

EQUIVARIANT SPECTRAL TRIPLES AND POINCARÉ DUALITY FOR $SU_q(2)$

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ABSTRACT. Let \mathcal{A} be the C^* -algebra associated with $SU_q(2)$, let π be the representation by left multiplication on the L_2 space of the Haar state and let D be the equivariant Dirac operator for this representation constructed by the authors earlier. We prove in this article that there is no operator other than the scalars in the commutant $\pi(\mathcal{A})'$ that has bounded commutator with D . This implies that the equivariant spectral triple under consideration does not admit a rational Poincaré dual in the sense of Moscovici, which in particular means that this spectral triple does not extend to a K -homology fundamental class for $SU_q(2)$. We also show that a minor modification of this equivariant spectral triple gives a fundamental class and thus implements Poincaré duality.

1. INTRODUCTION

In noncommutative geometry (NCG), spaces are described by a triple $(\mathcal{A}, \mathcal{H}, D)$, where \mathcal{A} is a $*$ -algebra closed under holomorphic functional calculus acting on a complex separable Hilbert space \mathcal{H} and D is an unbounded selfadjoint operator with compact resolvent that has bounded commutators with elements from the algebra \mathcal{A} . Such a triple is called a spectral triple. In this spectral point of view, one requires D to be nontrivial in the sense that the associated Kasparov module should give a nontrivial element in K -homology. One can also formulate the notion of Poincaré duality in this context. A pair of separable C^* -algebras (A, B) is said to be a **Poincaré dual pair** if there exist a class $\Delta \in KK(A \otimes B, \mathbb{C})$ and a class $\delta \in KK(\mathbb{C}, A \otimes B)$ with the properties $\delta \otimes_B \Delta = id_A \in KK(A, A)$ and $\delta \otimes_A \Delta = id_B \in KK(B, B)$. The element Δ is called a K -homology fundamental class for the pair (A, B) . Poincaré duality is said to hold for a separable C^* -algebra A if there is a K -homology fundamental class for the pair (A, A) . See section 4, chapter 6 in [6] for a detailed formulation, and [12] for an interesting application.

The existence of a fundamental class can often be deduced from abstract KK -theory arguments, using the properties of the C^* -algebra in question. But more interesting from the point of view of noncommutative geometry is an explicit geometric realization of this fundamental class, with possibly other nice features. An explicit geometric realization of a K -homology class is given by a spectral triple. Suppose we have a spectral triple (\mathcal{H}, π, D) for a C^* -algebra A , where π is faithful.

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If there is another faithful representation π' of A on \mathcal{H} such that

- (1) π' and π commute,
- (2) (\mathcal{H}, π', D) is a spectral triple for A , and
- (3) $(\mathcal{H}, \pi \otimes \pi', D)$ gives a K -homology fundamental class for $A \otimes A$,

then we say that the spectral triple (\mathcal{H}, π, D) extends to a K -homology fundamental class. If one replaces condition 3 above with a slightly weaker condition, then one says that the spectral triples (\mathcal{H}, π, D) and (\mathcal{H}, π', D) are rational Poincaré duals (see [12] for this notion). In an earlier paper ([3]), the authors constructed an equivariant spectral triple for the quantum $SU(2)$ group that was later analysed further by Connes in [8]. It is natural to ask whether the triple gives rise to a fundamental class for $SU_q(2)$. This is what we try to answer in this paper.

Let h be the Haar state for the quantum $SU(2)$ group and let π be the representation of $C(SU_q(2))$ on $L_2(h)$ by left multiplication. In section 2, we make a detailed analysis of the operators α and β on $L_2(h)$. We also introduce and study two operators $\hat{\alpha}$ and $\hat{\beta}$ that are compact perturbations of α and β , respectively, and obey the same commutation relations as α and β . These play an important role in the proof of the main result in section 4. In section 3, we compute the modular conjugation operator associated with the Haar state. This helps us describe elements of the commutator in terms of elements of the strong closure of $\pi(C(SU_q(2)))$. Denote by D the equivariant Dirac operator constructed by the authors in [3]. In section 4, we prove that there is no operator other than the scalars in the commutant of $\pi(C(SU_q(2)))$ that has bounded commutator with D . An important consequence of this is that the equivariant spectral triple does not give a K -homology fundamental class for $SU_q(2)$. In the final section, we show that Poincaré duality holds for $SU_q(2)$. We also give an explicit construction of a spectral triple that gives a fundamental class for $SU_q(2)$.

2. CLOSER LOOK AT THE L_2 SPACE

In what follows, we will be concerned with the quantum $SU(2)$ group, the spectral triple under consideration being the equivariant spectral triple constructed by the authors in [3]. To fix notation, let us recall a few things from that paper. Let q be a real number in the interval $(0, 1)$. Let \mathcal{A} denote the C^* -algebra of continuous functions on $SU_q(2)$, which is the universal C^* -algebra generated by two elements α and β subject to the relations

$$(2.1) \quad \alpha^* \alpha + \beta^* \beta = I = \alpha \alpha^* + q^2 \beta \beta^*, \quad \alpha \beta - q \beta \alpha = 0 = \alpha \beta^* - q \beta^* \alpha, \quad \beta^* \beta = \beta \beta^*.$$

Let h denote the Haar state on \mathcal{A} and let $\pi : \mathcal{A} \rightarrow \mathcal{L}(L_2(h))$ be the representation given by left multiplication by elements in \mathcal{A} . We will often identify an element $a \in \mathcal{A}$ with $\pi(a)$. α_r and β_r will stand for α^r and β^r , respectively, if $r \geq 0$, and for $(\alpha^*)^{-r}$ and $(\beta^*)^{-r}$ if $r < 0$. Let D be the operator given by $D : e_{ij}^{(n)} \mapsto d(n, i) e_{ij}^{(n)}$, where

$$(2.2) \quad d(n, i) = \begin{cases} 2n + 1 & \text{if } n \neq i, \\ -(2n + 1) & \text{if } n = i. \end{cases}$$

Then $(L_2(h), \pi, D)$ is an odd equivariant spectral triple of dimension 3 and with nontrivial K -homology class.

Our objective is to study commutators of the form $[D, T']$ with T' coming from the commutant $(\pi(\mathcal{A}))'$. Any such T' can be written as JTJ , where J is the

modular conjugation operator associated with the Haar state and T comes from the strong closure $(\pi(\mathcal{A}))''$ of $\pi(\mathcal{A})$. With this in mind, in this section we study the structures of the operators that constitute $(\pi(\mathcal{A}))''$. Recall (cf. [3]) that $L_2(h)$ has a natural orthonormal basis $\{e_{ij}^{(n)} : n \in \frac{1}{2}\mathbb{N}, i, j = -n, -n + 1, \dots, n\}$, and the left multiplication operators in this basis are given by

$$(2.3) \quad \alpha : e_{ij}^{(n)} \mapsto a_+(n, i, j)e_{i-\frac{1}{2}, j-\frac{1}{2}}^{(n+\frac{1}{2})} + a_-(n, i, j)e_{i-\frac{1}{2}, j-\frac{1}{2}}^{(n-\frac{1}{2})},$$

$$(2.4) \quad \beta : e_{ij}^{(n)} \mapsto b_+(n, i, j)e_{i+\frac{1}{2}, j-\frac{1}{2}}^{(n+\frac{1}{2})} + b_-(n, i, j)e_{i+\frac{1}{2}, j-\frac{1}{2}}^{(n-\frac{1}{2})},$$

where

$$\begin{aligned} a_+(n, i, j) &= \left(q^{2(n+i)+2(n+j)+2} \frac{(1 - q^{2n-2j+2})(1 - q^{2n-2i+2})}{(1 - q^{4n+2})(1 - q^{4n+4})} \right)^{\frac{1}{2}}, \\ a_-(n, i, j) &= \left(\frac{(1 - q^{2n+2j})(1 - q^{2n+2i})}{(1 - q^{4n})(1 - q^{4n+2})} \right)^{\frac{1}{2}}, \\ b_+(n, i, j) &= - \left(q^{2(n+j)} \frac{(1 - q^{2n-2j+2})(1 - q^{2n+2i+2})}{(1 - q^{4n+2})(1 - q^{4n+4})} \right)^{\frac{1}{2}}, \\ b_-(n, i, j) &= \left(q^{2(n+i)} \frac{(1 - q^{2n+2j})(1 - q^{2n-2i})}{(1 - q^{4n})(1 - q^{4n+2})} \right)^{\frac{1}{2}}. \end{aligned}$$

We will also need the following operators on $L_2(h)$:

$$(2.5) \quad \hat{\alpha} : e_{ij}^{(n)} \mapsto \hat{a}_+(n, i, j)e_{i-\frac{1}{2}, j-\frac{1}{2}}^{(n+\frac{1}{2})} + \hat{a}_-(n, i, j)e_{i-\frac{1}{2}, j-\frac{1}{2}}^{(n-\frac{1}{2})},$$

$$(2.6) \quad \hat{\beta} : e_{ij}^{(n)} \mapsto \hat{b}_+(n, i, j)e_{i+\frac{1}{2}, j-\frac{1}{2}}^{(n+\frac{1}{2})} + \hat{b}_-(n, i, j)e_{i+\frac{1}{2}, j-\frac{1}{2}}^{(n-\frac{1}{2})},$$

where

$$\begin{aligned} \hat{a}_+(n, i, j) &= q^{2n+i+j+1}, \\ \hat{a}_-(n, i, j) &= (1 - q^{2n+2i})^{\frac{1}{2}}(1 - q^{2n+2j})^{\frac{1}{2}}, \\ \hat{b}_+(n, i, j) &= -q^{n+j}(1 - q^{2n+2i+2})^{\frac{1}{2}}, \\ \hat{b}_-(n, i, j) &= q^{n+i}(1 - q^{2n+2j})^{\frac{1}{2}}. \end{aligned}$$

It is easy to see that $\hat{\alpha}$ and $\hat{\beta}$ are compact perturbations of α and β , respectively.

We will now decompose the space $L_2(h)$ as a direct sum of smaller subspaces and study the behaviour of the above operators with respect to this decomposition. Note that the set $\Lambda = \{(n, i, j) : n \in \frac{1}{2}\mathbb{N}, i, j = -n, -n + 1, \dots, n\}$ parametrizes the canonical orthonormal basis for $L_2(h)$. For each $n \in \frac{1}{2}\mathbb{Z}$, denote by Λ_n the minimal subset of Λ containing the point $(|n|, -n, -n)$ and closed under the translations

$$(a, b, c) \mapsto (a + \frac{1}{2}, b + \frac{1}{2}, c - \frac{1}{2}), \quad (a, b, c) \mapsto (a + \frac{1}{2}, b - \frac{1}{2}, c + \frac{1}{2}).$$

For $n, k \in \frac{1}{2}\mathbb{Z}$, denote by Λ_{nk} the minimal subset of Λ_n that contains $(|n|+|k|, -n+k, -n-k)$ and is closed under the translation $(a, b, c) \mapsto (a + 1, b, c)$. Thus all the Λ_{nk} 's are disjoint, $\Lambda = \bigcup_n \Lambda_n$, $\Lambda_n = \bigcup_k \Lambda_{nk}$. Figure 1 will make it easier to visualize these sets. Represent the lattice Λ as a pyramid, where the vertical axis is the n -axis, the top vertex is on the plane $n = 0$ and n increases downwards. Then Λ_n are precisely the vertical cross sections parallel to the plane ABD . The ones that intersect the triangle BCD correspond to nonnegative values of n , and the ones that intersect the triangle BDE correspond to nonpositive values of n .

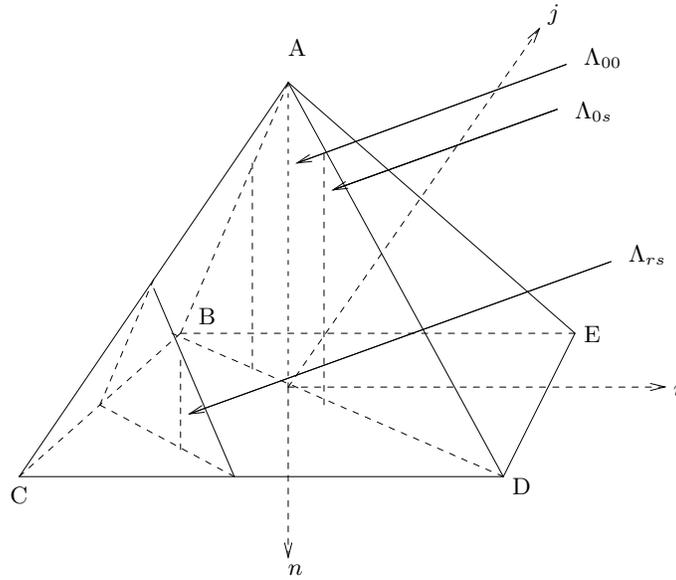


FIGURE 1

In particular, Λ_0 is the cross section given by the plane ABD . Similarly Λ_{nk} are vertical lines in the plane Λ_n . The lines that intersect the triangle CDE correspond to nonnegative values of k and the lines that intersect the triangle BCE correspond to nonpositive values of k . In particular, Λ_{n_0} are the lines that intersect the line CE .

Let us also note that the family of maps $\phi_n : \Lambda_n \rightarrow \Lambda_0$ given by

$$(2.7) \quad \phi_n(a, b, c) = (a - |n|, b + n, c + n)$$

gives bijections between Λ_n and Λ_0 whose restriction to Λ_{nk} yields a bijection from Λ_{nk} to Λ_{0k} .

Let \mathcal{H}_r denote the closed span of $\{e_{ij}^{(n)} : (n, i, j) \in \Lambda_r\}$, \mathcal{H}_{rs} denote the closed span of $\{e_{ij}^{(n)} : (n, i, j) \in \Lambda_{rs}\}$, P_r denote the projection onto \mathcal{H}_r and P_{rs} denote the projection onto \mathcal{H}_{rs} . For an operator T , denote by T_{rs} the restriction $P_{rs}TP_{rs}$ of T to \mathcal{H}_{rs} . Let U_n denote the unitary operator from \mathcal{H}_n to \mathcal{H}_0 induced by the bijection ϕ_n .

Proposition 2.1. *Let A stand for α or $\widehat{\alpha}$, and B stand for β or $\widehat{\beta}$. Then one has*

$$(2.8) \quad P_{n+\frac{1}{2}}AP_n = AP_n, \quad P_{r+\frac{1}{2},s}AP_{rs} = AP_{rs},$$

$$(2.9) \quad BP_n = P_nB, \quad P_{r,s+\frac{1}{2}}BP_{rs} = BP_{rs},$$

$$(2.10) \quad P_{rs}B^*B = B^*BP_{rs},$$

where $n, r, s \in \frac{1}{2}\mathbb{Z}$.

Moreover, for all $n \in \frac{1}{2}\mathbb{N}$, the operators $U_n\widehat{\alpha}U_n^*$ and $U_n\widehat{\beta}U_n^*$ are independent of n .

Proof. This is a simple consequence of equations (2.3)–(2.6). □

Lemma 2.2. $\widehat{\alpha}$ and $\widehat{\beta}$ satisfy the following commutation relations:

$$(2.11) \quad \widehat{\alpha}^* \widehat{\alpha} + \widehat{\beta}^* \widehat{\beta} = I, \widehat{\alpha} \widehat{\alpha}^* + q^2 \widehat{\beta} \widehat{\beta}^* = I, \widehat{\alpha} \widehat{\beta} - q \widehat{\beta} \widehat{\alpha} = 0, \widehat{\alpha} \widehat{\beta}^* - q \widehat{\beta}^* \widehat{\alpha} = 0, \widehat{\beta}^* \widehat{\beta} = \widehat{\beta} \widehat{\beta}^*.$$

Proof. The relations follow by direct computation from the actions of $\widehat{\alpha}$ and $\widehat{\beta}$ given in equations (2.5) and (2.6). \square

The following is a simple consequence of the above commutation relations.

Corollary 2.3. Let $\gamma = \beta^* \beta$ and $\widehat{\gamma} = \widehat{\beta}^* \widehat{\beta}$. Then $\sigma(\widehat{\gamma}) = \{q^{2k} : k \in \mathbb{N}\} \cup \{0\} = \sigma(\gamma)$, and $\ker \widehat{\alpha}^* = \{0\} = \ker \alpha^*$.

Note that the action of $\widehat{\gamma}$ on the basis vectors is given by

$$(2.12) \quad \widehat{\gamma} e_{ij}^{(n)} = c_+(n, i, j) e_{ij}^{(n+1)} + c_0(n, i, j) e_{ij}^{(n)} + c_-(n, i, j) e_{ij}^{(n-1)},$$

where

$$\begin{aligned} c_+(n, i, j) &= -q^{2n+i+j+1} (1 - q^{2n+2i+2})^{\frac{1}{2}} (1 - q^{2n+2j+2})^{\frac{1}{2}}, \\ c_0(n, i, j) &= (q^{2n+2j} (1 - q^{2n+2i}) + q^{2n+2i} (1 - q^{2n+2j+2})), \\ c_-(n, i, j) &= -q^{2n+i+j-1} (1 - q^{2n+2i})^{\frac{1}{2}} (1 - q^{2n+2j})^{\frac{1}{2}}. \end{aligned}$$

One can check, using (2.12) and (2.10), that $\ker \widehat{\gamma} = \{0\}$.

Lemma 2.4. Let $r \in \frac{1}{2}\mathbb{N}$ and $s \in \frac{1}{2}\mathbb{Z}$. The restriction $P_{rs} \widehat{\gamma} P_{rs}$ of $\widehat{\gamma}$ to \mathcal{H}_{rs} is compact, and the spectrum $\sigma(P_{rs} \widehat{\gamma} P_{rs})$ coincides with $\sigma(\widehat{\gamma})$.

Proof. Observe that for $r \in \frac{1}{2}\mathbb{N}$, $U_r(P_{rs} \widehat{\gamma} P_{rs}) U_r^* = P_{0s} \widehat{\gamma} P_{0s}$. So it is enough to prove the statement for $r = 0$.

It is easy to see that $P_{0s} \widehat{\gamma} P_{0s}$ is compact by using equation (2.12). This, along with the second equality in (2.9) and the fact that $\widehat{\beta}$ and $\widehat{\gamma}$ commute, tells us that $\sigma(P_{0s} \widehat{\gamma} P_{0s})$ is independent of s , and consequently $\sigma(P_{0s} \widehat{\gamma} P_{0s}) = \sigma(P_0 \widehat{\gamma} P_0)$ and in fact, this is the same as the essential spectrum $\sigma_{ess}(P_0 \widehat{\gamma} P_0)$.

Let us next show that $\sigma(P_0 \widehat{\gamma} P_0) = \sigma(\widehat{\gamma})$. Let K be the operator on \mathcal{H}_0 , given on the basis vectors $e_{i,-i}^{(n)}$ as follows:

$$(2.13) \quad K e_{i,-i}^{(n)} = c_+(n, i, -i) e_{i,-i}^{(n+1)} + (q^{2n+2|i|} - q^{4n} - q^{4n+2}) e_{i,-i}^{(n)} + c_-(n, i, -i) e_{i,-i}^{(n-1)}.$$

It is easy to see that K is compact, the restriction T of $P_0 \widehat{\gamma} P_0 - K$ to \mathcal{H}_{0s} is independent of s , and $\sigma(T) = \sigma(\widehat{\gamma})$. Hence $\sigma(P_0 \widehat{\gamma} P_0 - K) = \sigma_{ess}(P_0 \widehat{\gamma} P_0 - K) = \sigma(\widehat{\gamma})$. Since $\sigma_{ess}(P_0 \widehat{\gamma} P_0 - K) = \sigma_{ess}(P_0 \widehat{\gamma} P_0)$, the proof follows. \square

Lemma 2.5. The operator γ has trivial kernel. In particular, for all $r, s \in \frac{1}{2}\mathbb{Z}$, $\ker \gamma_{rs} = \{0\}$.

Proof. Let P be the projection onto $\ker \gamma$. Denote by ϕ the functional $\phi : T \mapsto \langle e_{00}^{(0)}, T e_{00}^{(0)} \rangle$. Notice that the restriction of ϕ to \mathcal{A} is the Haar state h , and therefore from appendix A1 in [15], we know that

$$\phi(\chi_{\{q^{2n}\}}(\gamma)) = (1 - q^2) q^{2n}.$$

Observe that $P = \chi_{\{0\}}(\gamma)$. Let $f(x) = \sum_{k=0}^n \chi_{\{q^{2k}\}}(x) + \chi_{\{0\}}(x)$. Then $\phi(f(\gamma)) = 1 - q^{2n+2} + \phi(P)$. Since $0 \leq f(\gamma) \leq 1$, we have $1 - q^{2n+2} + \phi(P) \leq 1$ for all $n \in \mathbb{N}$. Therefore $\phi(P) = 0$, so that $P e_{00}^{(0)} = 0$.

From the commutation relations (2.1), it follows that $P \in \pi(\mathcal{A})'$. Since the vector $e_{00}^{(0)}$ is cyclic for $\pi(\mathcal{A})$, it is separating for $\pi(\mathcal{A})'$. Therefore we have $P = 0$. \square

Proposition 2.6. *Let $r \in \frac{1}{2}\mathbb{N}$ and $s \in \frac{1}{2}\mathbb{Z}$. The operator γ_{rs} is compact and its spectrum $\sigma(\gamma_{rs})$ coincides with $\sigma(\gamma) = \{q^{2k} : k \in \mathbb{N}\} \cup \{0\}$. Moreover each $q^{2n} \in \sigma(\gamma_{rs})$ is an eigenvalue of multiplicity 1.*

Proof. Observe that $\gamma_{rs} = P_{rs}(\gamma - \widehat{\gamma})P_{rs} + P_{rs}\widehat{\gamma}P_{rs}$. Since $\gamma - \widehat{\gamma}$ is compact, and by Lemma 2.4, $P_{rs}\widehat{\gamma}P_{rs}$ is also compact, it follows that γ_{rs} is compact.

Next we claim that $\sigma(\gamma_{rs})$ is independent of s . Write T for the operator $P_{r,s+\frac{1}{2}}\beta P_{rs}$. Then from the commutation relations (2.8)–(2.10), it follows that $|T| = \gamma_{rs}^{\frac{1}{2}}$ and the partial isometry V_T appearing in the polar decomposition of T has initial space \mathcal{H}_{rs} and final space $\mathcal{H}_{r,s+\frac{1}{2}}$, so that it can be viewed as a unitary from \mathcal{H}_{rs} to $\mathcal{H}_{r,s+\frac{1}{2}}$. Again from the commutation relations (2.8)–(2.10), we have

$$\begin{aligned} \gamma_{r,s+\frac{1}{2}}T &= (P_{r,s+\frac{1}{2}}\gamma P_{r,s+\frac{1}{2}})(P_{r,s+\frac{1}{2}}\beta P_{rs}) \\ &= P_{r,s+\frac{1}{2}}\gamma\beta P_{rs} \\ &= P_{r,s+\frac{1}{2}}\beta\gamma P_{rs} \\ &= (P_{r,s+\frac{1}{2}}\beta P_{rs})(P_{r,s+\frac{1}{2}}\gamma P_{r,s+\frac{1}{2}}) \\ &= T\gamma_{r,s+\frac{1}{2}}. \end{aligned}$$

Therefore

$$\gamma_{r,s+\frac{1}{2}}V_T\gamma_{rs}^{\frac{1}{2}} = V_T\gamma_{rs}^{\frac{3}{2}}.$$

Since the range of $\gamma_{rs}^{\frac{1}{2}}$ is dense in \mathcal{H}_{rs} , it follows that $V_T^*\gamma_{r,s+\frac{1}{2}}V_T = \gamma_{rs}$. Thus $\gamma_{r,s+\frac{1}{2}}$ and γ_{rs} are unitarily equivalent. So their spectrums are the same.

Since $P_r\gamma P_r = \bigoplus_s \gamma_{rs}$, it follows that

$$(2.14) \quad \sigma(P_r\gamma P_r) = \sigma_{ess}(P_r\gamma P_r) = \sigma(\gamma_{rs}).$$

Our next claim is that $\sigma_{ess}(P_r\gamma P_r) = \sigma(\gamma)$. Let K be the operator in the proof of Lemma 2.4. We have seen that K is compact, $P_{0s}(P_0\widehat{\gamma}P_0 - K)P_{0s}$ is independent of s and

$$\sigma(P_{0s}(P_0\widehat{\gamma}P_0 - K)P_{0s}) = \sigma(\widehat{\gamma}).$$

Let $K_r := U_r^* K U_r$. Then

$$\begin{aligned} P_{rs}(P_r\widehat{\gamma}P_r - K_r)P_{rs} &= U_r^*(U_r P_{rs} U_r^*(U_r P_r U_r^* U_r \widehat{\gamma} U_r^* U_r P_r U_r^* - K)U_r P_{rs} U_r^*)U_r \\ &= U_r^*(P_{0s}(P_0\widehat{\gamma}P_0 - K)P_{0s})U_r. \end{aligned}$$

Therefore $P_{rs}(P_r\widehat{\gamma}P_r - K_r)P_{rs}$ is independent of s and

$$\sigma(P_{rs}(P_r\widehat{\gamma}P_r - K_r)P_{rs}) = \sigma(P_{0s}(P_0\widehat{\gamma}P_0 - K)P_{0s}) = \sigma(\widehat{\gamma}).$$

Hence

$$\sigma_{ess}(P_r\widehat{\gamma}P_r) = \sigma_{ess}(P_r\widehat{\gamma}P_r - K_r) = \sigma(\widehat{\gamma}) = \sigma(\gamma).$$

Finally,

$$\sigma_{ess}(P_r\gamma P_r) = \sigma_{ess}(P_r(\gamma - \widehat{\gamma})P_r + P_r\widehat{\gamma}P_r) = \sigma_{ess}(P_r\widehat{\gamma}P_r) = \sigma(\gamma).$$

It follows from (2.4) that

$$\gamma(e_{ij}^{(n)}) = k_{-1}(n, i, j)e_{ij}^{(n-1)} + k_0(n, i, j)e_{ij}^{(n)} + k_1(n, i, j)e_{ij}^{(n+1)},$$

where

$$\begin{aligned}
 k_1(n, i, j) &= -q^{2n+i+j+1} \\
 &\quad \times \left(\frac{(1 - q^{2n+2j+2})(1 - q^{2n-2i+2})(1 - q^{2n-2j+2})(1 - q^{2n+2i+2})}{(1 - q^{4n+2})(1 - q^{4n+4})(1 - q^{4n+4})(1 - q^{4n+6})} \right)^{\frac{1}{2}}, \\
 k_0(n, i, j) &= q^{2(n+j)} \frac{(1 - q^{2n-2j})(1 - q^{2n+2i})}{(1 - q^{4n})(1 - q^{4n+2})} \\
 &\quad + q^{2(n+i)} \frac{(1 - q^{2n+2j+2})(1 - q^{2n-2i+2})}{(1 - q^{4n+2})(1 - q^{4n+4})}, \\
 k_{-1}(n, i, j) &= -q^{2n+i+j-1} \left(\frac{(1 - q^{2n-2j})(1 - q^{2n+2i})(1 - q^{2n+2j})(1 - q^{2n-2i})}{(1 - q^{4n-2})(1 - q^{4n})(1 - q^{4n})(1 - q^{4n+2})} \right)^{\frac{1}{2}}.
 \end{aligned}$$

Therefore the operator $\gamma_{rs} - q^{2n}$ is a tridiagonal operator of the form

$$e_{s-r, -s-r}^{(|r|+|s|+k)} \mapsto \begin{cases} b_0 e_{s-r, -s-r}^{(|r|+|s|)} + c_0 e_{s-r, -s-r}^{(|r|+|s|+1)} & \text{if } k = 0, \\ a_k e_{s-r, -s-r}^{(|r|+|s|+k-1)} + b_k e_{s-r, -s-r}^{(|r|+|s|+k)} + c_k e_{s-r, -s-r}^{(|r|+|s|+k+1)} & \text{if } k > 0, \end{cases}$$

with all the coefficients a_k , b_k and c_k nonzero. It follows from this that the kernel of $\gamma_{rs} - q^{2n}$ can have dimension at most 1. Since γ_{rs} is compact, each $q^{2n} \in \sigma(\gamma_{rs})$ is an eigenvalue. Therefore each q^{2n} is an eigenvalue of multiplicity 1. \square

3. THE MODULAR CONJUGATION

We will compute the modular conjugation operator for the Haar state in this section.

Proposition 3.1. *Denote by S the operator $a \mapsto a^*$ on \mathcal{A} . Then viewed as an operator on $L_2(h)$, the set $\{e_{ij}^{(n)} : (n, i, j) \in \Lambda\}$ is contained in the domain of S and*

$$(3.1) \quad S e_{ij}^{(n)} = (-1)^{2n+i+j} q^{i+j} e_{-i, -j}^{(n)}.$$

Proof. Recall (equation 57, page 115, [10]) that if $t_{ij}^{(n)}$ denotes the ij^{th} matrix entry of the irreducible representation indexed by n , then $e_{ij}^{(n)}$'s are just the normalized $t_{ij}^{(n)}$'s, more specifically,

$$(3.2) \quad e_{ij}^{(n)} = q^{-n+i} \left(\frac{1 - q^{4n+2}}{1 - q^2} \right)^{1/2} t_{ij}^{(n)}.$$

Therefore $\{e_{i,j}^{(n)} : (n, i, j) \in \Lambda\}$ is contained in the domain of S and

$$\begin{aligned}
 S e_{i,j}^{(n)} &= \sum_{m,k,l} \langle e_{k,l}^{(m)}, S e_{i,j}^{(n)} \rangle e_{k,l}^{(m)} \\
 &= \sum_{m,k,l} \langle e_{i,j}^{(n)} e_{k,l}^{(m)}, 1 \rangle e_{k,l}^{(m)}.
 \end{aligned}$$

By the properties of the Clebsch-Gordon coefficients,

$$\langle e_{i,j}^{(n)} e_{k,l}^{(m)}, 1 \rangle = 0 \text{ for } n \neq m.$$

From the equation preceding equation (43), p. 74, [10], we get for $m = 0, 1, \dots, 2n$,

$$\sum_{l=0}^{2n} C_q(n, n, m; b-l+n, l-n, b) t_{a-k, b-l+n}^{(n)} t_{k, l-n}^{(n)} = C_q(n, n, m; a-k, k, a) t_{ab}^{(m)}.$$

Here $C_q(m, n, p; i, j, k)$ is the Clebsch-Gordon coefficient $\langle e_k^{(p)}, e_i^{(m)} \otimes e_j^{(n)} \rangle$. If we write

$$A_{mj}^{(b)} = C_q(n, n, m; b - j + n, j - n, b), \quad m, j = 0, 1, \dots, 2n,$$

then the above says that

$$\sum_{j=0}^{2n} A_{mj}^{(b)} t_{a-k, b-j+n}^{(n)} t_{k, j-n}^{(n)} = C_q(n, n, m; a - k, k, a) t_{ab}^{(m)}$$

for $m = 0, 1, \dots, 2n$. Therefore

$$t_{a-k, b-j+n}^{(n)} t_{k, j-n}^{(n)} = \sum_m (A^{(b)-1})_{jm} C_q(n, n, m; a - k, k, a) t_{ab}^{(m)}.$$

For $a = b = 0$, the coefficient of $t_{00}^{(0)}$ on the right hand side is

$$(A^{(0)-1})_{j0} C_q(n, n, 0; -k, k, 0),$$

where $A^{(0)}$ is the matrix $((A_{ij}^{(0)}))_{ij}$. Thus,

$$\langle t_{-k, -j+n}^{(n)} t_{k, j-n}^{(n)}, 1 \rangle = (A^{(0)-1})_{j0} C_q(n, n, 0; -k, k, 0).$$

From equation (3.2),

$$\|t_{ij}^{(n)}\| = q^{-i} (2n + 1)_q^{-\frac{1}{2}} = q^{-i} \left(\frac{1 - q^{4n+2}}{1 - q^2} \right)^{-\frac{1}{2}}.$$

Hence

$$(3.3) \quad \langle e_{-k, -j+n}^{(n)} e_{k, j-n}^{(n)}, 1 \rangle = q^{-2n} \left(\frac{1 - q^{4n+2}}{1 - q^2} \right) (A^{(0)-1})_{j0} C_q(n, n, 0; -k, k, 0).$$

Let us next find $(A^{(0)-1})_{j0}$. Using equation (73), page 81, [10], we get

$$A_{mj}^{(0)} = C_q(n, n, m; -j + n, j - n, 0) = (-1)^{2n+m} C_{q^{-1}}(n, n, m; j - n, -j + n, 0).$$

Write $D = ((d_{ij}))$ and $B = ((B_{ij}))$, where $B_{m,j} = C_{q^{-1}}(n, n, m; j - n, -j + n, 0)$ and $d_{ij} = \delta_{ij} (-1)^{2n+i}$. Then $A^{(0)} = DB$, so that $A^{(0)-1} = B^{-1}D$. Since B has real entries, it follows from equations (46) and (47), page 75, [10] that B is orthogonal. Therefore $A^{(0)-1} = B^t D$. Hence

$$\begin{aligned} (A^{(0)-1})_{j0} &= (B^t D)_{j0} = (B^t)_{j0} d_{00} = B_{0j} (-1)^{2n} \\ &= (-1)^{2n} C_{q^{-1}}(n, n, 0; j - n, -j + n, 0). \end{aligned}$$

Using equation (73), page 81, [10] and the equation preceding equation (68), page 81, [10], we get

$$(A^{(0)-1})_{j0} = C_q(n, n, 0; -j + n, j - n, 0) = (-1)^j q^{2n-j} \left(\frac{1 - q^{4n+2}}{1 - q^2} \right)^{-\frac{1}{2}}$$

and

$$C_q(n, n, 0; -k, k, 0) = (-1)^{n+k} q^{n-k} \left(\frac{1 - q^{4n+2}}{1 - q^2} \right)^{-\frac{1}{2}}.$$

Substituting these values in (3.3), we get

$$(3.4) \quad \langle e_{ij}^{(n)} e_{-i, -j}^{(n)}, 1 \rangle = (-1)^{2n+i+j} q^{i+j}.$$

Thus we have equation (3.1). □

In particular, it follows from the above proposition that the operator S is closable. Let \bar{S} denote the closure of S . Let J denote the antilinear operator, given on the basis elements by

$$(3.5) \quad J e_{ij}^{(n)} = (-1)^{2n+i+j} e_{-i,-j}^{(n)},$$

and let Δ be given by

$$(3.6) \quad \Delta e_{ij}^{(n)} = q^{2i+2j} e_{ij}^{(n)}.$$

Then it follows from (3.1) that $\bar{S} = J\Delta^{\frac{1}{2}}$, and $\text{Dom } \bar{S} = \text{Dom } \Delta^{\frac{1}{2}}$. By lemma 1.5, [14], it follows that J is the modular conjugation and Δ is the modular operator associated with the Haar state, and by theorem 1.19, [14], we have $\pi(\mathcal{A})' = J\pi(\mathcal{A})''J$.

4. THE MAIN THEOREM

Let D be the operator given by (2.2), and let $F = \text{sign } D$.

Theorem 4.1. *Let $T \in \pi(\mathcal{A})'$. If $[F, T]$ is compact, then T is a scalar.*

Let J be the modular conjugation operator computed in the previous section. Then the theorem says that if $T \in \pi(\mathcal{A})''$ and $[F, JTJ]$ is compact, then T must be a scalar. We will first prove the following special case of the above theorem.

Theorem 4.2. *Let f be a complex-valued function on $\sigma(\gamma)$. If $[F, Jf(\gamma)J]$ is compact, then $f(\gamma)$ must be a scalar.*

We will need the following simple lemma for the proof of this theorem.

Lemma 4.3. *Let A and B be two compact operators with trivial kernel such that $\sigma(A) = \sigma(B)$ and each nonzero element of $\sigma(A)$ is an eigenvalue of multiplicity 1 for both A and B . Let u be a unit eigenvector of A corresponding to an eigenvalue λ and let v be a unit eigenvector of B corresponding to the same eigenvalue λ . Assume that $|\langle u, v \rangle| < 1 - \epsilon$, where $\epsilon > 0$. Then there is a positive constant $c = c(\epsilon, \lambda, \sigma(A))$ such that*

$$\|A - B\| \geq c.$$

Proof. It follows from the given conditions that there is a unitary U such that $Uu = v$ and $B = UAU^*$. Let w be the projection of v onto u^\perp , i.e.

$$w = v - \langle u, v \rangle u.$$

Then

$$\|(A - B)v\| = \|(A - \lambda)v\| = \|(A - \lambda)w\|.$$

Since $(A - \lambda)$ is invertible on u^\perp , it follows that

$$\|w\| = \|((A - \lambda)|_{u^\perp})^{-1}(A - \lambda)w\| \leq \|((A - \lambda)|_{u^\perp})^{-1}\| \|(A - \lambda)w\|,$$

so that

$$\|A - B\| \geq \|(A - B)v\| = \|(A - \lambda)w\| \geq \|w\| \|((A - \lambda)|_{u^\perp})^{-1}\|^{-1}.$$

Observe that

$$\|((A - \lambda)|_{u^\perp})^{-1}\| = (\inf\{|\lambda - \mu| : \mu \in \sigma(A), \mu \neq \lambda\})^{-1} < \infty.$$

Since

$$1 = \|v\|^2 = |\langle u, v \rangle|^2 + \|w\|^2 < (1 - \epsilon)^2 + \|w\|^2,$$

the result follows. □

Proof of Theorem 4.2. Write $T := f(\gamma)$. Let $Q = \frac{I-JFJ}{2}$. Then compactness of $[F, J TJ]$ is equivalent to compactness of $[Q, T]$. Let $w_{rs} = e^{|r|+|s|} e_{s-r, -s-r}$. For an operator A on \mathcal{H} , denote by A_{rs} the operator $P_{rs}AP_{rs}$. The projection Q commutes with each P_{rs} and one has

$$Q_{rs} = \begin{cases} |w_{rs}\rangle\langle w_{rs}| & \text{if } r, -s \in \frac{1}{2}\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since γ also commutes with P_{rs} , we have

$$[Q, T] = \bigoplus_{r,s} [Q_{rs}, T_{rs}] = \bigoplus_{r, -s \in \frac{1}{2}\mathbb{N}} [Q_{rs}, T_{rs}].$$

Take $\ell \neq m \in \mathbb{N}$ and let $\lambda = f(q^{2\ell})$, $\mu = f(q^{2m})$. Recall that for each $r \in \frac{1}{2}\mathbb{N}$, $s \in \frac{1}{2}\mathbb{Z}$, one has $\sigma(\gamma_{rs}) = \{q^{2n} : n \in \mathbb{N}\} \cup \{0\}$, and each q^{2n} is an eigenvalue of multiplicity 1. Let u_{rs} and v_{rs} be eigenvectors of γ in \mathcal{H}_{rs} corresponding to the eigenvalues $q^{2\ell}$ and q^{2m} , respectively. Then $Tu_{rs} = \lambda u_{rs}$ and $Tv_{rs} = \mu v_{rs}$. Now

$$\begin{aligned} & \langle u_{rs}, [Q, T](u_{rs} - v_{rs}) \rangle \\ &= \langle u_{rs}, Q(\lambda u_{rs} - \mu v_{rs}) \rangle - \langle T^*u_{rs}, Q(u_{rs} - v_{rs}) \rangle \\ &= \lambda \langle u_{rs}, Qu_{rs} \rangle - \mu \langle u_{rs}, Qv_{rs} \rangle - \lambda \langle u_{rs}, Qu_{rs} \rangle + \lambda \langle u_{rs}, Qv_{rs} \rangle \\ &= (\lambda - \mu) \langle u_{rs}, Qv_{rs} \rangle. \end{aligned}$$

In particular,

$$(4.1) \quad \|[Q, T](u_{r0} - v_{r0})\|^2 \geq |\lambda - \mu|^2 |\langle u_{r0}, Qv_{r0} \rangle|^2$$

$$(4.2) \quad = |\lambda - \mu|^2 |\langle u_{r0}, w_{r0} \rangle \langle w_{r0}, v_{r0} \rangle|.$$

We will show that for some subsequence r_k ,

$$(4.3) \quad \lim_{k \rightarrow \infty} \langle u_{r_k 0}, w_{r_k 0} \rangle \neq 0,$$

$$(4.4) \quad \lim_{k \rightarrow \infty} \langle v_{r_k 0}, w_{r_k 0} \rangle \neq 0.$$

Observe that

$$\langle v_{r0}, w_{r0} \rangle = \langle U_r v_{r0}, U_r w_{r0} \rangle = \langle U_r v_{r0}, w_{00} \rangle.$$

Now let $U_r v_{00}$ be an eigenvector of $U_r \gamma_{r0} U_r^*$ corresponding to the eigenvalue q^{2m} . Let ξ be a unit eigenvector of $\widehat{\gamma}$ in \mathcal{H}_{00} corresponding to the eigenvalue q^{2m} . Note that

$$U_r \gamma_{r0} U_r^* - \widehat{\gamma}_{00} = U_r (\gamma_{r0} - \widehat{\gamma}_{r0}) U_r^* + U_r \widehat{\gamma}_{r0} U_r^* - \widehat{\gamma}_{00} = U_r (\gamma_{r0} - \widehat{\gamma}_{r0}) U_r^*.$$

Therefore

$$\|U_r \gamma_{r0} U_r^* - \widehat{\gamma}_{00}\| = \|\gamma_{r0} - \widehat{\gamma}_{r0}\| = \|P_{r0}(\gamma - \widehat{\gamma})P_{r0}\|.$$

Since $\gamma - \widehat{\gamma}$ is compact, it follows that

$$\lim \|U_r \gamma_{r0} U_r^* - \widehat{\gamma}_{00}\| = 0.$$

By Lemma 4.3,

$$\lim |\langle U_r v_{r0}, \xi \rangle| = 1.$$

Let $\theta_r \in [0, 2\pi]$ be such that $\exp(i\theta_r) = \langle U_r v_{r0}, \xi \rangle$. Write $\xi_r = \exp(-i\theta_r)\xi$. Then $\langle U_r v_{r0}, \xi_r \rangle = 1$ for each r . Choose a subsequence r_k such that θ_{r_k} converges, to say θ . Let $\zeta = \exp(-i\theta)\xi$. Then it follows that

$$\lim_k \langle U_{r_k} v_{r_k 0}, \zeta \rangle = 1.$$

Therefore

$$\lim_k \|U_{r_k} v_{r_k 0} - \zeta\| = 0.$$

Hence

$$\langle v_{r_k 0}, w_{r_k 0} \rangle = \langle U_{r_k} v_{r_k 0}, w_{00} \rangle = \langle U_{r_k} v_{r_k 0} - \zeta, w_{00} \rangle + \langle \zeta, w_{00} \rangle.$$

The first term converges to zero. Let us show that the second term is nonzero. Let

$$k = \min\{n \in \mathbb{N} : \langle \zeta, e_{00}^{(n)} \rangle \neq 0\}.$$

Then one has

$$\langle \zeta, \widehat{\gamma}^k w_{00} \rangle = \langle \zeta, \widehat{\gamma}^k e_{00}^{(0)} \rangle \neq 0.$$

Since $\widehat{\gamma}\zeta = q^{2m}\zeta$, we have

$$\langle \zeta, w_{00} \rangle = q^{-2mk} \langle \widehat{\gamma}^k \zeta, w_{00} \rangle = q^{-2mk} \langle \zeta, \widehat{\gamma}^k w_{00} \rangle \neq 0.$$

Thus we have (4.4). An identical proof shows that r_k will have a further subsequence, which we continue to denote by r_k by abuse of notation, for which we have both (4.4) and (4.3).

Since $[Q, T]$ is compact, $[Q, T](u_{r_k 0} - v_{r_k 0})$ converges to zero. Therefore by (4.2), we must have $\lambda = \mu$, i.e. $f(q^{2\ell}) = f(q^{2m})$. Since this is true for all $\ell \neq m \in \mathbb{N}$, $T = f(\gamma)$ must be a scalar. \square

Proposition 4.4. *Let $m, n \in \mathbb{Z}$ and let f be a complex-valued function on $\sigma(\gamma)$. Assume $m \neq 0$. If $[F, J\alpha_m \beta_n f(\gamma)J]$ is compact, then $\alpha_m \beta_n f(\gamma) = 0$.*

Proof. Assume $m > 0$, so that $\alpha_m = \alpha^m$. Compactness of $[F, J\alpha_m \beta_n f(\gamma)J]$ implies compactness of $[Q, \alpha_m \beta_n f(\gamma)]$. Since Q is selfadjoint, this implies that the operator $[Q, (f(\gamma))^* \beta_n^* (\alpha^m)^* \alpha^m \beta_n f(\gamma)]$ is compact. Now $\beta_n^* (\alpha^m)^* \alpha^m \beta_n$ is of the form $p(\gamma)$ for some polynomial p . By Theorem 4.2 it follows that $(f(\gamma))^* \beta_n^* (\alpha^m)^* \alpha^m \beta_n f(\gamma)$ is a scalar. Suppose it is nonzero. Since $(\alpha^m)^* \alpha^m$ is a polynomial in γ , we have

$$(f(\gamma))^* \beta_n^* (\alpha^m)^* \alpha^m \beta_n f(\gamma) = (f(\gamma))^* \beta_n^* \beta_n f(\gamma) (\alpha^m)^* \alpha^m.$$

It would then follow that the kernel of $(\alpha^m)^* \alpha^m$ is trivial. This implies that the kernel of α is trivial. But $\ker \alpha = \ker \alpha^* \alpha = \ker(1 - \gamma)$, which is infinite dimensional by Proposition 2.6. Therefore we must have $(f(\gamma))^* \beta_n^* \alpha_m^* \alpha_m \beta_n f(\gamma) = 0$, which implies $\alpha_m \beta_n f(\gamma) = 0$.

For $m < 0$, observe that $(\alpha_m \beta_n f(\gamma))^* = \alpha_{-m} \beta_{-n} g(\gamma)$ for some function g and use the above argument. \square

Proposition 4.5. *Let $n \in \mathbb{Z}$ and let f be a nonzero complex-valued function on $\sigma(\gamma)$. If $n \neq 0$, then $[F, J\beta_n f(\gamma)J]$ is not compact.*

Proof. As before, compactness of $[F, J\beta_n f(\gamma)J]$ is equivalent to the compactness of $[Q, \beta_n f(\gamma)]$. So it is enough to show that $[Q, \beta_n f(\gamma)]$ is not compact. Also it is enough to prove this for $n > 0$.

Since β and β^* both have trivial kernel, the partial isometry V appearing in the polar decomposition of β is unitary and $\beta_n f(\gamma) = V^n g(\gamma)$ for some function g . Let $m \in \mathbb{N}$ be such that $\lambda := g(q^{2m}) \neq 0$. For $r \in \frac{1}{2}\mathbb{N}$, let v_{r0} be a unit eigenvector of γ in \mathcal{H}_{r0} corresponding to the eigenvalue q^{2m} . For $s \in \frac{1}{2}\mathbb{Z}$, define

$$v_{rs} := V^{2s} v_{r0}.$$

Then v_{rs} is a unit vector in \mathcal{H}_{rs} , and since V commutes with γ , we have $\gamma v_{rs} = q^{2m}v_{rs}$. Therefore $g(\gamma)v_{rs} = \lambda v_{rs}$ for all r, s . We then have

$$\begin{aligned} \langle v_{r, \frac{n}{2}}, [Q, V^n g(\gamma)]v_{r0} \rangle &= \langle v_{r, \frac{n}{2}}, QV^n g(\gamma)v_{r0} \rangle - \langle v_{r, \frac{n}{2}}, V^n g(\gamma)Qv_{r0} \rangle \\ &= \lambda \langle v_{r, \frac{n}{2}}, QV^n v_{r0} \rangle - \langle g(\gamma)v_{r, \frac{n}{2}}, V^n Qv_{r0} \rangle \\ &= \lambda (\langle v_{r, \frac{n}{2}}, Qv_{r, \frac{n}{2}} \rangle - \langle v_{r0}, Qv_{r0} \rangle) \\ &= -\lambda \|Qv_{r0}\|^2 \\ &= -\lambda |\langle v_{r0}, w_{r0} \rangle|^2. \end{aligned}$$

From the proof of Theorem 4.2, there is a sequence r_k such that $\lim_k \langle v_{r_k 0}, w_{r_k 0} \rangle \neq 0$. Therefore the operator $[Q, V^n g(\gamma)]$ cannot be compact. \square

We now have all the ingredients ready for the proof of Theorem 4.1. In order to make use of these, we need to look at certain operator-valued Fourier coefficients.

Let τ be the action of $S^1 \times S^1$ on \mathcal{A} by automorphisms given by

$$\tau_{z,w} : \begin{cases} \alpha \mapsto z\alpha, \\ \beta \mapsto w\beta. \end{cases}$$

Let $V_{z,w} : L_2(h) \rightarrow L_2(h)$ be given by

$$V_{z,w} e_{ij}^{(n)} = z^{-i-j} w^{i-j} e_{ij}^{(n)}.$$

Then $\pi(\tau_{z,w}(a)) = V_{z,w}\pi(a)V_{z,w}^*$ for all $a \in \mathcal{A}$. Thus the action extends to a strongly continuous action of $S^1 \times S^1$ on the von Neumann algebra $\pi(\mathcal{A})''$. For $T \in \pi(\mathcal{A})''$ and $m, n \in \mathbb{Z}$, denote by $\mathcal{F}_{mn}(T)$ the following operator:

$$\mathcal{F}_{mn}(T) = \int_{S^1} \int_{S^1} z^{-m} w^{-n} \tau_{z,w}(T) dz dw.$$

Note that the above integral is defined in the strong sense. In case the integrand is norm continuous, it coincides with the corresponding integral in the norm sense.

Lemma 4.6. *Let T_k be a sequence of operators in $\pi(\mathcal{A})''$ that converges strongly to an operator T . Then for all $m, n \in \mathbb{Z}$, the sequence $\mathcal{F}_{mn}(T_k)$ converges strongly to $\mathcal{F}_{mn}(T)$.*

Proof. Take a vector $u \in L_2(h)$. Then

$$\begin{aligned} \mathcal{F}_{mn}(T_k)u &= \int \int z^{-m} w^{-n} V_{z,w} T_k V_{z,w}^* u dz dw, \\ \mathcal{F}_{mn}(T)u &= \int \int z^{-m} w^{-n} V_{z,w} T V_{z,w}^* u dz dw. \end{aligned}$$

Since T_k converges strongly to T , for each $z, w \in S^1$, we have

$$\lim_{k \rightarrow \infty} z^{-m} w^{-n} V_{z,w} T_k V_{z,w}^* u = z^{-m} w^{-n} V_{z,w} T V_{z,w}^* u$$

and

$$\begin{aligned} \|z^{-m} w^{-n} V_{z,w} T_k V_{z,w}^* u\| &\leq \|T_k\| \|u\| \\ &\leq (\sup_k \|T_k\|) \|u\|. \end{aligned}$$

Now an application of the Dominated Convergence Theorem for Banach space valued functions (Theorem 3, page 45, [9]) gives us the required result. \square

Lemma 4.7. *Let $T \in \pi(\mathcal{A})''$. If $\mathcal{F}_{mn}(T) = 0$ for all $(m, n) \neq (0, 0)$, then $T = f(\gamma)$ for some bounded measurable function f on $\sigma(\gamma)$.*

Proof. Let B be the $*$ -subalgebra of \mathcal{A} consisting of finite linear combinations of elements of the form $\alpha_m \beta_n \gamma^k$, where $m, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Clearly B is dense in \mathcal{A} . Observe that

- (1) for any $T \in B$, one has $\mathcal{F}_{mn}(T) = \alpha_m \beta_n p(\gamma)$ for some polynomial p ,
- (2) if $T = \alpha_m \beta_n p(\gamma)$ for some polynomial p , then

$$\mathcal{F}_{jk}(T) = \begin{cases} T & \text{if } j = m, k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $T \in \pi(\mathcal{A})''$ with $\mathcal{F}_{mn}(T) = 0$ for all $(m, n) \neq (0, 0)$. Take any two vectors u and v in \mathcal{H} and let $f : S^1 \times S^1 \rightarrow \mathbb{C}$ be the function given by $f(z, w) = \langle u, \tau_{z,w}(T)v \rangle$. Then by the above condition on T , it follows that all the Fourier coefficients $\hat{f}(m, n)$ are zero for all $(m, n) \neq (0, 0)$. This implies f is a constant function. Since this is true for any two vectors u and v , it follows that $(z, w) \mapsto \tau_{z,w}(T)$ is constant, so that for all $z, w \in S^1$, we have $\tau_{z,w}(T) = T$. Therefore $\mathcal{F}_{00}(T) = T$. Let T_k be a sequence in B that converges strongly to T . By Lemma 4.6, we have $\text{s-lim}_k \mathcal{F}_{mn}(T_k) = \mathcal{F}_{mn}(T)$ for all $m, n \in \mathbb{Z}$. In particular, we have $\text{s-lim}_k \mathcal{F}_{00}(T_k) = \mathcal{F}_{00}(T) = T$. Since each $\mathcal{F}_{00}(T_k)$ is of the form $p_k(\gamma)$ for some polynomial p_k , the operator T must be of the form $f(\gamma)$ for some bounded measurable function on $\sigma(\gamma)$. \square

Lemma 4.8. *Let $T \in \pi(\mathcal{A})''$. Then for $m, n \in \mathbb{Z}$, the operator $\mathcal{F}_{mn}(T)$ is of the form $\alpha_m \beta_n f(\gamma)$ for some function f on $\sigma(\gamma)$.*

Proof. Since $(\mathcal{F}_{mn}(T))^* = \mathcal{F}_{-m,-n}(T^*)$, it is enough to prove the statement for $m \leq 0$. So assume $m \leq 0$. Let T_k be a sequence in B that converges strongly to T . Then by Lemma 4.6, $\mathcal{F}_{mn}(T_k)$ converges strongly to $\mathcal{F}_{mn}(T)$. Each $\mathcal{F}_{mn}(T_k)$ is of the form $\alpha_m \beta_n p_k(\gamma) = (\alpha^*)^{|m|} \beta_n p_k(\gamma)$ for some polynomial p_k . Now

$$((\alpha^*)^{|m|} \beta_n)^* ((\alpha^*)^{|m|} \beta_n) = \alpha^{|m|} (\alpha^*)^{|m|} \gamma^{|n|},$$

and $\ker \alpha^* = \{0\} = \ker \beta = \ker \beta^*$. Therefore the operator $(\alpha^*)^{|m|} \beta_n$ has trivial kernel. Therefore the polar decomposition of $(\alpha^*)^{|m|} \beta_n$ is of the form $V \sqrt{r(\gamma)}$, where V is an isometry and r is a polynomial. Thus $V \sqrt{r(\gamma)} p_k(\gamma)$ converges strongly to $\mathcal{F}_{mn}(T)$. Therefore $\sqrt{r(\gamma)} p_k(\gamma)$ converges strongly to $V^* \mathcal{F}_{mn}(T)$. It follows that $V^* \mathcal{F}_{mn}(T) = f(\gamma)$ for some bounded function f and $\lim_k \sqrt{r(x)} p_k(x) = f(x)$ for all $x \in \sigma(\gamma)$. Define functions \tilde{p} and \tilde{p}_k on $\sigma(\gamma)$ as follows:

$$\tilde{p}(x) = \begin{cases} f(x)/\sqrt{r(x)} & \text{if } r(x) \neq 0, \\ 0 & \text{if } r(x) = 0, \end{cases} \quad \tilde{p}_k(x) = \begin{cases} p_k(x) & \text{if } r(x) \neq 0, \\ 0 & \text{if } r(x) = 0. \end{cases}$$

Then $\sqrt{r(x)} p_k(x) = \sqrt{r(x)} \tilde{p}_k(x)$ and $f(x) = \sqrt{r(x)} \tilde{p}(x)$. This means $\sqrt{r(\gamma)} p_k(\gamma) = \sqrt{r(\gamma)} \tilde{p}_k(\gamma)$ and $\sqrt{r(\gamma)} p_k(\gamma)$ converges strongly to $\sqrt{r(\gamma)} \tilde{p}(\gamma)$. Therefore the sequence $V \sqrt{r(\gamma)} p_k(\gamma)$ converges strongly to $V \sqrt{r(\gamma)} \tilde{p}(\gamma) = (\alpha^*)^{|m|} \beta_n \tilde{p}(\gamma)$. Hence $\mathcal{F}_{mn}(T) = (\alpha^*)^{|m|} \beta_n \tilde{p}(\gamma)$. \square

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. Compactness of $[F, JTJ]$ implies $[Q, T]$ is compact. Since $V_{z,w}[Q, T]V_{z,w}^* = [Q, \tau_{z,w}(T)]$, it follows that $[Q, \mathcal{F}_{mn}(T)]$ is compact for all m and n . Since the operator $\mathcal{F}_{mn}(T)$ is of the form $\alpha_m \beta_n f(\gamma)$ for some function f on $\sigma(\gamma)$, by Propositions 4.4 and 4.5 we get $\mathcal{F}_{mn}(T) = 0$ for all $(m, n) \neq (0, 0)$. An application of Lemma 4.7 now tells us that $T = f(\gamma)$ for some bounded function f . Hence using Theorem 4.2, we get that T is a scalar. \square

Remark 4.9. By the characterization of equivariant spectral triples in [3] (see the discussion preceding proposition 4.4, [3]), for any equivariant D , sign D has to be of the form $2P - I$ or $I - 2P$, where P is the projection onto the subspace spanned by $\{e_{ij}^{(n)} : n \in \frac{1}{2}\mathbb{N}, n - i \in E, j = -n, -n + 1, \dots, n\}$, E being some finite subset of \mathbb{N} . A slight modification in the proof of Theorem 4.2 will work for the sign of any such D .

Corollary 4.10. *Suppose $T \in \pi(\mathcal{A})'$. If $[D, T]$ is bounded, then T must be a scalar.*

Proof. Boundedness of $[D, T]$ implies compactness of $[F, T]$. Therefore the result follows from Theorem 4.1. \square

Remark 4.11. Suppose π is a faithful representation of a C^* -algebra \mathcal{A} on a Hilbert space \mathcal{H} and (\mathcal{H}, π, D) is a spectral triple for \mathcal{A} . If there is another C^* -algebra \mathcal{B} and a faithful representation ρ of \mathcal{B} on \mathcal{H} such that $\pi(a)$ and $\rho(b)$ commute for all $a \in \mathcal{A}, b \in \mathcal{B}$ and (\mathcal{H}, ρ, D) is a spectral triple for \mathcal{B} , then the pair (\mathcal{H}, D) together with the representation $\pi \otimes \rho : a \otimes b \mapsto \pi(a) \otimes \rho(b)$ gives rise to a spectral triple for $\mathcal{A} \otimes \mathcal{B}$ and hence an element in the K -homology of $\mathcal{A} \otimes \mathcal{B}$.

What the above corollary says is that for the Dirac operator constructed in [3] on $L_2(h)$ along with the representation by left multiplication, such a pair (\mathcal{B}, ρ) does not exist (other than the trivial one: $\mathcal{B} = \mathbb{C}$), thereby preventing one from turning it into a spectral triple for $\mathcal{A} \otimes \mathcal{B}$ in a natural manner. Thus the triple $(L_2(h), \pi, D)$ does not admit a rational Poincaré dual in the sense of Moscovici ([12]).

Remark 4.12. Note that the Dirac operator we have considered here is the one equivariant with respect to the right regular representation of the group $SU_q(2)$. Recall ([4]) that a generic Dirac operator equivariant with respect to the left regular representation is of the form $D : e_{ij}^{(n)} \mapsto d(n, i)e_{ij}^{(n)}$, where

$$d(n, i) = \begin{cases} 2n + 1 & \text{if } n \neq j, \\ -(2n + 1) & \text{if } n = j. \end{cases}$$

All the results in this section continue to hold for this Dirac operator as well. The proofs also go through verbatim, except the proof of Proposition 4.5, where one has to look at commutators $[F, J\beta_n f(\gamma)J]$ for $n < 0$.

5. K -THEORY FUNDAMENTAL CLASS

We will show in this section that even though the spectral triple we considered does not give a fundamental class, a little modification enables one to construct a fundamental class that gives Poincaré duality.

We start the section with the following straightforward but important observation.

Theorem 5.1. *Poincaré duality holds for \mathcal{A} .*

Proof. It follows from the description of the irreducible representations of \mathcal{A} ([11]) that it is a type I C^* -algebra. Both \mathcal{A} and $C(S^1)$ are separable type I C^* -algebras and have the same K_0 and K_1 groups. Therefore it follows from Rosenberg and Schochet ([13]) that \mathcal{A} and $C(S^1)$ are KK -equivalent. Poincaré duality holds for $C(S^1)$; hence it follows from lemma 3.4 in [2] that Poincaré duality holds for \mathcal{A} also. \square

One can see that Poincaré duality is just a consequence of the KK -theoretic properties of the underlying C^* -algebra. What is of greater interest is to get an explicit realization of the K -homology fundamental class. Thus we want to identify explicitly a class in $KK(\mathcal{A} \otimes \mathcal{A}, \mathbb{C})$ that will give us a K -homology fundamental class. As a first step, we will exhibit an element in $KK(\mathcal{A}, C(S^1))$ that will give us a KK -equivalence. We then compose this with the fundamental class for the torus to construct the desired class. This involves computing the Kasparov product of two elements, which can sometimes be difficult. As we will see, we avoid computing any nontrivial Kasparov product by exploiting the special form of the KK -equivalence we construct.

Lemma 5.2. $KK(\mathcal{A}, \mathbb{C}) \otimes KK(\mathbb{C}, C(S^1)) \cong KK(\mathcal{A}, C(S^1))$. (Here $KK(A, B)$ means $KK_0(A, B) \oplus KK_1(A, B)$.)

Proof. Observe that $KK(\mathcal{A}, \mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z}$ and $KK(\mathbb{C}, C(S^1)) = \mathbb{Z} \oplus \mathbb{Z}$. Thus both are torsion-free and by the Künneth theorem (due to Rosenberg and Schochet, theorem 23.1.2, [1]), the result follows. \square

Lemma 5.3. $KK(\mathcal{A}, C(S^1)) \cong M_2(\mathbb{Z})$.

Proof. Since $KK(\mathbb{C}, \mathcal{A}) = \mathbb{Z} \oplus \mathbb{Z}$, by the universal coefficient theorem (UCT) (theorem 23.1.1, [1]) it follows that

$$KK(\mathcal{A}, C(S^1)) \cong \text{Hom}(KK(\mathbb{C}, \mathcal{A}), KK(\mathbb{C}, C(S^1))) \cong \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}) \cong M_2(\mathbb{Z}).$$

Note that in the above isomorphism, an element $\eta \in KK(\mathcal{A}, C(S^1))$ is mapped to the homomorphism given by $\xi \mapsto \xi \otimes \eta$. \square

Lemma 5.4. Let $\xi_i \in K_i(\mathcal{A})$, $\zeta_i \in K^i(\mathcal{A})$ and $\eta_i \in K_i(C(S^1))$, $i = 0, 1$. Define

$$\gamma := \zeta_0 \otimes \eta_0 + \zeta_1 \otimes \eta_1.$$

Then the map $\xi \mapsto \xi \otimes \gamma$ takes ξ_0 to $\langle \xi_0, \zeta_0 \rangle \eta_0$ and ξ_1 to $\langle \xi_1, \zeta_1 \rangle \eta_1$.

Proof. The proof follows immediately from the observations that $\xi_0 \otimes \zeta_1 \equiv \langle \xi_0, \zeta_1 \rangle$ and $\xi_1 \otimes \zeta_0 \equiv \langle \xi_1, \zeta_0 \rangle$ are both zero, being elements of $K_1(\mathbb{C})$. \square

Proposition 5.5. Let σ denote the trivial grading on \mathbb{C} . Then the even Fredholm module $(\mathbb{C}, \sigma, \epsilon, 0)$ gives a generator for $K^0(\mathcal{A}) = \mathbb{Z}$.

Proof. Since \mathcal{A} and $C(S^1)$ are KK -equivalent, one has $K^0(\mathcal{A}) = \mathbb{Z}$. This together with the simple observation that the pairing $\langle [(\mathbb{C}, \sigma, \epsilon, 0)], [1] \rangle$ is 1 gives us the required result. \square

We now put together the two results above to produce a KK -equivalence.

Proposition 5.6. Let ζ_1 be the K -homology class of the equivariant triple for \mathcal{A} (under the $SU_q(2)$ action), i.e. $\zeta_1 = [(L_2(h), \pi, D)]$. Let η_1 be the element $[z]$ in $K_1(C(S^1))$. Let ζ_0 and η_0 be generators for $K^0(\mathcal{A})$ and $K_0(C(S^1))$, respectively. Then $\gamma := \zeta_0 \otimes \eta_0 + \zeta_1 \otimes \eta_1$ gives a KK -equivalence between \mathcal{A} and $C(S^1)$.

Proof. Recall that $KK(\mathbb{C}, \mathcal{A}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $KK(\mathbb{C}, C(S^1)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and γ corresponds to the element $\xi \mapsto \xi \otimes \gamma$ in $Hom(KK(\mathbb{C}, \mathcal{A}), KK(\mathbb{C}, C(S^1)))$. Using these identifications, it is now easy to see that γ maps the element $1 \oplus 0$ to $1 \oplus 0$ and $0 \oplus 1$ to $0 \oplus 1$. In other words, γ is the element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $KK(\mathcal{A}, C(S^1)) \cong M_2(\mathbb{Z})$. This, being an invertible element, gives a KK -equivalence. \square

Theorem 5.7. *The spectral triple $(L_2(h) \oplus L_2(h), \pi \oplus \epsilon, D \oplus D)$ gives a fundamental class for $\mathcal{A} = C(SU_q(2))$.*

Proof. Let ρ be the representation of $C(S^1) \otimes C(S^1)$ on $L_2(S^1)$ given by

$$\rho(f \otimes g)h = fgh,$$

and let $\partial = \partial_\theta$ be the derivative. Then $(L_2(S^1), \rho, \partial)$ gives the standard fundamental class for $C(S^1)$.

Write λ for the class of $(L_2(S^1), \rho, \partial)$ in $KK^1(C(S^1) \otimes C(S^1), \mathbb{C})$. Then $(\gamma \otimes \gamma) \otimes \lambda$ gives a K -homology fundamental class for \mathcal{A} . Now

$$\begin{aligned} \gamma \otimes \gamma &= (\zeta_0 \otimes \eta_0 + \zeta_1 \otimes \eta_1) \otimes (\zeta_0 \otimes \eta_0 + \zeta_1 \otimes \eta_1) \\ &= (\zeta_0 \otimes \eta_0) \otimes (\zeta_0 \otimes \eta_0) + (\zeta_0 \otimes \eta_0) \otimes (\zeta_1 \otimes \eta_1) \\ &\quad + (\zeta_1 \otimes \eta_1) \otimes (\zeta_0 \otimes \eta_0) + (\zeta_1 \otimes \eta_1) \otimes (\zeta_1 \otimes \eta_1) \\ &= (\zeta_0 \otimes \zeta_0) \otimes (\eta_0 \otimes \eta_0) + (\zeta_0 \otimes \zeta_1) \otimes (\eta_0 \otimes \eta_1) \\ &\quad + (\zeta_1 \otimes \zeta_0) \otimes (\eta_1 \otimes \eta_0) + (\zeta_1 \otimes \zeta_1) \otimes (\eta_1 \otimes \eta_1). \end{aligned}$$

Clearly $(\eta_0 \otimes \eta_0) \otimes \lambda$ and $(\eta_1 \otimes \eta_1) \otimes \lambda$ are zero. Taking η_0 to be $[1]$ and $\eta_1 = [z]$, it follows that $(\eta_0 \otimes \eta_1) \otimes \lambda$ and $(\eta_1 \otimes \eta_0) \otimes \lambda$ are both 1. Therefore

$$(\gamma \otimes \gamma) \otimes \lambda = \zeta_0 \otimes \zeta_1 + \zeta_1 \otimes \zeta_0.$$

Taking the spectral triples $(L_2(h), \pi, D)$ and $(\mathbb{C}, \sigma, \epsilon, 0)$ to represent the classes ζ_1 and ζ_0 , respectively, it follows that the triple given by (\mathcal{H}, ϕ, D_0) , where

$$\mathcal{H} = L_2(h) \oplus L_2(h), \quad \phi(a \otimes b) = \pi(a)\epsilon(b) \oplus \epsilon(a)\pi(b), \quad D_0 = D \oplus D,$$

gives the required class. Therefore the restriction of ϕ to the first copy of \mathcal{A} together with \mathcal{H} and D gives a fundamental class for \mathcal{A} . \square

Remark 5.8. The $(2\ell + 1)$ -dimensional quantum sphere $S_q^{2\ell+1}$ is given by the universal C^* -algebra $A_\ell := C(S_q^{2\ell+1})$ generated by elements $z_1, z_2, \dots, z_{\ell+1}$ satisfying the following relations:

$$\begin{aligned} z_i z_j &= q z_j z_i, & 1 \leq j < i \leq \ell + 1, \\ z_i^* z_j &= q z_j z_i^*, & 1 \leq i \neq j \leq \ell + 1, \\ z_i z_i^* - z_i^* z_i + (1 - q^2) \sum_{k>i} z_k z_k^* &= 0, & 1 \leq i \leq \ell + 1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

The K -theory and the K -homology groups for this algebra A_ℓ are known and by the same argument as in the proof of Theorem 5.1, A_ℓ is KK -equivalent to $C(S^1)$ and Poincaré duality holds for A_ℓ .

If one replaces the counit ϵ for $C(SU_q(2))$ by the functional

$$z_j \mapsto \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j \neq 1, \end{cases}$$

on A_ℓ , and replaces the equivariant spectral triple for $SU_q(2)$ by the spectral triple for $S_q^{2\ell+1}$ equivariant under the action of $SU_q(\ell+1)$ constructed in [5], then everything in this section goes through for the odd-dimensional quantum spheres.

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