal{MCG}(S)$, then it must stabilize a collection of pairwise disjoint simple closed curves on S . Hence up to a finite index, it must fix a curve γ on S . Then up to finite index, H is a subgroup of the direct product of the cyclic subgroup generated by the Dehn twist about γ and the restriction of H onto $S \setminus \gamma$, and we get a solvable subgroup of the mapping class group of a surface of smaller complexity (the complexity of S is, by definition, $3g + p - 3$, where g is the

genus and p is the number of punctures). Using an induction on the complexity, we deduce that up to a finite index, H is in the subgroup generated by Dehn twists about a collection of pairwise disjoint simple closed curves, so H is virtually Abelian (in fact the rank of the free Abelian subgroup of H does not exceed $3g + p - 3$, and the index does not exceed the maximal size of a finite subgroup of $\mathcal{MCG}(S)$).

Theorem 1.5 implies, in particular, the theorem of Farb-Kaimanovich-Masur [FM, KM96]: *the mapping class group of a surface does not contain lattices of semi-simple Lie groups of higher ranks*. Indeed for irreducible non-uniform lattices this is so because such lattices contain distorted cyclic subgroups [LMR93], while the mapping class groups do not, according to Farb, Lubotzky and Minsky [FLM01]. On the other hand, uniform higher rank lattices do not have cut-points in their asymptotic cones (see above). By Theorem 1.5, if such a lattice is a subgroup of a mapping class group, then it must stabilize a multi-curve.¹

The rest of the proof follows [FM]. Suppose that such a lattice Γ stabilizes a curve on S . Then a finite index subgroup of Γ would have a homomorphism with infinite image into the mapping class group of a surface of smaller complexity (the surface cut along the multi-curve). By Selberg's theorem, we can assume that Γ is torsion-free. By Margulis' normal subgroup theorem, the homomorphism must be injective, and we can proceed by induction on the complexity of S .

1.7. Applications to subgroups of relatively hyperbolic groups. Similar results are true for relatively hyperbolic groups. In this paper, when speaking about relatively hyperbolic groups we always mean strongly relatively hyperbolic groups, in the sense of [Gro87].

Recall that in such groups as well, a Tits alternative holds: a subgroup in a relatively hyperbolic group is either parabolic or it contains a free non-Abelian subgroup [Kou98]. Using the fact that every non-parabolic element in a relatively hyperbolic group is Morse ([DS05], [Osi06]), we immediately deduce

Theorem 1.6. *If H is an infinite finitely generated group without Morse elements, then every isomorphic copy of H in a finitely generated (strongly) relatively hyperbolic group is inside a parabolic subgroup.*

Note that this does not follow from the quasi-isometry rigidity result of [DS05] (stating that any quasi-isometric embedding of a subgroup without cut-points in some asymptotic cones stays in a tubular neighborhood of a parabolic subgroup) since we do not assume that the subgroup is undistorted and since the non-existence of Morse elements is weaker than the non-existence of cut-points in some asymptotic cones [OOS05].

2. GENERAL PRELIMINARIES

2.1. Definitions and general results. Throughout the paper we work with various discrete versions of paths and arcs. Let (X, dist) be a metric space. A *finite C -path* is a sequence of points a_1, a_2, \dots, a_n in X with $\text{dist}(a_i, a_{i+1}) \leq C$. *Infinite* and *bi-infinite C -paths* are defined similarly, with sets of indices $\pm\mathbb{N}$, respectively \mathbb{Z} .

¹Note that initially in [KM96] a similar conclusion was obtained as a corollary of a description of the Poisson boundaries of mapping class groups. Another proof, using quasi-morphisms, was found by Bestvina and Fujiwara in [BF02].

Let $L \geq 1$ and $C \geq 0$ be two constants. An (L, C) -quasi-geodesic is a map $\mathbf{p}: I \rightarrow X$, where I is an interval of the real line, such that

$$\frac{1}{L}|s - t| - C \leq \text{dist}(\mathbf{p}(s), \mathbf{p}(t)) \leq L|s - t| + C, \text{ for all } s, t \in I.$$

For a quasi-geodesic $\mathbf{p}: [0, d] \rightarrow X$ we call d its *quasi-length*. For a concatenation of quasi-geodesics, its *quasi-length* is the sum of the quasi-lengths of its components. If $I = [a, +\infty)$, then \mathbf{p} (or its image $\mathbf{p}(I)$) is called either an (L, C) -quasi-geodesic ray or an *infinite* (L, C) -quasi-geodesic. If $I = \mathbb{R}$, then \mathbf{p} (or its image) is called a *bi-infinite* (L, C) -quasi-geodesic. When the constants L, C are irrelevant they are not mentioned.

Quasi-geodesics may not be continuous, for instance C -paths may be (images of) quasi-geodesics. Since we tacitly identify finitely generated groups with sets of vertices in their Cayley graphs, we shall sometimes refer to sequences of elements in a group as “quasi-geodesics”, meaning that they compose a C -path which is the image of a quasi-geodesic.

We call $(L, 0)$ -quasi-isometries (quasi-geodesics) *L-bi-Lipschitz maps (paths)*, or simply *bi-Lipschitz maps (paths)*.

Let ω be an ultrafilter. All ultrafilters we use are assumed to be non-principal. A statement $P(n)$ depending on $n \in \mathbb{N}$ holds ω -almost surely (ω -a.s.) if the set of n 's where $P(n)$ holds belongs to the ultrafilter. The ω -limit $\lim_{\omega} x_n$ of a sequence of numbers (x_n) is the number x (possibly $\pm\infty$) such that for every neighborhood U of x , x_n is in U ω -almost surely.

An *asymptotic cone* $\text{Con}^{\omega}(X, (o_n), (d_n))$ of a metric space X , corresponding to a non-principal ultrafilter ω , a sequence of observation points (o_n) in X , and a sequence of positive scaling constants d_n such that $\lim_{\omega} d_n = +\infty$, is the quotient space of the set of sequences $\Pi_b X = \{(x_n) \mid \lim_{\omega} (\text{dist}(x_n, o_n)/d_n) \text{ finite}\}$ by the equivalence relation $(x_n) \equiv (y_n)$ if $\lim_{\omega} (\text{dist}(x_n, y_n)/d_n) = 0$. The equivalence classes composing the cone are denoted by $(x_n)^{\omega}$, and the distance function on the cone is defined by $\text{dist}((x_n)^{\omega}, (y_n)^{\omega}) = \lim_{\omega} (\text{dist}(x_n, y_n)/d_n)$. For details on asymptotic cones we refer the reader to [dDW84], [Gro93], [Dru02].

For every sequence of subsets A_n in a metric space X and for every asymptotic cone $\mathcal{C} = \text{Con}^{\omega}(X, (o_n), (d_n))$, the ω -limit $\lim^{\omega} (A_n)$ is defined as the set of all elements $(a_n)^{\omega} \in \mathcal{C}$ where $a_n \in A_n$. It is not difficult to check that the ω -limit of any sequence of geodesics $[a_n, b_n]$ of X is always either

- empty (if no point $(c_n)^{\omega}$, $c_n \in [a_n, b_n]$, is in \mathcal{C}), or
- a finite geodesic $[(a_n)^{\omega}, (b_n)^{\omega}]$ (if both $(a_n)^{\omega}$, $(b_n)^{\omega}$ are in \mathcal{C}), or
- a bi-infinite geodesic (if neither $(a_n)^{\omega}$ nor $(b_n)^{\omega}$ is in \mathcal{C} , but some point $(c_n)^{\omega}$, $c_n \in [a_n, b_n]$, is in \mathcal{C}), or
- a geodesic ray (if exactly one of the points $(a_n)^{\omega}, (b_n)^{\omega}$ is in \mathcal{C}).

A similar statement is true for quasi-geodesics. Only the ω -limit of a sequence of (finite) quasi-geodesics is either empty or a bi-Lipschitz embedded interval, ray or line.

In an asymptotic cone \mathcal{C} , a geodesic which is equal to the limit of a sequence of geodesics in X is called a *limit geodesic*.

Remark 2.1. Note that there exist examples of groups (see [Dru06]) such that “most” geodesics in their asymptotic cones are not limit geodesics.

For every subset A in a metric space, denote by $\mathcal{N}_\delta(A)$ the open δ -tubular neighborhood of A , i.e. the set of points x satisfying $\text{dist}(x, A) < \delta$; denote by $\overline{\mathcal{N}}_\delta(A)$ the closed δ -tubular neighborhood of A , i.e. the set of points x satisfying $\text{dist}(x, A) \leq \delta$.

Convention 2.2. In what follows, the setting is that of a geodesic metric space, and it is assumed that for some $\lambda \geq 1$ and $\kappa \geq 0$ a collection \mathcal{T} of (λ, κ) -quasi-geodesics is chosen, so that:

- (1) any two points in the metric space are joined by at least one quasi-geodesic in \mathcal{T} ;
- (2) any sub-quasi-geodesic of a quasi-geodesic in \mathcal{T} is also in \mathcal{T} .

A concatenation of k quasi-geodesics in \mathcal{T} is called a k -piecewise \mathcal{T} quasi-path. If a quasi-path \mathfrak{p} is obtained as the concatenation of finitely many quasi-geodesics $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ then we write $\mathfrak{p} = \mathfrak{p}_1 \sqcup \dots \sqcup \mathfrak{p}_k$. It is easily seen that when $\kappa = 0$ such a quasi-path is λ -Lipschitz. Therefore in this case we call \mathfrak{p} a piecewise \mathcal{T} path.

Lemma 2.3. *Suppose that $\kappa = 0$, i.e. \mathcal{T} consists of λ -bi-Lipschitz paths. Let Y be a geodesic metric space, let B be a closed set in Y and let x, y be in the same connected component of $Y \setminus B$. Then there exists a piecewise \mathcal{T} path \mathfrak{p} in Y connecting x and y and not intersecting B .*

Proof. The set $Y \setminus B$ is open. Hence for every point a in $Y \setminus B$ there exists an open ball $B(a, \epsilon)$ around a which is disjoint from B . Any two points inside $B(a, \epsilon/\lambda^2)$ are connected by a piecewise \mathcal{T} path (with at most two pieces) inside $B(a, \epsilon)$. Therefore the set Z_x of all points in $Y \setminus B$ reachable from x via piecewise \mathcal{T} paths is open. The set Z_x is also closed, since for every point $z \in Y \setminus B$ outside Z_x there exists an open ball $B(z, \delta)$ around z that does not intersect B . Then the ball $B(z, \delta/\lambda^2)$ cannot intersect Z_x , for otherwise there would be a piecewise \mathcal{T} path connecting x and z . Since Z_x is obviously connected, it coincides with the connected component of x in $Y \setminus B$. Hence $y \in Z_x$. \square

Lemma 2.4. *Let (C, dist) be a geodesic metric space, assume that \mathcal{T} is a collection of geodesics (i.e. $\lambda = 1$ and $\kappa = 0$) and let $\mathfrak{p} = [a, b] \cup [b, c]$ be a piecewise \mathcal{T} simple path in C joining points a and c . For every $x, y \in \mathfrak{p}$ denote by $\ell(x, y)$ the length of the shortest sub-arc of \mathfrak{p} of endpoints x, y . Let $b' \in [a, b]$, $b'' \in (b, c]$ and $C > 0$ be such that*

$$\text{dist}(x, y) \geq \frac{1}{C} \ell(x, y) \text{ for every } x \in [a, b'] \text{ and } y \in [b'', c].$$

If $d' \in [a, b']$ and $d'' \in [b'', c]$ minimize the distance, then $\mathfrak{p}' = [a, d'] \cup [d', d''] \cup [d'', c]$ is a C' -bi-Lipschitz path, where $C' = \max(C, 3)$ and $[d', d'']$ is a geodesic in \mathcal{T} joining d' and d'' .

Proof. Given $x, y \in \mathfrak{p}'$ denote by $\ell'(x, y)$ the length of the shortest sub-arc of \mathfrak{p}' of endpoints x, y . If $x \in [a, d']$ and $y \in [d'', c]$, then by hypothesis

$$\text{dist}(x, y) \geq \frac{1}{C} \ell(x, y) \geq \frac{1}{C} \ell'(x, y).$$

Hence it remains to study the case when one of the two points x and y is on $[d', d'']$. Assume it is y . Likewise assume that $x \in [a, d']$ (the other case is similar).

If $\text{dist}(x, y) < \text{dist}(d', y)$, then $\text{dist}(x, d'') < \text{dist}(d', d'')$, contradicting the choice of d', d'' . Thus, $\text{dist}(d', y) \leq \text{dist}(x, y)$; hence $\text{dist}(x, d') \leq 2\text{dist}(x, y)$ and $\ell'(x, y) = \text{dist}(x, d') + \text{dist}(d', y) \leq 3\text{dist}(x, y) \leq C' \text{dist}(x, y)$. \square

Lemma 2.5. *Let $(\mathcal{C}, \text{dist})$ be a geodesic metric space, assume that \mathcal{T} is a collection of geodesics, and let \mathbf{p} be a piecewise \mathcal{T} simple path in \mathcal{C} joining points A and B . Then for every δ small enough there exists a constant C and a piecewise \mathcal{T} simple path \mathbf{p}' at Hausdorff distance at most δ from \mathbf{p} and such that \mathbf{p}' is a C -bi-Lipschitz path.*

Proof. We denote the vertices of \mathbf{p} in consecutive order by $v_0 = A, v_1, \dots, v_k = B$, and by $[v_i, v_{i+1}]$ the consecutive edges of \mathbf{p} . For any two points x, y on \mathbf{p} we denote by $\ell(x, y)$ the length of the shortest sub-arc of \mathbf{p} of endpoints x, y .

Let $\phi : \mathbf{p} \times \mathbf{p} \setminus \Delta \rightarrow \mathbb{R}_+$ be the map defined by $\phi(x, y) = \frac{\text{dist}(x, y)}{\ell(x, y)}$, where $\Delta = \{(x, x) \mid x \in \mathbf{p}\}$. Note that the maximal value of ϕ is 1. If the infimum of ϕ is $1/K > 0$, then \mathbf{p} is a K -bi-Lipschitz path. We therefore assume that the minimal value of ϕ is zero.

As ϕ is a continuous function, if $x \in [v_i, v_{i+1}]$ and $y \in [v_j, v_{j+1}]$ with $\{v_i, v_{i+1}\} \cap \{v_j, v_{j+1}\} = \emptyset$, then $\phi(x, y) \geq C_{ij}$ for some constant $C_{ij} > 0$. Hence there must exist some $i \in \{1, 2, \dots, k - 1\}$ such that the infimum of ϕ on $[v_{i-1}, v_i] \times [v_i, v_{i+1}]$ is zero. We now show how \mathbf{p} can be slightly modified between v_{i-1} and v_{i+1} so that between these two vertices the function ϕ has a positive infimum. In what follows we always assume that $x \in [v_{i-1}, v_i]$ and $y \in [v_i, v_{i+1}]$.

Let $\delta > 0$ be such that the distance from every vertex v_i to $\mathbf{p} \setminus \{[v_{i-1}, v_i] \cup [v_i, v_{i+1}]\}$ is at least 2δ . Consider the distance from $[v_{i-1}, v_i] \setminus B(v_i, \delta)$ to $[v_i, v_{i+1}]$, the distance from $[v_{i-1}, v_i]$ to $[v_i, v_{i+1}] \setminus B(v_i, \delta)$, and let $\tau > 0$ be the minimum between the two distances.

There exist $x_0 \in [v_{i-1}, v_i]$ and $y_0 \in (v_i, v_{i+1})$ such that $\text{dist}(x_0, y_0) = \frac{\tau}{2}$. Clearly both x_0 and y_0 are in $B(v_i, \delta)$. Now let $x_1 \in [v_{i-1}, x_0]$ and $y_1 \in [y_0, v_{i+1}]$ be a pair of points minimizing the distance. As $\text{dist}(x_1, y_1) \leq \text{dist}(x_0, y_0) = \frac{\tau}{2}$ it follows that both points are again in $B(v_i, \delta)$. Consider \mathbf{p}' the piecewise \mathcal{T} path obtained by replacing in \mathbf{p} the sub-arc $[v_{i-1}, v_i] \cup [v_i, v_{i+1}]$ with $[v_{i-1}, x_1] \cup [x_1, y_1] \cup [y_1, v_{i+1}]$, where $[x_1, y_1]$ is in \mathcal{T} . The fact that $x_1, y_1 \in B(v_i, \delta)$ implies that \mathbf{p} and \mathbf{p}' are at Hausdorff distance at most δ . The fact that \mathbf{p}' is also simple follows from the choice of δ .

Let $\phi' : \mathbf{p}' \times \mathbf{p}' \setminus \Delta' \rightarrow \mathbb{R}_+$ be the function similar to ϕ defined with respect to \mathbf{p}' (it is distinct from ϕ even on common points, as the arc-lengths changed). Lemma 2.4 implies that the infimum of ϕ' is positive between v_{i-1} and v_{i+1} .

By eventually repeating the same modification in all vertices near which ϕ approaches the zero value, we obtain in the end a C -bi-Lipschitz path piecewise \mathcal{T} and at Hausdorff distance δ from \mathbf{p} . □

Lemma 2.6. *Let X be a geodesic metric space, let $\lambda \geq 1, \kappa \geq 0$, and let L be a collection of (λ, κ) -quasi-geodesics in X satisfying the conditions in Convention 2.2.*

Let $\mathcal{C} = \text{Con}^\omega(X, (o_n), (d_n))$ be an asymptotic cone of X , and let L_ω be the collection of limits of sequences of quasi-geodesics in L .

- (1) *Let \mathbf{p} be a piecewise L_ω path in \mathcal{C} joining distinct points $A = (a_n)^\omega$ and $B = (b_n)^\omega$. Then there exists k and D such that $\mathbf{p} = \lim^\omega(\mathbf{p}_n)$, where each \mathbf{p}_n is a k -piecewise L quasi-path joining a_n and b_n ; moreover, each \mathbf{p}_n is of quasi-length $\leq D \text{dist}(a_n, b_n)$.*
- (2) *Assume that L is a collection of geodesics, and let \mathbf{p} be as in (1); moreover, \mathbf{p} is a C -bi-Lipschitz path. Then there exists $C' = C'(C)$ and a natural*

number $k \geq 1$ such that $\mathbf{p} = \lim^\omega (\mathbf{p}_n)$, where each \mathbf{p}_n is a C' -bi-Lipschitz path joining a_n and b_n in X ; moreover, \mathbf{p}_n is a k -piecewise L path.

Proof. (1) If \mathbf{p} has m edges in L_ω , then a sequence \mathbf{p}_n with at most $2m$ edges in L can be easily constructed. The other properties of \mathbf{p}_n follow immediately.

(2) The path \mathbf{p} can be written as a limit $\lim^\omega (\mathbf{p}_n)$ of piecewise L paths \mathbf{p}_n with the same number k of edges. We now modify \mathbf{p}_n so that they become C' -bi-Lipschitz paths joining a_n and b_n in X .

As before $\ell(x, y)$ denotes the length distance on \mathbf{p} between two points x, y . Denote the consecutive vertices of \mathbf{p} by $v_0 = A, v_1, \dots, v_k = B$, where $v_i = (v_n^i)^\omega$, and denote by $[v_n^{i-1}, v_n^i]$ the geodesics in L whose limits compose \mathbf{p} .

Assume that for some i , both $[v_n^{i-1}, v_n^i]$ and $[v_n^i, v_n^{i+1}]$ have lengths of order d_n . Consider $[\bar{v}_n^{i-1}, \bar{v}_n^i] \subset [v_n^{i-1}, v_n^i]$ and $[\tilde{v}_n^i, \tilde{v}_n^{i+1}] \subset [v_n^i, v_n^{i+1}]$ maximal so that for any $x \in [v_n^{i-1}, \bar{v}_n^i]$ and $y \in [\tilde{v}_n^i, v_n^{i+1}]$ we have $\text{dist}(x, y) \geq \frac{1}{2C} \ell(x, y)$. By maximality we have that $\text{dist}(\bar{v}_n^i, \tilde{v}_n^i) = \frac{1}{2C} \ell(\bar{v}_n^i, \tilde{v}_n^i)$. This also implies that $\bar{v}_n^i, \tilde{v}_n^i$ are the only points to realize the distance between $[v_n^{i-1}, \bar{v}_n^i]$ and $[\tilde{v}_n^i, v_n^{i+1}]$, as any other pair of points on the two sub-segments are at a larger ℓ -distance, hence at a larger distance. The hypothesis that $\lim^\omega (\mathbf{p}_n)$ is a C -bi-Lipschitz path also implies that $\bar{v}_n^i, \tilde{v}_n^i$ are at distance $o(d_n)$ from v_n^i . We then modify \mathbf{p}_n by replacing $[\bar{v}_n^i, v_n^i] \sqcup [v_n^i, \tilde{v}_n^i]$ with a geodesic $[\bar{v}_n^i, \tilde{v}_n^i]$ in L .

Assume now that some edge $[v_n^i, v_n^{i+1}]$ has length $o(d_n)$. Then consider $[\bar{v}_n^{i-1}, \bar{v}_n^i] \subset [v_n^{i-1}, v_n^i]$ and $[\tilde{v}_n^{i+1}, v_n^{i+2}] \subset [v_n^{i+1}, v_n^{i+2}]$ maximal so that for any $x \in [\bar{v}_n^{i-1}, \bar{v}_n^i]$ and $y \in [\tilde{v}_n^{i+1}, v_n^{i+2}]$ we have $\text{dist}(x, y) \geq \frac{1}{2C} \ell(x, y)$. Then modify \mathbf{p}_n by replacing $[\bar{v}_n^i, v_n^i] \sqcup [v_n^i, v_n^{i+1}] \sqcup [v_n^{i+1}, \tilde{v}_n^{i+1}]$ with a geodesic $[\bar{v}_n^i, \tilde{v}_n^{i+1}]$ in L .

The piecewise L path \mathbf{p}_n thus modified is ω -almost surely a C' -bi-Lipschitz path according to a slight modification of the argument in Lemma 2.4, and clearly $\lim^\omega (\mathbf{p}_n) = \mathbf{p}$. □

3. CHARACTERIZATION OF ASYMPTOTIC CUT-POINTS AND MORSE GEODESICS

In this section we shall give internal characterizations of spaces all (some) of whose asymptotic cones have cut-points. The characterization is in terms of divergence functions. There are several possible definitions of divergence. Each of them estimates the “cost” of going from point a to point b while staying away from a ball around point c . The difference between various definitions is in the allowed position of c (how close can c be to a or b , and whether c belongs to a geodesic $[a, b]$). We show that these definitions give equivalent functions, in particular in the case of Cayley graphs of finitely generated groups. We also show that for finitely generated one-ended groups these functions are equivalent to the Gersten divergence function.

In the second part of the section, we characterize Morse geodesics in terms of divergence.

3.1. Divergence and asymptotic cut-points. Typically the spaces that we have in mind when defining divergence are finitely generated groups, geodesic metric spaces X quasi-isometric to finitely generated groups, and geodesic metric spaces X such that the action of their group of isometries is co-bounded; that is, the orbit of a ball under $Isom(X)$ covers X (for simplicity we call such spaces *periodic*).

We consider the usual relation on the set of functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f \preceq_C g$ if

$$f(n) \leq Cg(Cn) + Cn + C$$

for some $C > 1$ and all x . This defines the known equivalence relation on the set of functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f \equiv_C g$ if $f \preceq_C g$ and $g \preceq_C f$.

Most of the time we obliterate the constant C from the index and simply write $f \preceq g$ and $f \equiv g$.

We do not distinguish equivalent functions in this paper, so, for instance, all linear functions (including all constants) are equivalent.

Definition 3.1. Let (X, dist) be a geodesic metric space (one can formulate a similar definition for arbitrary length spaces), and let $0 < \delta < 1$. Let $a, b, c \in X$ with $\text{dist}(c, \{a, b\}) = r > 0$, where $\text{dist}(c, \{a, b\})$ is the minimum of $\text{dist}(c, a)$ and $\text{dist}(c, b)$. Define $\text{div}(a, b, c; \delta)$ as the infimum of the lengths of paths connecting a, b and avoiding the ball $B(c, \delta r)$. If no such path exists, take $\text{div}(a, b, c; \delta) = \infty$.

The behavior of the function $\text{div}(a, b, c; \delta)$ with respect to quasi-isometry is easily checked.

Lemma 3.2. *Let $q : X \rightarrow Y$ be an (L, C) -quasi-isometry between two geodesic metric spaces. Then for any three points a, b, c in X ,*

$$(1) \quad \text{div}(q(a), q(b), q(c); 2L\delta) \geq \frac{1}{2L} \text{div}(a, b, c; \delta) - C.$$

Definition 3.3. The *divergence function* $\text{Div}(n, \delta)$ of the space X is defined as the supremum of all numbers $\text{div}(a, b, c; \delta)$ with $\text{dist}(a, b) \leq n$.

Clearly if $\delta \leq \delta'$, then $\text{Div}(n; \delta) \leq \text{Div}(n; \delta')$ for every n .

Lemma 3.4. *If X is one-ended, admits a co-compact isometric action of a group, and every point belongs to a bi-infinite bi-Lipschitz path, then the function $\text{Div}(n, \delta)$ takes only finite values for a sufficiently small δ .*

In particular this holds if X is a Cayley graph of a finitely generated one-ended group, and one can take $\delta = \frac{1}{3}$ in this case.

Proof. First we prove that for a sufficiently small δ , the value $\text{div}(a, b, c; \delta)$ is always finite. Indeed, for every point $x \in X$ let \mathfrak{p}_x be a bi-infinite C_x -bi-Lipschitz path containing x where $C_x \geq 1$. Since the space admits a co-compact isometry action of a group, we can assume that $C_x = C$ does not depend on x . Let $\delta = \frac{1}{3C}$. Take $a, b, c \in X$, and let $r = \text{dist}(c, \{a, b\}) > 0$. We claim that one of the two connected components of $\mathfrak{p}_a \setminus \{a\}$ does not intersect $B(c, \delta r)$. Indeed, otherwise there would be two points on \mathfrak{p}_a at distance at most $u = 2\delta r = \frac{2r}{3C}$ from each other in X but at distance $v \geq 2(1 - \frac{1}{3C})r$ along \mathfrak{p}_a . This contradicts the assumption that \mathfrak{p}_a is C -bi-Lipschitz since $v/u = 3C - 1 > C$. It remains to note that since the space is one-ended, every two points on \mathfrak{p}_a and \mathfrak{p}_b , respectively, that are far enough from c , are connected by a path outside $B(c, \delta r)$.

Now we prove that for any n the value $\text{Div}(n, \delta)$ is finite. Take a, b, c such that $\text{dist}(a, b) \leq n$, and let $r = \text{dist}(c, \{a, b\})$. If a geodesic $[a, b]$ does not intersect $B(c, \delta r)$, then $\text{div}(a, b, c; \delta) \leq n$.

Assume that a geodesic $[a, b]$ intersects $B(c, \delta r)$. Then $r \leq \delta r + n$, hence $r \leq \frac{n}{1-\delta}$.

Without loss of generality we may assume that a is in a fixed compact. Then b and c are in tubular neighborhoods of this compact, which are other compacts. Each of these compacts is covered by finitely many balls of radius ϵ , with ϵ as

small as we choose. Therefore, to finish the proof, it suffices to show that for a, b, c fixed and ϵ small enough, all triples a', b', c' with $\text{dist}(a, a'), \text{dist}(b, b'), \text{dist}(c, c') \leq \epsilon$ have $\text{div}(a', b', c'; \delta)$ uniformly bounded. Take as previously a path of finite length ℓ joining a and b outside $B(c, \delta r + \epsilon)$. Then $\text{div}(a', b', c'; \delta) \leq \ell + 2\epsilon$. \square

Remark 3.5. If we do not require in Lemma 3.4 that every point belongs to a bi-infinite bi-Lipschitz path, then $\text{div}(a, b, c; \delta)$ can be infinite for every $\delta > 0$ and a, b arbitrarily far away from each other. Indeed, consider a homogeneous one-ended geodesic metric space Y , and construct a metric space X by attaching to each point y in Y a copy T_y of the same finite simplicial tree T with more than one point. Define a metric on X in the natural way: to get from point $p \in T_x$ to point $q \in T_y$, one needs to first get from p to x inside T_x , then from x to y inside Y , then from y to q inside T_y . The space X obviously admits a co-compact isometric group action. It has one end since Y is one-ended and T is finite. But if we take any point a in T_x , $a \neq x$, point b in Y , and $c = x$, then there is no path connecting a to b in $X \setminus B(c, \delta r)$ for any $\delta > 0$. So $\text{div}(a, b, c; \delta) = \infty$.

There are two equivalent and equally natural ways to define the divergence function Div for arbitrary one-ended length spaces admitting co-compact isometry group actions. In the first version, one chooses $C > 0$ and replaces paths in the definition of $\text{div}(a, b, c; \delta)$ by C -paths, as defined in Section 2.1. In the second version, one replaces the ball $B(c, \delta r)$ by the ball $B(c, \delta r - C)$ (again for some constant C), assuming that balls of negative radii are empty. Then it is natural to consider $\text{div}(a, b, c; \delta, C)$ and define $\text{Div}(n; \delta, C)$ in the same way as in Definition 3.3. One can easily check that all the statements in this section remain true for these more general functions (after some obvious modifications).

We now define a new divergence function, closer to the idea of divergence of rays that was inspiring the notion, in particular closer to the Gersten divergence.

Definition 3.6. Let $n_0 \geq 0$ and $\lambda \geq 2$. The *small divergence function* $\text{div}(n; \lambda, \delta)$ is defined as the supremum of all numbers $\text{div}(a, b, c; \delta)$ with $n_0 \leq \text{dist}(a, b) \leq n$ and

$$(2) \quad \lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b).$$

For simplicity in the sequel we do not mention n_0 most of the time.

Note that, unlike in Definition 3.3, in Definition 3.6 we do not allow c to get too close to a or b . The case $\lambda = 2$ is precisely the case when c is the midpoint of some geodesic connecting a and b .

The next lemma is obvious.

Lemma 3.7. *The following inequalities are true for every geodesic metric space X :*

- (1) *If $\delta \leq \delta'$, then $\text{div}(n; \lambda, \delta) \leq \text{div}(n; \lambda, \delta')$ for every n, λ .*
- (2) *If $\lambda \leq \lambda'$, then $\text{div}(n; \lambda, \delta) \leq \text{div}(n; \lambda', \delta)$ for every n, δ .*
- (3) *$\text{div}(n; \lambda, \delta) \leq \text{Div}(n; \delta)$, for every n, λ, δ .*

Since the function $\text{div}(n; \lambda, \delta)$ is non-decreasing, it only provides interesting information on the large scale geometry of the ambient space if it never takes the value $+\infty$. This accounts in particular for considering some n_0 and taking $\text{dist}(a, b) \geq n_0$:

some periodic metric spaces may have locally bad phenomena that are irrelevant to the study of divergence, as in the construction considered in Remark 3.5.

Proposition 3.8. (1) *The function $\text{div}(n; \lambda, \delta)$ takes only finite values if and only if for every n there exists d_n such that if for some a, b, c satisfying (2), $\text{div}(a, b, c; \delta) \geq d_n$, then either $\text{dist}(a, b) \leq n_0$ or $\text{dist}(a, b) \geq n$.*
 (2) *If (X, dist) is quasi-isometric to a group, then the existence of some $n_0 > 0, \delta \in (0, 1)$ and $\lambda \geq 2$ such that the function $\text{div}(\cdot; \lambda, \delta)$ takes only finite values is granted if and only if X is one-ended.*

Proof. The statement in (1) is an easy exercise.

In view of Lemma 3.2 and of (1), it suffices to prove (2) for $X = G$ a finitely generated group with a word metric. In this case we may take $n_0 = 0$.

If G has infinitely many ends, then for any $0 < \delta < 1$ and $\lambda \geq 2$, there exist a, b, c satisfying (2) and with $r = \text{dist}(c, \{a, b\})$ large enough so that $\text{div}(a, b, c; \delta) = +\infty$.

Assume that G is one-ended. Then all $\text{div}(a, b, c; \delta)$ are finite.

We rewrite (1) as follows: for every n there exists d_n such that if $\text{dist}(a, b) \in [0, n]$, then $\text{div}(a, b, c; \delta) \leq d_n$.

Assume then that $\text{dist}(a, b) \in [0, n]$. Without loss of generality we may assume that $a = 1$; hence $b \in B(1, n)$. Take an arbitrary vertex c satisfying (2), and take $r = \text{dist}(c, \{a, b\})$. If $B(c, \delta r)$ does not intersect all geodesics joining a, b , then $\text{div}(a, b, c; \delta) = \text{dist}(a, b) \leq n$. If it does, then $r \leq \delta r + \text{dist}(a, b)$; hence $r \leq \frac{n}{1-\delta}$. Thus there are finitely many possibilities for b, c ; hence the supremum of $\text{div}(a, b, c; \delta)$ over all a, b, c satisfying (2) and $\text{dist}(a, b) \leq n$ is finite. \square

Remark 3.9. When G has infinitely many ends, one may consider another definition of the divergence function. Indeed, in this case it is known by Stallings's theorem [Sta71, Theorems 4.A.6.5 and 5.A.9] that the group splits as a free product with amalgamation over a finite subgroup. When G is finitely presented it can moreover be written as the fundamental group of a finite graph of groups with finite edge groups and one-ended vertex groups [Dun85]. Then one can take the divergence function of G as the supremum function of the divergence functions of the vertex groups, and thus obtain a finite function.

Note that this new function will provide more information on a group with infinitely many ends, as it will provide an upper bound for divergences of factors. On the other hand it will no longer distinguish between the group \mathbb{Z}^2 and $\mathbb{Z}^2 * \mathbb{Z}^2$, for instance. Thus, it is appropriate only when restricting to the class of groups with infinitely many ends.

We need to introduce two more functions, further restricting the choice of c . We assume, as before, that X is a geodesic metric space. For every pair of points $a, b \in X$, we choose and fix a geodesic $[a, b]$ joining them. The next lemma will show that the definition we are about to give does not depend on the choice of the geodesic $[a, b]$. We say that a point c is *between* a and b if c is on the fixed geodesic segment $[a, b]$.

We define $\text{Div}'(n; \delta)$ and $\text{div}'(n; \lambda, \delta)$ as Div and div , but restricting c to the set of points between a and b . Clearly $\text{Div}'(n; \delta) \leq \text{Div}(n; \delta)$ and $\text{div}'(n; \lambda, \delta) \leq \text{div}(n; \lambda, \delta)$ for every λ, δ .

Lemma 3.10. *For every $a, b \in X$, every $\delta \in (0, 1)$ and every $\lambda \geq 2$, we have*

$$\begin{aligned}
 (3) \quad & \max_{\substack{c \in [a, b] \\ \lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b)}} \text{div}(a, b, c; \delta/3) \\
 & \leq \max_{\lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b)} \text{div}(a, b, c; \delta/3) \\
 & \leq \max_{\substack{c \in [a, b] \\ 2\lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b)}} \text{div}(a, b, c; \delta) + \text{dist}(a, b).
 \end{aligned}$$

Proof. The first inequality in (3) is obvious. We prove only the second inequality in (3).

Let $c \in X$, $r = \text{dist}(c, \{a, b\})$ and assume that (2) is satisfied. Suppose first that the distance between c and the geodesic segment $[a, b]$ is at least $\frac{\delta}{3}r$. Then $[a, b]$ avoids the ball $B(c, \frac{\delta}{3}r)$, and so $\text{div}(a, b, c; \delta/3) = \text{dist}(a, b)$, and the second inequality in (3) holds.

Suppose now that $\text{dist}(c, [a, b]) < \frac{\delta}{3}r$. Then there exists a point c' on $[a, b]$ at distance at most $\frac{\delta}{3}r$ from c . Note that the distance r' from c' to $\{a, b\}$ is at least $(1 - \frac{\delta}{3})r$. Since $\delta < 1$, we have

$$\frac{2}{3} \delta < \delta \left(1 - \frac{\delta}{3} \right).$$

Therefore the ball $B' = B(c', \delta r') \supseteq B(c', \delta(1 - \frac{\delta}{3})r)$ contains the ball $B = B(c, \frac{\delta}{3}r)$. Hence every path avoiding the ball B' also avoids the ball B . Hence $\text{div}(a, b, c'; \delta) \geq \text{div}(a, b, c; \frac{\delta}{3})$. This gives the second inequality in (3). \square

Lemma 3.11. (a) *For every $a, b \in X$, every $\delta \in (0, 1)$ and every $\lambda \geq 2$, we have*

$$(4) \quad \max_{c \in [a, b]} \text{div}(a, b, c; \delta/3) \leq \max_{c \in X} \text{div}(a, b, c; \delta/3) \leq \max_{c \in [a, b]} \text{div}(a, b, c; \delta) + \text{dist}(a, b);$$

$$(5) \quad \text{div}(n; \lambda, \delta) \leq \text{Div}(n; \delta), \quad \forall n \geq n_0.$$

(b) *If X is a space as in Lemma 3.4, then*

$$(6) \quad \text{Div}(n; \delta) \leq \text{div}(n; 2, \delta) + 2n + O(1).$$

Proof. (a) The first inequality in (4) and inequality (5) are obvious. The second inequality in (4) is proved in exactly the same manner in which it was proved for (3).

(b) In order to prove (6) it suffices to prove a similar inequality for Div' and div' according to (3) and (4). Take a, b with $\text{dist}(a, b) = n$, and take $c \in [a, b]$. Let $r = \text{dist}(c, \{a, b\})$ and assume that $r = \text{dist}(a, c)$. Let $b' \in [c, b]$ at distance r from c .

Then $\text{div}(a, b, c; \delta) \leq \text{div}(a, b', c; \delta) + n$. In particular, if $2r \leq n_0$, then this implies that $\text{div}(a, b, c; \delta) \leq \text{Div}(n_0; \delta) + n = n + O(1)$. If $2r \geq n_0$, then $\text{div}(a, b, c; \delta) \leq \text{div}(n; 2, \delta) + n$. \square

Lemmas 3.10 and 3.11 immediately imply.

Corollary 3.12. (1) *The functions $\text{div}'(n; \lambda, \delta)$ and $\text{Div}'(n; \delta)$ do not depend (up to the equivalence relation \equiv) on the choice of geodesics $[a, b]$ for every pair of points a, b .*

- (2) If X is a space as in Lemma 3.4, then the function $\text{Div}(n; \delta)$ is equivalent to $\text{div}(n; 2, \delta)$ as a function in n . Thus in order to estimate $\text{Div}(n, \delta)$ for a sufficiently small δ (say, $\delta < \frac{1}{3}$), it is enough to consider points a, b, c , where c is the midpoint of a (fixed) geodesic segment connecting a and b .

We now recall another definition of divergence function, due to S. Gersten ([Ger94b], [Ger94a]). Let X be a geodesic metric space, let x_0 be a fixed point and let $\rho \in (0, 1)$. Denote by S_r the sphere of center x_0 and radius r and by C_r the complementary set of the open ball $B(x_0, r)$.

For every $x, y \in S_r$ define $\delta_\rho(x, y)$ to be the shortest path distance in $C_{\rho r}$ between x, y . Then define $\delta_\rho(r)$ as the supremum of $\delta_\rho(x, y)$ over all $x, y \in S_r$ that can be connected by a path in $C_{\rho r}$.

The collection of functions $\delta = \{\delta_\rho \mid \rho \in (0, 1)\}$ is called *the divergence* of X . Such a collection is considered up to the equivalence relation \sim defined in what follows. We write $\delta \preceq \delta'$ if and only if there exist $\epsilon, \epsilon' \in (0, 1)$ and $C \geq 1$ such that for every $\rho < \epsilon$ there exists $\rho' < \epsilon'$ such that $\delta_\rho \preceq_C \delta'_{\rho'}$.

Then we define $\delta \sim \delta'$ by $\delta \preceq \delta'$ and $\delta' \preceq \delta$.

It can be easily checked that the collection of functions δ up to the equivalence relation \sim is independent of the basepoint x_0 , and it is a quasi-isometry invariant. Thus we may assume that x_0 is an arbitrary fixed basepoint.

The relationship between Gersten divergence and the small divergence function is the following.

- Lemma 3.13.** (1) Let X be a geodesic metric space. Then for every $\rho \in (0, 1)$, $\delta_\rho(n) \leq \text{div}(n; 2, \rho)$ for every n .
 (2) Let X be a one-ended geodesic metric space quasi-isometric to a finitely generated group. Then for every $\rho, \rho' \in (0, 1)$ with $\rho < \rho'$ there exists n_0 large enough so that

$$\text{div}'(n; 2, \rho) \leq \sup_{n_0 \leq x \leq n} \delta_{\rho'} \left(\frac{x}{2} + O(1) \right) + n.$$

We leave the proof to the reader. Note that if X is as in Lemma 3.13(2) and if the collection of functions δ is equivalent to a non-decreasing function f (i.e. to the collection $\delta'_{\rho'} = f$ for every ρ'), then the divergence function is equivalent to f . In all the cases when Gersten divergence is computed, this is precisely the case.

Notation. Let $(d_n), (d'_n)$ be two sequences of numbers, and let ω be an ultrafilter. We write $d'_n = O_\omega(d_n)$ if for some constant $C > 1$, $\frac{1}{C}d'_n < d_n < Cd'_n$ ω -a.s. for all n .

Recall that a finitely generated group G is called *wide* if none of its asymptotic cones has a cut-point; it is called *unconstricted* if one of its asymptotic cones does not have cut-points. We say that a closed ball $\bar{B} = \bar{B}(c, \delta)$ in a metric space X *separates* point u from point v if u and v are in different connected components of $X \setminus \bar{B}$.

- Lemma 3.14.** Let X be a periodic geodesic metric space. Let ω be any ultrafilter, and let (d_n) be a sequence of positive numbers such that $\lim_\omega d_n = \infty$.

Let $\mathcal{C} = \text{Con}^\omega(X, (o_n), (d_n))$, $A = (a_n)^\omega, B = (b_n)^\omega, C = (c_n)^\omega \in \mathcal{C}$. Let $r = \text{dist}(C, \{A, B\})$. The following conditions are equivalent for any $0 \leq \delta < 1$:

- (i) The closed ball $\overline{B}(C, \delta)$ in \mathcal{C} separates A from B .
- (ii) For every $\delta' > \delta$ the limit $\lim_\omega \frac{\text{div}(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ is ∞ .

Proof. Suppose that $\frac{\text{div}(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ is bounded by some constant M ω -almost surely. This means there exists (ω -a.s.) a path \mathbf{p}_n of length $O(d_n)$ connecting a_n, b_n and avoiding the ball $B(c_n, \frac{\delta'}{r}r_n)$ where $r_n = \text{dist}(c_n, \{a_n, b_n\})$. Since $\lim^\omega (r_n/d_n) = r$, the ω -limit of these balls in \mathcal{C} is the closed ball $\overline{B}(C, \delta')$. The limit \mathbf{p} of the paths \mathbf{p}_n in \mathcal{C} exists (because the length of \mathbf{p}_n is $O_\omega(d_n)$), connects A and B and avoids the open ball $B(C, \delta')$. Hence \mathbf{p} avoids every closed ball $\overline{B}(C, \delta)$ for $\delta < \delta'$. Thus if there exists a closed ball $\overline{B}(C, \delta)$ separating A from B , then for any $\delta' > \delta$ the set of numbers $\frac{\text{div}(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ cannot be bounded ω -almost surely. So (i) implies (ii).

Now suppose that property (ii) holds and that $\overline{B}(C, \delta)$ does not separate A from B . Then there exists a path \mathbf{p} in \mathcal{C} connecting A and B and avoiding the ball $\overline{B}(C, \delta)$. By Lemma 2.3 we can assume that \mathbf{p} is a piece-wise geodesic, avoiding $\overline{B}(C, \delta')$ for some $\delta' > \delta$, and that \mathbf{p} is an ω -limit of paths \mathbf{p}_n in the space X . Then the lengths of \mathbf{p}_n are $O_\omega(d_n)$ and these paths avoid (ω -a.s.) balls $B(c_n, \delta''r_n)$, where $r_n = \text{dist}(c_n, \{a_n, b_n\})$, $\delta < \delta'' < \delta'$. Therefore the set of numbers $\frac{\text{div}(a_n, b_n, c_n; \delta'')}{d_n}$ is bounded, which contradicts (ii). \square

Taking $\delta = 0$ in Lemma 3.14, we obtain the following

Lemma 3.15. *Let $X, \omega, (d_n)$ be as in Lemma 3.14. Then the following conditions are equivalent:*

- (i) The asymptotic cone $\text{Con}^\omega(X, (o_n), (d_n))$ has a cut-point.
- (ii) There exists a sequence of pairs of points (a_n, b_n) in X with

$$\text{dist}(a_n, b_n) = O_\omega(d_n) \text{ and } \frac{\text{dist}(a_n, o_n)}{d_n}, \frac{\text{dist}(b_n, o_n)}{d_n} \text{ bounded,}$$

and a sequence of midpoints c_n of geodesics $[a_n, b_n]$ such that the sequence $\text{div}(a_n, b_n, c_n; 2, \delta)$, $n \geq 1$, is superlinear ω -a.s. for every $\delta \in (0, 1)$.

Moreover, if that condition holds, then the point $C = (c_n)^\omega$ is a cut-point in $\text{Con}^\omega(X, (o_n), (d_n))$ separating $A = (a_n)^\omega$ from $B = (b_n)^\omega$.

Lemma 3.16. *Let X be as in Lemma 3.14; in addition X contains a bi-infinite quasi-geodesic. Suppose that one of its asymptotic cones \mathcal{C} has a closed ball $\overline{B}(C, \delta)$ separating A from B and $\text{dist}(C, \{A, B\}) > 3\delta$. Then X is not wide; that is, one of the asymptotic cones of X has a cut-point.*

Proof. Let $\overline{B}(C, \delta)$ be a cut-ball, $C = (c_n)^\omega$, and let $A = (a_n)^\omega, B = (b_n)^\omega$ be two points in \mathcal{C} separated by $\overline{B}(C, \delta)$. Since X contains a bi-infinite quasi-geodesic and it has a cobounded action of $\text{Isom}(X)$, then \mathcal{C} also contains a bi-infinite bi-Lipschitz line. Since \mathcal{C} is a homogeneous space, there exist bi-infinite bi-Lipschitz lines containing A and B , respectively. We proceed as in Lemma 3.4. The point A cuts the bi-Lipschitz line containing it into two bi-Lipschitz rays. Since $\text{dist}(C, \{A, B\}) > 3\delta$, one of these rays does not cross $\overline{B}(C, \delta)$. Let us denote it by \mathbf{p} . Similarly, there exists an infinite bi-Lipschitz ray \mathbf{p}' starting at B that does not cross $\overline{B}(C, \delta)$. Clearly

every point A' on \mathfrak{p} and every point B' on \mathfrak{p}' are separated by $\overline{B}(C, \delta)$. For every $n > 1$ let A_n be a point on \mathfrak{p} at distance n from A , and let B_n be a point on \mathfrak{p}' at distance n from B . Note that $\text{dist}(A_n, B_n) = O(n)$. Indeed, otherwise for a big enough n , any geodesic $[A_n, B_n]$ avoids $\overline{B}(C, \delta)$. Also note that $\text{dist}(C, \{A_n, B_n\}) = O(n)$. Now consider the asymptotic cone $\mathcal{C}' = \text{Con}^\omega(\mathcal{C}, (C), (n))$ of the space \mathcal{C} . Applying Lemma 3.14 to \mathcal{C} , we get that every closed ball in \mathcal{C}' around the point $(C)^\omega$ is a cut-ball. Hence $(C)^\omega$ is a cut-point in \mathcal{C}' . Since \mathcal{C}' is an asymptotic cone of X [DS05], X is not wide. \square

Lemma 3.17. *Let X be a one-ended space, admitting a co-compact isometric action of a group, and such that every point belongs to a bi-infinite bi-Lipschitz path (in particular X may be the Cayley graph of a one-ended group).*

The space X is wide if and only if any of the following conditions holds:

- (i) *The function $\text{Div}(n; \frac{1}{4})$ is bounded by a linear function for n large enough.*
- (ii) *The function $\text{Div}(n; \delta)$ is bounded by a linear function for every $\delta \leq \frac{1}{4}$.*
- (iii) *The sequence of numbers $\left\{ \frac{\text{div}(n; 2, \frac{1}{4})}{n}, n = 1, 2, \dots, \right\}$ is bounded.*
- (iv) *For some $\delta \leq \frac{1}{4}$ the sequence of numbers $\left\{ \frac{\text{div}(n; 2, \delta)}{n}, n = 1, 2, \dots, \right\}$ is bounded.*
- (v) *For some λ and some $\delta \leq \frac{1}{4}$ the sequence of numbers $\left\{ \frac{\text{div}(n; \lambda, \delta)}{n}, n = 1, 2, \dots, \right\}$ is bounded.*

Proof. The equivalence of (i)-(v) follows almost immediately from Lemmas 3.7, 3.10 and 3.11. Indeed, (i)→(ii) follows from the fact that $\text{Div}(n; \delta)$ is non-decreasing in δ ; (ii)→(iii) by Lemma 3.11 and the inequality in (5); (iii)→(iv) by Lemma 3.7; (iv)→(v) is immediate. Finally, (v)→(i) by the inequality in (6).

Suppose that X is not wide, and $\mathcal{C} = \text{Con}^\omega(X, (o_n), (d_n))$ has a cut-point. Then \mathcal{C} has a closed ball $\overline{B}(C, 0)$ (where $C = (c_n)^\omega$), separating a pair of points $A = (a_n)^\omega$ and $B = (b_n)^\omega$. By Lemma 3.14, for every $\delta > 0$ the limit $\lim_\omega \frac{\text{div}(a_n, b_n, c_n; \delta)}{d_n}$ is ∞ . Since ω -almost surely $\text{dist}(a_n, b_n) < \kappa d_n$ for some κ , $\text{Div}(\kappa d_n, \delta)$ is not bounded by any linear function in d_n for any $\delta > 0$ (including $\delta = \frac{1}{4}$).

Conversely, suppose that the function $\text{Div}(n, \frac{1}{4})$ is bounded by a linear function. By (5), the same holds for $\text{div}(n; 2, \delta)$; hence there exist $d_n \geq n$ such that $\text{div}(d_n; 2, \delta) \geq 2nd_n$. Consequently, there exists a sequence of triples of points a_n, b_n, c_n , where c_n is the midpoint of a geodesic $[a_n, b_n]$, such that $\text{dist}(a_n, b_n) \leq d_n$ and $\frac{\text{div}(a_n, b_n; 2, \frac{1}{4})}{d_n}$ is at least n . We can assume without loss of generality that $\text{dist}(a_n, b_n) = d_n$. In the asymptotic cone $\text{Con}^\omega(X, (c_n), (d_n))$ of X , the distance between $A = (a_n)^\omega$ and $B = (b_n)^\omega$ is 1, and the ball $\overline{B} = \overline{B}(C, \frac{1}{7})$ separates A from B (here $C = (c_n)^\omega$) by Lemma 3.14. By Lemma 3.16, X is not wide since $\frac{1}{2} = \text{dist}(C, \{A, B\}) > 3 \cdot \frac{1}{7}$. The lemma is proved. \square

3.2. Morse quasi-geodesics. Recall the definition of tree-graded spaces.

Definition 3.18 ([DS05]). Let \mathbb{F} be a complete geodesic metric space and let \mathcal{P} be a collection of closed geodesic subsets, called *pieces*. Suppose that the following two properties are satisfied:

- (T₁) Every two different pieces have at most one point in common.
- (T₂) Every simple non-trivial geodesic triangle in \mathbb{F} is contained in one piece.

Then we say that the space \mathbb{F} is *tree-graded with respect to \mathcal{P}* .

When there is no risk of confusion as to the set \mathcal{P} , we simply say that \mathbb{F} is *tree-graded*.

The topological arcs starting in a given point and intersecting each piece in at most one point compose a real tree called *transversal tree*. Some transversal trees may reduce to singletons.

We shall need the following general facts.

Lemma 3.19 ([DS05, Theorem 3.30]). *Let (X_n, \mathcal{P}_n) be a sequence of tree-graded spaces, and let ω be an ultrafilter. Let $\mathbb{F} = \lim^\omega (X_n, o_n)$ be the ω -limit of X_n with observation points o_n . Let $\tilde{\mathcal{P}}$ be the set of ω -limits $\lim^\omega (M_n)$, where $M_n \in \mathcal{P}_n$ (the same ultralimit is counted only once; the empty ultralimits, corresponding to $M_n \in \mathcal{P}_n$ such that $\lim^\omega (\text{dist}_n(o_n, M_n)) = \infty$, are not counted). Then \mathbb{F} is tree-graded with respect to $\tilde{\mathcal{P}}$.*

Proposition 3.20. *Let (X_n, \mathcal{P}_n) be a sequence of homogeneous unbounded tree-graded metric spaces with observation points o_n . Let ω be an ultrafilter. Then the ultralimit $\lim^\omega (X_n, o_n)$ has a tree-graded structure with a non-trivial transversal tree at every point.*

Proof. By [DS05, Lemma 2.31], we can first assume that pieces from \mathcal{P}_n do not have cut-points. We can also add all transversal trees to the collection of pieces and assume that transversal trees of X_n are trivial (each of them consists of one point); that is, we assume that every arc in X_n intersects at least one piece in a non-trivial sub-arc. By [DS05, Lemma 2.15] every path-connected subspace of X_n without cut-points is in one of the pieces of \mathcal{P}_n . Hence every isometry of X_n permutes the pieces of \mathcal{P}_n . If the ω -limit $\mathcal{L} = \lim^\omega (X_n, o_n)$ is an \mathbb{R} -tree, there is nothing to prove. So we can assume that one of the maximal subsets of \mathcal{L} without cut-points has non-zero (possibly infinite) diameter τ . In particular, ω -almost surely X_n is not an \mathbb{R} -tree.

We assume that $n > 2$. Since X_n is homogeneous and has cut-points, every point of X_n is a cut-point. Pick a number $c > 0$ smaller than the non-zero diameter of a subset of \mathcal{L} without cut-points. We can assume that \mathcal{P}_n contains a piece of diameter $\geq c$ and without cut-points. By homogeneity, there exists such a piece $M_{n,0}$ in \mathcal{P}_n containing o_n and such that a geodesic $\mathfrak{p}_{n,1} = [o_n, x_{n,1}]$ of length c/n is contained in $M_{n,0}$. Since $x_{n,1}$ is a cut-point, $X \setminus \{x_{n,1}\}$ has at least two connected components, one of which contains $M_{n,0} \setminus \{x_{n,1}\}$. Let y be a point in another connected component at distance at most c/n from $x_{n,1}$. By homogeneity, there is an isometric copy $M_{n,2}$ of $M_{n,0}$ containing y . Let $x_{n,2}$ be the projection of $x_{n,1}$ onto $M_{n,2}$. Note that $\text{dist}(x_{n,2}, x_{n,1}) \leq c/n$. We can assume (again by homogeneity) that there exists a geodesic $[x_{n,2}, x_{n,3}]$ in $M_{n,2}$ of length c/n . By induction, we can construct a sequence of points $o_n = x_{n,0}, x_{n,1}, \dots, x_{n,n}$ such that:

- $\text{dist}(x_{n,i}, x_{n,i+1}) \leq c/n$.
- $c \leq \sum \text{dist}(x_{n,i}, x_{n,i+1}) \leq 2c$.
- If i is even, then $[x_{n,i}, x_{n,i+1}]$ is contained in a piece $M_{n,i}$ from \mathcal{P}_n without cut-points, and all these pieces are isometric and different.
- If i is odd, then $x_{n,i+1}$ is the projection of $x_{n,i}$ onto $M_{n,i+1}$.

By [DS05, Lemma 2.28], this implies that the union of geodesic segments $\mathfrak{p}_n = \bigsqcup_{i=0}^{n-1} [x_{n,i}, x_{n,i+1}]$ is a geodesic intersecting every piece of \mathcal{P}_n in a sub-geodesic of length at most c/n .

Let \mathfrak{p} be the ω -limit of \mathfrak{p}_n . Then \mathfrak{p} is a geodesic in \mathcal{L} . Let us show that \mathfrak{p} is a transversal geodesic. Suppose that \mathfrak{p} contains two points $A = (u_n)^\omega, B = (v_n)^\omega$ belonging to a piece $\lim^\omega(N_n)$, where $N_n \in \mathcal{P}_n, u_n, v_n \in \mathfrak{p}_n$. Let $l = \text{dist}(A, B)$. That means $\lim^\omega(\text{dist}(u_n, N_n)) = \lim^\omega(\text{dist}(v_n, N_n)) = 0$, while $\lim^\omega(\text{dist}(u_n, v_n)) = l$. Let u'_n (resp. v'_n) be the projections of u_n (resp. v_n) onto N_n . These projections exist and are unique by [DS05, Lemma 2.6]. Applying the strong convexity of pieces in a tree-graded space [DS05, Corollary 2.10], we can conclude that any geodesic $[u'_n, v'_n]$ is in N_n . Moreover, a union of three geodesics $[u_n, u'_n] \sqcup [u'_n, v'_n] \sqcup [v'_n, v_n]$ is a topological arc. This follows from the fact that any two consecutive geodesics intersect only in their common endpoint, and $[u_n, u'_n] \cap [v_n, v'_n] = \emptyset$ since these geodesics are at a distance at least $l/2$ ω -almost surely.

According to [DS05, Proposition 2.17] the projections u'_n, v'_n are on \mathfrak{p}_n , in between u_n and v_n . Thus ω -almost surely $[u_n, v_n]$ intersects N_n in a sub-geodesic of length at least $l/2$. But by our construction it intersects N_n in a sub-geodesic of length at most c/n ω -almost surely, a contradiction.

Thus \mathcal{L} contains a non-trivial transversal geodesic. By homogeneity, every transversal tree of \mathcal{L} is non-trivial. \square

Remark 3.21. As noticed by Denis Osin, if the Continuum Hypothesis is true, Proposition 3.20 implies that every tree-graded asymptotic cone of a finitely generated group has a non-trivial transversal tree at every point. Indeed, by Corollary 5.5 in [KSTT05] every asymptotic cone of a finitely generated group is isometric to its ω -limit for every ultrafilter ω .

Definition 3.22. Let X be a metric space. Given a quasi-geodesic $\mathfrak{q} : [0, \ell] \rightarrow X$ we call the *middle third* of \mathfrak{q} its restriction to $[\ell/3, 2\ell/3]$. Note that when \mathfrak{q} is continuous and of finite length, the middle third in the above sense does not in general coincide with the middle third in the arc-length sense.

Definition 3.23. A bi-infinite quasi-geodesic \mathfrak{q} in a geodesic metric space (X, dist) is called a *Morse quasi-geodesic* if for every $L \geq 1, C \geq 0$, every (L, C) -quasi-geodesic \mathfrak{p} with endpoints on the image of \mathfrak{q} stays M -close to \mathfrak{q} , where M depends only on L, C .

Note that the above property is equivalent to the fact that \mathfrak{p} is at uniformly bounded Hausdorff distance from a sub-quasi-geodesic of \mathfrak{q} having the same endpoints.

Proposition 3.24 (Morse quasi-geodesics). *Let X be a metric space, and for every pair of points $a, b \in X$ let $L(a, b)$ be a fixed set of (λ, κ) -quasi-geodesics (for some constants $\lambda \geq 1$ and $\kappa \geq 0$) connecting a to b . Let $L = \bigcup_{a, b \in X} L(a, b)$.*

Let \mathfrak{q} be a bi-infinite quasi-geodesic \mathfrak{q} in X , and for every two points x, y on \mathfrak{q} denote by \mathfrak{q}_{xy} the maximal sub-quasi-geodesic of \mathfrak{q} with endpoints x and y .

The following conditions are equivalent for \mathfrak{q} :

- (1) *In every asymptotic cone of X , the ultralimit of \mathfrak{q} is either empty or contained in a transversal tree for some tree-graded structure.*
- (2) *\mathfrak{q} is a Morse quasi-geodesic.*
- (3) *For every $C \geq 1$ there exists $D \geq 0$ such that every path of length $\leq Cn$ connecting two points a, b on \mathfrak{q} at distance $\geq n$ crosses the D -neighborhood of the middle third of \mathfrak{q}_{ab} .*

- (4) For every $C \geq 1$ and natural $k > 0$ there exists $D \geq 0$ such that every k -piecewise L quasi-path \mathfrak{p} that:
 - connects two points $a, b \in \mathfrak{q}$,
 - has quasi-length $\leq C \text{dist}(a, b)$,
 crosses the D -neighborhood of the middle third of \mathfrak{q}_{ab} .
- (5) For every $C \geq 1$ there exists $D \geq 0$ such that for every $a, b \in \mathfrak{q}$ and every path \mathfrak{p} of length $\leq C \text{dist}(a, b)$ connecting a, b , the sub-quasi-geodesic \mathfrak{q}_{ab} is contained in the D -neighborhood of \mathfrak{p} .

Proof. We prove (1) \Leftrightarrow (2) and (1) \rightarrow (5) \rightarrow (3) \rightarrow (4) \rightarrow (1).

(1) \rightarrow (2), (1) \rightarrow (5). These two implications are proved in a similar manner, so we prove only the second one, leaving the first one to the reader. Assume that in every asymptotic cone \mathcal{C} , $\lim^\omega(\mathfrak{q})$ is in the transversal tree. Also, by contradiction, assume that for some $C \geq 1$ and every natural $n \geq 1$ there exists a sequence \mathfrak{p}_n of paths with endpoints a_n, b_n on \mathfrak{q} and lengths $\leq C \text{dist}(a_n, b_n)$ such that $\mathfrak{q}_n = \mathfrak{q}_{a_n b_n}$ is not in $\overline{\mathcal{N}}(\mathfrak{p}_n, n)$. Then there exists a point $z_n \in \mathfrak{q}_n$ at distance $\delta_n \geq n$ from \mathfrak{p}_n . We choose $z_n \in \mathfrak{q}_n$ so that $\delta_n = \text{dist}(z_n, \mathfrak{p}_n)$ is maximal. Thus \mathfrak{q}_n is in $\overline{\mathcal{N}}(\mathfrak{p}_n, \delta_n)$.

In the asymptotic cone $\text{Con}^\omega(X, (z_n), (\delta_n))$, the ultralimits $\mathfrak{q}_\omega = \lim^\omega(\mathfrak{q}_n)$ and $\mathfrak{p}_\omega = \lim^\omega(\mathfrak{p}_n)$ are paths such that \mathfrak{q}_ω is transversal, staying 1-close to \mathfrak{p}_ω and containing a point $z = (z_n)^\omega$ at distance 1 from \mathfrak{p}_ω . Note that since \mathfrak{p}_n can be parameterized so that $\mathfrak{p}_n : [0, \text{dist}(a_n, b_n)] \rightarrow X$ is C -Lipschitz, \mathfrak{p}_ω can also be seen as a C -Lipschitz path. Both \mathfrak{p}_ω and \mathfrak{q}_ω are either finite, or infinite or bi-infinite simultaneously, and with the same endpoints, if any.

Assume that both \mathfrak{p}_ω and \mathfrak{q}_ω are finite. Possibly by diminishing both, we may assume that they intersect only in their endpoints a, b . By [DS05, Corollary 2.11] applied to the piece $M = T_z$ the transversal tree in z , $\mathfrak{p}_\omega \setminus \{a, b\}$ projects onto T_z both in a and in b . This contradicts the unicity of the projection point also stated in [DS05, Corollary 2.11].

Assume that both \mathfrak{p}_ω and \mathfrak{q}_ω are infinite. On the infinite branch \mathfrak{q}' of \mathfrak{q} starting at z , consider a point t at distance 10 from z and the sub-path \mathfrak{q}'' of \mathfrak{q}' of endpoints z and t . Let t' be a nearest point to t on \mathfrak{p}_ω , and consider a geodesic $[t, t']$. Replacing \mathfrak{q}' by \mathfrak{q}'' on \mathfrak{q}_ω and the infinite branch on \mathfrak{p}_ω starting at t' by $[t', t]$, and using for the thus modified paths the argument in the finite case, we obtain that z must be contained in the modified \mathfrak{p}_ω . Since z is at a distance at least 9 from $[t', t]$, it cannot be contained in it, so z must be contained in the path \mathfrak{p}_ω . This contradicts the fact that z is at distance 1 from \mathfrak{p}_ω .

If \mathfrak{p}_ω and \mathfrak{q}_ω are bi-infinite, then the same operation as before may be performed on both sides of z , again obtaining a contradiction.

(2) \rightarrow (1). Let \mathfrak{q} be a bi-infinite quasi-geodesic satisfying (2). Any asymptotic cone in which the ultralimit of \mathfrak{q} is non-empty equals an asymptotic cone of the form $\text{Con}^\omega(X; (x_n), d)$, where x_n is a sequence of points on \mathfrak{q} and $d = (d_n)$ is a sequence of positive numbers with $\lim^\omega(d_n) = +\infty$. Take one of these asymptotic cones \mathcal{C} . Consider any point $M = (m_n)^\omega$ with m_n on \mathfrak{q} .

It is enough to show that the two halves of $\lim^\omega(\mathfrak{q})$, before M and after M , are in two different connected components of $\mathcal{C} \setminus \{M\}$. Indeed, we can then consider the tree-graded structure on \mathcal{C} with the maximal subsets of \mathcal{C} without cut-points as pieces [DS05, Lemma 2.31]. The limit $\lim^\omega(\mathfrak{q})$ cannot intersect any piece in a non-trivial arc; hence it is in a transversal tree of this tree-graded structure.

Suppose there exist two points $A = (x_n)^\omega, B = (y_n)^\omega$ in $\mathcal{C} \setminus \{M\}$, with $x_n, y_n \in \mathfrak{q}$ and $m_n \in \mathfrak{q}_{x_n y_n}$ such that A, B can be connected by a path \mathfrak{p} in $\mathcal{C} \setminus \{M\}$.

Let 2ϵ be the distance from M to \mathfrak{p} . By Lemma 2.3, we can assume that \mathfrak{p} is the concatenation of a finite number of limit geodesics. We can also assume that \mathfrak{p} is simple.

By Lemmas 2.5 and 2.6, there exists a constant $C \geq 1$, and a sequence of C -bi-Lipschitz paths \mathfrak{p}_n connecting x_n with y_n such that $\lim^\omega(\mathfrak{p}_n)$ is in $\overline{\mathcal{N}}_\epsilon(\mathfrak{p})$. By property (2), each path \mathfrak{p}_n must be contained in the D -neighborhood of $\mathfrak{q}_{x_n y_n}$ for some constant D . It follows easily that $\lim^\omega(\mathfrak{p}_n) = \lim^\omega(\mathfrak{q}_{x_n y_n})$ and that M is at distance $\leq \epsilon$ from \mathfrak{p} , a contradiction.

(5) \rightarrow (3) and (3) \rightarrow (4) are obvious. In (3) \rightarrow (4) one must use the fact that every quasi-geodesic is at a finite Hausdorff distance from a Lipschitz quasi-geodesic with the same endpoints and with quasi-length of the same order [BBI01, Proposition 8.3.4].

(4) \rightarrow (1). Let \mathfrak{q} be a bi-infinite quasi-geodesic satisfying (4). We argue as in (2) \rightarrow (1), and suppose there exist two points $A = (x_n)^\omega, B = (y_n)^\omega$ in $\mathcal{C} \setminus \{M\}$, with $x_n, y_n \in \mathfrak{q}$ and $m_n \in \mathfrak{q}_{x_n y_n}$ such that A, B can be connected by a path \mathfrak{p} in $\mathcal{C} \setminus \{M\}$, where $M = (m_n)^\omega$.

Let 2ϵ be the distance from M to \mathfrak{p} . By Lemma 2.3, we can assume that \mathfrak{p} is the concatenation of a finite number of limits of quasi-geodesics in L .

Let $A' = (x'_n)^\omega, B' = (y'_n)^\omega$ with $x'_n, y'_n \in \mathfrak{q}$ be such that $\mathfrak{q}_{A'B'} = \lim^\omega(\mathfrak{q}_{x'_n y'_n})$ contains M in its middle third, and this middle third is of a diameter at most ϵ . Consider $\mathfrak{p}' = \mathfrak{t}_{A'A} \sqcup \mathfrak{p} \sqcup \mathfrak{t}_{BB'}$, where $\mathfrak{t}_{A'A}$ and $\mathfrak{t}_{BB'}$ are limits of quasi-geodesics in L joining A', A and B, B' , respectively.

By Lemma 2.6(1), $\mathfrak{p}' = \lim^\omega(\mathfrak{p}_n)$, where each \mathfrak{p}_n is a k -piecewise L quasi-path joining x'_n and y'_n ; moreover each \mathfrak{p}_n is of quasi-length $\leq D' \text{dist}(a_n, b_n)$. By property (4), each path \mathfrak{p}_n must cross the D -neighborhood of the middle third of $\mathfrak{q}_{x'_n y'_n}$ for some constant D . Then \mathfrak{p}' must cross the middle third of $\mathfrak{q}_{A'B'}$. Neither $\mathfrak{t}_{A'A}$ nor $\mathfrak{t}_{BB'}$ can cross this middle third. Hence \mathfrak{p} contains a point from the middle third of $\mathfrak{q}_{A'B'}$, hence at a distance $\leq \epsilon$ from M . This contradicts the fact that \mathfrak{p} is at a distance 2ϵ of M . □

3.3. Morse elements. Recall that an element h of a finitely generated group G is called *Morse* if $(h^n)_{n \in \mathbb{Z}}$ is a Morse quasi-geodesic. Notice that the property of being a Morse element obviously does not depend on the choice of a finite generating set of G .

Lemma 3.25. *Suppose that h is a Morse element in G and that H is a finitely generated subgroup in G containing h . Then h is Morse in the group H (considered with its own word metric).*

Proof. Without loss of generality, we may assume that the generating set of H contains h and is inside the generating set of G . Since h is Morse in G , $\mathfrak{q} = \{h^n; n \in \mathbb{Z}\}$ is a Morse quasi-geodesic in G . In particular, $m \geq \text{dist}_G(h^n, h^{n+m}) \geq \frac{1}{L}m - C$ for every n, m and some constants C, L . Since $\text{dist}_H(u, v) \geq \text{dist}_G(u, v)$ for every $u, v \in H$, we deduce that \mathfrak{q} is a quasi-geodesic in H .

Consider an arbitrary path \mathfrak{p} in H connecting h^n with h^{n+m} , \mathfrak{p} of length $\leq C_1 m$ for some constant C_1 . Then \mathfrak{p} is a path in G having the same length. By Proposition 3.24(3), \mathfrak{p} comes D -close (in G) to the middle third of $[h^n, h^{n+m}] \subset \mathfrak{q}$; i.e. \mathfrak{p} contains a point x which is at a distance at most D from h^{n+i} where $\frac{m}{3} \leq i \leq \frac{2m}{3}$. Here D is a constant depending only on C_1 . Note that the set $B_G(h^i, D) \cap H$ is finite, so it is contained in some ball $B_H(h^i, D_1)$ in H where D_1 depends only on

$D = D(C_1)$. Hence $\text{dist}_H(x, h^i) \leq D_1$. Therefore \mathfrak{p} comes D_1 -close to the middle third of $[h^n, h^{m+n}]$. By Proposition 3.24, \mathfrak{q} is a Morse quasi-geodesic in H . \square

Lemma 3.25 immediately implies

Proposition 3.26. *Let H be a finitely generated subgroup of a finitely generated group G . Suppose that H does not have its own Morse elements. Then H cannot contain Morse elements of G .*

According to Proposition 3.24 a group H such that at least one asymptotic cone of it is without cut-points (an *unconstricted* group, in the terminology of [DS05]) does not contain Morse elements. Hence, Proposition 3.26 can be applied to the cases when H satisfies a non-trivial law or has an infinite cyclic central subgroup, or H is a co-compact lattice in a semi-simple Lie group of rank ≥ 2 , or H is $\text{SL}_n(\mathcal{O}_S)$ for $n \geq 3$, or H is a Cartesian product of two infinite subgroups. Also if H is a torsion group it cannot contain a Morse element. Among the groups G where Morse elements exist and play an important part can be counted hyperbolic or relatively hyperbolic groups, or mapping class groups of a surface.

4. CUT-POINTS IN ASYMPTOTIC CONES OF GROUPS ACTING ON HYPERBOLIC GRAPHS

In this section, we prove that groups acting acylindrically on trees and certain hyperbolic graphs have cut-points in their asymptotic cones and Morse quasi-geodesics. Recall that the action of a group G on a graph X is *acylindrical* if there exist l and N such that for any pair of vertices x, y in X with $\text{dist}(x, y) \geq l$ there are at most N distinct elements, $g \in G$, such that $gx = x, gy = y$ (in this case the action is called l -acylindrical).

4.1. Trees.

Theorem 4.1. *Suppose that a finitely generated group G acts acylindrically on a simplicial tree X . Then every element $g \in G$ that does not fix a point in X is Morse. In particular, every asymptotic cone of G has cut-points and non-trivial transversal trees.*

We remark that all the actions on trees we consider are such that there are no inversions of edges. One can always pass to this setting by splitting each edge into two edges by adding a new vertex at the middle of the edge.

Throughout the proof of this theorem, we suppose that G acts l -acylindrically on the simplicial tree X .

Since X is a tree, there exists a bi-infinite geodesic \mathfrak{p} in X stabilized by g . The element g acts on \mathfrak{p} with some translation number τ , and for every $x \in X$ we have that $\text{dist}(x, g \cdot x) \geq \tau$.

Consider a vertex $o \in \mathfrak{p}$. We can assume that the tree X is the convex hull of $G \cdot o$. The factor-graph X/G is finite, and X is the Bass-Serre tree of some finite graph of groups K (and G is the fundamental group of that graph of groups). For every $h \in G$, we denote $h \cdot o$ by $\pi(h)$. We consider G endowed with a word metric corresponding to a finite generating set U stable with respect to inversion. Let $B(1, R)$ be the ball of radius R in the group G with respect to the word metric. Let N be the upper bound on the cardinality of the stabilizers of arcs of length l in X .

- Lemma 4.2.** (1) For every $R \geq 0$ and every pair of vertices a, b in the orbit $G \cdot o$, the set $V_{a,b}$ of elements $g \in \pi^{-1}(a)$ such that $\text{dist}(g, \pi^{-1}(b)) \leq R$ is either empty or covered by at most $|\mathbb{B}(1, R)|$ right cosets of the stabilizer of the pair (a, b) .
- (2) In particular, if $\text{dist}(a, b) \geq l$, then there exists $D = D(R)$ such that $V_{a,b}$ has diameter at most D .

Proof. Without loss of generality we may assume that $a = o$. Let S_o be the stabilizer of o , and S_b be the stabilizer of b . Then $\pi^{-1}(a) = S_o$, $\pi^{-1}(b) = g_b S_o$ and $S_b = g_b S_o g_b^{-1}$, for some g_b such that $g_b \cdot o = b$.

(1) Every element $g \in V_{o,b}$ has the property that $g \in S_o$ and for some w in the ball $\mathbb{B}(1, R)$ of G , $gw \in \pi^{-1}(b)$. Therefore,

$$V_{o,b} \subseteq g_b S_o \mathbb{B}(1, R) \cap S_o = g_b S_o g_b^{-1} g_b \mathbb{B}(1, R) \cap S_o = S_b g_b \mathbb{B}(1, R) \cap S_o.$$

For every $v \in g_b \mathbb{B}(1, R)$, either $S_b v \cap S_o$ is empty or $v = h(v)g(v)$ for some $h(v) \in S_b$, $g(v) \in S_o$ and $S_b v \cap S_o = (S_b \cap S_o)g(v)$. Therefore the set $V_{o,b}$ is a union of at most $|\mathbb{B}(1, R)|$ right cosets of $S_b \cap S_o$.

(2) Since $\text{dist}(o, b) \geq l$, the stabilizer of (o, b) has cardinality $\leq N$; hence by (1), the diameter of $V_{o,b}$ is finite. Let $D = D(R)$ be the supremum over all diameters of $V_{o,b}$ with $b \in \mathbb{B}(1, R) \cdot o$, $\text{dist}(o, b) \geq l$. If an arbitrary vertex b with $\text{dist}(o, b) \geq l$ is such that $V_{o,b}$ is non-empty, then $b = gwo$, where $g \in S_o$ and $w \in \mathbb{B}(1, R)$. Therefore $V_{o,b} = gV_{o,wo}$ has diameter at most D . \square

Now we are ready to prove Theorem 4.1.

We choose a finite generating set U of G such that:

- (Int) for every generator u , the geodesic in X joining o with $u \cdot o$ intersects $G \cdot o$ only in its endpoints.

Indeed, consider a finite generating set S of G and, for every $s \in S$, consider the geodesic $[o, s \cdot o]$ connecting o and $s \cdot o$, and g_1, \dots, g_m a finite sequence of elements in G such that $o, g_1 \cdot o, g_2 \cdot o, \dots, g_{m-1} \cdot o, s \cdot o$ are the consecutive intersections of $[o, s \cdot o]$ with $G \cdot o$. Let $h_k = g_{k-1}^{-1}g_k$ for $k = 1, 2, \dots, m$, where we take $g_0 = 1$ and $g_m = s$. Then $s = h_1 h_2 \dots h_m$. It follows that the union of finite sets $\{h_1^{\pm 1}, \dots, h_m^{\pm 1}\}$ for all $s \in S$ composes a finite generating set satisfying (Int).

Let λ be a constant such that for every $u \in U$,

$$(7) \quad \text{dist}_X(o, u \cdot o) \leq \lambda.$$

Since g acts on X with translation number τ , the sequence $(g^n)_{n \in \mathbb{Z}}$ is a $(\frac{\lambda}{\tau}|g|, |g|)$ -quasi-geodesic, where λ is the constant in (7).

By Proposition 3.24, we need to show that for every $C > 0$ there exists $D \geq 0$ such that every path of length $\leq Cn$ in the Cayley graph of G joining g^{-3n} and g^{3n} passes within distance D from g^i for some i between $-n$ and n .

Consider an arbitrary constant $C, n > 1$, and a path q of length $< Cn$ connecting g^{-3n} with g^{3n} in the Cayley graph of G (n is large enough).

Denote the preimage $\pi^{-1}(\pi(g^i))$ by Y_i . Note also that $\text{dist}(\pi(g^{-n}), \pi(g^n)) \geq \tau n$.

Let h_1, h_2, \dots, h_m be the consecutive vertices of q . Then by (7),

$$\text{dist}_X(\pi(h_i), \pi(h_{i+1})) \leq \lambda.$$

Connecting $\pi(h_i)$ with $\pi(h_{i+1})$ by a geodesic in X , we get a path $\pi(q)$ on X connecting $\pi(g^{-3n})$ with $\pi(g^{3n})$ and intersecting $G \cdot o$ only in $\pi(h_i)$, $i = 1, 2, \dots, m$. Since X is a tree, $\pi(q)$ must cover the sub-interval $[\pi(g^{-3n}), \pi(g^{3n})]$ of \mathfrak{p} , hence

$\pi(\mathfrak{q}) \cap G \cdot o$ must contain all $\pi(g^i)$, $-3n \leq i \leq 3n$. Therefore \mathfrak{q} must cross all Y_i , $-3n \leq i \leq 3n$, on its way from g^{-3n} to g^{3n} .

Let $k - 1$ be the integral part of l/τ . We may assume that $n > k$.

The maximal sub-path of the path \mathfrak{q} joining a point in Y_{-n} with a point in Y_n can be divided into sub-paths joining Y_i to Y_{i+1} such that their lengths sum up to the length of \mathfrak{q} . There exists $i \in [-n, n]$ such that the sum of the lengths of the sub-paths of \mathfrak{q} between Y_j and Y_{j+1} with $i \leq j \leq i+k$ is at most $4Ck$ (otherwise the total length of \mathfrak{q} would be greater than Cn). Note that the distance between $\pi(g^i)$ and $\pi(g^{i+k})$ is at least $k\tau > l$, and the distance between $g^i \in Y_i$ and $g^{i+k} \in Y_{i+k}$ is at most $|g|k$. Let R be the maximum of $|g|k$ and $4Ck$. Let x be the start point of the first sub-path of \mathfrak{p} joining Y_i with Y_{i+1} . Applying Lemma 4.2, we find a constant $D = D(R)$ such that x is at a distance at most D from g^i , as required. \square

Remark 4.3. One cannot replace “simplicial tree” by “ \mathbb{R} -tree” in the formulation of Theorem 4.1. Indeed, the group \mathbb{Z}^2 acts freely on the real line, but the asymptotic cones of \mathbb{Z}^2 do not have cut-points.

4.2. Uniformly locally finite hyperbolic graphs. Theorem 4.1 can be easily generalized to actions on hyperbolic uniformly locally finite graphs. Let G be a group acting on a hyperbolic graph X . An element $g \in G$ is called *loxodromic* if its translation length is > 0 .

Theorem 4.4. *Let G be an infinite finitely generated group acting acylindrically on an infinite hyperbolic uniformly locally finite graph X . Suppose that for some $l > 0$ the stabilizer of any pair of points $x, y \in X$ with $\text{dist}(x, y) \geq l$ is finite of uniformly bounded size. Let g be a loxodromic element of G . Then the sequence $(g^n)_{n \in \mathbb{Z}}$ is a Morse quasi-geodesic in G . In particular, every asymptotic cone of G has cut-points.*

Proof. The proof is similar to the proof of Theorem 4.1. By [Bow, Lemma 3.4], some power g^m , $m > 0$, stabilizes a bi-infinite geodesic \mathfrak{p} in X . Since we can always replace g by $g^m \neq 1$, we can assume that g stabilizes \mathfrak{p} . Let o be a vertex on \mathfrak{p} . We also denote $g \cdot o$ by $\pi(g)$, for every $g \in G$.

The proof of Lemma 4.2 does not use the fact that X is a tree, so it holds in our case too.

The fact that X was a simplicial tree was used in the proof of Theorem 4.1 twice:

- (P1) In the choice of a finite generating set of G with property (Int) we used the uniqueness of a geodesic joining two points.
- (P2) We used the fact that X is a tree to deduce that $\pi(\mathfrak{q})$ must cover the interval $[\pi(g^{-3n}), \pi(g^{3n})]$ of \mathfrak{p} .

In this proof we do not need a finite generating set of G with property (Int). Instead of (P2) we can use the fact that \mathfrak{p} is a Morse quasi-geodesic (as is every bi-infinite geodesic in a Gromov hyperbolic space by, say, Proposition 3.24). Then by Proposition 3.24(5), the D_0 -tubular neighborhood of $\pi(\mathfrak{q})$ must contain the interval $[\pi(g^{-3n}), \pi(g^{3n})]$ (for some constant $D_0 = D_0(C)$).

Instead of preimages of points $Y_i = \pi^{-1}(\pi(g^i))$, let us consider the sets Y'_i which are π -preimages of balls of radius $D_0 + \lambda$ around $\pi(g^i)$. The path $\pi(\mathfrak{q})$ visits each ball $B(\pi(g^i), D_0)$, $-n \leq i \leq n$, so the path \mathfrak{q} must visit each Y'_i , $-n \leq i \leq n$.

We need a version of Lemma 4.2(2) for pairs of vertices a, b in $G \cdot o \subset X$ with $\text{dist}(a, b)$ large enough, and for the set $\tilde{V}_{a,b}$ of elements $g \in \pi^{-1}(B(a, D_0 + \lambda))$ at a distance at most R from $\pi^{-1}(B(b, D_0 + \lambda))$.

Consider a pair of vertices a, b at a distance $\geq l + 4(D_0 + \lambda)$ such that $\tilde{V}_{a,b}$ is non-empty. Then there exists $g \in G, w \in B(1, R)$ such that $g \cdot o \in B(a, D_0 + \lambda)$ and $gw \cdot o \in B(b, D_0 + \lambda)$. It follows that $\tilde{V}_{a,b} \subseteq g\tilde{V}'$, where \tilde{V}' is the set of elements $h \in \pi^{-1}(B(o, 2(D_0 + \lambda)))$ at a distance at most R from $\pi^{-1}(B(w \cdot o, 2(D_0 + \lambda)))$. In other words \tilde{V}' is covered by the sets $V_{x,y}$ with $x \in B(o, 2(D_0 + \lambda))$ and y at a distance at most $2(D_0 + \lambda)$ from a vertex $w \cdot o$ in $B(1, R) \cdot o$ satisfying $\text{dist}(o, w \cdot o) \geq l + 4(D_0 + \lambda)$. Since X is locally finite it follows that there are finitely many pairs of vertices x, y as above, all $V_{x,y}$ are finite, and hence \tilde{V}' has a finite diameter.

The rest of the proof of Theorem 4.1 carries without change. □

4.3. Bowditch’s graphs. Theorem 4.1 can also be generalized to actions on a family of hyperbolic graphs which was first introduced by Bowditch in [Bow, §3], and which here we call Bowditch graphs. This family of graphs includes the 1-skeleta of curve complexes of surfaces. As a result, we recover the theorem of Behrstock [Beh06] that asymptotic cones of mapping class groups have cut-points, and for every pseudo-Anosov mapping class g , the sequence $(g^n)_{n \in \mathbb{Z}}$ is a Morse quasi-geodesic.

Let us define Bowditch’s graphs.

Let \mathcal{G} be a δ -hyperbolic graph. For any two vertices a, b , choose a non-empty set of geodesics connecting a, b and call these geodesics *tight*.

For every $r \geq 0$, we denote by $T(a, b; r)$ the union of all the points on all tight geodesics connecting a point in the ball $B(a, r)$ with a point in $B(b, r)$.

Definition 4.5. We say that a hyperbolic graph \mathcal{G} equipped with a collection of tight geodesics as above is a *Bowditch graph* if it satisfies the following conditions:

- (F0) Every subpath of a tight geodesic is tight.
- (F1) For every $r > 0$ there exist $m = m(r)$ and $k = k(r)$ such that for every three vertices a, b, c in \mathcal{G} with $\text{dist}(c, \{a, b\}) \geq k$, the set $F_c(a, b; r) = B(c, r) \cap T(a, b; r)$ contains at most m points.

According to [Bow, Theorems 1.1 and 1.2], the 1-skeleton of a curve complex of a surface is a Bowditch graph.

The first statement of the following lemma is [Bow, Lemma 3.4], the second statement is obvious.

Lemma 4.6. *For every loxodromic isometry g of \mathcal{G} , there exists a bi-infinite geodesic \mathfrak{g} and a natural number $m \leq m_0$, where $m_0 = m_0(\mathcal{G})$, such that $g^m \mathfrak{g} = \mathfrak{g}$.*

If moreover the isometry g preserves tight geodesics, then for every vertex c in \mathfrak{g} , $gF_c(a, b; r) = F_{g \cdot c}(g \cdot a, g \cdot b; r)$.

Definition 4.7. Suppose that $G = \langle S \rangle$ acts on a Bowditch graph \mathcal{G} and on the set of vertices $V(\mathcal{G}) = G \cdot \Delta$ for some finite set of vertices Δ . Let $g, h \in G, o \in \mathcal{G}$. We say that a geodesic \mathfrak{g} in \mathcal{G} with endpoints $g \cdot o$ and $h \cdot o$ Δ -*shadows* a path p with vertices $p_1 = g, p_2, p_3, \dots, p_m = h$ in $\text{Cay}(G, S)$ if the sets $p_i \cdot \Delta, i = 1, \dots, m$, cover the set of vertices of the geodesic \mathfrak{g} .

Definition 4.8. We say that a group G acts *tightly* on \mathcal{G} if:

- (T1) G stabilizes the set of tight geodesics in \mathcal{G} .
- (T2) For every vertex o of \mathcal{G} , there exist a finite set of vertices Δ in \mathcal{G} , and numbers $\lambda \geq 1, \kappa \geq 0$ such that every $g, h \in G$ can be joined by a (λ, κ) -quasi-geodesic \mathfrak{q} in $\text{Cay}(G, S)$ that is Δ -shadowed by a tight geodesic connecting $g \cdot o, h \cdot o$.

Following the terminology from [MM00] for the mapping class group, we call (λ, κ) -quasi-geodesics \mathfrak{q} as in (T2) *hierarchy paths*.

It follows from [MM00] that the mapping class group of a punctured surface acts tightly on the curve complex of the surface (see Lemma 4.10 below).

Theorem 4.9. *Let G be a finitely generated group acting tightly and acylindrically on a Bowditch graph \mathcal{G} . Then every loxodromic element of G is Morse. In particular, if G has loxodromic elements, then every asymptotic cone of G has cut-points.*

Proof. The proof is similar to the proofs of Theorems 4.1 and 4.4. By Lemma 4.6 we can assume that g stabilizes a bi-infinite geodesic \mathfrak{p} and that it acts on it with translation length τ . Pick a vertex o on \mathfrak{g} , and let Δ be the set from (T2). Without loss of generality we may assume that $o \in \Delta$. Since $\langle g \rangle$ acts co-compactly on \mathfrak{g} , it has a finite fundamental domain, which we include for convenience in Δ as well. Hence \mathfrak{g} is covered by the sets $g^i \Delta$, $i \in \mathbb{Z}$.

Let k be a natural number. By Proposition 3.24(4), it is enough to show that if \mathfrak{q} is an arbitrary k -piecewise hierarchy path connecting g^{-3n} , g^{3n} , with $n \gg 1$, \mathfrak{q} at Hausdorff distance $\leq K_0$ from a path joining g^{-3n} , g^{3n} and of length $\leq K_0 n$, then \mathfrak{q} crosses the K -neighborhood of the quasi-geodesic $[g^{-n}, g^n]$, where K depends only on k and on g (but not on \mathfrak{q} nor on n).

Since \mathfrak{q} is a k -piecewise hierarchy path, by property (T2) it is shadowed by a k -piecewise tight geodesic $\pi(\mathfrak{q})$ in \mathcal{G} of length $\leq K_1 n$ (for some constant K_1) connecting $g^{-3n} \cdot o$ and $g^{3n} \cdot o$. The fact that geodesics in a hyperbolic graph are Morse and part (5) of Proposition 3.24 imply that the sub-arc $[g^{-3n} \cdot o, g^{3n} \cdot o]$ in \mathfrak{g} is contained in the D -tubular neighborhood of $\pi(\mathfrak{q})$ for some constant D . In particular $[g^{-n} \cdot o, g^n \cdot o]$ has a sub-arc \mathfrak{g}' of length $\geq K_2 n$ (for some constant K_2) contained in the D -tubular neighborhood of one of the tight geodesic subpaths \mathfrak{t} of $\pi(\mathfrak{q})$. Notice that the length $|\mathfrak{t}|$ is $\geq K_2 n - 2D \geq K_3 n$ for some constant K_3 (since $n \gg 1$).

Since \mathfrak{g}' and \mathfrak{t} are two geodesics in a hyperbolic space and $\text{dist}(\mathfrak{g}'_-, \mathfrak{t}_-) \leq D$, $\text{dist}(\mathfrak{g}'_+, \mathfrak{t}_+) \leq D$, the Hausdorff distance between these two geodesics is at most K_4 for some constant K_4 .

Since \mathfrak{t} is tight, by (T2), \mathfrak{t} is covered by sets $F_x = F_x(\mathfrak{g}'_-, \mathfrak{g}'_+; K_4)$ where $x \in \mathfrak{g}'$.

Let \mathfrak{q}' be the hierarchy sub-path in \mathfrak{q} shadowed by \mathfrak{t} . As in the proofs of Theorems 4.1 and 4.4, we find a subarc \mathfrak{q}'' with endpoints $h, h' \in G$ at distance $\leq K_5 l$ (for some constant K_5) which is shadowed by a sub-arc of \mathfrak{t} of length at least $l + K_6$ of \mathfrak{t} , where K_6 is any number exceeding the diameter of Δ plus $2K_4$ (recall that $2K_4$ bounds the diameters of $F_x, x \in \mathfrak{g}'$). Denote the endpoints of that sub-arc of \mathfrak{t} by x, y . Then there are vertices u, v in \mathfrak{g}' such that $x \in F_u, y \in F_v$, and there are two powers of g , say, g^i, g^j , $-n \leq i, j \leq n$, such that $u \in g^i \Delta, v \in g^j \Delta$ (recall that $g^m \Delta, m \in \mathbb{Z}$, covers \mathfrak{g}). By Lemma 4.6, the number of elements in $F_u \cup F_v$ is bounded by constants which do not depend on n . Hence h is contained in a union of a bounded (independently of n) number of subsets $V_{a,b}$ with $a, b \in \mathcal{G}$, $\text{dist}(a, b) \geq l$. It is easy to establish the natural generalization of Lemma 4.2 to Bowditch graphs. Hence the distance $\text{dist}(h, g^i)$ is bounded by a number that does not depend on n . \square

Lemma 4.10. *The mapping class group of a compact connected orientable surface acts tightly on the 1-skeleton of the curve complex.*

Proof. Let S be a surface of genus g with p punctures. The fact that the action of the mapping class group $\mathcal{MCG}(S)$ on the curve complex of S satisfies (T1) is proved in [MM00].

By [MM00], the mapping class group $\mathcal{MCG}(S)$ acts co-compactly on the so called marking complex $\mathcal{M}(S)$. Each marking $\mu \in \mathcal{M}(S)$ consists of a pair of data: a *base multicurve* denoted $\text{base}(\mu)$, which is a multicurve composed of $3g + p - 3$ disjoint curves, and a collection of *transversal curves*. The 1-skeleton of $\mathcal{M}(S)$ is a locally finite graph. So there exists a finite collection of markings Φ such that $\mathcal{MCG}(S) \cdot \Phi = \mathcal{M}(S)$. Let Δ be the union of all the base curves of all markings in Φ .

It is proved in [MM00] that there exists an $\mathcal{MCG}(S)$ -equivariant projection π of $\mathcal{M}(S)$ onto the curve complex of S such that for every two markings μ, ν , there exists an (L, C) -quasi-geodesic in $\mathcal{M}(S)$ connecting μ and ν and Δ -shadowed by a tight geodesic connecting $\pi(\mu), \pi(\nu)$, where L, C are uniform constants. Using the quasi-isometry of $\mathcal{MCG}(S)$ and $\mathcal{M}(S)$, we can pull the set of hierarchy paths of $\mathcal{M}(S)$ into $\mathcal{MCG}(S)$ and obtain a collection of hierarchy paths in $\mathcal{MCG}(S)$ satisfying (T2). \square

Lemma 4.10 and Theorem 4.9 immediately imply Behrstock's theorem from [Beh06] that every pseudo-Anosov element in the mapping class group is Morse since pseudo-Anosov elements are loxodromic (see, for example, [Bow]).

5. LATTICES IN SEMI-SIMPLE LIE GROUPS. THE \mathbb{Q} -RANK ONE CASE

5.1. Preliminaries.

5.1.A. *Horoballs and horospheres.* Let X be a CAT(0)-space and ρ a geodesic ray in X . The *Busemann function associated to ρ* is the function $f_\rho : X \rightarrow \mathbb{R}$, $f_\rho(x) = \lim_{t \rightarrow \infty} [d(x, \rho(t)) - t]$. A level hypersurface $H_a(\rho) = \{x \in X \mid f_\rho(x) = a\}$ is called a *horosphere*, a level set $Hb_a(\rho) := \{x \in X \mid f_\rho(x) \leq a\}$ is called a *closed horoball* and its interior, $\text{Hbo}_a(\rho) := \{x \in X \mid f_\rho(x) < a\}$, *an open horoball*. We use the notation $H(\rho)$, $Hb(\rho)$, $\text{Hbo}(\rho)$ for the horosphere, the closed and open horoball corresponding to the value $a = 0$.

For two asymptotic rays, their Busemann functions differ by a constant [BH99]. Thus the families of horoballs and horospheres are the same, and we shall call them horoballs and horospheres *of basepoint α* , where α is the common point at infinity of the two rays.

Lemma 5.1 ([Dru04, Lemma 2.C.2]). *Let X be a product of symmetric spaces and Euclidean buildings and $\alpha_1, \alpha_2, \alpha_3$ three distinct points in $\partial_\infty X$. If there exist three open horoballs Hbo_i of basepoints α_i , $i = 1, 2, 3$, which are mutually disjoint, then $\alpha_1, \alpha_2, \alpha_3$ have the same projection on the model chamber Δ_{mod} .*

5.1.B. *Spherical and Euclidean buildings.* Let Y be a Euclidean building.

An Euclidean building is called *c -thick* if every wall bounds at least c half-apartments with disjoint interiors.

Lemma 5.2 ([KL97], proof of Proposition 4.2.1). *Two geodesic rays r_1, r_2 in Y are asymptotic, respectively, to two rays r'_1, r'_2 bounding a Euclidean planar sector with angular value the Tits distance between $r_1(\infty)$ and $r_2(\infty)$.*

Let Σ be a spherical building of rank 2. Then

- Σ is a $CAT(1)$ spherical complex of dimension one, all its simplices are isometric (and called chambers), and isometric to an arc of a circle of length π/m ;
- all simplicial cycles are of simplicial length at least $2m$ (so of length at least 2π);
- Σ is of simplicial diameter $m + 1$;
- if two points in Σ are at a distance smaller than π , then there exists a unique geodesic joining them.

Definition 5.3. Let A be an apartment in Σ and let x be a point outside A . We call an *entrance point for x in A* any point y such that the geodesic joining x with y intersects A only in y .

Lemma 5.4. *Every entrance point in an apartment A for a point outside A is a vertex.*

Proof. If an entrance point y is not a vertex, then it is in the interior of a chamber \mathcal{W} . By the axioms of buildings \mathcal{W} and x are contained in one apartment. Obviously one of the vertices of \mathcal{W} is on the geodesic joining x to any interior point of \mathcal{W} , in particular with y . On the other hand the fact that the point y is in A implies that \mathcal{W} is in A . □

According to the result in Lemma 5.4, we shall speak from now on of *entrance vertices for a point in an apartment*.

Lemma 5.5. *Let Σ be a spherical building of rank 2, let x be a point in it, and let A be an apartment not containing x .*

If the distance from x to A is less than $\pi/2$, then there exists at most one entrance vertex for x in A which is at a distance less than $\pi/2$ from x .

If the distance from x to A is $\pi/2$, then there exist at most two entrance vertices for x in A which are at a distance $\pi/2$ from x . Moreover if two such vertices exist, then they must be opposite.

Proof. In the first case, since two vertices in A are at a distance at most π , the existence of two entrance vertices would imply the existence of a cycle in Σ of length $< 2\pi$.

The second case is proved similarly. □

5.1.C. *Lattices in semi-simple groups.* A lattice in a Lie group G is a discrete subgroup Γ such that $\Gamma \backslash G$ admits a finite G -invariant measure. We refer to [Mar91] or [Rag72] for a definition of \mathbb{Q} -rank for an arithmetic lattice in a semi-simple group.

Theorem 5.6 (Lubotzky-Mozes-Raghunathan, [LMR93], [LMR00]). *On any irreducible lattice of a semisimple group of rank at least 2, the word metrics and the induced metric are bi-Lipschitz equivalent.*

By means of removing open horoballs one can construct a subspace X_0 of the symmetric space $X = G/K$ on which the lattice Γ acts with a compact quotient. In the particular case of lattices of \mathbb{Q} -rank one, the family of open horoballs have the extra property of being disjoint.

Theorem 5.7 ([Rag72], [GR69]). *Let Γ be an irreducible lattice of \mathbb{Q} -rank one in a semisimple group G . Then there exists a finite set of geodesic rays $\{\rho_1, \rho_2, \dots, \rho_k\}$*

such that the space $X_0 = X \setminus \bigsqcup_{i=1}^k \bigcup_{\gamma \in \Gamma} \text{Hbo}(\gamma\rho_i)$ has a compact quotient with respect to Γ and such that any two of the horoballs $\text{Hbo}(\gamma\rho_i)$ are disjoint or coincide.

Let p be the projection of the boundary at infinity onto the model chamber Δ_{mod} . Lemma 5.1 implies that $p(\{\gamma\rho_i(\infty) \mid \gamma \in \Gamma, i \in \{1, 2, \dots, k\}\})$ is a singleton which we denote by θ and call the associated slope of Γ . We have the following property of the associated slope:

Proposition 5.8 ([Dru97], Proposition 5.7). *If Γ is an irreducible \mathbb{Q} -rank one lattice in a semi-simple group G of \mathbb{R} -rank at least 2, the associated slope, θ , is never parallel to a factor of $X = G/K$.*

In particular, if G decomposes into a product of rank one factors, θ is a point in $\text{Int } \Delta_{mod}$.

Since the action of Γ on X_0 has a compact quotient, Γ with the word metric is quasi-isometric to X_0 with the length metric (the metric defining the distance between two points as the length of the shortest curve between the two points). Thus, the asymptotic cones of Γ are bi-Lipschitz equivalent to the asymptotic cones of X_0 . Theorem 5.6 implies that one may consider X_0 with the induced metric instead of the length metric. We study the asymptotic cones of X_0 with the induced metric.

Theorem 5.9 ([KL97]). *Any asymptotic cone of a product X of symmetric spaces and Euclidean buildings, with X of rank $r \geq 2$, is a Euclidean building \mathbf{K} of rank r which is homogeneous and \aleph_1 -thick. The apartments of \mathbf{K} appear as limits of sequences of maximal flats in X . The same is true for Weyl chambers and walls, singular subspaces and Weyl polytopes of \mathbf{K} . Consequently, $\partial_\infty \mathbf{K}$ and $\partial_\infty X$ have the same model spherical chamber and model Coxeter complex.*

In the sequel, in any asymptotic cone \mathbf{K} of a product X of symmetric spaces and Euclidean buildings we shall consider the labelling on $\partial_\infty \mathbf{K}$ induced by a fixed labelling on $\partial_\infty X$. We denote the projection of $\partial_\infty \mathbf{K}$ on Δ_{mod} induced by this labelling by P and the associated Coxeter complex by S .

Concerning the asymptotic cone of a space X_0 obtained from a product of symmetric spaces and Euclidean buildings by deleting disjoint open horoballs, we have the following result

Theorem 5.10 ([Dru98], Propositions 3.10, 3.11). *Let X be a CAT(0) geodesic metric space and let $\mathcal{C} = \text{Con}^\omega(X, (x_n), (d_n))$ be an asymptotic cone of X .*

(1) *If (ρ_n) is a sequence of geodesic rays in X with $\frac{d(x_n, \rho_n)}{d_n}$ bounded and $\rho = [\rho_n]$ is its limit ray in \mathbf{K} , then $H(\rho) = [H(\rho_n)]$ and $Hb(\rho) = [Hb(\rho_n)]$.*

(2) *If $X_0 = X \setminus \bigsqcup_{\rho \in \mathcal{R}} \text{Hbo}(\rho)$ and $\frac{d(x_n, X_0)}{d_n}$ is bounded, then the limit set of X_0 (which is the same thing as the asymptotic cone of X_0 with the induced metric) is*

$$(2.1) \quad \mathcal{C}_0 = \mathcal{C} \setminus \bigsqcup_{\rho_\omega \in \mathcal{R}_\omega} \text{Hbo}(\rho_\omega),$$

where \mathcal{R}_ω is the set of rays $\rho_\omega = [\rho_n]$, $\rho_n \in \mathcal{R}$.

We note that if X is a product of symmetric spaces and Euclidean buildings, the disjointness of $\text{Hbo}(\rho)$, $\rho \in \mathcal{R}$, implies by Lemma 5.1 that $p(\{\rho(\infty) \mid \rho \in \mathcal{R}\})$ reduces to one point, θ , if $\text{card } \mathcal{R} \neq 2$. Then $P(\{\rho_\omega(\infty) \mid \rho_\omega \in \mathcal{R}_\omega\}) = \theta$.

We also need the following result.

Lemma 5.11. *Let X be a product of symmetric spaces and Euclidean buildings, X be of rank $r \geq 2$, and $\mathbf{K} = \text{Con}_\omega(X, x_n, d_n)$ be an asymptotic cone of it. Let F_ω and ρ_ω be an apartment and a geodesic ray in \mathbf{K} , F_ω asymptotic to ρ_ω . Let $\rho_\omega = [\rho_n]$, where ρ_n have the same slopes as ρ_ω . Then*

- (a) F_ω can be written as a limit set $F_\omega = [F_n]$ with F_n asymptotic to ρ_n ω -almost surely;
- (b) every geodesic segment $[x, y]$ in $F_\omega \setminus \text{Hbo}(\rho_\omega)$ may be written as a limit set of segments $[x_n, y_n] \subset F_n \setminus \text{Hbo}(\rho_n)$.

5.2. Lattices in \mathbb{Q} -rank one Lie groups of real rank ≥ 2 .

Theorem 5.12. *Let X be a product of symmetric spaces of non-compact type and Euclidean buildings of rank at least two, and let \mathcal{R} be a collection of geodesic rays in X , with no ray contained in a rank one factor, and such that if r, r' are two distinct elements of it, then the open horoballs $\text{Hbo}(r), \text{Hbo}(r')$ are disjoint. Then the space $X' = X \setminus \bigsqcup_{r \in \mathcal{R}} \text{Hbo}(r)$ has linear divergence $\text{Div}(n, \delta)$ for every $\delta \in (0, 1)$.*

Theorems 5.7 and 5.12 immediately imply

Corollary 5.13. *Every lattice of a semi-simple Lie group of \mathbb{Q} -rank 1 and \mathbb{R} -rank ≥ 2 has linear divergence and no cut-points in its asymptotic cones.*

In what follows X and \mathcal{R} will always be as in Theorem 5.12.

Lemma 5.14 (Proposition 3.A.1, [Dru04]). *Let Y be a Euclidean building of rank at least two, let r be a geodesic ray in it and let F be an apartment intersecting the horoball $\text{Hb}(r)$. The intersection of F with $\text{Hb}(r)$ is a convex polytope whose interior is $F \cap \text{Hbo}(r)$. In particular, if the interior of the polytope is empty, then it is in the horosphere $H(r)$.*

Lemma 5.15. *Let Y, r and F be as in Lemma 5.14. If $F \cap \text{Hb}(r)$ has infinite diameter, then the Tits distance between $\partial_\infty F$ and $r(\infty)$ is at most $\pi/2$.*

Proof. Let o be a fixed point in $F \cap \text{Hb}(r)$. Since $F \cap \text{Hb}(r)$ has infinite diameter and it is a polytope, it contains a geodesic ray.

According to Lemma 5.2 the rays ρ and r are asymptotic to the rays ρ' and r' bounding a Euclidean sector with angular value the Tits distance between $\rho(\infty)$ and $r(\infty)$.

Since ρ and ρ' are at finite Hausdorff distance it follows that ρ' is contained in $\text{Hb}_a(r)$ for some $a > 0$. The ray r' is asymptotic to r , hence $\text{Hb}_a(r)$ coincides with some $\text{Hb}_b(r')$, hence ρ' is contained in $\text{Hb}_b(r')$. This cannot happen if the angle between ρ' and r' is larger than $\pi/2$.

It follows that the Tits distance between $\rho(\infty)$ and $r(\infty)$ is at most $\pi/2$; hence the same holds for the Tits distance between $\partial_\infty F$ and $r(\infty)$. □

Lemma 5.16. *Let Y be a 4-thick Euclidean building, of rank 2, let r be a geodesic ray in Y , not contained in a factor of Y , let F be an apartment intersecting $\text{Hbo}(r)$, and let a, b be two distinct points in $F \cap H(r)$. Then there exists an apartment F' in Y containing a, b and a point from $\text{Hbo}(r)$, and such that the Tits distance from $\partial_\infty F'$ to $r(\infty)$ is larger than $\pi/2$.*

Proof. By [KL97, Proposition 4.2.1] the boundary at infinity $\partial_\infty Y$ endowed with the Tits metric is a spherical building of rank 2. All its chambers are isometric, and

isometric to an arc of a circle of angle $\theta = \pi/m$, with $m \in \mathbb{N}, m \geq 2$. The building Y is reducible if and only if $m = 2$ [KL97, Proposition 3.3.1].

Assume that Y is reducible. Then Y is a product of trees $T_1 \times T_2$, and $r(t) = (r_1(\sigma t), r_2(\tau t))$, where r_i is a ray in T_i , and $\sigma^2 + \tau^2 = 1$, $\sigma > 0$ and $\tau > 0$. Let $a = (a_1, a_2)$, let $b = (b_1, b_2)$. There exists a geodesic line L_i in T_i containing $[a_i, b_i]$ and not asymptotic to r_i . Then the apartment $L_1 \times L_2$ satisfies the hypothesis. Indeed, the Tits distance from $\partial_\infty(L_1 \times L_2)$ to $r(\infty)$ is larger than $\pi/2$, because all chambers in $\partial_\infty(L_1 \times L_2)$ are opposite of the chamber containing $r(\infty)$.

In order to prove that $L_1 \times L_2$ contains a point in $\text{Hbo}(r)$, note first that the value of the Busemann function $f_r(x_1, x_2)$ is equal to $\sigma f_{r_1}(x_1) + \tau f_{r_2}(x_2)$. Hence $\sigma f_{r_1}(a_1) + \tau f_{r_2}(a_2) = \sigma f_{r_1}(b_1) + \tau f_{r_2}(b_2) = 0$.

If $f_{r_1}(a_1) < f_{r_1}(b_1)$, then $f_{r_2}(a_2) > f_{r_2}(b_2)$, and the point (a_1, b_2) in $L_1 \times L_2$ is in $\text{Hbo}(r)$. If $f_{r_1}(a_1) = f_{r_1}(b_1)$, then $f_{r_2}(a_2) = f_{r_2}(b_2)$, and either $a_1 \neq b_1$ or $a_2 \neq b_2$. Assume that $a_1 \neq b_1$. The geodesic $[a_1, b_1]$ contains a point e_1 such that $f_{r_1}(e_1) < f_{r_1}(a_1)$. Then the point (e_1, a_2) in $L_1 \times L_2$ is in $\text{Hbo}(r)$.

Assume that Y is irreducible. Let F be an apartment in Y containing $e \in \text{Hbo}(r)$, and $a \neq b \in H(r)$. Assume that the Tits distance δ from $r(\infty)$ to $A = \partial_\infty F$ is smaller than $\pi/2$. Then we construct an apartment F' containing a, b, e and a point from $\text{Hbo}(r)$ such that $\partial_\infty F'$ is at Tits distance $\delta + \theta$ from $r(\infty)$.

Indeed, Lemma 5.5 implies that in this case there exists only one entrance vertex u for $r(\infty)$ in A at distance δ . All the other vertices in A are at a distance at least $\delta + \theta$ from $r(\infty)$. Let v, w be two opposite vertices in $A \setminus \{u\}$. The 0-sphere $\{v, w\}$ is the boundary at infinity of a singular line H , and we may assume that this line does not separate $\{a, b, e\}$, but separates the set $\{a, b, e\}$ from a geodesic ray with point at infinity u .

By the hypothesis of thickness there exists a half-apartment \mathcal{D} in Y of boundary H and with interior disjoint from F . Let $\mathcal{D}_i, i = 1, 2$, be the two half-apartments in F determined by H such that D_1 contains u (hence D_2 contains $\{a, b, e\}$). Note that $D \cup D_i, i = 1, 2$, is an apartment. Lemma 5.5 applied to the spherical apartment $\partial_\infty(D_1 \cup D)$ implies that all the vertices in $\partial_\infty D$ are at Tits distance at least $\delta + \theta$ from $r(\infty)$. It follows that all the vertices in $\partial_\infty(D_2 \cup D)$ are at Tits distance at least $\delta + \theta$ from $r(\infty)$. Take $F' = D_2 \cup D$.

Now we can assume the Tits distance from $r(\infty)$ to $A = \partial_\infty F$ is $\pi/2$. Then we construct an apartment F' containing a, b, e such that $\partial_\infty F'$ is at Tits distance $\pi/2 + \theta$ from $r(\infty)$.

Lemma 5.5 implies that $\partial_\infty F$ contains at most two entrance vertices for $r(\infty)$ in A , and that in case there are two, they must be opposite. Let H be the singular hyperplane in F containing either one or both of these entrance vertices in its boundary.

We may moreover assume that H does not separate a, b, e . Let H' be a singular hyperplane composing with H two opposite Weyl chambers, and which also does not separate a, b, e . By the irreducibility assumption on Y , H' is not orthogonal to H .

The line H' splits F into two half-apartments D_1, D_2 , with D_1 containing a, b, e . By 3-thickness there exist D_3 a half-apartment of the boundary of H' such that D_1, D_2, D_3 have pairwise disjoint interiors.

Assume that $\partial_\infty F$ contains two opposite entrance vertices $x, y \in A$ for $r(\infty)$ in A with $x \in \partial_\infty D_1$. If the apartment $\partial_\infty D_1 \cup D_3$ contains two entrance vertices

at distance $\pi/2$, then the second entry vertex must be in D_3 and opposite to x . The same point must also be opposite of y , but since x, y are not symmetric with respect to $\partial_\infty H'$, this gives a contradiction. Thus $\partial_\infty D_3$ is at Tits distance $\pi/2 + \theta$ from $r(\infty)$, and the apartment $F' = D_1 \cup D_3$ is such that $\partial_\infty F'$ contains only one entrance vertex x for $r(\infty)$ at distance $\pi/2$.

Thus we reduced to the case when $\partial_\infty F$ contains only one entrance vertex x for $r(\infty)$ at distance $\pi/2$. The line H' may be chosen such that it splits F into two halves, D_1, D_2 , with D_1 containing a, b, e and $\partial_\infty D_2$ containing x . By 4-thickness it also bounds two half apartments D_3, D_4 s.t. $D_i, i = 1, \dots, 4$, have disjoint interiors.

If $\partial_\infty D_3$ is at Tits distance $\pi/2 + \theta$ from $r(\infty)$, then $F' = D_1 \cup D_3$ is the required apartment.

Assume that $\partial_\infty D_3$ is at Tits distance $\pi/2$ from $r(\infty)$. Then the spherical apartment $\partial_\infty(D_2 \cup D_3)$ has two opposite entrance points for $r(\infty)$ at distance $\pi/2$ from $r(\infty)$. An argument as above implies that $\partial_\infty D_4$ is at distance $\pi/2 + \theta$ from $r(\infty)$. We can take $F' = D_1 \cup D_4$. \square

Lemma 5.17. *Let $[x, y]$ be a segment containing a point o in its interior, and let r be a ray in a Euclidean building with origin o . Then there exists $x' \in [x, o]$, $y' \in (o, y]$ such that r and $[x', y']$ are in the same apartment.*

Proof. For every $t \in [o, x]$ denote by $\theta(t)$ the angle between the segment $[t, x]$ and the ray of origin t asymptotic to r . According to [KL97, Lemmas 2.1.5 and 5.2.2] the map $t \mapsto \theta(t)$ is an increasing upper semi-continuous function with finitely many values. It follows that for some $x' \in [x, o]$ the function is constant on $[x', o]$. Similarly one can find a point $y' \in (o, y]$ with the angle function constant on $[o, y']$. Since $[x', y']$ is a geodesic, the segment $[x', y']$ and the ray r are in the same flat; hence in the same apartment. \square

Proposition 5.18. *Let Y be an Euclidean building, of rank at least 2, and let r be a geodesic ray not contained in a factor of Y . Let $H(r)$ be the horosphere in Y determined by r . Then for every three points a, b, c in $H(r)$ with $\text{dist}(a, c) = \text{dist}(b, c) = 1$, there is a path in $H(r)$ connecting a and b and avoiding c .*

Proof. CASE 1. Assume that Y is of rank 2. If the points a, b are contained in an apartment F intersecting $H(r)$, then by Lemmas 5.15 and 5.16 it can be assumed that $F \cap H(r)$ is the boundary of a finite convex polytope (i.e. a flat polygon since the rank is 2) with non-empty interior. Then $F \cap H(r)$ is a simple loop containing a, b , and one of the boundary paths of this loop connecting a, b does not pass through c .

Now suppose that a, b are contained in an apartment F such that $F \cap H(r)$ is empty. Then $F \cap H(r) = F \cap Hb(r)$ is a convex polytope of dimension 1, so it is a segment, a ray or a line. If c is not on the segment $[a, b] \subset F \cap H(r)$, then we are done.

Assume that $c \in [a, b]$. By Lemma 5.17 there exists an apartment F' which is asymptotic to r and which contains a sub-segment $[a', b']$ of $[a, b]$ having c as an interior point.

Note that the apartment F' intersects $H(r)$. Thus, the first part of the proof can be applied to show that a' and b' can be connected in $H(r)$ avoiding c . This finishes the proof of Case 1.

CASE 2. Assume that Y is of rank $n > 2$. Consider an apartment F containing a, b .

If F also intersects $Hb(r)$, then $F \cap Hb(r)$ is a polytope of non-empty interior and dimension $n \geq 3$. If this polytope has two non-parallel co-dimension one faces, then its boundary is connected, and so we can connect a and b by an arc on the boundary of the polytope avoiding c . So suppose that the boundary of the polytope has just two (parallel) co-dimension one faces. We can assume that a, b belong to different faces. By [Dru04, Lemma 3.C.2], we can find an apartment F' containing a, b such that $F' \cap H(r)$ contains two non-parallel co-dimension one faces, and we are done.

If F does not intersect $Hb(r)$, then $F \cap H(r) = F \cap Hb(r)$ is a convex polytope of dimension $d < n$. If $d \geq 2$, then this polytope cannot have cut-points. Assume that $d = 1$; hence $F \cap H(r)$ is a segment, a ray or a line. As above, if $c \notin [a, b]$, then we are done.

If $c \in [a, b]$, then by Lemma 5.17 there exists an apartment F' asymptotic to r and containing a sub-segment $[a', b']$ of $[a, b]$ with c in the interior. In particular $[a', b']$ is in $F' \cap H(r)$, which is a hyperplane of dimension $n - 1$ (because it is a horosphere of the flat F'). Clearly a', b' can be joined in this hyperplane by a path avoiding c . □

Proposition 5.19. *Let X be as in Theorem 5.12 and let r be a geodesic ray not contained in a factor of X . Let H be the horosphere in X corresponding to r . Then there exists a constant $\lambda \in (0, 1)$ and two positive constants D and L such that for every three points a, b, c in H with $\min \{ \text{dist}(a, b), \text{dist}(a, c), \text{dist}(b, c) \} \geq D$, there is a path of length at most $L \text{dist}(a, b)$ in H connecting a and b and avoiding the ball of radius $\lambda \min(\text{dist}(a, c), \text{dist}(b, c))$ around c .*

Proof. Assume by contradiction that there exists a sequence of triples a_n, b_n, c_n such that the minimum of $\text{dist}(a_n, b_n), \text{dist}(a_n, c_n), \text{dist}(b_n, c_n)$, denoted by D_n , diverges to infinity, and such that all paths joining a_n and b_n outside the ball of radius $\frac{1}{n} \min(\text{dist}(a_n, c_n), \text{dist}(b_n, c_n))$ around c_n have length at least $n \text{dist}(a_n, b_n)$. Without loss of generality we may assume that $\text{dist}(a_n, c_n) = \text{dist}(b_n, c_n) = R_n$. The assumptions imply that $\text{dist}(a_n, b_n)$ is at least $R_n/2$, otherwise any geodesic $[a_n, b_n]$ stays outside the ball of radius $R_n/4$ around c_n . Thus $2R_n \geq D_n \geq R_n/2$.

The asymptotic cone $X_\omega = \text{Con}^\omega(X, (c_n), (R_n))$ is a Euclidean building of rank at least two by [KL97]. By Theorem 5.10(2), the limit points $a = (a_n)^\omega, b = (b_n)^\omega$ and $c = (c_n)^\omega$ are on the horosphere $H(r_\omega)$, where r_ω is the limit of the ray r ; the points a and b are at distance 1 from c . By Proposition 5.18 there exists a path \mathfrak{g} in $H(r_\omega)$ connecting a and b and avoiding c . By Lemma 2.3, we can assume that \mathfrak{g} is a limit of paths \mathfrak{g}_n of lengths $O(R_n)$ connecting a_n, b_n in $H(r)$ and avoiding a ball of radius $O(R_n)$ around c_n . This contradicts the assumptions in the previous paragraph. □

Proof of Theorem 5.12. Let X and \mathcal{R} be as in Theorem 5.12. If \mathcal{R} has cardinality at least three, then according to Lemma 5.1 the set of points $\{r(\infty) ; r \in \mathcal{R}\}$ projects onto one point on the model chamber of $\partial_\infty X$. This implies that all horospheres $H(r)$ with $r \in \mathcal{R}$ are isometric. Let λ, D and L be the three constants provided by Proposition 5.19 for the above family of isometric horospheres.

If \mathcal{R} has cardinality at most 2, then take λ to be the minimum and D, L to be the maximum among the corresponding constants for these horospheres.

Since rays from \mathcal{R} are not parallel to a rank one factor of X , the horospheres corresponding to them are M -bi-Lipschitz embedded into X for some constant M (see [Dru97, Theorems 1.2 and 1.3]).

Since the rank of X is at least 2, it has linear divergence. Let \mathfrak{p} be a path of length $K\text{dist}(a, b)$ connecting a, b in X and avoiding the ball $B(e, R/2)$.

Let $C = 12ML/\lambda$, and let three points $a, b, e \in X$ where $R = \text{dist}(e, a) = \text{dist}(e, b) = \text{dist}(a, b)/2$. We want to connect a, b by a path in X' of length $O(\text{dist}(a, b))$ avoiding a ball of radius R/C around e . Let H be a horosphere corresponding to some ray $r \in \mathcal{R}$ crossed by \mathfrak{p} . Let a' and b' be the first and last points on $\mathfrak{p} \cap H$.

If $\text{dist}(a', b') < R/4M$, then (since the distortion of H is linear) a' and b' can be connected in H by a path of length at most $M\text{dist}(a', b') < R/4$. That path avoids the ball of radius $R/2$ around e . By replacing all such sub-paths $[a', b']$ in \mathfrak{p} by the corresponding paths on the horospheres, we obtain a path of length at most MK connecting a, b and avoiding the ball of radius $R/4$ around e .

So we can assume that $\text{dist}(a', b') \geq R/4M$. Let e' be the projection of e onto H (in a CAT(0)-space, for every point x and every convex set P , there exists at most one point x' in the set P that realizes the distance from x to P [BH99]). If $\text{dist}(e, e') > R/2C$, then any path in H joining a' and b' avoids the ball $B(e, R/2C)$.

Assume that $\text{dist}(e, e') \leq R/2C$. Then any path joining a', b' outside the ball $B(e', R/C)$ is also outside the ball $B(e, R/2C)$. By Proposition 5.19 a' and b' can be joined in H by a path of length $\leq L\text{dist}(a', b')$ outside the ball $B(e', \lambda R/4M)$. Since $\lambda R/4M > R/C$, we are done. \square

6. $\text{SL}_n(\mathcal{O}_S)$

Let \mathbf{k} be a number field, let $\mathcal{O} \subset \mathbf{k}$ be its ring of integers, and let \mathcal{S} be a finite set of places of \mathbf{k} containing all the archimedean ones. Let \mathcal{O}_S be the ring of \mathcal{S} -integer points of \mathbf{k} and let us denote $\mathbf{k}_S = \prod_{\nu \in \mathcal{S}} \mathbf{k}_\nu$. Note that we have a natural diagonal embedding of \mathbf{k} in \mathbf{k}_S . The image of \mathcal{O}_S under this embedding is a cocompact lattice. When speaking of an action of an element of \mathbf{k} (or more generally of a matrix over \mathbf{k}) on \mathbf{k}_S (respectively on a vector in \mathbf{k}_S^t) we will be implicitly using this diagonal embedding. For $x \in k$ we denote

$$|x|_S = \max\{|x|_\nu : \nu \in \mathcal{S}\}.$$

Similarly for vectors $v \in \mathbf{k}_S^t$ we have

$$\|v\|_S = \max\{\|v\|_\nu : \nu \in \mathcal{S}\}.$$

Theorem 6.1. *The asymptotic cones of $\Gamma = \text{SL}_d(\mathcal{O}_S)$, $d \geq 3$, do not have cut-points.*

Remark 6.2. When $|\mathcal{S}| > 1$, one may allow $d = 2$ by Theorem 5.12 (because in that case $\text{SL}_2(\mathcal{O}_S)$ is of \mathbb{Q} -rank one).

We shall prove the theorem only for $d = 3$. The case of $d > 3$ is similar (and easier). In fact one can easily deduce that case from the case $d = 3$ by using various embeddings of SL_3 into SL_d .

6.1. Notation and terminology.

- As usual, for a given set S generating Γ we shall denote by $\text{dist}_S(\cdot, \cdot)$ the word metric on Γ with respect to S .

- An entry a of $\gamma \in \mathrm{SL}_d(\mathcal{O}_S)$ is called *large* if

$$\log(1 + |a|_S) \geq C \log \sqrt{|\mathrm{tr} \gamma^* \gamma|_S}$$

for some fixed constant C .

- We shall use the notation $x \approx y$ to mean that for some constants (which we choose and fix for the given group Γ) $c_1, c_2 > 0$, $c_1 \leq (1 + |x|_S)/(1 + |y|_S) \leq c_2$.
- Two elements $\alpha, \beta \in \mathrm{SL}_d(\mathcal{O}_S)$ are said to be “of the same size” if $\mathrm{dist}_S(\alpha, e) \approx \mathrm{dist}_S(\beta, e)$
- Let $\kappa > 0$ be fixed. Let $\gamma_1, \gamma_2 \in \Gamma$. A κ -*exterior trajectory* from γ_1 to γ_2 is a path ω in the Cayley graph $\mathrm{Cay}(\Gamma, S)$ starting at γ_1 and ending at γ_2 such that:

- (1) The length of ω is comparable to $\mathrm{dist}_S(\gamma_1, \gamma_2)$, i.e. bounded by some constant (depending only on the group Γ) times $\mathrm{dist}_S(\gamma_1, \gamma_2)$.
- (2) The path ω remains outside a ball of center e and radius $\kappa \cdot \mathrm{dist}_S(e, \{\gamma_1, \gamma_2\})$.

We shall usually omit the constant κ , and we will speak about *exterior trajectory* where $\kappa > 0$ is implicit.

- Two elements $\gamma_1, \gamma_2 \in \Gamma$ will be said to be *exteriorly connected* if there exists an exterior trajectory connecting them.
- We shall use the notation $\|\gamma\|_S = \sqrt{|\mathrm{tr} \gamma^* \gamma|_S}$. Note that

$$\|\gamma\|_S \approx \max\{|a_{ij}|_S : \gamma = (a_{ij})\}.$$

- Recall (cf. [LMR93],[LMR00]) that $\mathrm{dist}_S(\gamma, e) \approx \log \|\gamma\|_S$.
- Let us denote the following subgroups of SL_3 :

$$L = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \right\}.$$

6.2. Some facts about \mathcal{O}_S . Abusing notation, we write \mathbb{Z}_S for the localization of \mathbb{Z} with respect to the non-Archimedean places in S restricted to \mathbb{Q} .

The next lemma contains well-known facts about \mathcal{O}_S and its ideals.

Lemma 6.3 ([AM69]). (i) *The ring \mathcal{O}_S is a finitely generated \mathbb{Z}_S module.*

(ii) *Every non-zero ideal of \mathcal{O}_S is a unique product of prime ideals and it is contained in finitely many prime ideals.*

Lemma 6.4. *There exists a constant $C > 1$ depending only on \mathcal{O} such that every principal ideal I of \mathcal{O} of norm k is generated by an element x of absolute value $|x| \leq C(k)^{\frac{1}{r}}$, where r is the number of Archimedean valuations of \mathcal{O} . In case $r > 1$ we actually have that for every $k' \geq k$, I is generated by an element x of absolute value $\frac{1}{C}(k')^{\frac{1}{r}} \leq |x| \leq C(k')^{\frac{1}{r}}$.*

Proof. Let r be the number of Archimedean valuations of \mathcal{O} . We distinguish the case where $r = 1$ from $r > 1$. When $r = 1$ we have only finitely many choices of a generator for the given principal ideal, and all satisfy the assertion. Assume $r > 1$. Consider the logarithmic map $\phi: \mathcal{O} \setminus \{0\} \rightarrow \mathbb{R}^r$ that takes x to $(\log |\nu_1(x)|, \dots, \log |\nu_r(x)|)$. By the Dirichlet theorem, the image of the group of units \mathcal{O}^* is a lattice in the subspace of \mathbb{R}^r given by the equation $x_1 + \dots + x_r = 0$. Let $c\mathcal{O}$ be a principal ideal of \mathcal{O} . Then the image under ϕ of the set of all generators of $c\mathcal{O}$ is the coset $\phi(c) + \phi(\mathcal{O}^*)$.

Let Δ be the (closed) fundamental domain of the lattice $\phi(\mathcal{O}^*)$ containing the point on the diagonal $x_1 = x_2 = \dots = x_r$. Let c' be a generator of $c\mathcal{O}$ such that $\phi(c')$ is in Δ . Then the difference between the maximal and minimal values of coordinates of $\phi(c')$ does not exceed a constant λ depending only on \mathcal{O} (we can take λ to be the diameter of Δ). Therefore for every i, j between 1 and r , $\frac{|\nu_i(c')|}{|\nu_j(c')|} \leq \exp(\lambda)$. Hence $|c'|_S = \max\{|\nu_i(c')|, i = 1, \dots, r\}$ does not exceed $(|\nu_1(c')| \cdots |\nu_r(c')|)^{\frac{1}{r}} \exp \lambda \leq k^{\frac{1}{r}} \exp \lambda$ and is at least $\exp(-\lambda)k^{\frac{1}{r}}$.

Suppose now that $k' > k$. Let $|\nu_i(c')|$ be the maximal number among all $|\nu_j(c')|$. By Dirichlet's theorem, there exists a unit ϵ of \mathcal{O} (depending only on \mathcal{O}) such that $|\nu_i(\epsilon)| > 1$ is the maximal number among all $|\nu_j(\epsilon)|$. Then there exists a constant C and an integer $u > 0$ such that $c'' = \epsilon^u c'$ satisfies the desired inequalities

$$\frac{1}{C}(k')^{\frac{1}{r}} \leq |c''|_S \leq C(k')^{\frac{1}{r}}.$$

□

Lemma 6.5. *Let $a, b, c \in \mathcal{O}_S$, $c \neq 0$, and let P_1, \dots, P_s be distinct prime ideals not containing $a\mathcal{O}_S$, but s.t. $c \in P_1, \dots, P_s$. Then there exists $m \in \mathcal{O}_S$ such that $b+ma$ is not contained in P_1, \dots, P_s and $|m|_S$ is bounded by a polynomial in $|a|_S, |b|_S, |c|_S$ (the polynomial depends only on \mathcal{O}_S).*

Proof. Let h be the class number of K . Without loss of generality assume that ideals P_1, \dots, P_u do not contain b but P_{u+1}, \dots, P_s contain b . Since $c \in P_1 \dots P_u$, the norm of $P_1 \dots P_u$ is smaller than the norm of the ideal $c\mathcal{O}_S$ which is bounded by $O(|c|_S^r)$, where r is the number of valuations in S .

Then the ideal $P = (P_1 \dots P_u)^h$ is principal, and its norm is bounded by $O(|c|_S^{rh})$. The intersection $P' = P \cap \mathcal{O}$ is also a principal ideal of \mathcal{O} with the same norm as P . By Lemma 6.4, there exists a generator m of P' with $|m|_S$ bounded by $O(|c|_S^{rh})$. This element m generates the ideal P as well.

If $1 \leq i \leq u$, then $m \in P_i$ but $b \notin P_i$, hence $b + ma \notin P_i$. If $u < i \leq s$, then $b \in P_i$ but neither m nor a is in P_i , so $ma \notin P_i$, hence $b + ma \notin P_i$. So $b + ma$ is not in P_i for any i . □

Lemma 6.6 (Effective stable range). *For every three elements $a, b, c \in \mathcal{O}_S$ such that $a\mathcal{O}_S + b\mathcal{O}_S + c\mathcal{O}_S = \mathcal{O}_S$, there exist two elements $k, m \in \mathcal{O}_S$ such that $(ma + b)\mathcal{O}_S + (ka + c)\mathcal{O}_S = \mathcal{O}_S$ and the absolute values $|k|_S, |m|_S$ are bounded by a polynomial in $|a|_S, |b|_S, |c|_S$ (for some polynomial depending only on \mathcal{O}_S).*

Proof. We can assume that $b, c \neq 0$. Let P_1, \dots, P_t be all the prime ideals containing $c\mathcal{O}_S$. Let P_1, \dots, P_s be the ideals that do not contain $a\mathcal{O}_S$, and let P_{s+1}, \dots, P_t be the ideals containing $a\mathcal{O}_S$. By Lemma 6.5 we can find m such that $(b + ma)\mathcal{O}_S$ is not contained in P_1, \dots, P_s and $|m|_S$ is polynomially bounded in terms of $|a|_S, |b|_S$ and $|c|_S$. We claim that $c\mathcal{O}_S + (ma + b)\mathcal{O}_S = \mathcal{O}_S$. Indeed, suppose that $c\mathcal{O}_S + (ma + b)\mathcal{O}_S \neq \mathcal{O}_S$. Then there exists a prime ideal P containing $c\mathcal{O}_S + (ma + b)\mathcal{O}_S$. Since $c \in P$, that prime ideal must be one of the $P_i, i = 1, \dots, t$. Since $ma + b$ is not in P_1, \dots, P_s , we have $i > s$. Then P contains a , so P contains a, b, c , which contradicts the equality $a\mathcal{O}_S + b\mathcal{O}_S + c\mathcal{O}_S = \mathcal{O}_S$. Hence $c\mathcal{O}_S + (ma + b)\mathcal{O}_S = \mathcal{O}_S$. □

Remark 6.7. The proof of Lemma 6.6 shows that one can always take $k = 0$ unless $c = 0$, in which case we can take $k = 1$.

Lemma 6.8. *For every $a, c \in \mathcal{O}_S$ there exists an element $a' \in a + c\mathcal{O}_S$ with $|a'|_S = O(|c|_S)$.*

Proof. Let $K_S = \bigoplus_{\nu \in S} K_\nu$. Fix a closed fundamental set (parallelepiped) \mathcal{P} for the lattice $\mathcal{O}_S < K_S$. Observe that the set $c\mathcal{P}$ contains a full set of representatives for $\mathcal{O}_S/c\mathcal{O}_S$. Hence we can find an element $a' \in a + c\mathcal{O}_S$ such that $|a'|_S \leq |c|_S \max_{x \in \mathcal{P}} |x|_S$. \square

6.3. Choice of a generating set. Since all the Cayley graphs $\text{Cay}(\text{SL}_3(\mathcal{O}_S), T)$ for various finite generating sets T are quasi-isometric, it will be convenient in the argument to have a sufficiently rich generating set.

6.3.A. Fix a finite set of generators $e_\ell, 1 \leq \ell \leq N$, of the \mathbb{Z} -module \mathcal{O} .

Let

$$S_0 = \{E_{i,j}(e_\ell) : i \neq j \in \{1, 2, 3\}, 1 \leq \ell \leq N\},$$

$$S_1 = \left\{ \begin{pmatrix} p & 0 & 0 \\ 0 & p^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}^{\pm 1} \right\},$$

where p ranges over the primes in \mathcal{S} restricted to \mathbb{Z} ,

$$S_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^{\pm 1} \right\}.$$

6.3.B. Recall that by the Dirichlet unit theorem the group of units of \mathcal{O}_S is a product of some finite Abelian group and $t = r_1 + r_2 - 1 + r_3$ cyclic groups, where r_1 is the number of places $\nu \in \mathcal{S}$ such that $\mathbf{k}_\nu = \mathbb{R}$, r_2 is the number of places $\nu \in \mathcal{S}$ such that $\mathbf{k}_\nu = \mathbb{C}$ and r_3 is the number of non-Archimedean places in \mathcal{S} . In case $t = 0$, i.e., $\mathcal{O}_S = \mathbb{Z}$, we shall choose a hyperbolic matrix $A \in \text{SL}_2(\mathbb{Z})$, and denote $\mathcal{A} = \{A\}$. When $t > 0$ let $\lambda_i, 1 \leq i \leq t$, be fixed generators of this product of cyclic groups. Let us denote by \mathcal{A} the following set of elements:

$$\mathcal{A} = \left\{ A(\lambda_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} : 1 \leq i \leq t \right\}.$$

Fix some constant $0 < c < 1$. For each $\nu \in \mathcal{S}$ let us denote by $NC_\nu(\mathcal{A})$ the following set:

$$NC_\nu(\mathcal{A}) = \{v \in \mathbf{k}_\nu^2 : \|vA\|_\nu > c\|v\|_\nu \ \forall A \in \langle \mathcal{A} \rangle \cup \{0\}\}.$$

Observe that for an appropriate $0 < c < 1$ we can find finitely many elements $\gamma_i, 1 \leq i \leq M$, such that if we denote $\mathcal{A}^{\gamma_i} = \{A^{-1}\gamma_i A : A \in \mathcal{A}\}$, then $\bigcup_{i=1}^M NC_\nu(\mathcal{A}^{\gamma_i}) = \mathbf{k}_\nu^2$, for each $\nu \in \mathcal{S}$.

Moreover, we may choose enough γ_i 's so that for any line in \mathbf{k}_ν^2 we will have some \mathcal{A}^{γ_i} so that each of its eigenspaces form an angle $\pi/3 < \psi < 2\pi/3$ with the given line. Let

$$S_3 = \left\{ \begin{pmatrix} A^{\pm 1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & A^{\pm 1} \\ 0 & 0 & 1 \end{pmatrix} : A \in \mathcal{A}^{\gamma_i}, 1 \leq i \leq M \right\}.$$

Let S_4 be a (finite) set of generators of $\text{SL}_2(\mathcal{O}_S)$ embedded into $\text{SL}_3(\mathcal{O}_S)$ as the lower right corner.

6.3.C. If $K = \mathbb{Q}[\sqrt{-d}]$ for some $d \geq 0$, then we also define S_5 and S_6 as follows (if K is not of this form, then $S_5 = S_6 = \emptyset$). Fix a geometrically finite fundamental domain \mathcal{F} of the action of $\mathrm{SL}_2(\mathcal{O}_S)$ on the corresponding symmetric space, which is the hyperbolic space \mathbb{H}^n of dimension $n = 2, 3$. Note that in this case $\mathrm{SL}_2(\mathcal{O}_S)$ is of real rank 1. For each face of \mathcal{F} we include in S_5 a generator taking \mathcal{F} to the neighboring domain. Let $\Omega_0, \dots, \Omega_\kappa$ be the points at infinity of \mathcal{F} . Let P_0, \dots, P_κ be the stabilizers of $\Omega_0, \dots, \Omega_\kappa$ in $\mathrm{SL}_2(K)$. We assume that P_0 is the group of upper triangular matrices in $\mathrm{SL}_2(K)$. Then for every $i = 1, \dots, \kappa$ there exist $\alpha_i \in \mathrm{SL}_2(K)$ conjugating $P_0 \cap \mathrm{SL}_2(\mathcal{O}_S)$ to $P_i \cap \mathrm{SL}_2(\mathcal{O}_S)$. Let $\alpha_0 = e$. In order to define S_6 , we need the following statement.

Lemma 6.9. *There exist a finite set of matrices $\mathcal{T} = \{T_1, \dots, T_\iota\} \subset \mathrm{SL}_3(\mathcal{O}_S)$ such that for every $L \in \mathrm{SL}_3(\mathcal{O}_S)$ there exists $T_j, 1 \leq j \leq \iota$, such that $\alpha_i L T_j \alpha_i^{-1} \in \mathrm{SL}_3(\mathcal{O}_S)$ for every $i = 0, \dots, \kappa$. We may also require that the identity belongs to \mathcal{T} .*

Proof. Every element of K is a fraction with its numerator and denominator from \mathcal{O}_S . Consider the generic matrix $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ and the matrices $\alpha_i X \alpha_i^{-1}$. The entries of these matrices are linear polynomials in x_{11}, \dots, x_{33} with coefficients from K . Let D be the product of denominators of all these coefficients. Then for every i and every $L \in \mathrm{SL}_3(\mathcal{O}_S)$ the matrix $\alpha_i(1 + DL)\alpha_i^{-1}$ belongs to $\mathrm{SL}_3(\mathcal{O}_S)$. Now it is enough to take T_1, \dots, T_ι to be representatives of the right cosets of the congruence subgroup of $\mathrm{SL}_3(\mathcal{O}_S)$ corresponding to D . \square

6.3.D. Now S_6 is defined by

$$S_6 = \left\{ \alpha_i T^{-1} s T' \alpha_i^{-1} : 0 \leq i \leq \kappa, T, T' \in \mathcal{T}, s \in \bigcup_{m=0}^5 S_m \right\} \cap \mathrm{SL}_3(\mathcal{O}_S).$$

Note that for any choice of $T \in \mathcal{T}$ and $s \in \bigcup_{m=0}^5 S_m$ there exists at least one T' so that the corresponding element $\alpha_i T^{-1} s T' \alpha_i^{-1}$ belongs to $\mathrm{SL}_3(\mathcal{O}_S)$.

Let us fix $S = S_0 \cup \dots \cup S_6$ as the set of generators of $\mathrm{SL}_3(\mathcal{O}_S)$.

6.4. Proof of Theorem 6.1. In order to show that asymptotic cones of $\Gamma = \mathrm{SL}_3(\mathcal{O}_S)$ do not have cut-points, it suffices to show that any two elements $\alpha, \beta \in \Gamma$ are exteriorly connected. This follows immediately from the following two lemmas:

Lemma 6.10. *Let $\gamma \in \mathrm{SL}_3(\mathcal{O}_S)$. There exists $\alpha \in M = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$ such that*

- (1) $\mathrm{dist}_S(\alpha, e) \approx \mathrm{dist}_S(\gamma, e)$; that is, γ and α are approximately of the same size.
- (2) γ is exteriorly connected to α .

Lemma 6.11. *Given any $\alpha = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & 0 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 & 0 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{pmatrix}$ with $u_i, v_i \in \mathcal{O}_S$, there exists an exterior trajectory connecting them.*

A basic tool in proving these lemmas is:

Lemma 6.12. *Let $\gamma \in \text{SL}_3(\mathcal{O}_S)$ be an element having some large entry in the first column. For any $\theta = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \in L$, $m, n \in \mathcal{O}_S$ there exists an exterior trajectory from γ to $\gamma\theta$.*

Proof. A path ω of length k in the Cayley graph connecting γ to $\gamma\theta$ corresponds to a word $\theta = s_1s_2 \dots s_k$, where each $s_i \in S$, $\omega(i) = \gamma s_1s_2 \dots s_i$, $0 \leq i \leq k$. Since we want the path to be an exterior trajectory, it should satisfy the following conditions:

- (E₁) $k = \text{length}(\omega) \approx \text{dist}_S(\theta, e)$.
- (E₂) $\forall 0 \leq i \leq k$, $\text{dist}_S(\omega(i), e) \geq \kappa \text{dist}_S(\gamma, e)$ for some constant $\kappa = \kappa(\Gamma)$. (Note that we have that $\text{dist}_S(\gamma\theta, e) \geq C \text{dist}_S(\gamma, e)$ for some constant C which depends on our notion of an “entry being large”.)

In [LMR93] it was shown, in the particular case where $m, n \in \mathbb{Z}$, how to construct for any $\theta = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ a word of length $O(\log(n^2 + m^2 + 1)) \approx \text{dist}_S(\theta, e)$, expressing it in terms of a given generating set. Let us describe the slightly modified argument for elements $\theta = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathcal{O}_S)$; see also (2.12) in [LMR00].

Let $H = \langle \mathcal{A} \rangle$ be the subgroup generated by the set \mathcal{A} defined in Subsection 6.3.B. That is, either H is a cyclic group generated by some hyperbolic matrix $A \in \text{SL}_2(\mathbb{Z})$ when $\mathcal{O}_S = \mathbb{Z}$, or otherwise

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathcal{O}_S^* \right\},$$

where \mathcal{O}_S^* is the group of units of \mathcal{O}_S . Consider the group $\Lambda = H \times \mathcal{O}_S^2$. Λ is a finitely generated group which is a co-compact lattice in the group $H \times \prod_{\nu \in S} K_\nu^2$, where H acts on each of the factors of $\prod_{\nu \in S} K_\nu^2$ via the corresponding embedding of $\mathcal{O}_S \in K$ into K_ν . Observe that [LMR00, §3.15 – 3.18] for each $\nu \in S$ the two dimensional vector space K_ν^2 is spanned by eigenspaces on which H acts with eigenvalues of absolute value strictly larger than 1. It follows as in [LMR00, section 3] that the restriction to K_ν^2 of the left invariant coarse path metric on $H \times \prod_{\nu \in S} K_\nu^2$

is such that the distance from the identity of $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where $x, y \in K_\nu$, is

$O(\log(|x|_S^2 + |y|_S^2 + 1))$. If we fix any $\gamma \in \Gamma$, then any $\theta = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ with $m, n \in$

\mathcal{O}_S can be expressed as a word $s_1s_2 \dots s_k$ of length $k = O(\log(|n|_S^2 + |m|_S^2 + 1))$ with respect to a set of generators of the form

$$S(\gamma_\ell) = \left\{ \begin{pmatrix} 1 & 0 & e_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_j \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} A & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1} \right\},$$

where $\{e_j\}$ is a finite set generating \mathcal{O} as a \mathbb{Z} -module and $\{A \in \mathcal{A}^{\gamma_\ell}\}$, where \mathcal{A} is as in Subsection 6.3.B and $1 \leq \ell \leq M$ (see the definition of S_3). In particular we have for any $1 \leq i \leq k$ that

$$s_1 s_2 \dots s_i \in L(H) = \left\{ \begin{pmatrix} A & s \\ 0 & 0 & 1 \end{pmatrix} : A \in H^{\gamma_\ell} \ s, t \in \mathcal{O}_S \right\}.$$

We recall that in our choice of a generating set S for $SL_3(\mathcal{O}_S)$ we have given ourselves several possible choices of generators using various (finitely many) conjugates of H .

Let us denote $\gamma = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. By our assumption, for some $1 \leq j \leq 3$

we have a_{j1} large. Let $\nu_0 \in \mathcal{S}$ be such that $|a_{j1}|_{\nu_0} > c_1 |a_{j1}|_S$ for some fixed $c_1 > 0$ depending on K and \mathcal{S} . By the choice of S in Section 6.3 there is some γ_ℓ , $1 \leq \ell \leq M$, so that $(a_{j1}, a_{j2}) \in NC_{\nu_0}(\mathcal{A}^{\gamma_\ell})$. We shall use $\mathcal{A}^{\gamma_\ell}$ for producing a short word representing the element θ . Note that for any $A \in H^{\gamma_\ell}$, $\|(a_{j1}, a_{j2})A\|_{\nu_0} \geq c_0 \|(a_{j1}, a_{j2})\|$, where c_0 is the constant in Section 6.3. This immediately implies that if we use $S(A) \subset S$ for expressing θ , then (E_2) is satisfied. \square

We now turn to the proof of Lemma 6.10. As we use the right Cayley graph, applying elements of S on the right corresponds to column operations.

Proof of Lemma 6.10.

Step 1. γ is exteriorly connected to some $\gamma_1 = \gamma s$ such that γ_1 has a large entry in the first column and

$$s \in \left\{ I, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}^{\pm 1} \right\}.$$

Proof of Step 1. If there is no large entry in the first column we can exchange columns (and reverse sign to keep the determinant 1). We can also assume (for the sake of simplicity) that the (1,1)-entry of the matrix is large.

Step 2. $\gamma_1 = \begin{pmatrix} a & b & c \\ * & * & * \\ * & * & * \end{pmatrix}$ is exteriorly connected to $\gamma_2 = \gamma_1 \begin{pmatrix} 1 & m & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b' & c' \\ * & * & * \\ * & * & * \end{pmatrix}$ so that $b' \mathcal{O}_S + c' \mathcal{O}_S = \mathcal{O}_S$, with m and n polynomially controlled by the size of γ_1 .

Proof of Step 2. This follows from Lemma 6.6 and Lemma 6.12 using

$$\begin{pmatrix} 1 & m & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Step 3. If $c' = 0$ we may jump to Step 7 (with clear changes of notation). Otherwise $\gamma_2 = \begin{pmatrix} a & b' & c' \\ * & * & * \\ * & * & * \end{pmatrix}$ is exteriorly connected to $\gamma_3 = \gamma_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix} =$

$\begin{pmatrix} a & b' & c' \\ * & * & * \\ * & * & * \end{pmatrix}$ so that b' is large, and k is polynomially controlled by the size of γ_2 .

Proof of Step 3. When $c' \neq 0$ we can find $k \in \mathcal{O}_S$ so that $b' = b'' + kc'$ is large and the size of k is polynomially controlled by the size of γ . Then apply Lemma 6.12,

where we use $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

Step 4. $\gamma_3 = \begin{pmatrix} a & b' & c' \\ * & * & * \\ * & * & * \end{pmatrix}$ is exteriorly connected to $\gamma_4 = \begin{pmatrix} b' & -a & c' \\ * & * & * \\ * & * & * \end{pmatrix} =$

$\gamma_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Step 5. $\gamma_4 = \begin{pmatrix} b' & -a & c' \\ * & * & * \\ * & * & * \end{pmatrix}$ is exteriorly connected to $\gamma_5 = \gamma_4 \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & v & 1 \end{pmatrix}$

$= \begin{pmatrix} b' & 1 & c' \\ * & * & * \\ * & * & * \end{pmatrix}$.

Proof of Step 5. Since $\mathcal{O}_S = b'\mathcal{O}_S + c'\mathcal{O}_S$, we have $u', v' \in \mathcal{O}_S$ so that $b'u' + c'v' = a + 1$. Observe that for any $u = u' + c'r \in u' + c'\mathcal{O}_S$ we have $v = v' - b'r \in \mathcal{O}_S$ such that $b'u + c'v = a + 1$. By Lemma 6.8 one can choose $u \in u' + c'\mathcal{O}_S$ which satisfies $|u|_S \leq \text{const} \cdot |c'|_S$, where the constant depends only on \mathcal{O}_S .

Step 6. $\gamma_5 = \begin{pmatrix} b' & 1 & c' \\ * & * & * \\ * & * & * \end{pmatrix}$ is exteriorly connected to $\gamma_6 = \gamma_5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c' \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} b' & 1 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$.

Proof of Step 6. It follows from Lemma 6.12.

Step 7. In Step 8 we are going to subtract $b' - 1$ times the second column from the first. To insure that the first column remains large, we first note that there exists some $x \in \mathcal{O}_S$ whose size is polynomially bounded in terms of the size of γ_6 and

such that γ_6 is exteriorly connected to $\gamma_7 = \gamma_6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}$, so that both γ_7 as

well as $\gamma_8 = \gamma_7 \begin{pmatrix} 1 & 0 & 0 \\ 1 - b' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ have a large first column.

Step 8. γ_7 is exteriorly connected to $\gamma_8 = \gamma_7 \begin{pmatrix} 1 & 0 & 0 \\ 1 - b' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$.

Step 9. $\gamma_8 = \begin{pmatrix} 1 & 1 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$ is exteriorly connected to

$$\gamma_9 = \gamma_8 \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & & B \\ y & & \end{pmatrix}.$$

Step 10. γ_9 is exteriorly connected to $\gamma_{10} = \gamma_9 \begin{pmatrix} 1 & 0 & 0 \\ 0 & B^{-1} \\ 0 & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{pmatrix}$.

Proof of Step 10. Let us distinguish two cases:

- (i) The rank of $SL_2(\mathcal{O}_S)$ is at least 2.
- (ii) The rank of $SL_2(\mathcal{O}_S)$ is 1, i.e., $\mathcal{O}_S = \mathbb{Q}(\sqrt{-d})$ for some integer $d \geq 0$, $S = \{\infty\}$.

We shall abuse terminology and identify SL_2 with its image under embedding into SL_3 as the lower right corner. In the case when $\text{rank}(SL_2(\mathcal{O}_S)) \geq 2$ we may

express (by the results of [LMR00]) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & B^{-1} \\ 0 & & \end{pmatrix}$ as a short word $s_1 s_2 \dots s_n$

using a subset of the generating set which generates $\begin{pmatrix} 1 & 0 & 0 \\ 0 & SL_2(\mathcal{O}_S) \\ 0 & & \end{pmatrix}$. For any

prefix of this word, we have that the first column of $\gamma_9 s_1 s_2 \dots s_i$ is the same as that of γ_9 .

In the case when $\text{rank}(SL_2(\mathcal{O}_S)) = 1$ we may write B^{-1} as a short word in terms of our generating set using the following procedure. Consider the symmetric space $X = \mathbb{H}^n$, $n = 2, 3$, associated with our rank 1 group $SL_2(K_\infty)$ tessellated by fundamental domains for the action of $SL_2(\mathcal{O}_S)$. We may choose an $SL_2(\mathcal{O}_S)$ -invariant collection of horoballs so that if one removes them from X , the resulting subset X_0 has a compact quotient modulo $SL_2(\mathcal{O}_S)$. Fix a point $O \in X_0$ and consider the geodesic \mathfrak{g} connecting O to $B^{-1}O$. Following that geodesic we obtain a word $s_1 s_2 \dots s_t$ in the generators S_5 (see Section 6.3) expressing B^{-1} . Combining letters corresponding to parts of the geodesic spent inside various horoballs into sub-words, we obtain a word of the form $W_1 W_2 \dots W_k$, where each W_i is either one of the generators s_j corresponding to the geodesic \mathfrak{g} passing between domains outside the collection of horoballs, or W_i corresponds to the part spent in some horoball. Note that if we manage to replace each sub-word W_i corresponding to a horoball by a word, w_i , whose length is comparable to the length of the part of the geodesic inside the horoball (namely comparable to $\log(1 + \|W_i\|)$), we shall obtain a word of the required length expressing B^{-1} . Let us show by induction that we can indeed find for each W_i a short word w_i representing it and such that the resulting trajectory does not get too close to the identity. Suppose we have already treated

W_1, W_2, \dots, W_{m-1} and denote $\gamma' = \gamma_9 W_1 W_2 \dots W_{m-1} = \begin{pmatrix} 1 & 0 & 0 \\ x & & B' \\ y & & \end{pmatrix}$. If

$W_m = s_j$ for some $1 \leq j \leq t$, i.e., it does not correspond to passing through a horoball, we let the word representing it to be simply $w_m = s_j$, and clearly the trajectory is an exterior one (since the first column which contains a large element

was not changed). Suppose W_m corresponds to passing via some horoball. Let Ω_{k_0} be the cusp of the fundamental domain \mathcal{F} corresponding to that horoball.

Conjugating by the element $\alpha = \alpha_{k_0}$ defined in Subsection 6.3.C we have that

$$W'_m = \alpha^{-1}W_m\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying Lemma 6.12 to $\gamma = \gamma'\alpha$ and $\theta = W'_m$ we obtain a short word $r_1r_2 \dots r_\ell = W'_m$ such that the corresponding trajectory from $\gamma = \gamma'\alpha$ to $\gamma'\alpha W'_m$ is an exterior trajectory. Notice that actually $\gamma'\alpha$ does not belong to $SL_3(\mathcal{O}_S)$ but to $SL_3(K)$, so the notion of “exterior trajectory” should be understood with respect to the metric induced from the Riemannian metric on the corresponding symmetric space. Denote $T_0 = e$. For each r_i , $1 \leq i \leq \ell - 1$, there is $T_i \in \mathcal{T}$ so that we have a generator $t_i = \alpha T_{i-1}^{-1}r_iT_i\alpha^{-1} \in S_6$ as in Subsection 6.3.D. Let $t_\ell = \alpha T_{\ell-1}^{-1}r_\ell\alpha^{-1}$. Now observe that we have

$$W_m = \alpha W'_m \alpha^{-1} = t_1 t_2 \dots t_\ell.$$

This in particular implies that t_ℓ is indeed an element of $SL_3(\mathcal{O}_S)$ and hence $t_\ell \in S_6$, and that we obtain an exterior trajectory connecting γ_9 to

$$\gamma_9 W_1 W_2 \dots W_m = \begin{pmatrix} 1 & 0 & 0 \\ x & B'' & \\ y & & \end{pmatrix}.$$

Repeating this process we shall obtain an exterior trajectory connecting γ_9 to

$$\gamma_{10} = \begin{pmatrix} 1 & 0 & 0 \\ x & I & \\ y & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{pmatrix}. \quad \square$$

Remark. Note that for each $1 \leq i \leq 9$ the sizes of γ_i and γ_{i+1} are polynomially comparable.

Proof of Lemma 6.11. We are given $\alpha = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ & 0 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ & 0 & 1 \end{pmatrix}$,

where $u, v \in \mathcal{O}_S^2$. Let us denote $w = v - u$. Our goal is to produce an exterior trajectory connecting α to β . We shall construct a word using generators belonging to a subset of $S_0 \cup S_3$. The argument proving Lemma 6.12 allows us to produce a

short word $s_1 s_2 \dots s_n$ representing $\begin{pmatrix} 1 & 0 & 0 \\ w & 1 & 0 \\ & 0 & 1 \end{pmatrix}$. This gives us a path \mathfrak{p} from

$\begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ & 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ & 0 & 1 \end{pmatrix}$ of the required length. However, this path

may get too close to the origin. To avoid getting too close to the origin we shall show that we can “shift the whole path away from the origin”. We fix an Archimedean place ν_0 of K . Look at the projection of the path \mathfrak{p} at this place in $K_{\nu_0}^2$, identified

with $\begin{pmatrix} 1 & 0 & 0 \\ K_{\nu_0}^2 & 1 & 0 \\ & 0 & 1 \end{pmatrix}$. There is a hyperbolic matrix $A \in SL_2(\mathbb{Z})$ such that

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & & A \\ 0 & & \end{pmatrix} \in S_3$, which has an eigen-direction so that when translating the path \mathbf{p} in this eigen-direction, \mathbf{p} is moved away from the origin. Choose a word $t_1 t_2 \dots t_m$ in the generators $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & & A \\ 0 & & \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$ which represents an element of the form $\begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ q & 0 & 1 \end{pmatrix}$ so that the vector $\begin{pmatrix} p \\ q \end{pmatrix}$ is close (actually within uniformly bounded distance) to the eigen-direction of A and whose size is comparable to the size of $\|u\|_{\mathcal{S}}$; see [LMR93]. We claim that the trajectory corresponding to

$$\begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} t_1 t_2 \dots t_m s_1 s_2 \dots s_n (t_1 t_2 \dots t_m)^{-1}$$

gives a trajectory from $\begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 0 \\ v & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which never gets too close to the origin. Indeed observe that as we go along the path corresponding to $t_1 t_2 \dots t_m$ we get further away at the ν_0 place from the origin. We might be getting closer to the origin at some other (Archimedean) place, but since the rate at which we move in any other Archimedean place is comparable to the rate at which we move at the ν_0 place and where any non-Archimedean place moving along this path does not change our distance from the origin, we conclude that we are always at a distance which is bounded below by a fixed positive fraction of $\|u\|_{\mathcal{S}}$. Once we have completed the path $t_1 t_2 \dots t_m$, the path along $s_1 s_2 \dots s_n$ is away from the origin by a distance comparable to $\|u\|_{\mathcal{S}}$, and finally, as before, moving back on $t_1 t_2 \dots t_m$ cannot get us too close to the identity. \square

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ADDED AFTER POSTING

Important changes need to be made in Section 3.1 that includes the definition of divergence and related notions (see the version of the paper posted in the arXiv on May 1 of 2009 preprint 0801.4141). Below is the correct version of this section. We are grateful to Curt Kent for finding inconsistencies in the original version of that section. In addition, the second author acknowledges the support of the Israeli Science Foundation.

3.1. Divergence and asymptotic cut-points. Typically the spaces that we have in mind when defining divergence are finitely generated groups, geodesic metric spaces X quasi-isometric to finitely generated groups, and geodesic metric spaces X such that the action of their group of isometries is co-bounded, that is the orbit of a ball under $Isom(X)$ covers X (for simplicity we call such spaces *periodic*).

We consider the usual relation on the set of functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f \preceq_C g$ if

$$f(n) \leq Cg(Cn) + Cn + C$$

for some $C > 1$ and all x . This defines the known equivalence relation on the set of functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f \equiv_C g$ if $f \preceq_C g$ and $g \preceq_C f$.

Most of the time we obliterate the constant C from the index and simply write $f \preceq g$ and $f \equiv g$.

We do not distinguish equivalent functions in this paper, so, for instance, all linear functions (including all constants) are equivalent.

Definition 3.1. Let (X, dist) be a geodesic metric space (one can formulate a similar definition for arbitrary length spaces), and let $0 < \delta < 1$ and $\gamma \geq 0$. Let $a, b, c \in X$ with $\text{dist}(c, \{a, b\}) = r > 0$, where $\text{dist}(c, \{a, b\})$ is the minimum of $\text{dist}(c, a)$ and $\text{dist}(c, b)$. Define $\text{div}_\gamma(a, b, c; \delta)$ as the infimum of the lengths of paths connecting a, b and avoiding the ball $B(c, \delta r - \gamma)$ (note that by definition a ball of non-positive radius is empty). If no such path exists, take $\text{div}_\gamma(a, b, c; \delta) = \infty$.

The behavior of the function $\text{div}_\gamma(a, b, c; \delta)$ with respect to quasi-isometry is easily checked.

Lemma 3.2. Let $q : X \rightarrow Y$ be an (L, C) -quasi-isometry between two geodesic metric spaces. Then for every $0 < \delta < 1$ and $\gamma \geq 0$ there exists $\gamma_1 = \gamma_1(\delta, \gamma, L, C) \geq \gamma$ such that for any three points a, b, c in X ,

$$(1) \quad \text{div}_\gamma(q(a), q(b), q(c); \delta) \geq \frac{1}{2L} \text{div}_{\gamma_1}(a, b, c; \delta/L^2) - C.$$

Definition 3.3. The divergence function $\text{Div}_\gamma(n, \delta)$ of the space X is defined as the supremum of all numbers $\text{div}_\gamma(a, b, c; \delta)$ with $\text{dist}(a, b) \leq n$.

Clearly if $\delta \leq \delta'$ and $\gamma \geq \gamma'$, then $\text{Div}_\gamma(n; \delta) \leq \text{Div}_{\gamma'}(n; \delta')$ for every n .

Lemma 3.4. If X is one-ended, proper, periodic, and every point is at distance less than κ from a bi-infinite bi-Lipschitz path, then there exists δ_0 such that for every $\gamma \geq 4\kappa$ the function $\text{Div}_\gamma(n, \delta_0)$ takes only finite values.

In particular this holds if X is a Cayley graph of a finitely generated one-ended group, and one can take $\delta_0 = \frac{1}{2}$ and $\kappa = \frac{1}{2}$ in this case.

Proof. Since the space is periodic and proper we may assume that there exist finitely many bi-infinite bi-Lipschitz paths $\mathbf{p}_1, \dots, \mathbf{p}_n$, such that every point in X is at distance less than κ from a path $g\mathbf{p}_i$ with $g \in \text{Isom}(X)$ and $i \in \{1, 2, \dots, n\}$. Let $C \geq 1$ be such that \mathbf{p}_i is C -bi-Lipschitz for every $i \in \{1, 2, \dots, n\}$. Let $\delta_0 = \frac{1}{C^2+1}$, let a be a point on a path $\mathbf{p}_a = g\mathbf{p}_i$ and b a point on a path $\mathbf{p}_b = g'\mathbf{p}_j$, where $g, g' \in \text{Isom}(X)$ and $i, j \in \{1, 2, \dots, n\}$. We prove that the value $\text{div}_0(a, b, c; \delta_0)$ is always finite.

Let $r = \text{dist}(c, \{a, b\}) > 0$. We claim that one of the two connected components of $\mathbf{p}_a \setminus \{a\}$ does not intersect $B(c, \delta_0 r)$. Indeed, otherwise there would be two points on \mathbf{p}_a at distance at most $u = 2\delta_0 r$ from each other in X but at distance $v > \frac{2}{C}(1 - \delta_0)r$ along \mathbf{p}_a . This contradicts the assumption that \mathbf{p}_a is C -bi-Lipschitz since $v/u > \frac{1-\delta_0}{C\delta_0} = C$. It remains to note that since the space is one-ended, every two points on \mathbf{p}_a and \mathbf{p}_b respectively that are far enough from c , are connected by a path outside $B(c, \delta_0 r)$.

Now we prove that for any n and any $\gamma \geq 4\kappa$ the value $\text{Div}_\gamma(n, \delta_0)$ is finite. Take a, b, c such that $\text{dist}(a, b) \leq n$, and let $r = \text{dist}(c, \{a, b\})$. If a geodesic $[a, b]$ does not intersect $B(c, \delta_0 r)$, then $\text{div}_\gamma(a, b, c; \delta_0) \leq n$.

Assume that a geodesic $[a, b]$ intersects $B(c, \delta_0 r)$. Then $r \leq \delta_0 r + n$, hence $r \leq \frac{n}{1-\delta_0}$.

Without loss of generality we may assume that a is in a fixed compact. Then b and c are in tubular neighborhoods of this compact, which are other compacts. Each of these compacts is covered by finitely many balls of radius κ and with center on some path gp_i with $g \in Isom(X)$ and $i \in \{1, 2, \dots, n\}$. Therefore, to finish the proof, it suffices to note that for a, b, c fixed, all triples a', b', c' with $\text{dist}(a, a'), \text{dist}(b, b'), \text{dist}(c, c') \leq \kappa$ satisfy

$$\text{div}_{4\kappa}(a', b', c'; \delta_0) \leq \text{div}_0(a, b, c; \delta_0) + 2\kappa.$$

□

Remark 3.5. For a space as in Lemma 3.4 and a fixed $\delta \in (0, 1)$, we must require at least that γ is larger than $\delta\kappa$, otherwise $\text{div}_\gamma(a, b, c; \delta)$ can be infinite for a, b arbitrarily far away from each other. Indeed, consider a Cayley graph Y of a finitely generated one-ended group, and construct a metric space X by attaching to each vertex y in Y a copy T_y of the same finite simplicial non-trivial tree T with basepoint v . Define a metric on X in the natural way: to get from a point $p \in T_x$ to a point $q \in T_y$, one needs to first get from p to x inside T_x , then from x to y inside Y , then from y to q inside T_y . The space X obviously admits a co-compact isometric group action. It has one end since Y is one-ended and T is finite. Every point in X is within distance $\leq \kappa$ of a bi-infinite geodesic, where κ is the Hausdorff distance between T and $\{v\}$. If we take a point a in T_x with $\text{dist}(a, x) = \kappa$, a point b in Y with $\text{dist}(b, x) \geq \kappa$, $c = x$ and $\gamma < \delta\kappa$, then there is no path connecting a to b in $X \setminus B(c, \delta r - \gamma)$. So $\text{div}_\gamma(a, b, c; \delta) = \infty$.

There is one equivalent and equally natural way to define divergence function Div_γ for arbitrary one-ended length spaces admitting co-compact isometry group actions. One chooses $C > 0$ and replaces paths in the definition of $\text{div}_\gamma(a, b, c; \delta)$ by C -paths, as defined in Section 2.1. One can easily check that all the statements in this section remain true for this more general function (after some obvious modifications).

We now define a new divergence function, closer to the idea of divergence of rays that was inspiring the notion, in particular closer to the Gersten divergence.

Definition 3.6. Let $\lambda \geq 2$. The *small divergence function* $\text{div}_\gamma(n; \lambda, \delta)$ is defined as the supremum of all numbers $\text{div}_\gamma(a, b, c; \delta)$ with $0 \leq \text{dist}(a, b) \leq n$ and

$$(2) \quad \lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b).$$

The next lemma is obvious.

Lemma 3.7. *The following inequalities are true for every geodesic metric space X .*

- (1) *If $\delta \leq \delta'$, then $\text{div}_\gamma(n; \lambda, \delta) \leq \text{div}_\gamma(n; \lambda, \delta')$ for every n, λ and γ .*
- (2) *If $\lambda \leq \lambda'$, then $\text{div}_\gamma(n; \lambda, \delta) \leq \text{div}_\gamma(n; \lambda', \delta)$ for every n, δ and γ .*

Proposition 3.8. (1) *The function $\text{div}_\gamma(n; \lambda, \delta)$ takes only finite values if and only if for every n there exists d_n such that if for some a, b, c satisfying (2), $\text{div}_\gamma(a, b, c; \delta) \geq d_n$, then $\text{dist}(a, b) \geq n$.*

- (2) If (X, dist) is a geodesic metric space quasi-isometric to a finitely generated group, then the existence of some $\delta \in (0, 1)$, $\gamma \geq 0$ and $\lambda \geq 2$ such that the function $\text{div}_\gamma(\cdot; \lambda, \delta)$ takes only finite values is granted if and only if X is one-ended.

Proof. The statement in (1) is an easy exercise.

In view of Lemma 3.2 and of (1), it suffices to prove (2) for $X = G$ finitely generated group with a word metric.

If G has infinitely many ends, then for any $0 < \delta < 1$, $\gamma > 0$ and $\lambda \geq 2$, there exist a, b, c satisfying (2) and with $r = \text{dist}(c, \{a, b\})$ large enough so that $\text{div}_\gamma(a, b, c; \delta) = \infty$.

Assume that G is one-ended. Then all $\text{div}_\gamma(a, b, c; \delta)$ with $\delta \in (0, 1/2]$ and $\gamma \geq 2$ are finite.

We rewrite (1) as follows: for every n there exists d_n such that if $\text{dist}(a, b) \in [0, n]$, then $\text{div}_\gamma(a, b, c; \delta) \leq d_n$.

Assume then that $\text{dist}(a, b) \in [0, n]$. Without loss of generality we may assume that $a = 1$, hence $b \in B(1, n)$. Take an arbitrary vertex c satisfying (2), and take $r = \text{dist}(c, \{a, b\})$. If $B(c, \delta r)$ does not intersect all geodesics joining a, b , then $\text{div}_\gamma(a, b, c; \delta) = \text{dist}(a, b) \leq n$. If it does, then $r \leq \delta r + \text{dist}(a, b)$, hence $r \leq \frac{n}{1-\delta}$. Thus there are finitely many possibilities for b, c , hence the supremum of $\text{div}_\gamma(a, b, c; \delta)$ over all a, b, c satisfying (2) and $\text{dist}(a, b) \leq n$ is finite. \square

Remark 3.9. When G has infinitely many ends, one may consider another definition of the divergence function. Indeed, in this case it is known by Stallings' theorem [Sta71, Theorems 4.A.6.5 and 5.A.9] that the group splits as a free product with amalgamation over a finite subgroup. When G is finitely presented it can be moreover written as the fundamental group of a finite graph of groups with finite edge groups and one-ended vertex groups [Dun85]. Then one can take the divergence function of G as the supremum function of the divergence functions of the vertex groups, and thus obtain a finite function.

Note that this new function will provide more information on a group with infinitely many ends, as it will provide upper bound for divergences of factors. On the other hand it will no longer distinguish between the group \mathbb{Z}^2 and $\mathbb{Z}^2 * \mathbb{Z}^2$, for instance. Thus, it is appropriate only when restricting to the class of groups with infinitely many ends.

We need to introduce two more functions, further restricting the choice of c . We assume, as before, that X is a geodesic metric space. For every pair of points $a, b \in X$, we choose and fix a geodesic $[a, b]$ joining them such that if x, y are points on a geodesic $[a, b]$ chosen to join a, b the sub-geodesic $[x, y] \subseteq [a, b]$ is chosen for x, y . The next lemmas will show that the definition we are about to give does not depend on the choice of the geodesic $[a, b]$. We say that a point c is *between* a and b if c is on the fixed geodesic segment $[a, b]$.

We define $\text{Div}'_\gamma(n; \delta)$ and $\text{div}'_\gamma(n; \lambda, \delta)$ as Div_γ and div_γ before, but restricting c to the set of points between a and b . Clearly $\text{Div}'_\gamma(n; \delta) \leq \text{Div}_\gamma(n; \delta)$ and $\text{div}'_\gamma(n; \lambda, \delta) \leq \text{div}_\gamma(n; \lambda, \delta)$ for every λ, δ .

Lemma 3.10. *For every $a, b \in X$, every $\delta \in (0, 1)$ and every $\lambda \geq 2$, we have*

$$\begin{aligned}
 (3) \quad \sup_{\substack{c \in [a, b] \\ \lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b)}} \text{div}_\gamma(a, b, c; \delta/3) &\leq \sup_{\lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b)} \text{div}_\gamma(a, b, c; \delta/3) \\
 &\leq \sup_{\substack{c \in [a, b] \\ 2\lambda \text{dist}(c, \{a, b\}) \geq \text{dist}(a, b)}} \text{div}_\gamma(a, b, c; \delta) \\
 &\quad + \text{dist}(a, b).
 \end{aligned}$$

Proof. The first inequality in (3) is obvious. We prove only the second inequality in (3).

Let $c \in X$, $r = \text{dist}(c, \{a, b\})$ and assume that (2) is satisfied. Suppose first that the distance between c and the geodesic segment $[a, b]$ is at least $\frac{\delta}{3}r$. Then $[a, b]$ avoids the ball $B(c, \frac{\delta}{3}r - \gamma)$, and so $\text{div}_\gamma(a, b, c; \delta/3) = \text{dist}(a, b)$, and the second inequality in (3) holds.

Suppose now that $\text{dist}(c, [a, b]) < \frac{\delta}{3}r$. Then there exists a point c' on $[a, b]$ at distance at most $\frac{\delta}{3}r$ from c . Note that the distance r' from c' to $\{a, b\}$ is at least $(1 - \frac{\delta}{3})r$. Since $\delta < 1$, we have

$$\frac{2}{3}\delta < \delta \left(1 - \frac{\delta}{3}\right).$$

Therefore the ball $B' = B(c', \delta r' - \gamma) \supseteq B(c', \delta(1 - \frac{\delta}{3})r - \gamma)$ contains the ball $B = B(c, \frac{\delta}{3}r - \gamma)$. Hence every path avoiding the ball B' also avoids the ball B . Hence $\text{div}_\gamma(a, b, c'; \delta) \geq \text{div}_\gamma(a, b, c; \frac{\delta}{3})$. This gives the second inequality in (3). \square

Lemma 3.11. (a) *For every $a, b \in X$, every $\delta \in (0, 1)$, $\gamma \geq 0$ and every $\lambda \geq 2$, we have*

$$\begin{aligned}
 (4) \quad \sup_{c \in [a, b]} \text{div}_\gamma(a, b, c; \delta/3) &\leq \sup_{c \in X} \text{div}_\gamma(a, b, c; \delta/3) \\
 &\leq \sup_{c \in [a, b]} \text{div}_\gamma(a, b, c; \delta) + \text{dist}(a, b);
 \end{aligned}$$

$$(5) \quad \text{div}_\gamma(n; \lambda, \delta) \leq \text{Div}_\gamma(n; \delta), \quad \forall n, \lambda \geq 2, \delta \in (0, 1), \gamma \geq 0.$$

(b) *For every $\delta \in (0, 1)$ and $\gamma \geq 0$ we have*

$$(6) \quad \text{Div}_\gamma\left(n; \frac{\delta}{3}\right) \leq \text{div}_\gamma(n; 2, \delta) + 2n, \quad \forall n.$$

(c) *Let X be a space as in Lemma 3.4, and let δ_0 and $\gamma_0 = 4\kappa$ be the constants provided by the proof of that lemma. For every $0 < \delta' \leq \delta \leq \delta_0$ and $\gamma' \geq \gamma \geq \gamma_0$, $\text{Div}_\gamma(n; \delta) \equiv \text{Div}_{\gamma'}(n; \delta')$.*

Proof. (a) The first inequality in (4) and inequality (5) are obvious. The second inequality in (4) is proved exactly in the same manner in which it was proved for (3).

(b) In order to prove (6) it suffices to prove a similar inequality for Div'_γ and div'_γ according to (3) and (4). Take a, b with $\text{dist}(a, b) \leq n$, and take $c \in [a, b]$. Let $r = \text{dist}(c, \{a, b\})$ and assume that $r = \text{dist}(a, c)$. Let $b' \in [c, b]$ at distance r from c .

Then $\text{div}_\gamma(a, b, c; \delta) \leq \text{div}_\gamma(a, b', c; \delta) + n \leq \text{div}_\gamma(n; 2, \delta) + n$.

(c) We prove that $\text{Div}_\gamma(n; \delta) \leq \text{Div}_{\gamma'}(n; \delta')$. As in the proof of Lemma 3.4 we may assume there exist $\mathbf{p}_1, \dots, \mathbf{p}_n$, bi-infinite C -bi-Lipschitz paths such that every

point in X is at distance $\leq \kappa$ from a path gp_i with $g \in Isom(X)$ and $i \in \{1, 2, \dots, n\}$; we take $\delta_0 = \frac{1}{C^2+1}$.

Let $a, b \in X$ be such that $\text{dist}(a, b) \leq n$, and for $c \in X$ let $r = \text{dist}(c, \{a, b\}) = \text{dist}(c, a)$. Assume that $B(c, \delta r - \gamma)$ intersects $[a, b]$ (otherwise $\text{div}_\gamma(a, b, c; \delta) \leq n$) whence $r \leq \frac{n}{1-\delta}$.

The points a, b are at distance $\leq \kappa$ from points a' respectively b' on paths $\mathbf{p}_a = gp_i, \mathbf{p}_b = g'p_j$. Note that $\text{dist}(a', c) \geq r - \kappa > \delta r - \gamma$ and the same for $\text{dist}(b', c)$. An argument as in the proof of Lemma 3.4 implies that one of the two connected components of $\mathbf{p}_a \setminus \{a'\}$ does not intersect $B(c, \delta r - \gamma)$. We denote it by \mathbf{c}_a . Likewise we obtain \mathbf{c}_b ray in \mathbf{p}_b . Let r' be such that $\delta' r' - \gamma' = \delta r - \gamma$. There exists a point a_1 on $[a, a'] \cup \mathbf{c}_a$ at distance r' from c , and a point b_1 on $[b, b'] \cup \mathbf{c}_b$ at distance $\geq r'$ from c . Moreover $\text{dist}(a, a_1) \leq r + r'$ and $\text{dist}(b, b_1) \leq r + n + r'$, whence $\text{dist}(a_1, b_1) \leq 2(r + r') + n \leq Dn + D$. Then $\text{div}_\gamma(a, b, c; \delta) \leq \text{div}_{\gamma'}(a_1, b_1, c; \delta') + 2\kappa + CDn + CD \leq \text{Div}_{\gamma'}(Dn + D; \delta') + 2\kappa + CDn + CD$. \square

Lemmas 3.10 and 3.11 immediately imply.

Corollary 3.12. *Let X be a space as in Lemma 3.4, and $\delta_0, \gamma_0 = 4\kappa$ the constants provided by the proof of that lemma.*

- (1) *The functions $\text{div}'_\gamma(n; \lambda, \delta)$ and $\text{Div}'_\gamma(n; \delta)$ with $\delta \leq \delta_0, \gamma \geq \gamma_0$, do not depend (up to the equivalence relation \equiv) on the choice of geodesics $[a, b]$ for every pair of points a, b .*
- (2) *For every $\delta \leq \delta_0, \gamma \geq \gamma_0$, and $\lambda \geq 2$*
- (7) $\text{Div}_\gamma(n; \delta) \equiv \text{Div}'_\gamma(n; \delta) \equiv \text{div}_\gamma(n; \lambda, \delta) \equiv \text{div}'_\gamma(n; \lambda, \delta)$.

Moreover all functions in (7) are independent of $\delta \leq \delta_0$ and $\gamma \geq \gamma_0$ (up to the equivalence relation \equiv).

- (3) *The function $\text{Div}_\gamma(n; \delta)$ is equivalent to $\text{div}'_\gamma(n; 2, \delta)$ as a function in n . Thus in order to estimate $\text{Div}_\gamma(n, \delta)$ for $\delta \leq \delta_0$ it is enough to consider points a, b, c where c is the midpoint of a (fixed) geodesic segment connecting a and b .*

We now recall another definition of divergence function, due to S. Gersten ([Ger94b],[Ger94a]). Let X be a geodesic metric space, let x_0 be a fixed point and let $\rho \in (0, 1)$. Denote by S_r the sphere of center x_0 and radius r and by C_r the complementary set of the open ball $B(x_0, r)$.

For every $x, y \in S_r$ define $\text{DG}_\rho(x, y)$ to be the shortest path distance in $C_{\rho r}$ between x, y . Then define $\text{DG}_\rho(r)$ as the supremum of $\text{DG}_\rho(x, y)$, over all $x, y \in S_r$ that can be connected by a path in $C_{\rho r}$.

The collection of functions $\Delta = \{\text{DG}_\rho \mid \rho \in (0, 1)\}$ is called *the divergence* of X . Such a collection is considered up to the equivalence relation \sim defined in what follows. We write $\Delta \preceq \Delta'$ if and only if there exist $\epsilon, \epsilon' \in (0, 1)$ and $C \geq 1$ such that for every $\rho < \epsilon$ there exists $\rho' < \epsilon'$ such that $\text{DG}_\rho \preceq_C \text{DG}'_{\rho'}$.

Then we define $\Delta \sim \Delta'$ by $\Delta \preceq \Delta'$ and $\Delta' \preceq \Delta$.

It can be easily checked that the collection of functions Δ up to the equivalence relation \sim is independent of the basepoint x_0 , and it is a quasi-isometry invariant. Thus we may assume that x_0 is an arbitrary fixed basepoint.

The relationship between Gersten divergence and the small divergence function is the following.

- Lemma 3.13.** (1) *Let X be a geodesic metric space. Then for every $\rho \in (0, 1)$, $DG_\rho(n) \leq \text{div}_0(\pi n; 2, \rho)$ for every n .*
 (2) *Let X be a space as in Lemma 3.4, and $\delta_0, \gamma_0 = 4\kappa$ the constants provided by the proof of that lemma. Then for every $0 < \rho < \rho' \leq \delta_0$ and $\gamma \geq \gamma_0$*

$$\text{div}'_\gamma(n; 2, \rho) \leq \sup_{x \leq n} DG_{\rho'} \left(\frac{x}{2} + O(1) \right) + n + O(1).$$

We leave the proof to the reader. Note that if X is as in Lemma 3.13, (2), and if the collection of functions Δ is equivalent to a non-decreasing function f (i.e. to the collection $DG_{\rho'} = f$ for every ρ') then the divergence function is equivalent to f . In all the cases when Gersten divergence is computed this is precisely the case.

Notation. Let $(d_n), (d'_n)$ be two sequences of numbers, and let ω be an ultrafilter. We write $d'_n = O_\omega(d_n)$ if for some constant $C > 1$, $\frac{1}{C}d'_n < d_n < Cd'_n$ ω -a.s. for all n .

Recall that a finitely generated group G is called *wide* if none of its asymptotic cones has a cut-point; it is called *unconstricted* if one of its asymptotic cones does not have cut-points. We say that a closed ball $\bar{B} = \bar{B}(c, \delta)$ in a metric space X *separates* point u from point v if u and v are in different connected components of $X \setminus \bar{B}$.

Lemma 3.14. *Let X be a geodesic metric space. Let ω be any ultrafilter, and let (d_n) be a sequence of positive numbers such that $\lim_\omega d_n = \infty$. Let $\mathcal{C} = \text{Con}^\omega(X, (o_n), (d_n))$, $A = (a_n)^\omega, B = (b_n)^\omega, C = (c_n)^\omega \in \mathcal{C}$. Let $r = \text{dist}(C, \{A, B\})$. The following conditions are equivalent for any $0 \leq \delta < 1$.*

- (i) *The closed ball $\bar{B}(C, \delta)$ in \mathcal{C} separates A from B .*
- (ii) *For every $\delta' > \delta$ and every (some) $\gamma \geq 0$ the limit $\lim_\omega \frac{\text{div}_\gamma(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ is ∞ .*

Proof. Suppose that $\frac{\text{div}_\gamma(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ is bounded by some constant M ω -almost surely. This means there exists (ω -a.s.) a path \mathfrak{p}_n of length $O(d_n)$ connecting a_n, b_n and avoiding the ball $B(c_n, \frac{\delta'}{r}r_n)$ where $r_n = \text{dist}(c_n, \{a_n, b_n\})$. Since $\lim^\omega (r_n/d_n) = r$, the ω -limit of these balls in \mathcal{C} is the closed ball $\bar{B}(C, \delta')$. The limit \mathfrak{p} of the paths \mathfrak{p}_n in \mathcal{C} exists (because the length of \mathfrak{p}_n is $O_\omega(d_n)$), connects A and B and avoids the open ball $B(C, \delta')$. Hence \mathfrak{p} avoids every closed ball $\bar{B}(C, \delta)$ for $\delta < \delta'$. Thus if there exists a closed ball $\bar{B}(C, \delta)$ separating A from B , then for any $\delta' > \delta$ the set of numbers $\frac{\text{div}_\gamma(a_n, b_n, c_n; \frac{\delta'}{r})}{d_n}$ cannot be bounded ω -almost surely. So (i) implies (ii).

Now suppose that property (ii) holds and that $\bar{B}(C, \delta)$ does not separate A from B . Then there exists a path \mathfrak{p} in \mathcal{C} connecting A and B and avoiding the ball $\bar{B}(C, \delta)$. By Lemma 2.3 we can assume that \mathfrak{p} is a piece-wise geodesic, avoiding $\bar{B}(C, \delta')$ for some $\delta' > \delta$, and that \mathfrak{p} is an ω -limit of paths \mathfrak{p}_n in the space X . Then the lengths of \mathfrak{p}_n are $O_\omega(d_n)$ and these paths avoid (ω -a.s.) balls $B(c_n, \delta''r_n)$ where $r_n = \text{dist}(c_n, \{a_n, b_n\})$, $\delta < \delta'' < \delta'$. Therefore the set of numbers $\frac{\text{div}_\gamma(a_n, b_n, c_n; \delta'')}{d_n}$ is bounded, which contradicts (ii). \square

Taking $\delta = 0$ in Lemma 3.14, we obtain the following

Lemma 3.15. *Let $X, \omega, (d_n)$ be as in Lemma 3.14. Then the following conditions are equivalent:*

- (i) *The asymptotic cone $\text{Con}^\omega(X, (o_n), (d_n))$ has a cut-point.*
- (ii) *There exists a sequence of pairs of points (a_n, b_n) in X with*

$$\text{dist}(a_n, b_n) = O_\omega(d_n) \text{ and } \frac{\text{dist}(a_n, o_n)}{d_n}, \frac{\text{dist}(b_n, o_n)}{d_n} \text{ bounded,}$$

and a sequence of midpoints c_n of geodesics $[a_n, b_n]$ such that the sequence of numbers $\text{div}_\gamma(a_n, b_n, c_n; 2, \delta)$, $n \geq 1$, is superlinear ω -a.s. for every $\delta \in (0, 1)$ and $\gamma \geq 0$.

Moreover, if that condition holds, then the point $C = (c_n)^\omega$ is a cut-point in the cone $\text{Con}^\omega(X, (o_n), (d_n))$ separating $A = (a_n)^\omega$ from $B = (b_n)^\omega$.

Lemma 3.16. *Let X be a periodic geodesic metric space which contains a bi-infinite quasi-geodesic. Suppose that one of its asymptotic cones \mathcal{C} has a closed ball $\overline{B}(C, \delta)$ separating A from B and $\text{dist}(C, \{A, B\}) > 3\delta$. Then X is not wide; that is one of the asymptotic cones of X has a cut-point.*

Proof. Let $\overline{B}(C, \delta)$ be a cut-ball, $C = (c_n)^\omega$, and let $A = (a_n)^\omega, B = (b_n)^\omega$ be two points in \mathcal{C} separated by $\overline{B}(C, \delta)$. Since X contains a bi-infinite quasi-geodesic and it has a cobounded action of $\text{Isom}(X)$, the cone \mathcal{C} contains a bi-infinite bi-Lipschitz line and it is a homogeneous space. Therefore there exist bi-infinite bi-Lipschitz lines containing A and B respectively. We proceed as in Lemma 3.4. The point A cuts the bi-Lipschitz line containing it into two bi-Lipschitz rays. Since $\text{dist}(C, \{A, B\}) > 3\delta$, one of these rays does not cross $\overline{B}(C, \delta)$. Let us denote it by \mathfrak{p} . Similarly, there exists an infinite bi-Lipschitz ray \mathfrak{p}' starting at B that does not cross $\overline{B}(C, \delta)$. Clearly every point A' on \mathfrak{p} and every point B' on \mathfrak{p}' are separated by $\overline{B}(C, \delta)$. For every $n > 1$ let A_n be a point on \mathfrak{p} at distance n from A , B_n be a point on \mathfrak{p}' at distance n from B . Note that $\text{dist}(A_n, B_n) = O(n)$. Indeed otherwise for a big enough n , any geodesic $[A_n, B_n]$ avoids $\overline{B}(C, \delta)$. Also note that $\text{dist}(C, \{A_n, B_n\}) = O(n)$. Now consider the asymptotic cone $\mathcal{C}' = \text{Con}^\omega(\mathcal{C}, (C), (n))$ of the space \mathcal{C} . Applying Lemma 3.14 to \mathcal{C} , we get that $(C)^\omega$ is a cut-point in \mathcal{C}' . Since \mathcal{C}' is an asymptotic cone of X [DS05], X is not wide. \square

Lemma 3.17. (i) *Let X be a geodesic metric space. If there exists $\delta \in (0, 1)$ and $\gamma \geq 0$ such that the function $\text{Div}_\gamma(n; \delta)$ is bounded by a linear function then X is wide.*

- (ii) *Let X be a periodic geodesic metric space which contains a bi-infinite quasi-geodesic. If X is wide then for every $0 < \delta < \frac{1}{54}$ and every $\gamma \geq 0$, the function $\text{Div}_\gamma(n; \delta)$ is bounded by a linear function.*

Proof. (i) Suppose that X is not wide, and $\mathcal{C} = \text{Con}^\omega(X, (o_n), (d_n))$ has a cut-point. Then \mathcal{C} has a closed ball $\overline{B}(C, 0)$ (where $C = (c_n)^\omega$), separating a pair of points $A = (a_n)^\omega$ and $B = (b_n)^\omega$. By Lemma 3.14, for every $\delta > 0$ and $\gamma \geq 0$ the limit $\lim_\omega \frac{\text{div}_\gamma(a_n, b_n, c_n; \delta)}{d_n}$ is ∞ . Since ω -almost surely $\text{dist}(a_n, b_n) < \kappa d_n$ for some κ , $\text{Div}_\gamma(\kappa d_n, \delta)$ is not bounded by any linear function in d_n .

(ii) Suppose that for some $\delta < \frac{1}{54}$ and $\gamma \geq 0$ the function $\text{Div}_\gamma(n, \delta)$ is superlinear. By (3) and (6), the same holds for $\text{div}'_\gamma(n; 4, 9\delta)$, hence there exist $d_n \geq n$ such that $\text{div}'_\gamma(d_n; 4, 9\delta) \geq 2nd_n$. Consequently, there exists a sequence of triples of points a_n, b_n, c_n , where c_n is on a geodesic $[a_n, b_n]$, $\text{dist}(c_n, \{a_n, b_n\}) \geq \frac{\text{dist}(a_n, b_n)}{4}$, such that $\text{dist}(a_n, b_n) \leq d_n$ and $\frac{\text{div}_\gamma(a_n, b_n, c_n; 9\delta)}{d_n}$ is at least n . We can assume without loss of generality that $\text{dist}(a_n, b_n) = d_n$. In the asymptotic cone $\text{Con}^\omega(X, (c_n), (d_n))$ of X , the distance between $A = (a_n)^\omega$ and $B = (b_n)^\omega$ is 1 and the ball $\bar{B} = \bar{B}(C, \frac{9\delta}{2})$ separates A from B (here $C = (c_n)^\omega$) by Lemma 3.14. By Lemma 3.16, X is not wide since $\text{dist}(C, \{A, B\}) \geq \frac{1}{4} > 3 \cdot \frac{9\delta}{2}$. The lemma is proved. \square

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