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UNDISTORTED SOLVABLE LINEAR GROUPS

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Abstract. We discuss distortion of solvable linear groups over a locally compact field and provide necessary and sufficient conditions for a subgroup to be undistorted when the field is of characteristic zero.

1. Introduction

Distortion of subgroups of solvable groups is the main object of our study. We concentrate on the case of linear solvable groups. In this case Gromov [6, 3. D_2' Remark] gives a natural condition for a subgroup to be undistorted using its eigenvalues: no eigenvalues of absolute value one in the adjoint representation (see Example 6.6 below for a discussion of this condition). We give necessary and sufficient conditions in our main theorem, Theorem 3.2, that a linear solvable group over a characteristic zero local field is undistorted in its ambient general linear group.

2. Preliminaries

2.1. A *pseudometric* on X is a function $d : X \times X \to \mathbb{R}$ which has all the properties of a metric, i.e., non–negativity, symmetry, triangle inequality, and zero on the diagonal, but not necessarily the property that $d(x, y) = 0$ implies $x = y$. A *pseudometric space* is a pair (X, d) consisting of a set X and a pseudometric d on X.

2.2. Let (X, d_X) and (Y, d_Y) be pseudometric spaces. A map $f : X \to Y$ is called a *submetry* if there are real constants $C_1 > 0$ and C_2 such that

$$d_Y(f(x), f(x')) \leq C_1 d_X(x, x') + C_2$$

for every pair x, x' of points of X. If $X = Y$ and the identity map of X is a submetry from (X, d_1) to (X, d_2), we write

$$d_1 \prec d_2$$

and say that d_1 is dominated by d_2. A submetry $f : (X, d_X) \to (Y, d_Y)$ is called a *quasi–isometry* if there is a submetry $g : (Y, d_Y) \to (X, d_X)$ such that both $g \circ f$ and $f \circ g$ are of bounded distance from the identity, i.e., if there is a real constant C_3 such that $d_X(x, g \circ f(x)) \leq C_3$ and $d_Y(y, f \circ g(y)) \leq C_3$ for every $x \in X$ and every $y \in Y$. This definition is equivalent to the usual definition of a quasi–isometry f

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from (X, d_X) to (Y, d_Y); namely requiring that there be real constants $C_1 > 0$ and C_2 such that

$$C_1^{-1} d_X(x, x') - C_2 \leq d_Y(f(x), f(x')) \leq C_1 d_X(x, x') + C_3$$

for every pair x, x' of points in X and that there be, for every $y \in Y$, a point $x \in X$ such that $d_Y(y, f(x)) \leq C_2$. Since then the map g mapping $y \in Y$ to such an $x \in X$ gives the required quasi–inverse g from (Y, d_Y) to (X, d_x).

Two pseudometrics d_1 and d_2 on the same set X are called *quasi–equivalent* if the identity map of X is a quasi–isometry from (X, d_1) to (X, d_2), or from (X, d_2) to (X, d_1), which is the same condition.

Thus d_1 and d_2 are quasi–equivalent iff they dominate each other, i.e., $d_1 \prec d_2$ and $d_2 \prec d_1$. We then write

$$d_1 \not\prec d_2.$$

2.3. The word metric. Let G be a locally compact topological group. Suppose G has a compact set E of generators. Let ℓ_E be the corresponding *word length*, so

$$\ell_E(g) = \min\{q\,;\, g = a_1^{\varepsilon_1} \dots a_q^{\varepsilon_q} \text{ with } \ a_i \in E \text{ and } \ \varepsilon_i \in \{+1, -1\}\}.$$

Let d_E be the corresponding left invariant metric; that is, $d_E(g, h) = \ell_E(g^{-1}h)$. The metric d_E is called the *word metric* for the set E of generators of G. It is well known that the word metrics d_E and $d_{E'}$ are quasi–equivalent if E and E' are sets of generators of G which are either compact or relatively compact neighborhoods of the identity. We thus have a unique quasi–equivalence class of pseudometrics on G, and we call every pseudometric in this quasi–equivalence class a (or sometimes even the) word metric on G. Whenever we speak of G as a pseudometric space, we mean one of these pseudometrics, unless a special warning is given.

2.4. The word metric d is a proper metric. This means that for every $x \in G$ and every $R \in \mathbb{R}$ the ball $B_R(x) = \{g \in G\,;\, d(g, x) < R\}$ of radius R with center x has compact closure.

Proof. If d and d' are quasi–equivalent pseudometrics, then one of them is proper iff the other one is as well. We thus may assume that $d = d_E$ for a compact set E of generators of G. If $R \leq N \in \mathbb{N}$, then $B_R(e) \subset B_N(e) = (E \cup E^{-1})^N$ consists of words in E of length $\leq N$ and is hence relatively compact. Then $B_R(x)$ also has compact closure since it is the left translate of $B_R(e)$ by x. □

2.5. Undistorted subgroups. Let G and H be two compactly generated locally compact topological groups. A homomorphism $f : G \to H$ is called *undistorted* if the word metric d_G of G and the pull back $d_H \circ (f \times f)$ of the word metric of H are quasi–equivalent.

a) If $f : G \to H$ is a continuous homomorphism, then f is a submetry. Thus $d_H \circ (f \times f) \prec d_G$.

b) If $f : G \to H$ is a continuous surjective homomorphism, then for appropriately chosen generating sets, for any $h \in H$ there is a $g \in G$ such that $f(g) = h$ and $\ell_G(g) = \ell_H(h)$.

c) If $f : G \to H$ is a continuous undistorted homomorphism, then f is a proper mapping. In particular, the kernel of f is a compact subgroup of G, f is a closed mapping, f induces an isomorphism of topological groups $G/\ker f \to \mathrm{Im} f$, and $\mathrm{Im} f$ is a closed subgroup of H.

Proof. Part a) follows easily since we may assume that our compact generating set for H contains the f–image of our compact generating set for G. To see part b) take for the generating set of H the image of a generating set for G and then take for a given h the corresponding shortest word g in the generators for G. Part c) follows from Section 2.4. □

A subgroup G of a compactly generated locally compact topological group H is called *undistorted* if G endowed with the induced topology from H is locally compact and compactly generated and if the inclusion map $i : G \to H$ is undistorted.

2.6. Every undistorted subgroup is closed. In fact, every locally compact subgroup of a locally compact topological group is closed since a locally closed subgroup of a topological group is closed; see [4].

2.7. The word metric on the general linear group. Let k be a local field. Let $|\cdot|$ be the absolute value on k. We endow k^n, $n \in \mathbb{N}$, with the product topology. Given a finite dimensional vector space V over k, there is a unique topology on V such that every k–linear isomorphism $V \to k^n$ is a homeomorphism. The general linear group $GL_n(k)$, considered as an open subset of the k–vector space of k–linear endomorphisms of k^n, is a locally compact topological group.

In order to see that $GL_n(k)$ is compactly generated, and for other applications, we use the *Cartan decomposition*. Let K be the following maximal compact subgroup of $GL_n(k)$:

$$K = \begin{cases} O(n) & \text{for } k = \mathbb{R}, \\ U(n) & \text{for } k = \mathbb{C}, \\ GL_n([2009/06/22v3.00EulerFraktur]\mathfrak{o}) & \text{for } k \text{ non–Archimedean.} \end{cases}$$

Here $O(n)$ is the orthogonal group, $U(n)$ is the unitary group and $\mathfrak{o} = \{x \in k\,;\, |x| \leq 1\}$ is the ring of integers in k, and $\mathfrak{o}^* = \{x \in k\,;\, |x| = 1\}$ is the group of units of $\mathfrak{o}$ and $GL_n(\mathfrak{o}) = \{g \in GL_n(k) \cap \mathfrak{o}^{n^2}\,;\, \det g \in \mathfrak{o}^*\}$. Let $T_n(k)$ be the subgroup of diagonal matrices in $GL_n(k)$.

2.8. Cartan decomposition

$$GL_n(k) = K \cdot T_n(k) \cdot K.$$

Since we shall need facts from the proof we give a sketch of a proof. The statement is well known for the Archimedean fields. For non–Archimedean k let $g \in GL_n(k)$ and let a_{ij} be an entry of the matrix g with maximal absolute value, so $|a_{ij}| \geq |a_{k\ell}|$ for every entry $a_{k\ell}$ of g. By multiplying g with a permutation matrix on the left and one on the right we may assume that $i = j = n$. Note that the group of permutation matrices is a subgroup of K. Then multiplying g with a matrix in K from the right and one from the left, as in the Gauss elimination procedure, we get a matrix with $a_{in} = 0$ and $a_{ni} = 0$ for every $i < n$. This reduces our claim to $GL_{n-1}(k)$ and thus finishes the proof by induction.

Proposition 2.9. *$GL_n(k)$ has a compact set of generators, and its word metric is quasi–equivalent to the following pseudometric:*

$$d(g,h) = \max\{\left|\log \|g^{-1}h\|\right|, \left|\log \|h^{-1}g\|\right|\}.$$

Here we choose a vector space norm $\|\cdot\|$ on k^n and take for elements of $GL_n(k)$ the operator norm on k^n, also denoted $\|\cdot\|$. The logarithm may be taken with respect to any basis > 1. Different bases of the logarithm give quasi–equivalent pseudometrics. The absolute value signs outside the logarithm denote the absolute value in $\mathbb{R}$. Different norms $\|\cdot\|$ lead to quasi–equivalent pseudometrics. Actually the constant C_1 in the formula in Section 2.2 can be chosen to be equal to 1. As the operator norm is submultiplicative, every term under the maximum fulfills the triangle inequality. So the formula in Proposition 2.9 in fact defines a left invariant pseudometric on $GL_n(k)$.

Proof of the proposition. The multiplicative group $k^* = GL_1(k)$ is generated by $\{x \in k\,;\, |\log|x|| < R\}$ for R sufficiently large. This shows our claim for $n = 1$. Since $T_n(k)$ is isomorphic to $(k^*)^n$, it follows that $T_n(k)$ is compactly generated and the word metric on $T_n(k)$ is given by the formula in Proposition 2.9 for $g, h \in T_n(k)$ if we choose for $\|\cdot\|$ the sup–norm on k^n. It follows from the Cartan decomposition that $GL_n(k)$ has a compact set of generators.

Let d_T and d_G be the word metrics on $T = T_n(k)$ and $G = GL_n(k)$ for appropriately chosen compact sets of generators and let d be the pseudometric defined by the formula in Proposition 2.9. Then if $h = k_1 \cdot t \cdot k_2$ with $h \in G$, $t \in T$ and $k_1, k_2 \in K$, we have

$$d_G(e,h) \le d_G(e,t) + 2C_1 \le d_T(e,t) + 2C_1 \asymp d(e,t) + 2C_1 \le d(e,h) + 2C_1 + 2C_2,$$

where $C_1 = \max_{k\in K} d_G(e,k)$, $C_2 = \max_{k\in K} d(e,k)$ and the symbol $\asymp$ means that the two expressions dominate each other. The first inequality follows from the triangle inequality for d_G, the second one from Section 2.5 a), the third one by the first part of this proof and the last one by the triangle inequality for d. The converse

$$d(e,h) \le d(e,t) + 2C_2 \asymp d_T(e,t) + 2C_2 \le d_G(e,t) + 2C_2 \le d_G(e,h) + 2C_1 + 2C_2$$

follows by the same arguments.

Note that we have actually proved the following:

Corollary 2.10. *$T_n(k)$ is undistorted in $GL_n(k)$.*

A similar statement holds for the word metric on any reductive group. The proofs are similar to those above; see [1].

Corollary 2.11. *Let G be a reductive group over a local field k. Then every maximal split torus in G is undistorted. Every faithful k–representation $\rho : G \to GL_n(k)$ is undistorted.*

Corollary 2.12. *Let k' be a local field containing the local field k. Then $[k' : k]$ is finite and $GL_n(k)$ is undistorted in $GL_n(k')$.*

Proof. It suffices to prove undistortedness of the inclusion $T_n(k) \subset T_n(k')$ by Corollary 2.10, and hence of $k^* \subset k'^*$. But the distance from e is given by the absolute value of the valuations of k and k', respectively, and the restriction of the latter to k^* is a multiple of the former. □

Let $B_n(k)$ be the subgroup of upper triangular matrices in $GL_n(k)$.

Proposition 2.13. *$B_n(k)$ is undistorted in $GL_n(k)$.*

Since $GL_n(k)/B_n(k)$ is compact, this follows from Lemma 7.8 in the appendix.

3. Main result

3.1. Let G be a closed subgroup of $B_n(k)$, the group of upper triangular invertible matrices with entries from the local field k. Let $\lambda : B_n(k) \to T_n(k)$ be the homomorphism which sends every upper triangular matrix to its diagonal part. The kernel of λ is the group $U_n(k)$ of unipotent upper triangular matrices. The kernel of $\lambda \mid G$ is $N := G \cap U_n(k)$. The group G acts by inner automorphisms on N and hence on the Zariski closure N^* of N and on N'^*, the Zariski closure of N', where N' denotes the commutator subgroup of N. The action of N on the vector space $V := N^*/N'^*$ is trivial; we thus have a representation ρ of G/N on V. The weights of this representation are of the form $\chi \circ \lambda$, where χ is a weight of T_n on U_n, as is easy to see; see Lemma 3.8 below. So V decomposes into a direct sum of primary subspaces $V = \bigoplus V_\mu$, where for every weight μ we put

$$V_\mu = \{v \in V;\ (\rho(g) - \mu(g)Id)^n v = 0 \text{ for every } g \in G\}.$$

Let V^0 be the sum of those subspaces V_μ, where $\mu(g)$ has absolute value 1 for every $g \in G$. We then have a projection map

$$p^0 : N \to N^*/N'^* = V \to V^0.$$

We now state the main result; the proof will be given below.

Theorem 3.2. *Suppose k has characteristic zero. Then G is undistorted in $GL_n(k)$ iff the following two conditions hold:*
a) $\lambda(G)$ is closed in $T_n(k)$.
b) $p^0(N)$ has compact closure in V^0.

In the case of a locally compact field of positive characteristic one might apply similar techniques to prove our main theorem for algebraic groups except in characteristic two, but we leave this for another time.

We now make some comments concerning the different types of local fields.

Remarks 3.3. *If k is Archimedean, the only subgroup of the vector space V^0 having compact closure is the zero subgroup. Thus, for Archimedean k condition b) is equivalent to condition b′) $V^0 = 0$.*

Compare this with the following theorem of Borel and Tits [3, 13.4].

Theorem 3.4. *If k is p-adic, i.e., a finite extension of a field $\mathbb{Q}_p$, and G is Zariski closed, then G is compactly generated iff $V^0 = 0$.*

3.5. An undistorted subgroup is by definition compactly generated. This raises the question of whether the property of being undistorted is stronger than that of being compactly generated. For Archimedean k, every closed subgroup G of $B_n(k)$ is compactly generated by a theorem of Mostow; see Theorem 7.4. So there is a wide gap.

In the p–adic case, however, the fact that compact generation implies undistortedness follows from the next result.

Theorem 3.6. *If k is p–adic, then G is compactly generated only if a) and b) hold.*

This result is proven below in Theorem 4.10 and Corollary 4.16. For the special case that G is Zariski closed and k is p–adic, the equivalence of being undistorted and compactly generated has already been proven by Mustapha [8].

Theorem 3.2 is in a sense a part of the philosphy that whereas qualitiative results differ for Archimedean and non-Archimedean fields, this distinction disappears for quantitative results.

3.7. We now prove our claim concerning the weights of ρ.

Lemma 3.8. *Every weight of the representation ρ is of the form $\chi \circ \lambda | G$, where $\lambda : B_n(k) \to T_n(k)$ was defined above and χ is one of the weights of the adjoint representation of $T_n(k)$ on $U_n(k)$, so of the form $\chi = \delta_j - \delta_i$ with $j < i$ and δ_i is the projection of T_n to its i–th diagonal entry.*

Proof. $U = U_n(k)$ is filtered by the subgroups $U^{(i)} = \{(a_{k\ell})_{k,\ell} \in U \ ; \ a_{k\ell} = 0$ if $0 < \ell - k < i\}$. Thus $U^{(1)} = U$ and $[U^{(i)}, U^{(j)}] = U^{i+j}$. The groups $U^{(i)}/U^{(i+1)}$ are vector spaces over k in a natural way, the isomorphism $k^{n-i} \to U^{(i)}/U^{(i+1)}$ being given by $(\alpha_1, \ldots, \alpha_{n-i}) \to 1 + \alpha_1 E_{1,i+1} + \cdots + \alpha_{n-i} E_{n-i,n} \mod U^{(i+1)}$. The group B_n acts on U by conjugation leaving each $U^{(i)}$ invariant and hence on $U^{(i)}/U^{(i+1)}$. The above isomorphism then identifies $U^{(i)}/U^{(i+1)}$ with the direct sum of $n - i$ 1–dimensional representation spaces with associated characters $\delta_\ell - \delta_{\ell+i}$, $\ell = 1, \ldots, n - i$. It follows that the representation of G on $U^{(i)}/U^{(i+1)}$ is semisimple, since every endomorphism is semisimple and the operators commute. The associated weights are of the form $\chi \circ \lambda$. The filtration of U induces a filtration of N^*, and hence of N^*/N'^*, whose factors are submodules of the $U^{(i)}/U^{(i+1)}$. $\square$

4. Necessity of conditions

4.1. **Condition a).**

4.1. Let $\lambda : B_n(k) \to T_n(k)$ be the homomorphism which sends every upper triangular matrix to its diagonal part. So if $g = (g_{ij})_{i,j} \in B_n(k)$, then $\lambda(g) = (t_{ij})_{i,j}$ with $t_{ij} = 0$ if $i \neq j$ and $t_{ii} = g_{ii}$ for $i = 1, \ldots, n$. The kernel of λ is $U = U_n(k)$, the group of upper triangular unipotent matrices.

4.2. Suppose G is a closed subgroup of $B := B_n(k)$. We want to show that $\lambda(G)$ is a closed subgroup of $T := T_n(k)$ if G is undistorted in B. More generally, we analyze the homomorphism $\lambda \mid G : G \to T$ for any closed subgroup G of B. The kernel of $\lambda \mid G$ is $G \cap U$. Let H be the locally compact topological group $G/G \cap U$. We have a continuous bijective homomorphism $\overline{\lambda} : H \to \lambda(G)$ induced by λ. In particular, H is abelian. Another way of looking at this situation is the following. The abstract group $\lambda(G)$ has two topologies, one coming from G which is locally compact. This topological group will always be called H, and $\lambda(G)$ has a topology induced from T, which may be strange. This topological group will always be called $\lambda(G)$. We distinguish three cases, called α, β and γ. Every compactly generated closed subgroup G of B falls under either case α or case γ, not both, by Lemma 4.7. Case β is an easier subcase of case γ.

Lemma 4.3. Case α) *The following statements are equivalent:*
i) $\lambda(G)$ is locally compact.
ii) $\lambda(G)$ is a closed subgroup of T.
iii) $\overline{\lambda} : H \to \lambda(G)$ is an isomorphism of topological groups.

A fourth equivalent condition will be formulated later on; see Lemma 4.7.

Proof. *i)* $\Longrightarrow$ *ii)* If $\lambda(G)$ is locally compact, then $\lambda(G)$ is a locally closed subgroup of T. But every locally closed subgroup of a topological group is closed; see [4].

ii) $\Longrightarrow$ *iii)* If $\lambda(G)$ is closed in T, then $\lambda(G)$ is a countable union of compact subsets. The same holds for G and hence for H. So by the Baire category argument there is a compact neighborhood of the identity, say K, in H, such that $\lambda(K)$ contains a non–empty open subset in $\lambda(G)$, and hence $\lambda(L)$ is a neighborhood of the identity in $\lambda(G)$ for $L = K \cdot K^{-1}$. Then $\overline{\lambda} : L \to \lambda(L)$ is a bijective continuous map of compact spaces and hence is a homeomorphism. Thus λ is an open mapping near the identity elements of H and $\lambda(G)$, and hence everywhere.

iii) $\Longrightarrow$ *i)* is trivial. □

4.4. Suppose now that G is compactly generated and still a closed subgroup of B. Then $H = G/G \cap U$ is also compactly generated, and hence H has a unique maximal compact subgroup, say K, and H/K is isomorphic to a cocompact closed subgroup of a finite dimensional real vector space, $H/K \otimes \mathbb{R} =: V$, by Pontryagin's structure theorem [10]. We identify H/K with this subgroup of V. Then H/K is undistorted in V; see Lemma 7.8. Let $\|\cdot\|$ be a norm on V. We have

$$\ell_{H/K} \asymp \ell_V \,|H/K \asymp \|\cdot\|\,|_{H/K}\,.$$

Hence if $\overline{\overline{\lambda}} : G \to H/K$ denotes the natural map

$$G \to H = G/G \cap U \to H/K,$$

we have

$$\ell_G(g) \succ \ell_V\left(\overline{\overline{\lambda}}(g)\right) \asymp \|\overline{\overline{\lambda}}(g)\|. \tag{4.12}$$

Let L be the maximal compact subgroup of T. So L is the group of those diagonal matrices whose diagonal entries have absolute value 1. Also, L is the intersection of T with the maximal compact subgroup of $GL(n,k)$ exhibited above; see Section 2.7. The homomorphism $\bar{\lambda}$ induces a homomorphism

$$\mu : H/K \to T/L.$$

Lemma 4.5. Case β) *Suppose μ is not injective. So there is an element $g \in G$ such that $h := g(G \cap U) \in H$ is a non–compact element of H but $\lambda(g)$ is a compact element of T. Then*

$$\ell_G(g^n) \asymp n \text{ and } \ell_B(g^n) \prec \log(n).$$

So G is (at least) exponentially distorted. If k is non–Archimedean, then $\ell_B(g^n)$ is actually bounded.

Proof. Recall that an element g of a locally compact topological group is called compact if the cyclic subgroup generated by g has compact closure; otherwise g is called non–compact. Our first claim follows from the inequality (4.12). The second claim follows from the Lemma 7.18 in the appendix, or in an elementary way as follows. Write g as $d + u$, where d is diagonalizable, u is nilpotent, and d and u commute. So $g = d+u$ is the additive Jordan decomposition of g. Note that $\lambda(g)$ is a diagonal matrix whose entries are the eigenvalues of g. So the Jordan decomposition of g can be done over k. By assumption $\lambda(g) \in L$, so all the eigenvalues of g are of absolute value 1.

So with respect to the sup–norm corresponding to an eigenbasis for g, we have $\|d^m\| = 1$ for every $m \in \mathbb{Z}$. Then for every $m \in \mathbb{N}$ we have

$$g^m = (d+u)^m = d^m + \binom{m}{1} d^{m-1}u + \cdots + \binom{m}{n-1} d^{m-n+1}u^{n-1},$$

and hence $\|g^m\|$ is bounded above by a polynomial in m of degree at most $n-1$. The same holds for g^{-1} and hence our second claim.

If k is non–Archimedean, the Jordan normal form of g is in $GL(n, \mathfrak{o})$ and hence contained in a compact subgroup of $GL(n, k)$. □

4.6. We may think of μ as the restriction of the $\mathbb{R}$–linear map

$$\mu \otimes \mathbb{R} : V = H/K \otimes \mathbb{R} \to T/L \otimes \mathbb{R}.$$

Lemma 4.7. Case α) continued. *The following statements are equivalent:*

iii) $\overline{\lambda} : H \to \lambda(G)$ is an isomorphism of topological groups.

iv) $\mu \otimes \mathbb{R}$ is injective.

Proof. In the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\overline{\lambda}} & T \\ \downarrow & & \downarrow \\ H/K & \xrightarrow{\mu} & T/L \end{array}$$

the vertical arrows are proper surjections, so $H \to T$ is a proper map iff $\mu : H/K \to T/L$ is a proper map iff $\mu \otimes \mathbb{R}$ is an injective $\mathbb{R}$–linear map, since both H/K and T/L are cocompact closed subgroups of the finite dimensional real vector spaces $H/K \otimes \mathbb{R}$ and $T/L \otimes \mathbb{R}$, respectively. □

It may happen that μ is injective but $\mu \otimes \mathbb{R}$ is not injective; e.g. if H/K is discrete and μ maps a basis of H/K to elements in T/L which are linearly independent over $\mathbb{Q}$ but not over $\mathbb{R}$. For an example see Section 6.2. This can only happen if k is Archimedean, by the following lemma.

Lemma 4.8. *If k is non–Archimedean, either $\overline{\lambda} : H \to \lambda(G)$ is an isomorphism of topological groups or μ is not injective. Thus if μ is injective, then $\overline{\lambda} : H \to \lambda(G)$ is an isomorphism of topological groups.*

Proof. A locally compact topological group is totally disconnected iff there is a neighborhood base of the identity consisting of open compact subgroups; see [4]. Thus, since G is totally disconnected, so are H and H/K. But H/K is the direct sum of a vector group isomorphic to some $\mathbb{R}^p$ and a group isomorphic to $\mathbb{Z}^m$ for some m, by Pontryagin's structure theorem. So $H/K \cong \mathbb{Z}^m$ and $T/L \cong (k^*/\mathfrak{o}^*)^n \cong \mathbb{Z}^n$. We thus may think of μ as a homomorphism from $\mathbb{Z}^m$ to $\mathbb{Z}^n$. Thus μ is injective iff $\mu \otimes \mathbb{R}$ is injective. □

One might think that we need not consider case γ discussed below since it does not occur if k is non–Archimedean or if k is Archimedean and G is connected. In the latter case $\mu = \mu \otimes \mathbb{R}$, and it follows from Theorem 7.5 of the appendix that every group G we consider is virtually contained in a closed connected group as a cocompact subgroup. However, condition a) is not preserved under passing to cocompact subgroups or supergroups, as Section 6.3 shows.

Lemma 4.9. Case γ) *Suppose $\mu \otimes \mathbb{R}$ is not injective. Then there is a sequence of elements $g_j \in G$ such that*

$$\ell_G(g_j) \prec\!\!\sim j \text{ and } \ell_B(g_j) \prec \log(j).$$

Thus G is (at least) exponentially distorted.

Proof. The proof is similar to that of Case β), just technically more involved. The diagonal part remains bounded, not only for the g_j themselves, but on the entire way leading to g_j. As a consequence the unipotent part grows at most logarithmically.

Now to the proof itself. By introducing appropriate coordinates we may assume that $H/K \otimes \mathbb{R} = \mathbb{R}^m$ and H/K contains $\mathbb{Z}^m$. Let $\|\cdot\|$ be the sup norm on $\mathbb{R}^m$, $\|(x_1, \dots, x_m)\| = \sup\{|x_i|\,;\, i = 1, \dots, m\}$. Choose a ray $R = \{tv\,;\, t \in [0, \infty)\}$ in $\mathrm{Ker}(\mu \otimes \mathbb{R})$. For every $j \in \mathbb{N}$ there is a point $v_j \in R$ with $\|v_j\| = j$ and a point $w_j \in \mathbb{Z}^m$ such that $\|v_j - w_j\| \leq 1$. So $\|w_{j+1} - w_j\| \leq 3$. Let $\pi : G \to G/G \cap U = H \to H/K \cong \mathbb{Z}^m$ be the composed natural projection. There is a compact subset C of G such that $\pi(C)$ contains all the elements $w \in \mathbb{Z}^m$ with $\|w\| \leq 3$. Choose for every $j \in \mathbb{N}$ an element $c_j \in C$ such that $\pi(c_j) = w_j - w_{j-1}$. Here we put $w_0 = 0$. We define

$$g_j = c_1 c_2 \dots c_j.$$

Then

$$\ell_G(g_j) \prec\!\!\sim j,$$

since on one hand $\ell_G \succ \ell_{H/K} \prec\!\!\sim \|\cdot\|$ and on the other hand the length of g_j is at most j when computed with respect to any generating set of G containing C.

To compute $\ell_B(g_j)$ we need not stay inside G. Note that C is contained in a subset of $B = B_n(k)$ of the form $D \cdot A$ where D and A are compact subsets of T and U, respectively. Put

$$c_j = d_j \cdot a_j\,,\ d_j \in D \subset T, a_j \in A \subset U.$$

Then

$$d_j = \lambda(c_j),$$

since d_j is the diagonal part of c_j, and hence

$$d_j \bmod L = \mu\pi(c_j) = \mu(w_j - w_{j-1}).$$

Now

$$g_j = d_1 a_1 d_2 a_2 \dots d_j a_j = d_1 \dots d_j e_{1j}^{-1} a_1 e_{1j} e_{2j}^{-1} a_2 e_{2j} \dots e_{jj}^{-1} a_j e_{jj},$$

where

$$e_{\ell,j} = d_{\ell+1} \dots d_j \text{ for } \ell = 1, \dots, j.$$

Thus the diagonal part of g_j is

$$\lambda(g_j) = d_1 \dots d_j$$

and

$$\begin{aligned} \lambda(g_j) \bmod L &= \sum_{\ell=1}^{j} \mu(w_\ell) - \mu(w_{\ell-1}) \\ &= \mu(w_j) \end{aligned}$$

is a compact subset of T/L since w_j have bounded distance from the kernel of $\mu\otimes\mathbb{R}$. The same holds for the $e_{\ell,j}$'s since

$$\lambda(e_{\ell,j}) \bmod L = \sum_{k=\ell+1}^{j} \mu(w_k) - \mu(w_{k-1}) = \mu(w_j) - \mu(w_\ell).$$

The a_ℓ's belong to the compact subset A of U, hence there is a common compact subset A' of U containing all the $e_{\ell,j}^{-1} a_\ell e_{\ell,j}$ for all $j \in \mathbb{N}$ and all $\ell = 1, \dots, j$. It follows that

$$\left\| \prod_{\ell=1}^{j} \left(e_{\ell,j}^{-1} a_\ell e_{\ell,j} \right) \right\| \prec \log(j);$$

see Lemma 7.16 in the appendix. □

Theorem 4.10. *If k is non–Archimedean and G is a compactly generated closed subgroup of $B_n(k)$, then $\lambda(G)$ is a closed subgroup of $T_n(k)$.*

Proof. Again let L be the maximal compact subgroup of $T_n(k)$. So L is the group of diagonal matrices with diagonal entries from the group $\mathfrak{o}^*$ of units of $\mathfrak{o}$. It suffices to show that $\lambda(G) \cap L$ is compact, since L is an open subgroup of $T_n(k)$. We will show that there is a compact subgroup H of G such that $\lambda(H) = \lambda(G) \cap L$. Let us write $D = \lambda^{-1}(L)$. So D is the group of those matrices in $B_n(k)$ whose diagonal entries are in $\mathfrak{o}^*$. Let K be a compact generating subset of G. We can write K in the form $K = \bigcup t_i K_i$ with a finite number of elements $t_i \in K$ and a finite number of compact subsets $K_i \subset D \cap G$, since $B_n(k)/D \cong T_n(k)/L \cong \mathbb{Z}^n$ is discrete. Let A be the subgroup of G generated by the t_i's. Note that we do not know (or need) that A is abelian or closed. Every element g of G can be written as a product of an element $a \in A$ and a product of conjugates of $K' = \bigcup K_i$ by elements of A. Thus $\lambda(g) \in L$ iff $\lambda(a) \in L$ iff $a \in \ker \pi \circ \lambda$, where $\pi : T_n(k) \to T_n(k)/L \cong \mathbb{Z}^n$ is the natural homomorphism. The group $\pi \circ \lambda(A)$ is isomorphic to a subgroup of $\mathbb{Z}^n$ and hence has a finite presentation. So finitely many relations between the $\pi \circ \lambda(t_i)$ imply all the relations between them. Thus there is a finitely generated subgroup A' of A such that the smallest normal subgroup of A containing A' is the kernel of $\pi \circ \lambda | A$.

Let H be the closure of the subgroup of D generated by A' and K'. Then H is a subgroup of $D \cap G$, by construction, and is compact, by Corollary 7.17, since it is the closure of a subgroup of D generated by a compact set. We claim that $\lambda(H) = \lambda(G) \cap L$. We show that in fact $D \cap G = \lambda^{-1}(L) \cap G = \ker(\pi \circ \lambda \mid G)$ is the smallest normal subgroup — let us call it H_1 — of G which contains H. This implies our claim since $\lambda(G)$ is abelian. Clearly, $H_1 \subset D \cap G$. To see the converse recall that every element $g \in G$ can be written as a product $a \cdot b$, where $a \in A$ and b is a product of conjugates of elements of K'; hence $b \in H_1$. Furthermore $g \in G$ is in the kernel of $\pi \circ \lambda$ if and only if $a \in A$ is in the kernel of $\pi \circ \lambda$ if and only if a is in the smallest normal subgroup of A containing A'; hence $a \in H_1$. □

Thus we have proven the necessity of condition a) for undistortedness in the non-Archimedean case using Theorem 4.10 and in the Archimedean case using Lemmas 4.3, 4.7, and 4.9.

4.2. **Necessity of condition b).**

The statements and proofs for the two cases — k Archimedean and p–adic k — are so different that we state them as two distinct results.

Again, we assume our standing hypotheses and notation, as introduced in the beginning of Section 3.

Archimedean case:

Proposition 4.11. *If k is Archimedean and V^0 is non–zero, then G is exponentially distorted.*

Proof. We know by Theorem 7.4 that G is compactly generated. Next, we may assume that $k = \mathbb{C}$. We may furthermore assume that N is a connected real Lie subgroup of $U_n(\mathbb{C})$ or, equivalently, Zariski closed over $\mathbb{R}$ for the following reason. Regard B_n as an algebraic group over $\mathbb{R}$. Let N_1 be the $\mathbb{R}$–Zariski closure of N. Then G normalizes N_1 and N_1/N is compact. We may thus pass from G to $G \cdot N_1$. Let $\mathfrak{u}$ and $\mathfrak{n}$ be the Lie algebras of $U_n(\mathbb{C})$ and N, respectively. Let $\|\cdot\|$ be a vector space norm on $\mathfrak{u}$. Then we have

$$\ell \circ \exp \asymp \log \|\cdot\| \text{ on } \mathfrak{u}$$

for the length function ℓ of $B_n(\mathbb{C})$ or $GL_n(\mathbb{C})$, by Proposition 2.9, since the exponential map $\mathfrak{u} \to U_n(\mathbb{C})$ is a polynomial isomorphism in the standard coordinates. We will show that if $V^0 \neq \{0\}$, then there is a real line $\{tx; t \in \mathbb{R}\}$ in $\mathfrak{n}$ such that

$$\ell_G(\exp(tx))^2 \succ |t| \text{ for } t \in \mathbb{R},$$

thus proving our claim.

The representation ρ of G on V^0 induced from the conjugation action has the property that every eigenvalue of every element $\rho(g) \in GL(V^0)$ has absolute value 1. Then there is a $\rho(G)$–invariant subspace V_2 of V^0 such that $V_2 \neq V^0$, and for the induced representation ρ_1 of G on $V_1 := V^0/V_2$ the image $\rho_1(G)$ is contained in a compact subgroup of $GL(V_1)$; see [5]. Let $V^{\neq 0}$ be the complementary $\rho(G)$–invariant subspace of V^0 in V (see Section 3.1). Let N_2 be the inverse image of $V^{\neq 0} \oplus V_2$ under the natural map $N \to N/N' = V$. Then N_2 is an $\mathbb{R}$–Zariski closed subgroup of the $\mathbb{R}$–Zariski closure G^* of G. Let H be the closure — with respect to the Euclidean topology — of the image of G under the composed homomorphism φ:

$$G \hookrightarrow G^* \to G^*/N_2.$$

We can identify the kernel of $G^*/N_2 \to G^*/N$ with V_1. So V_1 is a closed normal subgroup of H with the property that for every compact subset K of V_1 the set of conjugates $\bigcup_{g \in G} g\,K g^{-1}$ of K by elements of G, or rather of G/N_2, is relatively compact. Then the same is true for the H–conjugates $\bigcup_{h \in H} h\,K h^{-1}$. So all the assumptions of Lemma 7.18 are fulfilled for $V_1 \triangleleft H$, and hence

$$\ell_{V_1} \prec (\ell_H \mid V_1)^2.$$

Since $V_1 \neq \{0\}$ there is an element $x \in \mathfrak{n}$ such that the φ–image of $\exp x$ in V_1 is non–zero. We thus have

$$|t| \asymp \|\varphi \exp tx\| = \ell_{V_1}(\varphi \exp tx) \prec \ell_H(\varphi \exp tx)^2 \prec \ell_G(\exp tx)^2.$$

The first equivalence holds for every norm $\|\cdot\|$ on V_1, the second for every real vector space, the third inequality follows from Lemma 7.18 and the last one from Section 2.5 a). $\square$

The p–adic case:

Proposition 4.12. *Let k be a p–adic field. Let A be a compact subgroup of $U_n(k)$. Let $\mathfrak{a}$ be the smallest Lie algebra over $\mathbb{Q}_p$ containing $\log A$. Then A is an open subgroup of $\exp \mathfrak{a}$.*

Proof. It suffices to show that A contains a neighborhood of e in $\exp \mathfrak{a}$. We may thus assume that A is compact by intersecting with the compact open subgroup $U_n(\mathfrak{o})$ of $U_n(k)$. The image of A under the composition of the maps $\log : \exp \mathfrak{a} \to \mathfrak{a}$ and the projection $p : \mathfrak{a} \to \mathfrak{a}^{ab} = \mathfrak{a}/\mathfrak{a}'$ is a compact subgroup of the additive group of the vector space $\mathfrak{a}^{ab}$ over $\mathbb{Q}_p$ and spans this vector space, hence is an open subgroup of $\mathfrak{a}^{ab}$, since it is a $\mathbb{Z}_p$–submodule. Let $\mathfrak{a}^n, n = 1, 2, \ldots,$ be the descending central series of the Lie algebra $\mathfrak{a}$ over $\mathbb{Q}_p$, and let $p_n : \mathfrak{a} \to \mathfrak{a}/\mathfrak{a}^n$ be the natural projection. Let us set $A_n = p_n \circ \log(A)$. We claim that A_n is an open subgroup of $\mathfrak{a}/\mathfrak{a}^n$ where we endow $\mathfrak{a}/\mathfrak{a}^n$ with the Campbell–Hausdorff multiplication. The case $n = 1$ was proved above. We first show by induction on n that $A_n \cap \mathfrak{a}^{n-1}/\mathfrak{a}^n$ is open in $\mathfrak{a}^{n-1}/\mathfrak{a}_n$. The group commutator induces a $\mathbb{Q}_p$–bilinear map $\mathfrak{a}/\mathfrak{a}^1 \times \mathfrak{a}^{n-1}/\mathfrak{a}^n \to \mathfrak{a}^n/\mathfrak{a}^{n+1}$ whose image spans $\mathfrak{a}^n/\mathfrak{a}^{n+1}$. Then $A_{n+1} \cap \mathfrak{a}^n/\mathfrak{a}^{n+1}$ contains the image of $A_1 \times (A_n \cap \mathfrak{a}^{n-1}/\mathfrak{a}^n)$, hence also spans $\mathfrak{a}^n/\mathfrak{a}^{n+1}$ over $\mathbb{Q}_p$ and is a $\mathbb{Z}_p$–submodule, and hence is open. It then follows by induction on n that A_n is open compact in $\mathfrak{a}/\mathfrak{a}^n$, by the following lemma. □

Lemma 4.13. *Let A be a closed normal subgroup of the locally compact topological group B. Suppose B_1 is a compact subgroup of B with the following properties: $A \cap B_1$ is open in A, and the image of B_1 under the natural map $B \to B/A$ is open in B/A. Then B_1 is open in B.*

The proof is straightforward and left to the reader.

Corollary 4.14. *With notation and hypotheses as in Proposition 4.12, suppose there is an automorphism α of the Lie algebra $\mathfrak{a}$ such that $\lim_{n\to\infty} \alpha^n(x) = 0$ for every $x \in \mathfrak{a}$. Suppose further that $\alpha(\log A) = \log A$. Then $A = \exp \mathfrak{a}$.*

The necessity of condition b) in the p-adic case follows as a corollary from the next result.

Proposition 4.15. *Let G be a closed subgroup of $B_n(k)$ and $N = G \cap U_n(k)$. For every $g \in G$ let $N(g) = \{u \in N; \lim_{n\to\infty} g^n u g^{-n} = e\}$ and let E be the subgroup of N generated by $\bigcup_{g\in G} N(g)$. Then the $N(g), g \in G$, and E are closed Lie subgroups over $\mathbb{Q}_p$ contained in $U_n(k)$ and E is normal G. If G is compactly generated, then N/E is compact.*

Proof. For every element $g \in B_n(k)$ define

$$E(g) = \{x \in B_n(k); \lim_{n\to\infty} g^n x g^{-n} = e\}.$$

The arguments in the proof of Theorem 7.23 show that

a) $E(g) \subset U_n(k)$.

b) $E(g) = \exp \mathfrak{e}(g)$, where

$$\mathfrak{e}(g) = \{x \in \mathfrak{u}_n(k); \lim_{n\to\infty} \mathrm{Ad}(g^n)x = 0\}$$

and $\mathfrak{e}(g)$ is a Lie subalgebra of $\mathfrak{u}_n(k)$ over k.

c) $E(g) = \{e\}$ if $g \in U_n(k)$.

Let $\mathfrak{n}$ be the smallest Lie algebra over $\mathbb{Q}_p$ containing $\log N$. We regard $\mathfrak{n}$ as a group with respect to the Campbell–Hausdorff multiplication. To simplify our notation let us identify N with $\log N$. Set $\mathfrak{n}(g) = \mathfrak{n} \cap \mathfrak{e}(g)$. Then $\mathfrak{n}(g)$ is a Lie subalgebra of $\mathfrak{n}$ over $\mathbb{Q}_p$. We know by Proposition 4.12 that N is open in $\mathfrak{n}$ and hence $N(g)$ is open in $\mathfrak{n}(g)$ and even $N(g) = \mathfrak{n}(g)$, by Corollary 4.14. The subgroup E of N generated by $\bigcup_{g\in G} E(g)$ is normal in G and is the exponential of the Lie algebra $\mathfrak{e}$, which we define as the Lie algebra over $\mathbb{Q}_p$ generated by $\bigcup_{g\in G} \mathfrak{n}(g)$.

Now let $\rho : G \to \mathrm{Aut}(\mathfrak{n}/\mathfrak{e})$ be the representation of G induced by the adjoint representation. By construction, every eigenvalue of $\rho(g)$, $g \in G$, has absolute value 1. We want to conclude that $\rho(g)$ has compact closure.

For the adjoint representation of $B_n(k)$ on $\mathfrak{u}_n(k)$ there is a filtration by $B_n(k)$–invariant subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathfrak{u}_n(k)$ such that the induced representation of $B_n(k)$ on the quotient space V_i/V_{i-1} is semisimple and the image group in $GL(V_i/V_{i-1})$ is abelian for every i. The same then holds if we regard $\mathfrak{u}_n(k)$ as a representation space for G and hence also for $\mathfrak{n}$ and for ρ on $\mathfrak{n}/\mathfrak{e}$. Furthermore, all the eigenvalues of $\rho(g)$, $g \in G$, are of absolute value 1. If G is compactly generated, so is $\overline{\rho(G)}$. It then follows from Corollary 7.17 that $\overline{\rho(G)}$ is compact.

Again, if G is compactly generated, then so is G/E. On the other hand, G/N is compactly generated abelian. The subgroup N/E is an open, hence closed subgroup of $\exp(\mathfrak{n}/\mathfrak{e})$, and for every compact subset K of this group the set of its conjugates $\bigcup_{g\in G} gKg^{-1}$ has compact closure, since its log is $\rho(G)\log K$ and is actually contained in a compact subgroup of the unipotent group $\exp(\mathfrak{n}/\mathfrak{e})$, by Lemma 7.16. Hence N/E is compact. □

As a consequence of the proof we obtain the desired result.

Corollary 4.16. *If G is compactly generated, then $p^0(N)$ has compact closure in V^0.*

Proof. Since G is compactly generated, $p^0(N)$ is the image under the natural map $N/E \to V^0$. □

5. Sufficiency of the conditions

Lemma 5.1. *Let V be a finite dimensional vector space over the local field k and let $g \in GL(V)$ have all eigenvalues of absolute value < 1. Let H be the split extension of V by $\mathbb{Z}$ defined by the action of g on V. Then*

$$\ell_H(v) \prec \log \|v\|.$$

Proof. Let k' be a field of finite degree over k which contains all the eigenvalues of g. For every eigenvalue λ of g let

$$V_\lambda = \{v \in V \otimes k' \ ; \ (g - \lambda(g)\mathrm{Id})^d v = 0\}$$

be the corresponding primary subspace, where $d = \dim V$. Then $g \mid V_\lambda$ is of the form $\lambda\,\mathrm{Id} + u$, with u nilpotent. Hence

$$(\lambda + u)^n = \lambda^n + n \cdot \lambda^{n-1} u + \cdots + \binom{n}{d-1} \lambda^{n-d+1} n^{d-1}$$

and thus

$$\|(\lambda + u)^n\| \le |\lambda|^n \cdot P(n),$$

where P is a polynomial in n of degree at most $d-1$. Let

$$|\lambda_{\max}| = \max\{|\lambda| \ ; \ \lambda \text{eigenvalue of } g\}.$$

It follows that

$$\|g^n\| \leq |\lambda_{\max}|^n \cdot P_1(n)$$

for the operator norm of g^n on $V \otimes k'$ and hence on V, and thus our claim. □

It remains to show

Proposition 5.2. *If G fulfills conditions a) and b) of Theorem 3.2, then G is undistorted in $GL_n(k)$.*

Proof. We first deal with the case that k is Archimedean. We may assume that $k = \mathbb{C}$ and N is Zariski closed over $\mathbb{R}$ by the arguments at the beginning of the proof of Proposition 4.11. Conditions a) and b) are still valid. By hypothesis b), the space $V = N/N'$ is, considered as a vector space with the representation ρ of G on V, the direct sum of $\rho(G)$–invariant subspaces V_i such that for every i there is an element $g \in G$ with all eigenvalues of absolute value less than 1. It follows from Lemma 5.1 that

$$\ell_{G/N'}(v) \prec \log \|v\|,$$

since for $v_i \in V_i$ we have

$$\ell_{G/N'}(v_i) \prec \ell_{<g> \ltimes V_i}(v_i) \prec \log \|v_i\|,$$

and hence

$$\ell_{G/N'}(v) = \ell_{G/N'}(v_1, \ldots, v_m) \leq \sum_{i=1}^{m} \ell_{G/N'}(v_i) \prec \Sigma \log \|v_i\| \leq m \log \|v\|$$

for $v \in N$.

This is the case $i = 1$ of the inductive claim that

$$\ell_{G/N^i}(\exp v) \prec \log \|v\|,$$

where N^i is the i–th term of the descending central series of N and v is an element of the Lie algebra $\mathfrak{n}/\mathfrak{n}^i$ of N/N^i.

Note first that the group commutator map $a, b \longmapsto (a, b) = aba^{-1}b^{-1}$ induces a bilinear map $N/N' \times N^{i-1}/N^i \to N^i/N^{i+1}$ whose image generates N^i/N^{i+1}. We may identify N/N' with $\mathfrak{n}/\mathfrak{n}'$ and N^i/N^{i+1} with $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ via the exponential maps. Then the group commutator map is identified with the map $\mathfrak{n}/\mathfrak{n}' \times \mathfrak{n}^{i-1}/\mathfrak{n}^i \xrightarrow{\Phi} \mathfrak{n}^i/\mathfrak{n}^{i+1}$ induced by the Lie bracket. It follows that if $\ell_{G/N'}(u) \prec \log \|u\|$ for $u \in \mathfrak{n}/\mathfrak{n}'$ and $\ell_{G/N^i}(v) \prec \log \|v\|$ for $v \in \mathfrak{n}^{i-1}/\mathfrak{n}^i$, then $\ell_{G/N^{i+1}}((u, v)) \prec \log \|u\| + \log \|v\| = \log \|u\| \cdot \|v\|$, which implies

$$\ell_{G/N^{i+1}}(\exp w) \prec \log \|w\| \text{ for } w \in \mathfrak{n}^i/\mathfrak{n}^{i+1}$$

by our inductive hypothesis, since Φ is bilinear and its image spans $\mathfrak{n}^i/\mathfrak{n}^{i+1}$.

Now let $u \in \mathfrak{n}/\mathfrak{n}^{i+1}$. By our inductive hypothesis and Section 2.5 b) there is an element $v \in \mathfrak{n}/\mathfrak{n}^{i+1}$ such that $\ell_{G/N^{i+1}}(\exp v) \prec \log \|u\|$ and $u \equiv v \mod \mathfrak{n}^i$. For the norm of v we have the upper bound

$$\log \|v\| \prec \ell_{G/N^i}(\exp v) \prec \log \|u\|.$$

The first inequality holds since G/N^i is a solvable linear group, and the claimed bound holds for every element of its normal subgroup of unipotent elements by Proposition 2.9 and Section 2.5 a).

For the element $\exp w := \exp(-v) \cdot \exp u \in N^i/N^{i+1}$, $w \in \mathfrak{n}^i/\mathfrak{n}^{i+1}$, we have a bound for $\|w\|$ which is a polynomial in $\|u\|$ and $\|v\|$ of total degree $i+1$, by the Campbell–Hausdorff formula. Thus

$$\begin{aligned}\log \|w\| &\leq i(\log \|u\| + \log \|v\|)\\ &\prec 2i \log \|u\|.\end{aligned}$$

It now follows from our upper bound for $\ell_{G/N^{i+1}}(\exp w)$, $w \in \mathfrak{n}^i/\mathfrak{n}^{i+1}$, that

$$\begin{aligned}\ell_{G/N^{i+1}}(\exp u) &\leq \ell_{G/N^{i+1}}(\exp v) + \ell_{G/N^{i+1}}(\exp(-v)\exp u)\\ &\prec \log \|u\| + 2i \log \|u\|\\ &\asymp \log \|u\|.\end{aligned}$$

This proves our inductive claim. It follows that $\ell_G(\exp u) \prec \log \|u\|$ for every $u \in \mathfrak{n}$.

Finally, $\lambda(G)$ is closed in the abelian compactly generated group $T_n(\mathbb{C}) \cong (\mathbb{C}^*)^n$, by hypothesis a), and hence undistorted in $T_n(\mathbb{C})$ by Proposition 7.13. It follows that

$$\ell_{\lambda(G)}(t) \asymp \max\{\log \|t\|, \log \|t^{-1}\|\}.$$

So for every element $g \in G$ there is by Section 2.5 b) an element $h \in G$ of length $\ell_G(h) = \ell_{\lambda(G)}(\lambda(g))$ such that $h \equiv g \mod N$. For the norm of h we have

$$\log \|h\| \leq \ell_{GL_n}(h) \prec \ell_G(h) = \ell_{\lambda(G)}(\lambda(g))$$

and

$$\log \|h^{-1}\| \leq \ell_{GL_n}(h) \prec \ell_G(h) = \ell_{\lambda(G)}(\lambda(g)),$$

by Section 2.8. Hence

$$\begin{aligned}&\log \|h^{-1}g\| \leq \log \|h^{-1}\| + \log \|g\|\\ &\prec \max\{\log \|\lambda(g)\|, \log \|\lambda(g^{-1})\|\} + \log \|g\|\\ &\leq 2\max\{\log \|g\|, \log \|g^{-1}\|\}\end{aligned}$$

and similarly

$$\log \|g^{-1}h\| \prec 2\max\{\log \|g\|, \log \|g^{-1}\|\},$$

and hence $\ell_{GL_n}(g^{-1}h) = \max\{\log \|g^{-1}h\|, \log \|h^{-1}g\|\} \prec 2\ell_{GL_n}(g)$. As $g^{-1}h \in N$, the first part of our proof yields

$$\ell_G(g^{-1}h) \prec \log \|g^{-1}h\| \prec 2\ell_{GL_n}(g),$$

which together with

$$\ell_G(h) = \ell_{\lambda(G)}(\lambda(g)) \prec \ell_{GL_n}(g)$$

gives

$$\ell_G(g) \prec \ell_{GL_n}(g).$$

Suppose now that k is a p–adic field. Let us abbreviate $\ell := \ell_{GL_n(k)}$. Lemma 5.1 shows that $\ell_G|N(g) \asymp \ell|N(g)$. The "nilpotent technology" used for the Archimedian case then shows that $\ell_G \mid E \asymp \ell|E$ and hence $\ell_G|N \asymp \ell|N$, since N/E is compact by the assumed condition b). This then implies that $\ell_G \asymp \ell|G$ as above. Here we use condition a) just as in the Archimedean case. $\square$

6. Examples

6.1. Let $A \in B_n(k)$ and let G be the cyclic group generated by A. Then G is undistorted in $GL_n(k)$ iff G is either finite or at least one eigenvalue of A has absolute value different from 1.

6.2. Let A be the scalar 2×2 complex matrix $\alpha \cdot \mathrm{Id}$ with $|\alpha| > 1$, let B be the matrix $\beta \cdot \mathrm{Id} + E_{12}$ with $|\beta| > 1$, and suppose $\log|\alpha|$, $\log|\beta|$ are linearly independent over $\mathbb{Q}$. Let G be the group generated by A and B. Then G is discrete and free abelian of rank 2, and is distorted iff α/β is a real number. But in any case, every cyclic subgroup of G is undistorted.

The group G is also an example of the phenomenon pointed out before Lemma 4.8, namely that the map μ is injective but possibly $\mu \otimes \mathbb{R}$ is not. In our case we have in the notation of Lemma 4.8: $G = H = H/K \cong \mathbb{Z}^2$, and $\mu = \lambda : G \to T_+ = T/L$ will have its image in a subgroup isomorphic to $\mathbb{R}$ when α and β are positive real numbers; so μ is injective but $\mu \otimes \mathbb{R}$ is not.

6.3. Let A, B and G be as in Section 6.2 and suppose furthermore that A and B are real matrices with non–negative entries. Then G is a cocompact subgroup of the connected subgroup H of $B_2(\mathbb{R})$ of those matrices whose diagonal part is a positive scalar matrix. H is distorted as well, but fulfills condition a) but not b), whereas G fulfills condition b) but not a). Note that in this case H is the Zariski closure of G.

The next two examples show that the situation for discrete groups is more complicated than for real Lie groups. A group can be undistorted, even if no element of N is contracted; see Section 6.4. But in Section 6.4 every non–identity element is exponentially expanded. But even if every non–identity element of N is exponentially expanded, this does not imply that G is undistorted, as Section 6.5 shows.

6.4. Let A be a matrix in $SL_2(\mathbb{Z})$ which has two real eigenvalues λ, λ^{-1} with $|\lambda| > 1$. Regard the split extension $G = \mathbb{Z} \ltimes \mathbb{Z}^2$ defined by the action of A on $\mathbb{Z}^2$ as a subgroup of $SL_3(\mathbb{Z})$. Then G is undistorted in $GL_3(\mathbb{R})$, although no element u of $N = \mathbb{Z}^2$ is contracted by A, rather $\|A^n u\| \underset{n\to\infty}{\longrightarrow} \infty$ for every $u \in N, u \neq 0$. In fact, $\sqrt[n]{\|A^n u\|}$ converges to $|\lambda|$ for $n \to \infty$.

6.5. Let λ be a Salem number; that is, a real algebraic integer greater than 1 whose other algebraic conjugates have absolute value at most 1 and which has at least one algebraic conjugate of absolute value 1. Then λ^{-1} is a conjugate of λ, and all the other conjugates of λ are of absolute value 1 and can be paired with their complex conjugates. There is a Salem number of degree 4. Let G be the split extension $\mathbb{Z} \ltimes \mathbb{Z}[\lambda]$ with the action of $\mathbb{Z}$ on $\mathbb{Z}[\lambda]$ defined by multiplication with λ. Then $\mathbb{Z}[\lambda]$ is as a group isomorphic to $\mathbb{Z}^d$, $d =$ degree of λ, with a linear map A whose characteristic polynomial is the minimal polynomial of λ. Thus, with respect to the linear map A the vector space $\mathbb{R}^d$ decomposes into the two eigenlines V_λ and $V_{\lambda^{-1}}$ for λ and λ^{-1}, respectively, and a $d-2$–dimensional A–invariant subspace V_1 on which A acts orthogonally. Thus the word length on $\widetilde{G} = \mathbb{Z} \ltimes \mathbb{R}^d$ induces on $\mathbb{R}^d$ the following length function up to quasi-isometry:

$$\ell(v) = \log \|p_{\lambda,\lambda^{-1}}(v)\| + \|p_1(v)\|$$

for $v \in \mathbb{R}^d$, where $\|\cdot\|$ is a vector space norm on $\mathbb{R}^d$ and $p_{\lambda,\lambda^{-1}}$ and p_1 are the projections associated with the decomposition $\mathbb{R}^d = (V_\lambda \oplus V_{\lambda^{-1}}) \oplus V_1$. Now embed $\widetilde{G}$ into a linear group such that $\mathbb{R}^d$ becomes unipotent and $\mathbb{Z}$ becomes diagonal. Then $\widetilde{G}$ is distorted. Note that for $u \in \mathbb{Z}^d \simeq \mathbb{Z}[\lambda]$, $u \neq 0$, we have

$$\lim_{n\to\infty} \sqrt[n]{\|A^n u\|} = \lambda,$$

but $\ell(nu) \not\prec |n|$.

Example 6.6. In [6, 3.D_2' Remark] Gromov claims that if G is a simply connected solvable Lie group and at the same time a split extension $A \ltimes N$ with A abelian, then G is undistorted in every overgroup of G if the adjoint action of every $a \in A$ on the Lie algebra of N has at least one eigenvalue λ satisfying $|\lambda| \neq 1$. This fails to be true, e.g., if G is the subgroup of $B_n, n \geq 3$, consisting of those matrices having first diagonal entry arbitrary positive and all other diagonal entries equal to 1. The group G is distorted but fulfills Gromov's condition.

On the other hand it is not necessary for G to be undistorted so that all the subrepresentations of the adjoint representation of A on N are non-trivial. An example is the group of those matrices in $B_n(k), n \geq 3$, whose first and last diagonal entry equals 1.

7. Appendix

7.1. Topological preparations.

7.1. Pontryagin's structure theorem. *Let A be a locally compact compactly generated abelian topological group. Then A has a unique maximal compact subgroup K and A/K is isomorphic to $\mathbb{R}^p \times \mathbb{Z}^q$ for some p and q.*

Corollary 7.2. *Let N be a locally compact compactly generated nilpotent topological group. Then N has a unique maximal compact subgroup K and N/K is isomorphic to a cocompact closed subgroup of a simply connected nilpotent real Lie group. This Lie group is uniquely determined by N up to isomorphism and will be called $N \otimes \mathbb{R}$.*

7.3. *In case N is a closed unipotent subgroup of $GL(n,\mathbb{R})$, then N has no compact subgroup $\neq \{e\}$ and $N \otimes \mathbb{R}$ can be described in several equivalent ways, as follows:*

- *Regard $GL(n,\mathbb{R})$ as a real algebraic group. Then $N \otimes \mathbb{R}$ is the Zariski closure of N.*
- *Let M be the group of all unipotent elements of $GL(n,\mathbb{R})$, a positive power of which is contained in N. Then $N \otimes \mathbb{R}$ is the closure of M. The group M could be called $N \otimes \mathbb{Q}$, or the Malcev completion of N.*
- *Let $\mathfrak{n}$ be the set of nilpotent real $n \times n$–matrices A such that $\exp(A) \in N$. The $\mathbb{R}$–vector space $\mathfrak{n} \otimes \mathbb{R}$ spanned by $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ and $N \otimes \mathbb{R} = \exp(\mathfrak{n} \otimes \mathbb{R})$.*

Theorem 7.4 (Mostow [11])**.** *Every closed subgroup of a connected solvable real Lie group is compactly generated.*

Theorem 7.5. *Every closed solvable subgroup G of $GL(n,\mathbb{C})$ has a closed subgroup of finite index which is contained as a cocompact subgroup in a closed connected solvable subgroup of $GL(n,\mathbb{C})$. The same holds for the groups $GL(n,\mathbb{R})$, $B(n,\mathbb{C})$ and $B(n,\mathbb{R})$ instead of $GL(n,\mathbb{C})$.*

Note that Theorem 7.5 implies Theorem 7.4 for the case that G is linear, by Lemma 7.8. The proof of Theorem 7.5 will be given below; see Section 7.7.

It is convenient to use the "virtual" parlance, as follows. We say that a topological group G has a certain property virtually, if G has a closed subgroup of finite index which has the property.

7.6. Warning. It is not true that every closed subgroup H of a connected solvable real Lie group G is virtually contained cocompactly in a connected closed subgroup of G, even if G is simply connected. Here is an example.

Example. Let $\mathbb{R}$ act on $\mathbb{C}$ by rotations, $t \longmapsto e^{2\pi i t}$. Let G be the corresponding split extension $\mathbb{R} \ltimes \mathbb{C}$. So G is isomorphic to the following subgroup of $GL(4, \mathbb{C})$. It consists of all matrices of the form

$$g(t,z) := \begin{pmatrix} e^{2\pi i t} & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ t \in \mathbb{R}, z \in \mathbb{C}.$$

Let H be the subgroup of elements with $t \in \mathbb{Z}$ and $z \in \mathbb{Z}$. Then H is abelian but G is the only connected subgroup of G containing any subgroup of finite index of H, since every such group projects under the natural homomorphism $G = \mathbb{R} \ltimes \mathbb{C} \to \mathbb{R}$ onto $\mathbb{R}$, hence acts by the full group of rotations on $\mathbb{C}$ and hence contains a subgroup of finite index of $\mathbb{Z} \oplus i\mathbb{Z}$. But every connected subgroup of G containing $\mathbb{Z} \oplus i\mathbb{Z}$ also contains $\mathbb{C}$. Thus there is no connected subgroup of G containing a subgroup of H of finite index as a cocompact subgroup. Still, there is a closed connected subgroup of $GL(n, \mathbb{C})$ which contains the isomorphic image of H as a cocompact subgroup. Also note that the above subgroup G of $GL_4(\mathbb{C})$ is connected but $G \cap U$ is not connected.

7.7. Proof of Theorem 7.5. Every Zariski–connected solvable subgroup of $GL(n, \mathbb{C})$ is conjugate to a subgroup of $B(n, \mathbb{C})$ by the Lie–Kolchin theorem. We may thus assume that G is a closed subgroup of $B(n, \mathbb{C})$ by passing from G to a subgroup of finite index, if necessary. Regard $GL(n, \mathbb{C})$ as an algebraic group over $\mathbb{R}$. Let G^* be the Zariski closure of G. Let N be the subgroup of unipotent elements of G. So $N = G \cap U(n, \mathbb{C})$, where $U(n, \mathbb{C})$ is the group of upper triangular unipotent matrices in $GL(n, \mathbb{C})$. Then N is a closed normal subgroup of G and $[G, G] \subset G \cap U(n, \mathbb{C}) = N$. It follows that the Zariski closure N^* of N is a closed normal subgroup of G^* and $[G^*, G^*] \subset N^*$, so G^*/N^* is abelian. On the other hand, N^* contains N cocompactly and is connected by Section 7.3. It follows that the group $G \cdot N^*$ is a closed subgroup of G^*, which contains G cocompactly. Let $p : G^* \to G^*/N^*$ be the natural homomorphism. Then G^*/N^* is an abelian Lie group with a finite number of components. The group $p(G) = p(G \cdot N^*)$ is a closed subgroup. Thus, passing to a subgroup of $p(G)$ of finite index, if necessary, $p(G)$ is cocompactly contained in a closed connected subgroup H of G^*/N^*. Then $p^{-1}(H)$ is a closed connected solvable subgroup of G^*, hence of $GL(n, \mathbb{C})$, containing G virtually as a cocompact subgroup.

If $G \subset GL(n, \mathbb{R})$, again let N be the subgroup of unipotent matrices in G. Then G/N is virtually abelian. Let G^* and N^* be the Zariski closure of G and N in $GL(n, \mathbb{R})$. Then N is cocompactly contained in the connected unipotent closed subgroup N^*. The group G^*/N^* has a closed subgroup of finite index which is

an abelian connected Lie group and virtually contains the closed subgroup $p(G) = p(G \cdot N^*)$. The result then follows as above. In the cases of the upper triangular groups $B(n, \mathbb{C})$ and $B(n, \mathbb{R})$, the groups G^* are contained in these subgroups, which implies the claims. □

Lemma 7.8. *Let H be a closed subgroup of the locally compact topological group G. Suppose G/H is compact. Then H has a compact set of generators iff G does. Furthermore, then H is undistorted in G.*

Proof. There is a compact subset C of G such that $H \cdot C = G$. If E is a compact set of generators of H, then clearly $E \cup C$ is a compact set of generators of G. Conversely, suppose F is a compact set of generators of G. We may assume that the identity element e is contained in C and in F. If $h = g_1 \dots g_n$ with $g_i \in F \cup F^{-1}$, construct inductively $h_i \in H$ and $c_i \in C$ for $i = 1, \dots, n-1$ such that $g_1 = h_1 c_1, h_1 \in H, c_1 \in C$. Hence $h_1 \in (F \cup F^{-1})C^{-1} \cap H c_i g_{i+1} = h_{i+1} c_{i+1}, h_i \in H, c_{i+1} \in C$, and hence $h_{i+1} \in C \cdot (F \cup F^{-1}) \cdot C^{-1} \cap H$.

Then

$$\begin{aligned} h = g_1 \cdots g_n &= h_1 c_1 g_2 \cdots g_n = h_1 h_2 c_2 g_3 \cdots g_n = \cdots \\ &= h_1 h_2 \cdots h_{n-1} c_{n-1} g_n = h_1 h_2 \cdots h_n \end{aligned}$$

if we set $h_n = c_{n-1} g_n \in C(F \cup F^{-1}) \cap H$. It follows that H is generated by the compact set $C(F \cup F^{-1})C^{-1} \cap H$, and the word length of an element h in H is actually at most equal to the word length of h in G. Since we already know that $d_G|H \prec d_H$ (see Section 2.5 a)), this implies our claims. □

7.9. Warning. Suppose H is a closed subgroup of the locally compact topological group G and suppose that there is a compact subset C of G such that $G = CHC$. If G is compactly generated, it does not follow that H is compactly generated. As well, even if H is compactly generated, it does not follow that H is undistorted. Here are examples.

Example 7.10. Let H be an infinite torsion free discrete subgroup of $G = SL_2(\mathbb{R})$. Then there is a compact subset C of G with the property that $CHC = G$. Such an H may be neither finitely generated (e.g. an infinitely generated free subgroup of $SL_2(\mathbb{Z})$) nor, if finitely generated, undistorted (e.g. the group of integral unipotent upper triangular matrices).

Proof. We may assume that H is infinite cyclic, say generated by h. We work in the upper half plane $\mathbb{H}$ of $\mathbb{C}$ with its hyperbolic metric d, and identify $\mathbb{H}$ with G/K, $K = SO_2(\mathbb{R})$, and its base point i with K. We will show that there is a compact subset C of G such that $\bigcup_{n=0}^{\infty} Ch^n i = \mathbb{H}$; hence $\bigcup_{n=0}^{\infty} Ch^n K = G$. Take a path $g(t)$, $t \in [0,1]$, in G such that $g(0) = e$ and $g(1) = h$. Define $g(t) = g(t-n) \cdot h^n$ if $t \in [n, n+1]$. This defines a path $g : [0, \infty) \to G$. The distance $d(g(t)i, i)$ goes to infinity with $n \to \infty$, since H is discrete. Therefore it intersects every circle $C_R = \{x \in \mathbb{H}; d(x, i) = R\}$ in the hyperbolic metric. But these circles are precisely the orbits of $K = SO_2(\mathbb{R})$ in $\mathbb{H}$. So for every circle C_R there is a point $g(t)$ of our path such that $C_R = K \cdot g(t)$. Thus $\mathbb{H} = \bigcup K g(t) h^n$ with $t \in [0,1]$, $n = 0, 1, \dots$, as was to be shown. □

Lemma 7.11. *Suppose G and H are locally compact topological groups, and let $f : G \to H$ be an open continuous surjective homomorphism with compact kernel. Then G is compactly generated iff H is so. In this case, f is undistorted.*

Proof. If C is a compact generating subset of G, then so is $f(C)$ for H, and $\ell_{f(C)}(f(x)) \leq \ell_C(x)$ for $x \in G$. Conversely, if D is a compact generating set for H, so is $f^{-1}(D)$ for G and $\ell_{f^{-1}(D)}(x) \leq \ell_D(f(x))$. These two statements prove our claim. □

Corollary 7.12. *Suppose G and H are locally compact topological groups, and let $f : G \to H$ be a proper continuous homomorphism with cocompact image. Then G is compactly generated iff H is so. In this case, f is undistorted.*

Proof. Under our hypotheses, if G or H is compactly generated, then both are σ–compact and hence so is $f(H)$. Then $f : G \to f(H)$ is an open homomorphism by a well-known Baire category argument. The corollary then follows from Lemmas 7.8 and 7.11. □

7.2. **Word length in abelian or unipotent groups.**

Proposition 7.13. *Let G be an abelian locally compact topological group. By Pontryagin's structure theorem there is a finite dimensional real vector space V and a proper continuous homomorphism $f : G \to V$ with cocompact image.*

a) If $\|\cdot\|$ is any vector space norm on V, then $f : (G, \ell_G) \to (V, \|\cdot\|)$ is a quasi–isometry.

b) If H is a closed subgroup of G, then H is undistorted in G.

Proof. a) follows from Corollary 7.12 and the fact that $\|\cdot\|$ is a word length on V, namely for the unit ball $\{v \in V \ ; \ \|v\| \leq 1\}$ as a compact generating set of V.

b) reduces by Corollary 7.12 to the special case of a closed subgroup H of the vector space V. But H is cocompact in its span $H \otimes \mathbb{R}$, and the restriction of $\|\cdot\|$ to $H \otimes \mathbb{R}$ is again a norm. □

Remark 7.14. It is well known that the same is not true for nilpotent groups. E.g. let G be the Heisenberg group $G = U_3(\mathbb{R})$ and let H be its center. Then H is isomorphic to the additive group $\mathbb{R}$ via the exponential map $\exp : \mathfrak{h} \to H$, $\dim \mathfrak{h} = 1$; hence $\ell_H \circ \exp \not\prec \|\cdot\|$ for any norm $\|\cdot\|$ on $\mathfrak{h}$. But

$$(\ell_G \mid H) \circ \exp \succ \sqrt{\|\cdot\|}.$$

More generally, the following, also well-known, results imply that every closed subgroup H of a nilpotent locally compact compactly generated group G is at most polynomially distorted, i.e., there is a natural number d such that

$$(\ell_G \mid H)^d \succ \ell_H.$$

7.15. The word length on unipotent groups. Let u, v be elements of the unipotent group $U_n(k)$, thus $u = (u_{ij})$ with $u_{ii} = 1$ and $u_{ij} = 0$ for $i > j$, and similarly for $v = (v_{ij})$. Then $uv = w = (w_{ik})$ with

$$\begin{aligned} w_{ik} &= \sum_{i \leq j \leq k} u_{ij} v_{jk} \\ &= v_{ik} + u_{i,i+1} v_{i+1,k} + \cdots + u_{ik}. \end{aligned} \tag{7.15}$$

Thus, if $|u_{ij}| \leq C^{j-i}$ and $|v_{ij}| \leq D^{j-i}$ for all i, j, then

$$\begin{aligned} |w_{ik}| &\leq D^{k-1} + C \cdot D^{k-i-1} + \cdots + C^{k-i} \\ &\leq (C + D)^{k-i}. \end{aligned}$$

Thus, if we put

$$|||u||| = \sup |u_{ij}|^{\frac{1}{j-i}},$$

then

$$|||uv||| \leq |||u||| + |||v|||.$$

Lemma 7.16. *Consider the norm*

$$\|u\| = \sup\{|u_{ij}| \ ; \ 1 \leq i, j \leq n\}$$

on $U_n(k)$. For every compactly generated subgroup G of $U_n(k)$ we have

$$\|\cdot\| \prec (\ell_G)^{n-1}.$$

If k is non–Archimedean, then every closed compactly generated subgroup G of $U_n(k)$ is even compact.

Proof. If K is a compact symmetric generating subset of G, let

$$C = \sup\{|||u|||, u \in K\}.$$

Then for every element $g \in G$ of length ℓ with respect to K we have $|||u||| \leq \ell \cdot C$, hence $|u_{ij}| \leq C^{j-i} \cdot \ell^{j-i}$ for $i < j$, and hence the claim. If k is non–Archimedean, the formula (7.15) shows that $\{u : |||u||| \leq C\}$ is actually a compact subgroup of $U_n(k)$. □

Corollary 7.17. *If G is a compactly generated subgroup of $B_n(k)$ and every eigenvalue of every element $g \in G$ has absolute value* 1*, then the conclusions of Lemma* 7.16 *hold for G as well.*

Proof. Let L be the subgroup of $T_n(k)$ consisting of those diagonal matrices whose diagonal entries have absolute value 1. So L is the maximal compact subgroup of $T_n(k)$. Let K be a compact generating subset of G. Then there is a compact subset A of $U_n(k)$ such that $K \subset L \cdot A$. Let H be the subgroup of $U_n(k)$ generated by the compact subset $A_1 = \bigcup_{\ell \in L} \ell A \ell^{-1}$. Then L normalizes H. We show that G is contained in $L \cdot H$, which implies the corollary. Let $g \in G$ and write $g = k_1, \ldots, k_n$ with $k_i \in K$. Every k_i can be written in the form $k_i = \ell_i \, a_i$ with $\ell_i \in L$ and $a_i \in A$. Then

$$\begin{aligned} g &= \ell_1 a_1 \ldots \ell_n a_n \\ &= m_1 \cdot m_2^{-1} a_1 m_2 \cdot m_3^{-1} a_2 m_3 \cdot \ldots \cdot m_n^{-1} a_{n-1} m_n \cdot a_n \\ \text{with} \ \ m_i &= \ell_i \ldots \ell_n \in L, \text{which implies the claim.} \end{aligned}$$

□

Lemma 7.18. *Let G be a locally compact topological group and let A be a closed normal subgroup of G. We denote G/A by B. We make the following assumptions:*

a) A and B are compactly generated.

b) B is abelian.

c) For every compact subset K of A the set $\bigcup_{g \in G} gKg^{-1}$ of conjugates in G of elements of K is relatively compact.

Then

$$\ell_A \prec (\ell_G \mid A)^2.$$

So A is at most polynomially distorted in G and the degree of the distortion polynomial is at most two.

Proof. B has a unique maximal compact subgroup, say K_B, and B/K_B is isomorphic to a closed cocompact subgroup of the finite dimensional real vector space $\mathbb{R} \otimes B/K_B$. Let d be the dimension of this vector space. Let $p : G \to B/K_B$ be the natural homomorphism. We may assume that B/K_B is discrete and hence isomorphic to $\mathbb{Z}^d$ by passing to a cocompact closed subgroup of G, if necessary. Let $e_1, \dots, e_d$ be a basis of B/K_B. Choose for every e_i an element $g_i \in G$ such that $p(g_i) = e_i$. There is a compact subset L of G such that L maps under $G \to G/A$ onto K_B. Then every element of G can be written in the form

$$g = a \cdot \ell \cdot g_1^{t_1} \cdots g_d^{t_d}$$

with $a \in A$, $\ell \in L$ and $(t_1, \dots, t_d) \in \mathbb{Z}^d$. Note that $t = (t_1, \dots, t_d) \in \mathbb{Z}^d$ is uniquely determined by g. We may identify t with $p(g)$. Also $a \cdot \ell \in \ker p$ is uniquely determined by g, but a and ℓ may not be uniquely determined by g. We choose a compact symmetric generating set A_S of A with the following properties. A_S is closed under conjugation by elements of G and contains all the commutators (g_i, g_j), $1 \le i, j \le d$, and (g_j, ℓ), $\ell \in L$, $j = 1, \dots, d$, and contains $L^2 \cap A$. Here we use the notation

$$(a, b) = aba^{-1}b^{-1}$$

for the commutator of a and b. Then $S = A_S \cup L \cup \{g_1, \dots, g_d\}$ is a compact set of generators of G. We claim that if an element $g \in G$ has length n with respect to S, then there is a decomposition

$$g = a \cdot \ell \cdot g_i^{t_1} \dots g_d^{t_d}$$

of d such that

$$\sum_{\ell=1}^{d} |t_i| \le n \text{ and } \ell_{A_S}(a) \le n^2.$$

The first claim is obvious. We prove the second claim by induction on n. Thus let h be an element of G of length $n+1$. Then there is an element $s \in S$ and an element $g \in G$ of length n such that $sg = h$. We write g in the form above and assume our claim holds for g. The second claim is then clear if $s \in A_S$. If $s \in L$, then

$$sa\ell = sas^{-1}s\ell,$$

and $\ell_{A_S}(sas^{-1}) = \ell_{A_S}(a)$ and $\ell_{A_S}(s\ell) \le 1$, since $L^2 \cap A \subset A_S$. If $s = g_j$, pulling g_j across a costs nothing and pulling it across ℓ costs an A_S–length at most 1. This means in precise mathematical terms that

$$g_j a\ell = (g_j a g_j^{-1}) \cdot (g_j, \ell) \cdot \ell g_j, \ \ell_{A_S}(g_j a g_j^{-1}) = \ell_{A_S}(a) \text{ and } \ \ell_{A_S}(g_j, \ell) \le 1.$$

Pulling g_j across g_i^t costs ℓ_{A_S}–length at most $|t|$, since

$$g_j g_i^t = (g_j, g_i^t) g_i^t g_j \text{ and } \ell_{A_S}(g_j, g_i^t) \le |t|,$$

which follows by induction on $|t|$ using the formulas

$$(a, bc) = (a, b)(a, c)^{b^{-1}} \text{ and } (a, b^{-1}) = (a, b)^b,$$

where we use the notation

$$x^y = y^{-1}xy.$$

Pulling the element $a_1 = (g_j, g_i^t) \in A$ across the preceding $g_k^{t_k}$, $k < i$, does not cost anything, since

$$\ell_{A_S}(a_1^g) = \ell_{A_S}(a_1).$$

The same holds for pulling it across ℓ. Thus

$$h = g_j \cdot g = g_j(a \cdot \ell \cdot g_1^{t_1} \dots g_d^{t_d}) = a' \cdot b_0 \cdot b_1 \dots b_{j-1} \cdot \ell \cdot g_1^{t_1} \dots g_j^{t_j+1} \dots g_d^{t_d},$$

where $a' = a^{g_j^{-1}}$, $b_0 = (g_j, \ell)$ and b_k is a conjugate of $(g_j, g_k^{t_k})$ for $1 \le k < j$. Thus our new a–entry of h has length at most

$$\ell_{A_S}(a' \cdot b_0 \cdot b_1 \dots b_{j-1}) \le \ell_A(a) + 1 + \Sigma_{k<j}|t_k| \le \ell_A(a) + 1 + n \le (n+1)^2.$$

□

Corollary 7.19 (of the proof). *We make the same hypotheses as in the preceding Lemma* 7.18 *except that we replace hypothesis a) by the following two hypotheses:*
a′) B is compactly generated.
a″) Every compact subset of A is contained in a compact subgroup of A.
Then G is compactly generated (if and) only if A is compact.

Proof. Let T be a compact generating subset of G. Using the notation of the preceding proof, we have that $p(T)$ is a finite subset of the (w.l.o.g.) discrete group B/K_B. Hence there is a compact subset A_1 of A and a finite subset F of the discrete set $B_1 = \{g_1^{t_1} \dots g_d^{t_d}; (t_1, \dots, t_d) \in \mathbb{Z}^d\}$ such that $T \subset A_1 \cdot L \cdot F$. Then also $A_1 \cup L \cup \{g_1, \dots, g_d\}$ generates G. Now choose a compact symmetric subset A_S of A with the following properties: $A_1 \subset A_S$, A_S is closed under conjugation and contains all the commutators (g_i, g_j), $1 \le i, j \le d$, and all the (g_j, ℓ), $\ell \in L$, $1 \le j \le d$, and contains $L^2 \cap A$. Let K be a compact subgroup of A containing A_S. Then the proof above shows that $G \subset K \cdot L \cdot B_1$; hence $\ker p = A \cdot L$ is compact. □

Remark 7.20. The preceding corollary also follows from the following two results. Here we make use of the notion of compact presentability of a locally compact topological group; see [2].

7.21. *Every abelian locally compact compactly generated topological group has a compact presentation.*

7.22. *Let G be a compactly generated locally compact topological group and let A be a closed normal subgroup of G. If G/A has a compact presentation, then A is compactly generated as a normal subgroup of G. This means that there is a compact subset K of A such that A is the smallest normal subgroup of G containing K.*

7.3. **Exponential radicals.** The notion of the exponential radical may be relevant to our question. The notion is due to Osin [9]. Let G be a simply connected solvable real Lie group. Osin defines the exponential radical $\mathrm{Exp}(G)$ as the set of those elements $g \in G$ which are (strictly) exponentially distorted, i.e., such that

$$\ell_G(g^n) \prec \log(|n| + 1).$$

He proves the following two theorems.

Theorem 7.23. *a)* $\mathrm{Exp}(G)$ *is a closed connected normal subgroup of G.*
b) $\mathrm{Exp}(G)$ *is contained in the nilradical of G.*
c) $\mathrm{Exp}(G)$ *is strictly exponentially distorted in G.*
d) $\mathrm{Exp}(G/\mathrm{Exp}(G)) = \{1\}$.

Theorem 7.24. $\mathrm{Exp}(G) = \{1\}$ *iff G has polynomial growth.*

Furthermore, he gives an inductive construction of the exponential radical based on the fact that if $\mathrm{Exp}(G) \neq \{1\}$, then the intersection of $\mathrm{Exp}(G)$ with the center of the nilradical of G is $\neq \{1\}$. So polynomial growth is very important here. We thus recall the following result of [7].

Theorem 7.25. *Let G be a connected solvable real Lie group. The following conditions are equivalent:*

a) G has polynomial (volume-) growth.

b) All the eigenvalues of the adjoint representation of G on its Lie algebra $\mathfrak{g}$ have absolute value 1.

c) All the eigenvalues of the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ are purely imaginary.

d) For no element $x \neq e$ in G the closure of the conjugacy class $\{gxg^{-1}; g \in G\}$ of x contains e.

e) For no element $x \neq 0$ of $\mathfrak{g}$ the closure of the orbit $\mathrm{Ad}(G)x$ under the adjoint representation contains 0.

f) There is no pair of elements $x \neq e$, g in G such that $g^n x g^{-n}$ converges to e as n tends to infinity.

g) If $x \neq 0$ in $\mathfrak{g}$ and $g \in G$, then the sequence $\mathrm{Ad}(g^n)x$ does not converge to 0 for $n \to \infty$.

This, together with the results of Osin, gives the following descriptions of the exponential radical.

Theorem 7.26. *Let G be a simply connected solvable real Lie group.*

a) $\mathrm{Exp}(G)$ is the set of exponentially distorted elements in G.

b) $\mathrm{Exp}(G)$ is the smallest closed normal connected subgroup of G such that $G/\mathrm{Exp}(G)$ has polynomial growth.

c) $\mathrm{Exp}(G)$ is the subgroup of G generated by the elements $x \in G$ whose conjugacy class has e in its closure.

d) $\mathrm{Exp}(G)$ is the subgroup of G generated by the elements $x \in G$ for which there is an element $g \in G$ such that $\lim_{n\to\infty} g^n x g^{-n} = 1$.

e) The Lie algebra $\mathrm{Exp}(\mathfrak{g})$ of $\mathrm{Exp}(G)$ is the Lie subalgebra of $\mathfrak{g}$ generated by the elements $x \in \mathfrak{g}$ for which 0 is contained in the closure of $\mathrm{Ad}(G)x$.

f) The Lie algebra $\mathrm{Exp}(\mathfrak{g})$ of $\mathrm{Exp}(G)$ is the Lie subalgebra of $\mathfrak{g}$ generated by the elements $x \in \mathfrak{g}$ for which there is an element $g \in G$ such that the sequence $\mathrm{Ad}(g^n)x$ converges to 0.

Proof. Let $E_a - E_d$ be the groups described under a) — d) and let $\mathfrak{e}_e$, $\mathfrak{e}_f$ be the Lie algebras described under e) and f). We sometimes write $E_a(G), \ldots, \mathfrak{e}_f(\mathfrak{g})$ in order to emphasize the dependence on G and $\mathfrak{g}$, respectively. Note that it is not yet clear that there is a smallest group as in b), but this is easy to see in our case and will also follow from the proof we will give. In case G is abelian all our groups and Lie algebras $E_a(G) \ldots \mathfrak{e}_f(\mathfrak{g})$ are trivial. Furthermore, if $f : G \to H$ is a surjective homomorphism of solvable simply connected real Lie groups, then $f(E_x(G)) \subset E_x(H)$, $x = a, \ldots, d$, and $df(\mathfrak{e}_x(\mathfrak{g})) \subset \mathfrak{e}_x(\mathfrak{h})$ for $x = g, h$. Hence $E_x(G)$ is contained in the commutator group of G and in particular in the nilradical N of G, and similarly $\mathfrak{e}_x(\mathfrak{g})$ is contained in the Lie algebra $\mathfrak{n}$ of the nilradical N. Note that, since G is simply connected, N is simply connected as well and nilpotent. Hence every connected subgroup of N is a closed Lie subgroup. Furthermore, the exponential map $\mathfrak{n} \to N$ is a diffeomorphism. It follows that the generating sets under c) and d) are unions of one–parameter subgroups of N, and hence E_c and

E_d are closed connected Lie subgroups of N, obviously normal in G. Similarly, $\mathfrak{e}_f$ and $\mathfrak{e}_g$ are ideals of $\mathfrak{g}$ contained in $\mathfrak{n}$, $\mathfrak{e}_f$ is the Lie algebra of E_c, and $\mathfrak{e}_g$ is the Lie algebra of E_d. Clearly, $E_d \subset E_c$ and $\mathfrak{e}_g \subset \mathfrak{e}_f$. Furthermore, E_c is contained in the kernel of every surjective homomorphism from G to a group of polynomial growth, by Theorem 7.23 d). It thus suffices to show that G/E_d has polynomial growth, since this implies the existence of the group E_b and the equations $E_b = E_c = E_d$. The equation $E_a = E_b$ then follows from Osin's results, Theorems 7.23 d) and 7.24, together with the trivial remark that an exponentially distorted element is mapped to an exponentially distorted element by every continuous homomorphism.

To see that G/E_d has polynomial growth, we take a closer look at $\mathfrak{e}_g$. For every element $g \in G$ we consider the primary space decomposition of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ for $\mathrm{Ad}(g)$. Thus for every complex number λ let

$$\mathfrak{g}(g, \lambda) = \{x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}; (\mathrm{Ad}(g) - \lambda \ \mathrm{Id})^d x = 0\},$$

where $d = \dim \mathfrak{g}$. Then $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}(g, \lambda)$. For $r \in \mathbb{R}$ let $\mathfrak{g}(g, r) = \mathfrak{g} \cap \bigoplus_{|\lambda|=r} \mathfrak{g}(g, \lambda)$. Thus $\mathfrak{g} = \bigoplus_{r \in \mathbb{R}} \mathfrak{g}(g, r)$.

We have $\{x \in \mathfrak{g}; \lim_{n\to\infty} \mathrm{Ad}(g^n)x = 0\} = \bigoplus_{r<1} \mathfrak{g}(g, r) := \mathfrak{g}^<(g)$. So $\mathfrak{e}_g$ is the Lie subalgebra of $\mathfrak{g}$ generated by the vector subspaces $\mathfrak{g}^<(g)$, $g \in G$. Since $\mathfrak{e}_g$ is an ideal in $\mathfrak{g}$, it is in particular an $\mathrm{Ad}(g)$–invariant vector subspace which contains both $\mathfrak{g}^<(g)$ and $\mathfrak{g}^<(g^{-1})$. So the only possible eigenvalues of $\mathrm{Ad}(d)$ on $\mathfrak{g}/\mathfrak{e}_g$ are those of absolute value 1. It follows that the Lie algebra $\mathfrak{g}/\mathfrak{e}_g$ satisfies the criterion of Theorem 7.25 b) for polynomial growth, so G/E_d has polynomial growth. $\square$

Warning. Although $\mathrm{Exp}(G)$ intersects the center Z of the nilradical N of G non–trivially, if $\mathrm{Exp}(G)$ is non–trivial, it may happen that Z contains none of the generators of $\mathrm{Exp}(G)$ described in Theorem 7.26 c), much less those of Theorem 7.26 d). In fact, it may happen that $\mathrm{Exp}(G) = N$, but Z is central in G. An example is the group of all upper triangular real 3×3 matrices with the first and last diagonal entry equal to 1 and the second diagonal entry arbitrary positive.

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