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TANGENT BUNDLE OF A COMPLETE INTERSECTION

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Abstract. Let X be a Fano variety of Picard number one defined over an algebraically closed field. We give conditions under which the tangent bundle of a complete intersection on X is stable or strongly stable.

1. Introduction

Let X be an irreducible smooth projective variety, defined over an algebraically closed field k, such that $\dim X =: n_0 > 2$, and the anti–canonical line bundle K_X^{-1} is ample. The characteristic of k will be denoted by p (we allow $p = 0$). We impose the following conditions on X:

- The tangent bundle TX is semistable, and if the characteristic of k is positive, then TX is strongly semistable.
- The rank of the Néron-Severi group $\mathrm{NS}(X)$ is one.
- If the characteristic p of k is positive, then X lifts to a smooth proper scheme over the ring of Witt vectors of length two over k, and also $p \geq n_0$.

Since the rank of $\mathrm{NS}(X)$ is one, the stability and semistability conditions for vector bundles over X do not depend on the choice of the polarization on X.

Fix an ample line bundle ξ over X such that ξ generates the torsionfree part of $\mathrm{NS}(X)$. This condition determines ξ uniquely up to tensoring with a numerically trivial line bundle. Define $\tau(X)$ to be the index of X. Therefore, the image of K_X^{-1} in the torsionfree part of $\mathrm{NS}(X)$ is $\tau(X)$–times the image of ξ.

We prove the following theorem (see Theorem 4.1 and Proposition 4.2):

Theorem 1.1. *Let $H \subset X$ be a reduced smooth hypersurface from the complete linear system $|\xi^{\otimes d}|$ such that the restriction homomorphism $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(H)$ is an isomorphism. Assume that*

$$d > \frac{\tau(X)(n_0 - 2)}{2n_0 - 3}\,.$$

If $\tau(X) \neq d$, then the tangent bundle TH of H is stable.

If the characteristic of k is positive and $\tau(X) > d > \frac{\tau(X)(n_0-2)}{2n_0-3}$, then TH is in fact strongly stable.

If $d = \tau(X)$, then TH is semistable. If the characteristic of k is positive and $\tau(X) = d$, then TH is strongly semistable.

Theorem 1.1 can be extended to complete intersections using induction on the number of hyperplanes. We prove the following theorem (see Theorem 5.1).

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Theorem 1.2. *Take any integer $\ell \geq 2$ such that $n_0 - \ell \geq 2$. For each $i \in [1, \ell]$, let Z_i be an irreducible smooth hypersurface on X from the complete linear system $|\xi^{\otimes d_i}|$ such that*

$$\sum_{i=1}^{\ell-1} d_i < \tau(X)$$

and $d_\ell \neq \tau(X) - \sum_{i=1}^{\ell-1} d_i$. Assume that for each $j \in [1, \ell]$, the subvariety

$$\widehat{Z}_j := \bigcap_{i=1}^{j} Z_i \subset M$$

is a smooth complete intersection of codimension j. Also, assume the following:

(1) *the inclusion $\widehat{Z} := \widehat{Z}_\ell \hookrightarrow M$ induces an isomorphism of Picard groups (this condition is automatically satisfied when $n_0 - \ell \geq 3$) and*
(2) *the inequality*

$$d_j > \frac{(\tau(X) - \sum_{i=0}^{j-1} d_i)(n_0 - j - 1)}{2(n_0 - j) - 1}$$

holds for each $j \in [1, \ell]$ with the convention $d_0 = 0$.

Then the tangent bundle of $\widehat{Z}$ is stable.

If $d_\ell = \tau(X) - \sum_{i=1}^{\ell-1} d_i$ (the remaining conditions remain unchanged), then $T\widehat{Z}$ is semistable.

If the characteristic of k is positive and $\sum_{i=1}^{\ell} d_i < \tau(X)$, then $T\widehat{Z}$ is in fact strongly stable.

Similarly, if the characteristic of k is positive and $\sum_{i=1}^{\ell} d_i = \tau(X)$, then $T\widehat{Z}$ is strongly semistable.

2. Preliminaries

As before, k is an algebraically closed field of characteristic p and X is an irreducible smooth projective variety defined over k of dimension n_0, with $n_0 \geq 3$. Fix an ample line bundle ξ on X.

Take any positive integer r such that $\xi^{\otimes r}$ is very ample. For a torsionfree coherent sheaf V over X, consider the degree of the restriction of V to a general complete intersection curve obtained by intersecting $n_0 - 1$ hypersurfaces in X from the complete linear system $|\xi^{\otimes r}|$. We will denote this integer by $\delta(V, r)$. Now define the *degree* of V as

$$\text{degree}(V) := \delta(V, r)/r^{n_0 - 1} \in \mathbb{Q},$$

which is in fact independent of the choice of r.

Let

$$H \overset{\iota}{\hookrightarrow} X$$

be any reduced smooth hypersurface. Take any positive integer r such that $\iota^*\xi^{\otimes r}$ is very ample. For a torsionfree coherent sheaf V on H, let $\delta_H(V, r)$ denote the degree of the restriction of V to a general complete intersection curve obtained by intersecting hypersurfaces in H from the complete linear system $|\iota^*\xi^{\otimes r}|$ on H. It is easy to see that $\delta_H(V, r)/r^{n_0 - 2}$ is independent of the choice of r (recall that $n_0 \geq 3$). We define the degree of V as

$$\text{degree}(V) := \delta_H(V, r)/r^{n_0 - 2} \in \mathbb{Q}.$$

A torsionfree coherent sheaf V is called *stable* (respectively, *semistable*) if for all coherent subsheaves F of V with $0 < \mathrm{rank}(F) < \mathrm{rank}(V)$, the inequality

$$\mu(F) := \mathrm{degree}(F)/\mathrm{rank}(F) < \mathrm{degree}(V)/\mathrm{rank}(V) =: \mu(V)$$

(respectively, $\mu(F) \leq \mu(V)$) holds.

If the characteristic of the base field k is positive, then by F_X we will denote the absolute Frobenius morphism $X \longrightarrow X$. For any integer $j \geq 1$, let

$$F_X^j := F_X \circ \cdots \circ F_X : X \longrightarrow X$$

be the j–fold composition of the self–map F_X of X.

A vector bundle V over X is called *strongly stable* (respectively, *strongly semistable*) if $(F_X^j)^*V$ is stable (respectively, semistable) for all j. Strongly stable (respectively, strongly semistable) vector bundles over a smooth hypersurface in X are defined similarly.

The *degree* of a hypersurface $H \subset X$ is defined to be

$$\mathrm{degree}(H) := \mathrm{degree}(\mathcal{O}_X(H))/\mathrm{degree}(\xi)\,.$$

For the sake of convenience, we will employ the following notation:

For a coherent sheaf V over X, and for any integer m, the coherent sheaf $V \otimes \xi^{\otimes m}$ will also be denoted by $V(m)$. Similarly, for any coherent sheaf W over a smooth hypersurface $H \overset{\iota}{\hookrightarrow} X$ and any integer m, the coherent sheaf $W \otimes \iota^*\xi^{\otimes m}$ over H will be denoted by $W(m)$.

3. Tangent bundle of a hypersurface

We continue with the notation of the previous section. The following assumptions on X will be invoked throughout.

Assumptions. *The variety X is assumed to satisfy the following conditions:*

1. *The anti–canonical line bundle K_X^{-1} is ample.*
2. *The tangent bundle TX is semistable, and if the characteristic of k is positive, then TX is strongly semistable.*
3. *The rank of the Néron-Severi group $NS(X)$ is one.*
4. *If the characteristic of k is positive, then X lifts to a smooth proper scheme over the ring of Witt vectors of length two over k, and also $p \geq n_0 = \dim X$.*

The ring of Witt vectors of length two over k is denoted by $W_2(k)$ (see [DI]).

Proposition 3.1. *Assume that the characteristic of p of k is positive. Let $D \subset X$ be a reduced irreducible smooth hypersurface. Then D lifts to a smooth proper scheme over $W_2(k)$.*

Proof. Let $\widetilde{X}$ be a smooth proper scheme over $W_2(k)$ lifting X. (We already assumed that X lifts to $W_2(k)$.) We have a natural short exact sequence of sheaves on $\widetilde{X}$,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\widetilde{X}}^* \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

Using this it follows that the obstruction for lifting a line bundle over X to $\widetilde{X}$ lies in $H^2(X, \mathcal{O}_X)$.

We have assumed that K_X^{-1} is ample and $p \geq \dim X = n_0 \geq 3$. Hence a theorem due to Deligne, Illusie and Raynaud says that

$$H^2(X, \mathcal{O}_X) = H^{n_0-2}(X, K_X)^* = 0$$

(see [DI, p. 257, Corollaire 2.8]). Consequently, any line bundle over X lifts to a line bundle over $\widetilde{X}$.

Let $L = \mathcal{O}_X(D)$ be the line bundle over X associated to D. Let

$$\sigma \in H^0(X, L) \tag{3.1}$$

be the image of the constant function 1 in $H^0(X, \mathcal{O}_X(D))$.

We have shown that any line bundle over X lifts to a line bundle over $\widetilde{X}$. Let $\widetilde{L}$ be a line bundle over $\widetilde{X}$ that lifts L. The proof of the proposition will be completed by showing that σ in (3.1) lifts to a section of $\widetilde{L}$.

From the short exact sequence of sheaves on $\widetilde{X}$,

$$0 \longrightarrow L \longrightarrow \widetilde{L} \longrightarrow L \longrightarrow 0,$$

it follows that the obstruction for lifting a section of L to a section of $\widetilde{L}$ lies in $H^1(X, L)$.

Since $\mathrm{rank}(\mathrm{NS}(X)) = 1$, any hypersurface on X is ample. Consequently, the line bundle $L \otimes K_X^{-1}$ is ample. Hence from the theorem of Deligne, Illusie and Raynaud we conclude that

$$H^1(X, L) = H^1(X, L \otimes K_X^{-1} \otimes K_X) = 0\,.$$

Therefore, the section σ in (3.1) lifts to a section of $\widetilde{L}$. This completes the proof of the proposition. □

We choose the ample line bundle ξ in Section 2 such that ξ generates the torsionfree part of $\mathrm{NS}(X)$ (recall that the torsionfree part of $\mathrm{NS}(X)$ is by assumption isomorphic to $\mathbb{Z}$). We note that ξ is uniquely determined up to tensoring with a numerically trivial line bundle.

Define

$$\tau(X) := \mathrm{degree}(K_X^{-1})/\mathrm{degree}(\xi) \in \mathbb{N}^+ \tag{3.2}$$

(it is an integer since ξ generates the torsionfree part of $\mathrm{NS}(X)$).

Lemma 3.2. *Let $H \subset X$ be a reduced irreducible smooth hypersurface from the complete linear system $|\xi^{\otimes d}|$. Fix integers $\ell, j \in \mathbb{Z}$ such that $\ell + d > 0$ and $0 < j < \frac{n_0(\tau(X)+\ell)}{\tau(X)} - 1$, where $\tau(X)$ is defined in* (3.2)*. For any line bundle L over X of degree zero,*

$$H^{n_0-1}(H, \Omega_H^j(\ell) \otimes L) = 0\,.$$

Proof. Let

$$\iota : H \hookrightarrow X$$

be the inclusion map. Since $H \in |\xi^{\otimes d}|$, we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_H(-d) \longrightarrow \iota^*\Omega_X^1 \longrightarrow \Omega_H^1 \longrightarrow 0 \tag{3.3}$$

on H. Taking exterior powers, this exact sequence yields a short exact sequence of vector bundles

$$0 \longrightarrow \Omega^j_H(-d) \longrightarrow \iota^*\Omega^{j+1}_X \longrightarrow \Omega^{j+1}_H \longrightarrow 0 \tag{3.4}$$

for all $j \geq 0$.

Let L be a line bundle over X with degree$(L) = 0$. The condition that the rank of NS(X) is one implies that L is numerically trivial. Let

$$\overline{L} := \iota^* L \tag{3.5}$$

be the pull back to H. Tensoring this exact sequence in (3.4) with $\mathcal{O}_H(\ell + d) \otimes \overline{L}$ (see (3.5)), we obtain the short exact sequence of vector bundles

$$0 \longrightarrow \Omega^j_H(\ell) \otimes \overline{L} \longrightarrow \iota^*\Omega^{j+1}_X(\ell + d) \otimes \overline{L} \longrightarrow \Omega^{j+1}_H(\ell + d) \otimes \overline{L} \longrightarrow 0 \tag{3.6}$$

on H. Let

$$\begin{aligned} &\longrightarrow H^{n_0-2}(H, \Omega^{j+1}_H(\ell + d) \otimes \overline{L}) \longrightarrow H^{n_0-1}(H, \Omega^j_H(\ell) \otimes \overline{L}) \\ &\longrightarrow H^{n_0-1}(H, \iota^*\Omega^{j+1}_X(\ell + d) \otimes \overline{L}) \end{aligned} \tag{3.7}$$

be the corresponding long exact sequence of cohomologies; as before, $n_0 = \dim X$.

If the characteristic p of the base field k is positive, we know from Proposition 3.1 that the hypersurface H lifts to a smooth proper scheme over $W_2(k)$. The line bundle $\iota^*(\xi^{\otimes(\ell+d)} \otimes L)$ is ample because L in (3.5) is numerically trivial, and we have $n_0 - 2 + j + 1 > \dim Z$ because $j > 0$. Therefore, using the Akizuki–Nakano vanishing theorem (see [Kob, p. 74, Theorem 3.11]) and its positive characteristic version due to Deligne, Illusie and Raynaud (see [DI, p. 257, Corollaire 2.8]), we conclude that

$$H^{n_0-2}(H, \Omega^{j+1}_H(\ell + d) \otimes \overline{L}) = 0\,. \tag{3.8}$$

We now consider the short exact sequence of sheaves on X,

$$0 \longrightarrow \Omega^{j+1}_X(\ell) \otimes L \longrightarrow \Omega^{j+1}_X(\ell + d) \otimes L \longrightarrow \iota_*\iota^*(\Omega^{j+1}_X(\ell + d) \otimes L) \longrightarrow 0 \tag{3.9}$$

(see (3.5)), which is obtained by tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_H \longrightarrow 0$$

with $\Omega^{j+1}_X(\ell + d) \otimes L$ and the corresponding long exact sequence of cohomologies

$$\begin{aligned} H^{n_0-1}(X, \Omega^{j+1}_X(\ell + d) \otimes L) &\longrightarrow H^{n_0-1}(H, \iota^*(\Omega^{j+1}_X(\ell + d) \otimes L)) \\ &\longrightarrow H^{n_0}(X, \Omega^{j+1}_X(\ell) \otimes L)\,. \end{aligned} \tag{3.10}$$

The line bundle $\xi^{\otimes(\ell+d)} \otimes L$ is ample, and $n_0 - 1 + j + 1 > \dim X$ because $j > 0$. Therefore, from the Akizuki–Nakano vanishing theorem and its positive characteristic version due to Deligne, Illusie and Raynaud (see [DI, p. 257, Corollaire 2.8]), we know that

$$H^{n_0-1}(X, \Omega^{j+1}_X(\ell + d) \otimes L) = 0\,. \tag{3.11}$$

We will show that $H^{n_0}(X, \Omega^{j+1}_X(\ell) \otimes L)$ in (3.10) vanishes.

Note that degree$(K_X) = -\tau(X) \cdot$ degree(ξ). Since NS(X) is of rank one, we know that

$$K_X = \mathcal{O}_X(-\tau(X)) \otimes L_0\,, \tag{3.12}$$

where L_0 is a numerically trivial line bundle over X.

Recall that TX is semistable, and if the characteristic of k is positive, then TX is strongly semistable. Therefore, the vector bundle $\bigwedge^{j+1} TX$ is semistable [RR, p. 285, Theorem 3.18]. Hence

$$(\bigwedge^{j+1} TX)(-\ell - \tau(X)) \otimes L^* \otimes L_0 \,=\, \xi^{\otimes(-\ell-\tau(X))} \otimes L^* \otimes L_0 \otimes \bigwedge^{j+1} TX$$

is semistable, where $\tau(X)$ is defined in (3.2) and L_0 is the line bundle in (3.12). From (3.12) we have

$$(\Omega_X^{j+1}(\ell) \otimes L)^* \otimes K_X \,=\, (\bigwedge^{j+1} TX)(-\tau(X) - \ell) \otimes L^* \otimes L_0\,.$$

Hence the Serre duality gives

$$H^{n_0}(X,\, \Omega_X^{j+1}(\ell) \otimes L) \,=\, H^0(X,\, (\bigwedge^{j+1} TX)(-\tau(X) - \ell) \otimes L^* \otimes L_0)^*\,.$$

A semistable vector bundle of negative degree does not admit any nonzero sections. Therefore, to prove that $H^{n_0}(X,\, \Omega_X^{j+1}(\ell) \otimes L) \,=\, 0$ it suffices to show that

$$\text{degree}(\bigwedge^{j+1} TX(-\tau(X) - \ell)L^* \otimes L_0) \,=\, \text{degree}(\bigwedge^{j+1} TX(-\tau(X) - \ell)) \,<\, 0 \tag{3.13}$$

(recall that $\text{degree}(L^* \otimes L_0) \,=\, 0$). It is easy to see that

$$\mu(\bigwedge^{j+1} TX(-\tau(X) - \ell)) = \,((j+1)\tau(X)/n_0 - (\tau(X) + \ell)) \cdot \text{degree}(\xi) \tag{3.14}$$

for all $j \,\in\, [0\,, n_0]$.

We have $\Omega_H^i \,=\, 0$ for all $i \,>\, \dim H \,=\, n_0 - 1$. Therefore, the lemma is automatically valid if $j \,>\, n_0 - 1$. Hence we assume that $j \,\leq\, n_0 - 1$.

From the given inequality $j \,<\, n_0(\tau(X) + \ell)/\tau(X) - 1$ in the lemma we know that

$$(j+1)\tau(X)/n_0 - (\tau(X) + \ell) \,<\, 0\,.$$

Since $\text{degree}(\xi) \,>\, 0$, from (3.14) we have $\mu(\bigwedge^{j+1} TX(-\tau(X) - \ell)) \,<\, 0$. Therefore, the inequality (3.13) holds, implying $H^{n_0}(X,\, \Omega_X^{j+1}(\ell) \otimes L) \,=\, 0$ (as noted before). Substituting this and (3.11) the exact sequence in (3.10), we conclude that

$$H^{n_0-1}(H,\, \iota^*(\Omega_X^{j+1}(\ell + d) \otimes L)) \,=\, 0\,. \tag{3.15}$$

Finally, substituting (3.15) and (3.8) in the exact sequence in (3.7), we conclude that $H^{n_0-1}(H,\, \Omega_H^j(\ell)) \,=\, 0$. This completes the proof of the lemma. □

4. Stability of the tangent bundle of a hypersurface

We continue with the notation of the previous section. As in Lemma 3.2, let

$$\iota \,:\, H \,\longrightarrow\, X$$

be a reduced smooth hypersurface. From Grothendieck's Lefschetz theory it follows that the homomorphism

$$\iota^\sharp \,:\, \text{Pic}(X) \,\longrightarrow\, \text{Pic}(H) \tag{4.1}$$

defined by $\zeta \,\longmapsto\, \iota^*\zeta$ is injective. Furthermore, the homomorphism $\iota^\sharp$ is an isomorphism if $\dim H \,>\, 2$; see [Gr, Exposé X].

Theorem 4.1. *Let*

$$\iota : H \longrightarrow X$$

be a reduced smooth hypersurface from the complete linear system $|\xi^{\otimes d}|$ *such that the pull back homomorphism* $\iota^\sharp$ *in* (4.1) *is an isomorphism. Assume that*

$$d > \tau(X)(n_0 - 2)/(2n_0 - 3)\,. \tag{4.2}$$

If $\tau(X) \neq d$*, the tangent bundle* TH *of* H *is stable. If* $d = \tau(X)$*, then* TH *is semistable.*

Proof. Assume that TH is not stable. Let

$$E \subset TH \tag{4.3}$$

be a subsheaf such that $1 \leq \operatorname{rank}(E) < n_0 - 1 = \dim H$ and $\mu(E) \geq \mu(TH)$. Define

$$\delta := \operatorname{degree}(E)/\operatorname{degree}(\iota^*\xi) \in \mathbb{Z} \tag{4.4}$$

and $j := \operatorname{rank}(E)$. Therefore, the above inequality gives

$$\frac{\operatorname{degree}(E)}{\operatorname{degree}(\iota^*\xi)\cdot\operatorname{rank}(E)} = \frac{\delta}{j} \geq \frac{\tau(X)-d}{n_0-1} = \frac{\operatorname{degree}(TH)}{\operatorname{degree}(\iota^*\xi)\cdot\operatorname{rank}(TH)}\,, \tag{4.5}$$

where $\tau(X)$ is defined in (3.2).

The homomorphism $\iota^\sharp$ in (4.1) is an isomorphism, and the rank of $\mathrm{NS}(X)$ is one. Hence from the definition of δ (see (4.4)) it follows that $\bigwedge^j E = \mathcal{O}_H(\delta)\otimes\iota^*L_1$, where L_1 is a line bundle over X of degree zero. Consequently, the subsheaf E in (4.3) defines a nonzero section

$$0 \neq \sigma \in H^0(H, (\bigwedge^j E)^* \otimes \bigwedge^j TH) = H^0(H, (\bigwedge^j TH)(-\delta)\otimes\iota^*L_1^*)\,. \tag{4.6}$$

From the Poincaré adjunction formula we have

$$K_H = \iota^*K_X(d)$$

(see (3.3)). Hence from (3.12) it follows that

$$K_H = \mathcal{O}_H(d-\tau(X))\otimes\iota^*L_0\,. \tag{4.7}$$

Therefore, the Serre duality gives

$$H^0(H, (\bigwedge^j TH)(-\delta)\otimes\iota^*L_1^*) = H^{n_0-1}(H, \Omega^j_H(\delta-\tau(X)+d)\otimes\iota^*(L_1\otimes L_0))^*\,. \tag{4.8}$$

First assume that $d > \tau(X)$. We have $1 \leq j < n_0 - 1$. Hence from (4.5) it follows that $\delta/(\tau(X)-d) \leq j/(n_0-1) < 1$. Hence $\delta - \tau(X) + d > 0$. In other words, the line bundle $\mathcal{O}_H(\delta-\tau(X)+d)\otimes\iota^*(L_1\otimes L_0)$ is ample. (Recall that $L_1\otimes L_0$ is numerically trivial.) Therefore, from the Akizuki–Nakano vanishing theorem and its positive characteristic version due to Deligne, Illusie and Raynaud, it follows that

$$H^{n_0-1}(H, \Omega^j_H(\delta-\tau(X)+d)\otimes\iota^*(L_1\otimes L_0)) = 0\,. \tag{4.9}$$

Hence from (4.8),

$$H^0(H, (\bigwedge^j TH)(-\delta)\otimes\iota^*L_1^*) = 0\,. \tag{4.10}$$

But this contradicts the condition that the section σ in (4.6) is nonzero.

Consequently, the tangent bundle TH is stable if $d > \tau(X)$.

Now assume that $d = \tau(X)$.

Since $d = \tau(X)$, we have $\delta - \tau(X) + d = \delta$. On the other hand, from (4.5) we have $\delta \geq 0$. Hence

$$\delta - \tau(X) + d \geq 0\,. \tag{4.11}$$

First assume that $\delta - \tau(X) + d > 0$.

We have noted above that (4.9) holds if $\delta - \tau(X) + d > 0$. Hence from (4.8) we conclude that (4.10) is valid. But this contradicts the condition that $\sigma \neq 0$.

Now we assume that $\delta = 0$.

Since $d = \tau(X)$, from (4.7) we know that degree$(TH) = 0$. Hence degree$(E) =$ degree$(TH) = 0$. This immediately implies that TH is semistable.

Therefore, if $d = \tau(X)$, then TH is semistable.

Henceforth, in the proof of the theorem we will assume that $d < \tau(X)$.

In view of Lemma 3.2, to prove the theorem it suffices to show that the following two inequalities hold:

$$\delta - \tau(X) + d + d = \delta - \tau(X) + 2d > 0 \tag{4.12}$$

and

$$j < n_0(\tau(X) + \delta - \tau(X) + d)/\tau(X) - 1 = n_0(\delta + d)/\tau(X) - 1\,. \tag{4.13}$$

Indeed, if both (4.12) and (4.13) are valid, then Lemma 3.2 says that

$$H^{n_0-1}(H, \Omega^j_H(\delta - \tau(X) + d) \otimes \iota^*(L_1 \otimes L_0)) = 0\,.$$

Hence using (4.8) it follows that there is no nonzero section σ as in (4.6).

From (4.2) we have

$$(\tau(X) - d)(n_0 - 1) > \tau(X) - 2d\,.$$

Also,

$$j(\tau(X) - d)/(n_0 - 1) \geq \tau(X) - d/(n_0 - 1)$$

as $j \geq 1$ and $\tau(X) > d$. Hence

$$j(\tau(X) - d)/(n_0 - 1) > \tau(X) - 2d\,.$$

Now using (4.5) we conclude that the inequality in (4.12) holds.

We will now prove the inequality in (4.13). In view of (4.5), to prove (4.13) it is enough to show that

$$(j + 1)\tau(X)/n_0 - d < j(\tau(X) - d)/(n_0 - 1)\,. \tag{4.14}$$

The inequality in (4.14) is equivalent to the inequality

$$(j + 1)(d(n_0 - 1) + d - \tau(X)) < n_0(d(n_0 - 1) + d - \tau(X))\,. \tag{4.15}$$

Indeed, both of these inequalities are equivalent to the inequality

$$j(\tau(X) - d)n_0 + dn_0(n_0 - 1) - (j + 1)(n_0 - 1)\tau(X) > 0\,.$$

Since we have $j < n_0 - 1$, the inequality in (4.15) holds provided $d(n_0 - 1) + d - \tau(X) > 0$.

A theorem due to Kobayashi and Ochiai in [KO] says that

$$\tau(X) \leq n_0 + 1\,. \tag{4.16}$$

From [Kol, p. 243, Theorem 1.6.1] we know that the Fano variety X admits a rational curve C such that $-K_X \cdot C \leq n_0 + 1$. This immediately implies that $\tau(X) \leq n_0 + 1$ (see the proof of Theorem 1.11.1 in [Kol, p. 245]).

Using (4.16) we conclude that

$$d(n_0 - 1) + d - \tau(X) > 0 \tag{4.17}$$

if $d \geq 2$. Since $n_0 > j > 0$, we conclude from (4.17) the following: the inequality in (4.15) holds if $d \geq 2$. Hence (4.14) holds if $d \geq 2$. Therefore, (4.13) holds if $d \geq 2$.

Henceforth assume that $d = 1$.

Since $d = 1$, from the given inequality $d > \tau(X)(n_0 - 2)/(2n_0 - 3)$ in (4.2) and the assumption that $n_0 \geq 3$, it follows that

$$2 \geq \tau(X)\,.$$

Since $\tau(X) \geq 1$, we have

$$2 \geq \tau(X) \geq 1\,.$$

Consequently,

$$d(n_0 - 1) + d - \tau(X) = n_0 - \tau(X) > 0\,.$$

This implies that (4.15) holds because $n_0 > j > 0$.

Hence (4.14) holds if $d = 1$. Therefore, (4.13) holds if $d = 1$. This completes the proof of the theorem. □

Let Y be a smooth projective variety, defined over k, equipped with a polarization. We recall that a vector bundle V over Y is called *strongly stable* (respectively, *strongly semistable*) if $(F_Y^j)^*V$ is stable (respectively, semistable) for all j, where

$$F_Y^j : Y \longrightarrow Y$$

is the j–fold iteration of an absolute Frobenius morphism

$$F_Y : Y \longrightarrow Y$$

for Y.

Proposition 4.2. *Assume that the characteristic p of k is positive. Let*

$$\iota : H \longrightarrow X$$

be a smooth hypersurface of degree d such that the homomorphism $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(H)$ *defined by $\zeta \longmapsto \iota^*\zeta$ is an isomorphism. If*

$$\tau(X) > d > \frac{\tau(X)(n_0 - 2)}{2n_0 - 3}\,,$$

then the tangent bundle TH is strongly stable.

If $d = \tau(X)$, then TH is strongly semistable.

Proof. Let Y be a smooth projective variety, defined over k, equipped with a polarization. Assume that the tangent bundle TY is semistable and $\mathrm{degree}(TY) \geq 0$. Let

$$F_Y : Y \longrightarrow Y$$

be the Frobenius morphism of the variety Y. Then for any semistable vector bundle E over Y, the pull back F_Y^*E is also semistable [MR, p. 316, Theorem 2.1(1)]. Furthermore, if $\mathrm{degree}(TY) > 0$ and E is stable, then the vector bundle F_Y^*E is stable [MR, p. 316, Theorem 2.1(2)].

If $H \subset X$ is a smooth hypersurface of degree d as in the proposition, then

$$\mathrm{degree}(TH) = (\tau(X) - d) \cdot \mathrm{degree}(L)\,.$$

Now the proposition follows from Theorem 4.1 combined with the earlier mentioned results of [MR]. □

5. Stability of the tangent bundle of a complete intersection

Let $\iota : Z \hookrightarrow X$ be a reduced smooth hypersurface from the complete linear system $|\xi^{\otimes d}|$, with $d < \tau(X)$. Assume that $d_0 \geq 4$. If the inequality in (4.2) holds and the homomorphism $\iota^\sharp$ in (4.1) is an isomorphism, then combining Theorem 4.1, Proposition 3.1 and Proposition 4.2, it follows immediately that Z satisfies all the conditions that were imposed on X. Therefore, in that case we may replace X by Z. Now Theorem 4.1 gives a criterion under which a hypersurface in Z has a stable tangent bundle.

We have the following generalization of Theorem 4.1.

Theorem 5.1. *Take any integer $\ell \geq 2$ such that $n_0 - \ell \geq 2$. For each $i \in [1, \ell]$, let Z_i be an irreducible smooth hypersurface on X from the complete linear system $|\xi^{\otimes d_i}|$ such that*

$$\sum_{i=1}^{\ell-1} d_i < \tau(X)$$

and $d_\ell \neq \tau(X) - \sum_{i=1}^{\ell-1} d_i$. Assume that for each $j \in [1, \ell]$, the subvariety

$$\widehat{Z}_j := \bigcap_{i=1}^{j} Z_i \subset M \tag{5.1}$$

is a smooth complete intersection of codimension j. Also, assume that the following two hold:

1. *the inclusion $\widehat{Z} := \widehat{Z}_\ell \hookrightarrow M$ induces an isomorphism of Picard groups (this condition is automatically satisfied when $n_0 - \ell \geq 3$) and*
2. *the inequality*

$$d_j > \frac{(\tau(X) - \sum_{i=0}^{j-1} d_i)(n_0 - j - 1)}{2(n_0 - j) - 1} \tag{5.2}$$

holds for each $j \in [1, \ell]$ with the convention $d_0 = 0$.

Then the tangent bundle of $\widehat{Z}$ is stable.

If $d_\ell = \tau(X) - \sum_{i=1}^{\ell-1} d_i$ (the remaining conditions remain unchanged), then $T\widehat{Z}$ is semistable.

If the characteristic of k is positive and $\sum_{i=1}^{\ell} d_i < \tau(X)$, then $T\widehat{Z}$ is in fact strongly stable.

Similarly, if the characteristic of k is positive and $d_\ell = \tau(X) - \sum_{i=1}^{\ell-1} d_i$, then $T\widehat{Z}$ is strongly semistable.

Proof. Let Z be an irreducible smooth ample hypersurface in a smooth projective variety Y defined over k. Let

$$\mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(Z) \tag{5.3}$$

be the homomorphism given by the inclusion map $Z \hookrightarrow Y$. From Grothendieck's Lefschetz theory it follows that the homomorphism in (5.3) is an isomorphism if $\dim Z \geq 3$; see [Gr, Expose X]. The homomorphism in (5.3) is injective if $\dim Z \geq 2$.

Take any $j \in [1\,,\ell]$. The two homomorphisms

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widehat{Z}_j) \tag{5.4}$$

(see (5.1)) and

$$\mathrm{Pic}(\widehat{Z}_j) \longrightarrow \mathrm{Pic}(\widehat{Z})\,, \tag{5.5}$$

induced by the inclusion maps $\widehat{Z}_j \hookrightarrow X$ and $\widehat{Z} \hookrightarrow \widehat{Z}_j$ respectively, are injective. Condition (1) in the statement of the theorem implies that their composition

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widehat{Z}_j) \longrightarrow \mathrm{Pic}(\widehat{Z})$$

is an isomorphism. Hence both the homomorphisms in (5.4) and (5.5) are in fact isomorphisms.

First assume that $\sum_{i=1}^{\ell} d_i \neq \tau(X)$. Therefore, by Theorem 4.1, the tangent bundle of the hypersurface Z_1 is stable. The given condition $\sum_{i=1}^{\ell-1} d_i < \tau(X)$ ensures that $K_{Z_1}^{-1}$ is ample. The inequality in (5.2) ensures that the inequality in (4.2) holds for the hypersurface $Z_1 \cap Z_2$ of Z_1.

Now substituting the pair $(Z_1\,, Z_1 \cap Z_2)$ for the pair $(X\,, Z)$ in Theorem 4.1 we conclude that the tangent bundle of $Z_1 \cap Z_2$ is stable. Finally, we get inductively that the tangent bundle of $\widehat{Z}_j$ is stable for all $j \in [1\,,\ell]$. The given condition $\sum_{i=1}^{\ell-1} d_i < \tau(X)$ implies that $K_{\widehat{Z}_j}^{-1}$ is ample for all $j \in [1\,,\ell-1]$. The inequality in (5.2) ensures that the inequality in (4.2) holds at each pair $(\widehat{Z}_j\,, \widehat{Z}_{j+1})$, where $j \in [1\,,\ell-1]$.

If the characteristic of k is positive, then by using Proposition 4.2 instead of Theorem 4.1 in the above argument, we conclude that $T\widehat{Z}_j$ is strongly stable for all $j \in [1\,,\ell]$. Note that the condition $\sum_{i=1}^{\ell} d_i < \tau(X)$ ensures that $K^{-1}\widehat{Z}_j$ is ample for all $j \in [1\,,\ell]$.

Now assume that $\sum_{i=1}^{\ell} d_i = \tau(X)$. In view of the second part of Theorem 4.1 which asserts that TZ is semistable, the above argument for the stability of $T\widehat{Z}_j$ shows that $T\widehat{Z}_j$ is stable for all $j \in [1\,,\ell-1]$, and $T\widehat{Z}_\ell$ is semistable.

Similarly, if the characteristic of k is positive and $d_\ell = \tau(X) - \sum_{i=1}^{\ell-1} d_i$, using the first part of Proposition 4.2 we conclude that $T\widehat{Z}_j$ is strongly stable for all $j \in [1\,,\ell-1]$, and using the second part of Proposition 4.2 it follows that $T\widehat{Z}_\ell$ is strongly semistable. This completes the proof of the theorem. □

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