

ALGEBRAS WITH CYCLE-FINITE GALOIS COVERINGS

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ABSTRACT. We prove that the finite dimensional algebras over an algebraically closed field which admit cycle-finite Galois coverings with torsion-free Galois groups are of tame representation type, and derive some consequences.

INTRODUCTION

Throughout the article by an algebra we mean a finite dimensional associative K -algebra with an identity over a fixed algebraically closed field K . By a module over an algebra A we mean a right A -module of finite dimension over K , if not specified otherwise.

From Drozd's remarkable Tame and Wild Theorem [Dro] (see also [CB1]) the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d , in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional algebras over K (see [SS2, Chapter XIX]). Hence, a classification of the finite dimensional modules is only feasible for tame algebras. A distinguished class of tame algebras is formed by the algebras of finite type, having only finitely many isoclasses of indecomposable modules, for which the representation theory is rather well understood (see [BGRS], [Bo2], [BoG], [BrG]). On the other hand, the representation theory of arbitrary tame algebras is still only emerging. At present the most accessible seem to be the (tame) algebras of polynomial growth, for which there exists an integer m such that the number of one-parameter families of indecomposable modules is bounded, in each dimension d , by d^m . This class of algebras has been the subject of intensive research over the last 30 years.

The main methods of the representation theory of algebras have been developed best for the triangular algebras (the Gabriel quiver has no oriented cycles), having finite global dimension (see [ASS], [Ri], [SS1], [SS2]). In order to deal with arbitrary algebras, covering techniques were introduced and developed at the beginning of the 1980s (see [BoG], [DS1], [DS3], [Ga]). Frequently, an algebra A admits a Galois covering $R \rightarrow R/G = A$, where R is a triangular locally bounded category and

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G is a torsion-free group acting freely on the objects of R , which allows us to reduce the representation theory of A to that for the corresponding algebras of finite global dimension inside R . Moreover, Geiss proved in [Ge] (see also [CB3]) that if an algebra A admits a tame degeneration (in the variety of algebras of a fixed dimension), then A is tame. Hence, a convenient way to determine whether a given algebra A is tame consists in finding a suitable tame degeneration of A . This was successfully applied in [CB3] to prove the tameness of arbitrary biserial algebras and in [BPS] to show that a strongly simply connected algebra is tame if and only if its Tits quadratic form is weakly nonnegative. It is expected that every algebra Λ of polynomial growth is Morita equivalent to an algebra A (basic algebra of Λ) which admits a canonical degeneration \bar{A} (standard form of A) of polynomial growth, having a triangular Galois covering $R \rightarrow R/G = \bar{A}$ with a torsion-free group G , and such that the representation theories of A and \bar{A} are very close. This is the case for all algebras of finite type (see [BGRS], [Bo2], [BrG]) and all self-injective algebras of polynomial growth (see [Sk1], [Sk11]).

A prominent role in the representation theory of algebras is played by cycles of modules, more generally cycles of complexes of modules (see [AS3], [Sk5], [Sk7]). A cycle in the category $\text{mod } A$ of finite dimensional modules over an algebra A is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

of nonzero nonisomorphisms between indecomposable modules in $\text{mod } A$, and the cycle is said to be finite if the homomorphisms f_1, \dots, f_s do not belong to the infinite Jacobson radical of $\text{mod } A$. Then an algebra A is said to be *cycle-finite* if all cycles in $\text{mod } A$ are finite [AS4]. It was shown in [Sk6] that every cycle-finite algebra is of polynomial growth. The class of cycle-finite algebras is wide and contains the algebras of finite type, the tame tilted algebras [Ke], the tame double tilted algebras [RS1], the tame generalized double tilted algebras [RS2], the tame quasi-tilted algebras [LS], [Sk10], the tame generalized multicoil algebras [MS], and the strongly simply connected algebras of polynomial growth [Sk8]. It has also been proved in [AS1], [AS2], [AS3] that the class of algebras A for which the derived category $D^b(\text{mod } A)$ of bounded complexes over $\text{mod } A$ is cycle-finite coincides with the class of algebras which are tilting-cotilting equivalent to the hereditary algebras of Dynkin type, hereditary algebras of Euclidean type, or tubular algebras (in the sense of [Ri]), and consequently these algebras A are also cycle-finite. We also mention that by a result due to Peng and Xiao [PX] and the second named author [Sk4], the Auslander-Reiten quiver of any algebra A has at most finitely many D Tr-orbits containing directing modules (not lying on a cycle in $\text{mod } A$). Moreover, Ringel has shown that the support algebra of a directing module is a tilted algebra [Ri]. The support algebras of directing modules over tame algebras have been investigated in [Bo1], [Dre] (finite type case) and in [Pe1], [Pe2] (representation-infinite case).

The main objective of this paper is to prove that every algebra A which admits a Galois covering $R \rightarrow R/G = A$ with R a cycle-finite locally bounded category and a torsion-free admissible group G of automorphisms of R is tame and to describe the indecomposable finite dimensional A -modules. Moreover, we show that, for such Galois coverings, the algebra A is of polynomial growth if and only if the number of G -orbits of isoclasses of indecomposable locally finite dimensional R -modules with nontrivial stabilizers is finite. As a consequence of the main theorem we also obtain

that Brenner’s conjecture on the number of terms in the middle of almost split sequences over tame algebras is true for the (tame) algebras with cycle-finite Galois coverings.

The main results of the paper extend results proved in [Bo2] for algebras of finite type, in [DS3], [ES], [Po], [PoS] for biserial algebras, and in [Sk9] for algebras with strongly simply connected (in the sense of [Sk3]) Galois coverings.

The paper is organized as follows. In Section 1 we recall basic facts on Galois coverings of algebras essential for further considerations. Section 2 contains results on cycle-finite algebras applied in the proofs of our main results. Section 3 is devoted to the proof of the main result and its immediate consequences. In Section 4 we establish a criterion for polynomial growth (respectively, domestic type) of algebras with cycle-finite Galois coverings. In the final section, Section 5, we exhibit examples illustrating our main results.

For basic background on the representation theory of algebras, we refer to the books [ASS], [Ri], [SS1], [SS2].

1. GALOIS COVERINGS OF ALGEBRAS

Following [BoG], by a *locally bounded category* we mean a K -category R which is isomorphic to a factor category KQ/I , where Q is a locally finite quiver and I is an admissible ideal of the path category KQ of Q . The quiver Q is said to be the *quiver* of R and will be denoted by Q_R . A locally bounded category R is said to be *connected* if its quiver Q_R is connected, and *triangular* if the quiver Q_R has no oriented cycles. An algebra A will be considered as a *finite category*, that is, a locally bounded category given by a finite quiver. A locally bounded category R is said to be *interval-finite* if, for any two vertices x and y of Q_R , the number of pairwise different vertices of Q_R lying on paths from x to y in Q_R is finite. We note that this is a much weaker property than the interval-finiteness defined in [BrG, (2.6)], which does not force R to be triangular. A full subcategory C of a locally bounded category R is said to be *convex* if any path in Q_R with source and target in Q_C lies entirely in Q_C . Observe that the convex hull of a finite subcategory of an interval-finite category R is a finite convex subcategory of R .

Throughout this section we denote by R a fixed locally bounded category (over K). By an R -module we mean a contravariant functor M from R to the category $\text{MOD } K$ of all vector spaces over K [BoG]. An R -module M is called *finite dimensional* (respectively, *locally finite dimensional*) if $\dim M = \sum_{x \in R} \dim_K M(x) < \infty$ (respectively, $\dim_K M(x) < \infty$ for any object x of R). We denote by $\text{MOD } R$ the category of all R -modules, by $\text{Mod } R$ (respectively, $\text{mod } R$) the category of all locally finite dimensional (respectively, all finite dimensional) R -modules, and by $\text{Ind } R$, (respectively, $\text{ind } R$) the full subcategory of $\text{Mod } R$ (respectively, $\text{mod } R$) formed by all indecomposable modules. The *support* $\text{supp } M$ of an R -module M is the full subcategory of R given by all objects x such that $M(x) \neq 0$.

Let G be a group of K -linear automorphisms of R acting freely on the objects of R . Then following [Ga] we may consider the orbit category R/G defined as follows. The objects of R/G are the G -orbits of the objects of R , and, for any two objects a and b of R/G , the morphism K -space $(R/G)(a, b)$ is defined as

$$(R/G)(a, b) = \left\{ (f_{y,x}) \in \prod_{(x,y) \in a \times b} R(x, y) \mid gf_{y,x} = f_{gy,gx} \quad \forall_{g \in G, x \in a, y \in b} \right\}$$

and the composition ef of morphisms $f : a \rightarrow b$ and $e : b \rightarrow c$ in R/G is defined by $(ef)_{zx} = \sum_{y \in b} e_{zy} f_{yx}$. Then we have a canonical *Galois covering functor*

$$F : R \longrightarrow R/G$$

which assigns to any object x of R its G -orbit Gx and maps a morphism $f \in R(x, y)$ onto the family $F(f) \in (R/G)(Gx, Gy)$ such that $F(f)_{hy, gx} = gf$ or 0 according as $h = g$ or $h \neq g$. Moreover, F induces the K -linear isomorphisms

$$\bigoplus_{F(y)=a} R(x, y) \xrightarrow{\sim} (R/G)(F(x), a), \quad \bigoplus_{F(y)=a} R(y, x) \xrightarrow{\sim} (R/G)(a, F(x)),$$

for all objects x of R and a of R/G . For a full subcategory D of R we denote by gD the full subcategory of R formed by the objects gx , $x \in D$, and its stabilizer $G_D = \{g \in G \mid gD = D\}$. Then we may consider the locally bounded category D/G_D . The group G acts on $\text{MOD } R$ by the translations ${}^g(-)$ which assign to each R -module M the R -module ${}^gM = M \circ g^{-1}$. For each R -module M , we denote by G_M the stabilizer $\{g \in G \mid {}^gM \cong M\}$ of M . Following [DS3], a module Y in $\text{Ind } R$ is said to be *weakly G -periodic* if $\text{supp } Y$ is infinite and $(\text{supp } Y)/G_Y$ is a finite category. Observe that in such a case, G_Y is infinite. We mention that in the later papers of Dowbor [Do1], [Do2], [Do3], [Do4], the weakly G -periodic modules are called infinite G -atoms.

Now assume that G is a group of K -linear automorphisms of R acting freely on the isoclasses of modules in $\text{ind } R$. Clearly, then G acts freely on the objects of R , since G acts freely on the isoclasses of indecomposable projective R -modules $R(-, x)$, $x \in R$. Consider the associated Galois covering functor $F : R \rightarrow R/G$. We denote by $F_\bullet : \text{MOD } R/G \rightarrow \text{MOD } R$ the *pull-up functor*, which assigns to an R/G -module M the R -module $M \circ F$, and by $F_\lambda : \text{MOD } R \rightarrow \text{MOD } R/G$ the *push-down functor*, left adjoint to F_\bullet (see [BoG, (3.2)]). Since G acts freely on the isoclasses in $\text{ind } R$, F_λ induces an injection from the set $(\text{ind } R / \cong) / G$ of G -orbits of isoclasses in $\text{ind } R$ into the set $(\text{ind } R/G) / \cong$ of isoclasses in $\text{ind } R/G$ [Ga, (3.5)]. We denote by $\text{mod}_1 R/G$ the full subcategory of $\text{mod } R/G$ consisting of all modules isomorphic to $F_\lambda(M)$ for some module M in $\text{mod } R$, and by $\text{mod}_2 R/G$ the full subcategory of $\text{mod } R/G$ formed by all modules without nonzero direct summands from $\text{mod}_1 R/G$. It was shown in [DS3, (2.2) and (2.3)] that a module X from $\text{mod } R/G$ belongs to $\text{mod}_1 R/G$ (respectively, $\text{mod}_2 R/G$) if and only if $F_\bullet(X)$ is a direct sum of finite dimensional R -modules (respectively, weakly G -periodic R -modules). We denote by $\text{ind}_1 R/G$ (respectively, $\text{ind}_2 R/G$) the full subcategory of $\text{mod}_1 R/G$ (respectively, $\text{mod}_2 R/G$) formed by the indecomposable modules. Following [DS3] the modules from $\text{ind}_1 R/G$ (respectively, $\text{ind}_2 R/G$) are called *indecomposable modules of the first kind* (respectively, *indecomposable modules of the second kind*). The category R is said to be *G -exhaustive* if $\text{mod } R/G = \text{mod}_1 R/G$ [DS3]. From [DS3, (2.5)] we know that R is G -exhaustive provided R is *locally support-finite*, that is, for each object x of R , the full subcategory of R consisting of the objects of all supports $\text{supp } M$, where M is a module in $\text{ind } R$ with $M(x) \neq 0$, is a finite category. An important class of locally support-finite categories is formed by the locally representation-finite locally bounded categories playing a fundamental role in the representation theory of representation-finite algebras (see [BGRS], [Bo2], [BoG], [Ga]). We also mention that if R is G -exhaustive, then the Auslander-Reiten quiver $\Gamma_{R/G}$ of R/G is the orbit translation quiver $\Gamma_{R/G}$ of the Auslander-Reiten

quiver Γ_R under the induced action of G on Γ_R , because the push-down functor $F_\lambda : \text{mod } R \rightarrow \text{mod } R/G$ preserves the almost split (Auslander-Reiten) sequences [Ga, (3.6)]. The problem whether the G -exhaustivity of R forces R to be a locally support-finite category still remains open, although it was confirmed in some important cases (see [AS5], [Sk9]). We refer also to [Do1], [Do2], [Do3], [Do4], [DS3] for some results on indecomposable R/G -modules of the second kind.

A crucial role in the proof of our main result will be played by a reduction theorem from [DS3], allowing us to describe the indecomposable modules of the second kind for a class of Galois coverings.

Assume that R is not G -exhaustive. Following [DS3, (3.1)], a family \mathcal{S} of full subcategories of R is called *separating* (with respect to G) if \mathcal{S} satisfies the following conditions:

- (i) for each $L \in \mathcal{S}$ and $g \in G$, $gL \in \mathcal{S}$;
- (ii) for each $L \in \mathcal{S}$ and each G -orbit \mathcal{O} of R , $\mathcal{O} \cap L$ is contained in finitely many G_L -orbits;
- (iii) for any two different $L, L' \in \mathcal{S}$, $L \cap L'$ is locally support-finite;
- (iv) for each weakly G -periodic R -module Y , there exists an $L \in \mathcal{S}$ such that $\text{supp } Y \subseteq L$.

For an additive category C and a full subcategory C_0 of C , we denote by $C/[C_0]$ the factor category of C by the ideal $[C_0]$ of all morphisms in C factorized through a direct sum of some objects of C_0 .

For a convex subcategory L of a locally bounded category R we have a canonical embedding $E^L : \text{MOD } L \rightarrow \text{MOD } R$ such that, for a module M in $\text{MOD } L$, $E^L(N)$ is an R -module such that $E^L(N)(x) = N(x)$ for any object x of L , $E^L(N)(f) = N(f)$ for any morphism f in L , and $E^L(N)(y) = 0$ for any object y of R which is not in L . Moreover, for any group G of K -linear automorphisms of R acting freely on the objects of R , we have a commutative diagram of functors

$$\begin{CD} \text{MOD } L @>E^L>> \text{MOD } R \\ @V F_\lambda^L VV @VV F_\lambda V \\ \text{MOD } L/G_L @>E_\lambda^L>> \text{MOD } R/G, \end{CD}$$

where F_λ^L is the push-down functor associated to the Galois covering $F^L : L \rightarrow L/G_L$, F_λ is the push-down functor associated to the Galois covering $F : R \rightarrow R/G$, and E_λ^L assigns to a module X in $\text{MOD } L/G_L$ the module $E_\lambda^L(X)$ in $\text{MOD } R/G$ such that $F_\bullet E_\lambda^L(X) = \bigoplus_{g \in U_L} {}^g F_\bullet^L(X)$, where $F_\bullet : \text{MOD } R/G \rightarrow \text{MOD } R$ and $F_\bullet^L : \text{MOD } L/G_L \rightarrow \text{MOD } L$ are the pull-up functors associated to F and F^L , and U_L is a fixed set of representatives of the cosets of G modulo G_L (see [DS3, (2.4), (3.2)]).

The following theorem is a special (convex) version of [DS3, Theorem 3.1].

Theorem 1.1. *Let R be a locally bounded K -category and G a group of K -linear automorphisms of R acting freely on the isoclasses in $\text{ind } R$. Let \mathcal{S} be a separating family of convex subcategories of R with respect to G and let \mathcal{S}_0 be a fixed set of representatives of G -orbits in \mathcal{S} . Then the functors $E_\lambda^L : \text{mod } L/G_L \rightarrow \text{mod } R/G$, $L \in \mathcal{S}_0$, induce a natural K -linear equivalence of categories*

$$E : \coprod_{L \in \mathcal{S}_0} (\text{mod } L/G_L)/[\text{mod}_1 L/G_L] \longrightarrow (\text{mod } R/G)/[\text{mod}_1 R/G].$$

In particular, the Auslander-Reiten quiver $\Gamma_{R/G}$ of R/G is the disjoint union of the translation quivers

$$\Gamma_{R/G} = (\Gamma_{R/G}) \sqcup \left(\coprod_{L \in \mathcal{S}_0} (\Gamma_{L/G_L})_2 \right),$$

where $(\Gamma_{L/G_L})_2$ is the union of all connected components of Γ_{L/G_L} formed by the indecomposable L/G_L -modules of the second kind.

In this paper we are interested in coverings of algebras by locally bounded categories of tame representation type. Recall that, following Drozd [Dro], a finite category R is called *tame* if, for any dimension d , there exists a finite number of $K[T]$ - R -bimodules M_i , $1 \leq i \leq n_d$, which are finitely generated and free as left $K[T]$ -modules, and all but a finite number of isoclasses of indecomposable R -modules of dimension d are of the form $K[T]/(T - \lambda) \otimes_{K[T]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \dots, n_d\}$. Let $\mu_R(d)$ be the least number of $K[T]$ - R -bimodules satisfying the above condition for d . Then R is said to be of *polynomial growth* (respectively, *domestic*) if there is a positive integer m such that $\mu_R(d) \leq d^m$ (respectively, $\mu_R(d) \leq m$) for any $d \geq 1$ (see [Sk2], [CB2]). An arbitrary locally bounded category R is said to be *tame* (respectively, *of polynomial growth*, *domestic*) if so is every finite full subcategory of R . We refer to [DS2] for results characterizing tame locally bounded categories. It has been proved in [DS1] that if R is a locally bounded category, G a group of automorphisms of R acting freely on the objects of R and R/G is tame, then R is tame. The inverse implication is still an open problem.

The following solution of the stabilizer conjecture by Dowbor in [Do4] is important for applications of covering techniques to tame algebras.

Theorem 1.2. *Let R be a tame locally bounded K -category and G a torsion-free automorphism group of R acting freely on the objects of R . Then the stabilizer G_Y of any weakly G -periodic R -module Y is an infinite cyclic group.*

The following consequence of (see [DS3, (2.4), (3.6)]) and [Ga, pp. 94-95] shows the importance of the above theorem for Galois coverings of algebras by tame locally bounded categories.

We note that the restriction imposed on the group to be torsion-free is natural because of the results of [Ga], [GP].

Proposition 1.3. *Let R be a locally bounded K -category, G a group of K -linear automorphisms of R acting freely on the objects of R , and Y a weakly G -periodic R -module whose stabilizer G_Y is an infinite cyclic group. Then $F_\lambda(Y)$ carries a canonical structure of a KG_Y - R/G -bimodule which is a free module of finite rank as a left module over the group algebra KG_Y of G_Y . In particular, we have a canonical functor*

$$\Phi^Y = - \otimes_{KG_Y} F_\lambda(Y) : \text{mod } KG_Y \longrightarrow \text{mod } R/G$$

whose image is contained in $\text{mod}_2 R/G$.

We also note that, if G_L is infinite cyclic, then KG_L is isomorphic to the algebra $K[T, T^{-1}]$ of Laurent polynomials.

Let R be a locally bounded K -category and G a group of K -linear automorphisms of R acting freely on the objects of R . A *line* in R is a convex subcategory L of R

which is isomorphic to the path category KQ of a linear quiver Q of type $\mathbb{A}_n, \mathbb{A}_\infty$ or ${}_\infty\mathbb{A}_\infty$. A line L in R is said to be G -periodic if its stabilizer G_L is nontrivial. Clearly, in this case, the quiver Q_L of L is of type

$${}_\infty\mathbb{A}_\infty : \dots \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots$$

and has a G_L -periodic orientation. With each G -periodic line L of R we may associate a canonical weakly G -periodic R -module M_L by setting $M_L(x) = K$ for any vertex x of Q_L , $M_L(y) = 0$ for all vertices y of $Q_R \setminus Q_L$, and $M_L(\gamma) = \text{id}_K$ for each arrow γ of Q_L . Since $G_{M_L} = G_L = \mathbb{Z}$, we then obtain a canonical functor

$$\Phi^L = - \otimes_{K[T, T^{-1}]} F_\lambda(M_L) : \text{mod } K[T, T^{-1}] \longrightarrow \text{mod } R/G,$$

where $\text{mod } K[T, T^{-1}]$ denotes the category of finite dimensional modules over $K[T, T^{-1}]$.

2. CYCLE-FINITE ALGEBRAS

In this section we describe properties of cycle-finite algebras applied in the proof of our main theorem and its consequences.

By a *tame concealed algebra* we mean a tilted algebra $C = \text{End}_H(T)$, where H is the path algebra KQ of a quiver Q of Euclidean type $\tilde{\mathbb{A}}_m (m \geq 1), \tilde{\mathbb{D}}_n (n \geq 4),$ or $\tilde{\mathbb{E}}_p (6 \leq p \leq 8),$ and T is a (multiplicity-free) preprojective tilting H -module. The tame concealed algebras have been completely classified by quivers and relations in [Bo3] and [HV]. Recall also that the Auslander-Reiten quiver Γ_C of a tame concealed algebra C is of the form

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C,$$

where \mathcal{P}^C is a preprojective component containing all indecomposable projective C -modules, \mathcal{Q}^C is a preinjective component containing all indecomposable injective C -modules, and \mathcal{T}^C is a $\mathbb{P}_1(K)$ -family $\mathcal{T}_\lambda^C, \lambda \in \mathbb{P}_1(K),$ of pairwise orthogonal standard stable tubes, all but a finite number of them of rank one (see [Ri, Chapter 4] and [SS1] for further properties of the module category of a tame concealed algebra).

By a *tubular algebra* we mean a tubular extension (equivalently, tubular coextension) of a tame concealed algebra of tubular type $(2, 2, 2, 2), (3, 3, 3), (2, 4, 4),$ or $(2, 3, 6),$ as defined in [Ri]. Recall that a tubular algebra B admits two different tame concealed convex subcategories C_0 and C_∞ such that the Auslander-Reiten quiver Γ_B of B is of the form

$$\Gamma_B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \right) \vee \mathcal{T}_\infty^B \vee \mathcal{Q}_\infty^B,$$

where \mathcal{P}_0^B is the preprojective component \mathcal{P}^{C_0} of $\Gamma_{C_0}, \mathcal{T}_0^B$ is a $\mathbb{P}_1(K)$ -family of pairwise orthogonal standard ray tubes, obtained from the stable tubes of \mathcal{T}^{C_0} by ray insertions, \mathcal{Q}_∞^B is the preinjective component \mathcal{Q}^{C_∞} of $\Gamma_{C_\infty}, \mathcal{T}_\infty^B$ is a $\mathbb{P}_1(K)$ -family of pairwise orthogonal standard coray tubes, obtained from the stable tubes of \mathcal{T}^{C_∞} by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers), \mathcal{T}_q^B is a $\mathbb{P}_1(K)$ -family of pairwise orthogonal standard stable tubes (we refer to [Ri, Chapter 5] for more properties of module categories of tubular algebras).

Recall also that a module M over an algebra A is called *sincere* if $\text{supp } M = A$.

The following characterization of tame concealed and tubular algebras was established in [Sk6, Theorem 4.1].

Theorem 2.1. *Let A be an algebra. The following conditions are equivalent:*

- (i) *A is cycle-finite and admits a sincere indecomposable module lying in a stable tube of Γ_A .*
- (ii) *A is either tame concealed or tubular.*

Moreover, we have the following consequence of the above theorem and a result of Crawley-Boevey [CB1, Corollary E], proved in [Sk6, Theorem 4.3].

Theorem 2.2. *Let A be a cycle-finite algebra. Then A is of polynomial growth.*

An algebra A is said to be *minimal representation-infinite* if A is representation-infinite but every proper convex subcategory of A is representation-finite. Then we have the following characterization of minimal representation-infinite cycle-finite algebras established in [Sk6, Corollary 4.4].

Theorem 2.3. *For an algebra A the following conditions are equivalent:*

- (i) *A is minimal representation-infinite and cycle-finite.*
- (ii) *A is tame concealed.*

We also mention the following characterization of domestic cycle-finite algebras, proved in [Sk6, Theorem 5.1].

Theorem 2.4. *Let A be a cycle-finite algebra. The following conditions are equivalent:*

- (i) *A is domestic.*
- (ii) *A does not contain a tubular algebra as a convex subcategory.*
- (iii) *All but finitely many components of Γ_A are stable tubes of rank one.*

Let A be a cycle-finite algebra. The support $\text{supp } M$ of a module M in $\text{ind } A$ is not necessarily a convex subcategory of A . We denote by $c(M)$ the number of pairwise different tame concealed convex subcategories of the convex hull of $\text{supp } M$ in A .

The following fact has recently been established in our joint work with Malicki [MPS, Theorem B].

Theorem 2.5. *Let A be a cycle-finite algebra and M be an indecomposable finite dimensional A -module. Then $c(M) \leq 3$.*

We note that there are many cycle-finite algebras A having indecomposable finite dimensional modules M with $c(M) = 3$. For example, the iterated tubular algebras studied in [PTo] have this property.

We also mention that there are (discrete) indecomposable finite dimensional modules over cycle-finite algebras with a very complicated structure, for example the modules lying in generalized multicoils of tame generalized multicoil algebras (see [MS]).

We end this section with a result on the number of terms in the middle of almost split sequences over cycle-finite algebras.

For an algebra A and a nonprojective indecomposable module X in $\text{mod } A$, consider an almost split sequence

$$0 \longrightarrow \tau_A X \longrightarrow \bigoplus_{i=1}^{s(X)} Y_i \longrightarrow X \longrightarrow 0,$$

where all Y_i are indecomposable A -modules. It has been proved by Bautista and Brenner [BB] (see also [Li]) that if A is representation-finite, then $s(X) \leq 4$, and if $s(X) = 4$, then one of the modules Y_i is projective-injective. It has been conjectured by Brenner that, for a tame algebra A , $s(X) \leq 5$ for every nonprojective indecomposable finite dimensional A -module X . This has been confirmed for some classes of tame algebras but for arbitrary tame algebras it is still an open and interesting problem.

For cycle-finite algebras this was shown in [PTa, Theorem 3] as follows.

Theorem 2.6. *Let A be a cycle-finite algebra and*

$$0 \longrightarrow \tau_A X \longrightarrow \bigoplus_{i=1}^{s(X)} Y_i \longrightarrow X \longrightarrow 0$$

be an almost split sequence in $\text{mod } A$, where the modules Y_i are indecomposable. Then $s(X) \leq 5$. Moreover, if $s(X) = 5$, then one of the modules Y_i is projective-injective.

We recall that an almost split sequence in $\text{mod } A$ whose middle term contains an indecomposable projective-injective module P is isomorphic to the almost split sequence of the form

$$0 \longrightarrow \text{rad } P \longrightarrow (\text{rad } P / \text{soc } P) \oplus P \longrightarrow P / \text{soc } P \longrightarrow 0$$

(see [ASS, Proposition 3.11]).

3. THE MAIN RESULT AND CONSEQUENCES

A locally bounded K -category R is said to be *cycle-finite* if R is interval-finite and every finite convex subcategory of R is cycle-finite. A group G of K -linear automorphisms of a locally bounded category R is said to be *admissible* if its action on the objects of R is free and has finitely many orbits. In such a case, R/G is a finite category (algebra). For a locally bounded category R and an admissible group G of automorphisms of R , we denote by $\mathcal{A}(R, G)$ a fixed set of representatives of isoclasses of weakly G -periodic R -modules closed under the induced action of G , and by $\mathcal{A}_0(R, G)$ a fixed set of representatives of G -orbits in $\mathcal{A}(R, G)$. Moreover, we identify the group algebra $K\mathbb{Z}$ of the infinite cyclic group \mathbb{Z} with the algebra $K[T, T^{-1}]$ of Laurent polynomials.

The following theorem is the main result of the paper.

Theorem 3.1. *Let R be a connected cycle-finite locally bounded K -category over an algebraically closed field K , G a torsion-free admissible group of K -linear automorphisms of R , $A = R/G$, $\mathcal{A} = \mathcal{A}(R, G)$, and $\mathcal{A}_0 = \mathcal{A}_0(R, G)$. Then the functors $\Phi^Y = - \otimes_{K[T, T^{-1}]} F_\lambda(Y) : \text{mod } K[T, T^{-1}] \rightarrow \text{mod } A$, $Y \in \mathcal{A}_0$, induce a K -linear equivalence of categories*

$$\Phi : \coprod_{\mathcal{A}_0} \text{mod } K[T, T^{-1}] \xrightarrow{\sim} \text{mod } A / [\text{mod}_1 A].$$

Moreover, the following statements hold:

- (i) A is tame.
- (ii) Every indecomposable finite dimensional A -module X is isomorphic either to $F_\lambda(M)$ for some indecomposable finite dimensional R -module M

or to $\Phi^Y(V)$ for some $Y \in \mathcal{A}_0$ and some indecomposable finite dimensional $K[T, T^{-1}]$ -module V .

(iii) The Auslander-Reiten quiver Γ_A of A has the disjoint union decomposition

$$\Gamma_A = (\Gamma_R/G) \sqcup \left(\coprod_{\mathcal{A}_0} \Gamma_{K[T, T^{-1}]} \right),$$

where $\Gamma_{K[T, T^{-1}]}$ is the Auslander-Reiten quiver of the category of finite dimensional $K[T, T^{-1}]$ -modules.

Proof. Since G is torsion-free and acts freely on the objects of R , it acts freely on the isoclasses of modules in $\text{ind } R$. Then the push-down functor $F_\lambda : \text{mod } R \rightarrow \text{mod } A$ preserves the Auslander-Reiten sequences and induces an injection from the set of G -orbits of isoclasses in $\text{ind } R$ into the set of isoclasses in $\text{ind } A$ [Ga, Section 3]. We will show that the functors $\Phi^Y, Y \in \mathcal{A}_0$, induce a K -linear equivalence of categories

$$\Phi : \coprod_{\mathcal{A}_0} \text{mod } K[T, T^{-1}] \longrightarrow \text{mod } A / [\text{mod}_1 A],$$

applying Theorems 1.1, 2.3 and 2.5, and results on representation-finite algebras. Observe that this will imply that every indecomposable module X in $\text{ind}_2 A$ is isomorphic to a module $\Phi^Y(V)$ for some $Y \in \mathcal{A}_0$ and some indecomposable finite dimensional $K[T, T^{-1}]$ -module V , and consequently (ii) will be proved.

In the first step of our proof we describe the structure of modules in $\text{Ind } R$ with nontrivial stabilizer, applying the theory of fundamental sequences developed in [DS3, Section 4]. We need a family of finite convex subcategories of R .

Following [DS1], for a full subcategory C of R , we denote by \widehat{C} the full subcategory of R formed by all objects x such that $R(x, y) \neq 0$ or $R(y, x) \neq 0$ for some object y of C . Observe that, if C is finite, then \widehat{C} is also finite, because R is locally bounded. For an R -module Z and a full subcategory C of R , we denote by $Z|_C$ the restriction of Z to C .

Fix a family $C_n, n \in \mathbb{N}$, of finite convex subcategories of R such that:

- (1) For each $n \in \mathbb{N}$, C_{n+1} is the convex hull of \widehat{C}_n in R .
- (2) $R = \bigcup_{n \in \mathbb{N}} C_n$.

Since R is connected, locally bounded and interval-finite, such a family exists. We shall identify a C_n -module M with an R -module, by setting $M(x) = 0$ for all objects x of R which are not in C_n .

Let Y be a module in $\text{Ind } R$ with G_Y nontrivial. We will show that Y is a weakly G -periodic R -module and describe its support $\text{supp } Y$. Since R is cycle-finite, hence tame, Theorem 1.2 will then imply that G_Y is an infinite cyclic group.

Let $m \in \mathbb{N}$ be the least number such that $Y|_{C_m} \neq 0$. We define a family of modules $Y_n \in \text{ind } C_n, n \in \mathbb{N}$, as follows. We set $Y_n = 0$ for $n < m$, and let Y_m be an arbitrary indecomposable direct summand of $Y|_{C_m}$. Then there exist $Y_{m+1} \in \text{ind } C_{m+1}$ and a splittable monomorphism $u_m : Y_m \rightarrow Y_{m+1}|_{C_m}$ such that Y_{m+1} is a direct summand of $Y|_{C_{m+1}}$. Repeating this procedure we can find, for all $n \geq m$, $Y_n \in \text{ind } C_n$ and splittable monomorphisms $u_n : Y_n \rightarrow Y_{n+1}|_{C_n}$ such that Y_n is a direct summand of $Y|_{C_n}$. We obtain a sequence $(Y_n, u_n)_{n \in \mathbb{N}}$, called in [DS3, (4.1)], a *fundamental R -sequence produced by Y* . Since $C_n, n \in \mathbb{N}$, are finite convex subcategories of R and Y is a locally finite dimensional R -module, it is in fact a sequence of finite dimensional indecomposable R -modules. Now invoking the facts

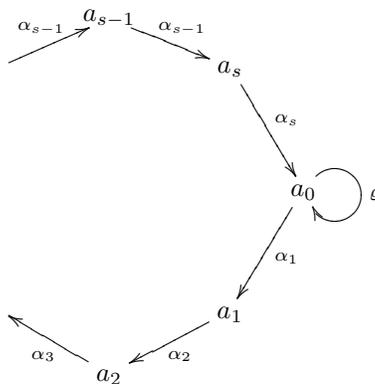
that $Y \in \text{Ind } R$ and $Y \cong {}^g Y$, we have the following consequences of [DS3, (4.3), (4.4), (4.5)]:

- (a) $Y = \varinjlim Y_n$.
- (b) For each $n \in \mathbb{N}$, there exists $p \geq n$ such that $Y_p|_{C_n} \cong Y|_{C_n}$.
- (c) For each $g \in G_Y$ and $n \in \mathbb{N}$, there exists $q \geq n$ such that $gC_n \subseteq C_q$ and ${}^g Y_n$ is a direct summand of $Y_q|_{gC_n}$.

For $n \geq m$, we denote by B_n the convex hull of $\text{supp } Y_n$ in R . Clearly, $B_n \subseteq C_n$ for any $n \geq m$. Moreover, since Y is indecomposable, infinite dimensional, locally finite dimensional, and C_{n+1} contains \widehat{C}_n , for each $n \in \mathbb{N}$, applying [DS1, Lemma 2], we conclude that, for any $n \geq m$, B_n is not contained in C_{n-1} .

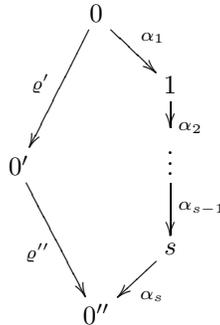
We now claim that all categories $B_n, n \geq m$, are representation-finite. Assume that B_p , for some $p \geq m$, is representation-infinite. Then, by Theorem 2.3, B_p contains a tame concealed algebra D as a convex subcategory, because $B_p \subseteq C_p$ and C_p is cycle-finite, as a convex subcategory of R . Take an arbitrary $r \in \mathbb{N}$ and consider the tame concealed convex subcategories $D, gD, \dots, g^r D$, for a fixed $g \in G_Y \setminus \{1_D\}$. Since the group G is torsion-free and acts freely on the objects of R , the categories $D, gD, \dots, g^r D$ are pairwise different. It follows from (c) that, for each $i \in \{0, \dots, r\}$, there exists $t_i \geq p$ such that $g^i C_p \subseteq C_{t_i}$ and $g^i Y_p$ is a direct summand of $Y_{t_i}|_{g^i C_p}$. Take $q \in \mathbb{N}$ such that C_q contains all categories $C_{t_i}, i \in \{0, \dots, r\}$. Then Y_{t_i} is a direct summand of $Y_q|_{C_{t_i}}$, and hence $g^i Y_p$ is a direct summand of $Y_q|_{g^i C_p}$, for any $i \in \{0, \dots, r\}$. Therefore, Y_q is an indecomposable finite dimensional R -module for which the convex hull B_q of $\text{supp } Y_q$ contains pairwise different tame concealed convex subcategories $D, gD, \dots, g^r D$. On the other hand, C_q is a cycle-finite algebra containing B_q as a convex subcategory, and so, applying Theorem 2.5, we conclude that B_q contains at most 3 pairwise different tame concealed convex subcategories. Hence, taking $r \geq 3$, we get a contradiction. Indeed this shows that all categories $B_n, n \geq m$, are representation-finite.

Our next claim is to show that all categories $B_n, n \geq m$, are standard, that is, are isomorphic to their standard forms \bar{B}_n (see [BoG, (5.1)] and [BrG, (3.1)]). Assume that B_p , for some $p \geq m$, is nonstandard. Then, by [BGRS, Theorem 9.6], K is of characteristic 2 and B_p contains a convex subcategory P which is a penny-farthing (Riedtmann-contour) KQ/I given by the quiver Q of the form



for some $s \geq 2$, and I is an ideal of the path algebra KQ of Q generated by the elements $\alpha_1 \dots \alpha_s - \rho^2$, $\alpha_s \alpha_1 - \alpha_s \rho \alpha_1$, and $\alpha_{i+1} \dots \alpha_n \rho \alpha_1 \dots \alpha_{f(i)}, 1 \leq i \leq s - 1$,

where f is a nondecreasing function from $\{1, \dots, s - 1\}$ to $\{1, \dots, s\}$, which is not constant with value 1. Moreover, we may assume that $p \geq m + 14$. Take an arbitrary positive integer $r \geq 3$ and consider the convex subcategories $P, gP, \dots, g^r P$ of P , which are clearly pairwise different, since G is torsion-free and acts freely on the objects of R . We conclude as above that there exists $q \geq p$ such that the convex hull B_q of $\text{supp } Y_q$ in R contains all the categories $P, gP, \dots, g^r P$ as convex subcategories. It has been proved by Bongartz in [Bo4, Corollary 3] that every representation-finite algebra Λ which admits a faithful indecomposable module is standard. Therefore, the indecomposable B_q -module Y_q is an indecomposable module over the standard algebra \bar{B}_q of B_q in which (see [BGRS, 2.7] and [BrG, (3.1)]) the nonstandard convex subcategories $P, gP, \dots, g^r P$ of B_q are replaced by their standard forms $\bar{P}, g\bar{P}, \dots, g^r \bar{P}$, where $\bar{P} = KQ/\bar{I}$ is given by the same penny-farthing quiver Q , and \bar{I} is the ideal in KQ generated by the elements $\alpha_1 \dots \alpha_s - \varrho^2$, $\alpha_s \alpha_1$, and $\alpha_{i+1} \dots \alpha_n \varrho \alpha_1 \dots \alpha_{f(i)}$, $1 \leq i \leq s - 1$. According to [BrG, Theorem 3.1], $\bar{B}_q = KQ_{\bar{B}_q}/I_{\bar{B}_q}$, where $Q_{\bar{B}_q}$ is the quiver of \bar{B}_q and the ideal $I_{\bar{B}_q}$ of $KQ_{\bar{B}_q}$ is generated by all nonstable paths of $Q_{\bar{B}_q}$ and the differences $v - w$, where (v, w) ranges over the stable contours of \bar{B}_q (see [BrG, (1.5)]). Now the representation-finite standard algebra $\Lambda_q = \bar{B}_q$ admits a simply connected locally representation-finite universal Galois covering $F^q : \tilde{\Lambda}_q \rightarrow \tilde{\Lambda}_q/G_q = \Lambda$ with the free fundamental group $G_q = \Pi_1(Q_{\bar{B}_q}, I_{\bar{B}_q})$, where $\tilde{\Lambda}_q = \tilde{Q}_{\bar{B}_q}/I_{\bar{B}_q}$, as defined in [MP, Corollary 1.5] (see also [Ga, (2.1)]). Since $\tilde{\Lambda}_q$ is locally representation-finite, the associated push-down functor $F_\lambda^q : \text{mod } \tilde{\Lambda}_q \rightarrow \text{mod } \Lambda_q$ is dense. In particular, the indecomposable Λ_q -module Y_q is isomorphic to $F_\lambda^q(M_q)$ for some indecomposable finite dimensional $\tilde{\Lambda}_q$ -module M_q . Moreover, the support $\text{supp } M_q$ of M_q is a finite convex subcategory of $\tilde{\Lambda}_q$ containing at least 14 objects (by our choice of p) and contains pairwise different convex subcategories E_0, E_1, \dots, E_r each of them isomorphic to the path category $K\Delta/J$, where Δ is the quiver



and J is the ideal of $K\Delta$ generated by $\alpha_1 \dots \alpha_s - \rho' \rho''$. This contradicts the shapes of the quivers on the Bongartz list [Bo1, (2.4)] of 24 families of algebras forming the support algebras of all indecomposable modules over the representation-finite simply connected algebras having quivers with at least 14 vertices (see also [Ri, (6.3)]). Therefore, indeed all algebras B_n , $n \geq m$, are representation-finite and standard.

We will now describe the convex hull $B = \bigcup_{n \geq m} B_n$ of the support $\text{supp } Y$ of $Y = \varinjlim Y_n$. Take $p = m + 14$. For each $n \geq p$, consider the universal

Galois covering $F^n : \tilde{B}_n \rightarrow \tilde{B}_n/G_n = B_n$, given by the standard form presentation $B_n = KQ_{B_n}/I_{B_n}$ described in [BrG, (3.1)], the fundamental group $G_n = \Pi_1(Q_{B_n}, I_{B_n})$ of the bound quiver (Q_{B_n}, I_{B_n}) , and $\tilde{B}_n = K\tilde{Q}_{B_n}/\tilde{I}_{B_n}$ the associated simply connected, locally representation-finite, locally bounded category. Since the push-down functors $F_\lambda^n : \text{mod } \tilde{B}_n \rightarrow \text{mod } B_n$, are dense, for each $n \geq p$, we have $Y_n = F_\lambda^n(M_n)$ for an indecomposable finite dimensional \tilde{B}_n -module M_n . Observe also that, for any $n \geq p$, the support $D_n = \text{supp } M_n$ of M_n in \tilde{B}_n has at least $p = m + 14 \geq 14$ objects. It follows from the property (c) of the fundamental R -sequence $(Y_n, u_n)_{n \in \mathbb{N}}$ produced by Y that, for each $g \in G_Y \setminus \{1\}$ and $n \geq p$, there exists $r \geq n$ such that $gC_n \subseteq C_r$, gY_n is a direct summand of $Y_r|_{gC_n}$, and obviously Y_n is a direct summand of $Y_r|_{C_n}$. Again using the structure of indecomposable sincere modules over Bongartz's 24 families of algebras from [Bo1, (2.4)] and the fact that M_n are sincere indecomposable D_n -modules with $F_\lambda^n(M_n) = Y_n$, we conclude that all categories D_n , $n \geq p$, are finite convex lines of \tilde{B}_n , and M_n are isomorphic to the canonical linear indecomposable modules M_{D_n} associated to the lines D_n , respectively. Moreover, we obtain $B_n = F^n(D_n) = F^n(\text{supp } M_n) = \text{supp } Y_n$, for any $n \geq p$, and hence $B_n = \text{supp } Y_n$ for any $n \geq m$. This implies that $B = \text{supp } Y$ is a convex subcategory which admits a simply connected (not locally representation-finite) Galois covering $F^B : \tilde{B} \rightarrow \tilde{B}/H = B$ with a free group H such that $Y \cong F_\lambda^B(M_L)$ for the canonical linear module M_L over a convex line L of type ${}_\infty\mathbb{A}_\infty$ of \tilde{B} with nontrivial stabilizer $H_L = \{h \in H \mid hL = L\}$. As a consequence we obtain that $\text{supp } Y/G_L = B/G_L$ is a finite category, since L/H_L is a finite category. In particular, we conclude that Y is a representative of the unique isoclass of modules Z in $\text{Ind } R$ with nontrivial stabilizer G_Z and $\text{supp } Z \subseteq B = \text{supp } Y$. Obviously, Y is a weakly G -periodic R -module.

Summing up, we have proved that the modules Y in $\text{Ind } R$ with nontrivial stabilizer G_Y are weakly G -periodic R -modules, G_Y are infinite cyclic groups, $B_Y = \text{supp } Y$ are convex subcategories of R with $G_{B_Y} = G_Y$, uniquely determined by the isoclass of Y , and Y is isomorphic to the push-down $F_\lambda^{B_Y}(M_L)$ of the canonical linear indecomposable module M_L , where $F_\lambda^{B_Y} : \text{Mod } \tilde{B}_Y \rightarrow \text{Mod } B_Y$ is the push-down functor associated to a universal Galois covering $F^{B_Y} : \tilde{B}_Y \rightarrow \tilde{B}_Y/H_Y = B_Y$, and L is an H_Y -periodic line in \tilde{B}_Y . Moreover, every finite convex subcategory of $B_Y = \text{supp } Y$ is representation-finite.

Assume now that R is not G -exhaustive, equivalently that there exist weakly G -periodic R -modules. Denote by \mathcal{S} the family of the supports B_Y of all weakly G -periodic R -modules Y . We claim that \mathcal{S} is a separating family of convex subcategories of R , and hence Theorem 1.1 can be applied. Clearly, if Y is a weakly G -periodic R -module and $g \in G$, then gY is a weakly G -periodic R -module with $B_{{}^gY} = \text{supp } ({}^gY) = gB_Y = g \text{supp } Y$ and $G_{{}^gY} = gG_Y g^{-1}$, and hence the condition (i) is satisfied. For any G -orbit \mathcal{O} of R , the intersection $\mathcal{O} \cap B_Y$ is contained in finitely many G_Y -orbits, because B_Y/G_Y is finite, so (ii) holds. Further, for two different categories B_Y and $B_{Y'}$ in \mathcal{S} , the intersection $B_Y \cap B_{Y'}$ is a finite, and hence representation-finite, convex subcategory of B_Y (and of $B_{Y'}$), and so (iii) holds. The condition (iv) is trivially satisfied.

Now consider a fixed set $\mathcal{A} = \mathcal{A}(R, G)$ of representatives of isoclasses of weakly G -periodic R -modules closed under the induced action of G , and a fixed set $\mathcal{A}_0 = \mathcal{A}_0(R, G)$ of representatives of G -orbits in $\mathcal{A}(R, G)$. Observe that then the supports

$B_Y = \text{supp } Y$ of all modules Y in \mathcal{A}_0 form a set \mathcal{S}_0 of representatives of G -orbits of the separating family \mathcal{S} of convex subcategories of R defined above. Therefore, applying Theorem 1.1, we have a commutative diagram of functors

$$\begin{CD} \coprod_{\mathcal{A}_0} \text{mod } B_Y/G_Y @>(E_\lambda^L)>> \text{mod } A \\ @V(\Pi^Y)VV @VV\Pi V \\ \coprod_{\mathcal{A}_0} (\text{mod } B_Y/G_Y)/[\text{mod}_1 B_Y/G_Y] @>E>> \text{mod } A/[\text{mod}_1 A], \end{CD}$$

where $E_\lambda^L : \text{mod } B_Y/G_Y \rightarrow \text{mod } A = \text{mod } R/G$ are the functors induced by the embeddings $E^Y : \text{MOD } B_Y \rightarrow \text{MOD } R$ (described in Section 1), E is a K -linear equivalence of categories, and $\Pi^Y : \text{mod } B_Y/G_Y \rightarrow (\text{mod } B_Y/G_Y)/[\text{mod}_1 B_Y/G_Y]$ and $\Pi : \text{mod } A \rightarrow (\text{mod } A)/[\text{mod}_1 A]$ are the canonical quotient functors.

Now take a module Y in \mathcal{A}_0 and consider the universal Galois covering functor $F^{B_Y} : \tilde{B}_Y \rightarrow \tilde{B}_Y/H_Y = B_Y$ of the support $B_Y = \text{supp } Y$ of Y , and the associated push-down functor $F_\lambda^{B_Y} : \text{MOD } \tilde{B}_Y \rightarrow \text{MOD } B_Y$. Then $Y \cong F_\lambda^{B_Y}(M_L)$ for the canonical linear \tilde{B}_Y -module M_L associated to an H_Y -periodic line L of \tilde{B}_Y . It follows from the above discussion that there is only one H_Y -orbit of H_Y -periodic lines in \tilde{B}_Y , so we may fix an H_Y -periodic line L_Y in \tilde{B}_Y and the linear \tilde{B}_Y -module $M_Y = M_{L_Y}$ such that $Y = F_\lambda^{B_Y}(M_Y)$ and $B_Y = F^{B_Y}(L_Y)$. Denote by $F^Y : B_Y \rightarrow B_Y/G_Y$ the Galois covering functor induced by the action of G_Y on B_Y and by $F_\lambda^Y : \text{MOD } B_Y \rightarrow \text{MOD } B_Y/G_Y$ the associated push-down functor. Consider the composed Galois covering functor

$$\mathbb{F}^Y = F^Y \circ F^{B_Y} : \tilde{B}_Y \rightarrow B_Y/G_Y = \tilde{B}_Y/\mathbb{G}_Y,$$

which is given by an admissible group \mathbb{G}_Y of K -linear automorphisms of \tilde{B}_Y having H_Y as a normal subgroup with $\mathbb{G}_Y/H_Y = G_Y$ (see [Ga]). Moreover, \tilde{B}_Y is a simply connected locally bounded category whose every finite convex subcategory is representation-finite. Then, applying [DS3, Proposition 5.1], we obtain that every weakly \mathbb{G}_Y -periodic \tilde{B}_Y -module Z is linear. Clearly, the push-down $F_\lambda^{B_Y}(Z)$ of such a weakly \mathbb{G}_Y -periodic \tilde{B}_Y -module Z is a weakly H_Y -periodic B_Y -module, and hence is isomorphic to Y . Hence, the \mathbb{G}_Y -orbit of the linear H_Y -periodic \tilde{B}_Y -module M_Y consists of representatives of the isoclasses of all weakly \mathbb{G}_Y -periodic \tilde{B}_Y -modules. Therefore, by [DS3, Theorem 3.6], the tensor product functor

$$- \otimes_{K[T, T^{-1}]} \mathbb{F}^Y(M_Y) : \text{mod } K[T, T^{-1}] \longrightarrow \text{mod } B_Y/G_Y,$$

composed with the functor Π^Y , gives a natural K -linear equivalence of categories

$$\text{mod } K[T, T^{-1}] \longrightarrow (\text{mod } B_Y/G_Y)/[\text{mod}_1 B_Y/G_Y].$$

Observe also that $\mathbb{F}^Y(M_Y) = F^Y(F^{B_Y}(M_Y)) = F^Y(Y)$. Finally, it follows from [DS3, (2.4), (3.2)] that, for the tensor product functors

$$\Psi^Y = - \otimes_{K[T, T^{-1}]} F^Y(Y) : \text{mod } K[T, T^{-1}] \longrightarrow \text{mod } B_Y/G_Y$$

and

$$\Phi^Y = - \otimes_{K[T, T^{-1}]} F_\lambda(Y) : \text{mod } K[T, T^{-1}] \longrightarrow \text{mod } A,$$

the following diagram of functors is commutative:

$$\begin{array}{ccc}
 & \text{mod } K[T, T^{-1}] & \\
 \Psi^Y \swarrow & & \searrow \Phi^Y \\
 \text{mod } B_Y/G_Y & \xrightarrow{E_\lambda^Y} & \text{mod } R/G = \text{mod } A.
 \end{array}$$

Therefore, the functor $(\Phi^Y) : \coprod_{\mathcal{A}_0} \text{mod } K[T, T^{-1}] \rightarrow \text{mod } A$ composed with the functor $\Pi : \text{mod } A \rightarrow \text{mod } A/[\text{mod}_1 A]$ gives a natural K -linear equivalence of categories

$$\Phi : \coprod_{\mathcal{A}_0} \text{mod } K[T, T^{-1}] \longrightarrow \text{mod } A/[\text{mod}_1 A].$$

In particular, we conclude that every module X in $\text{ind}_2 A$ is isomorphic to $\Phi^Y(V) = V \otimes_{K[T, T^{-1}]} F_\lambda(Y)$ for some indecomposable finite dimensional $K[T, T^{-1}]$ -module V . This proves statement (ii).

It follows from [DS3, Theorem 3.6] that the equivalences of categories $\text{mod } K[T, T^{-1}] \rightarrow (\text{mod } B_Y/G_Y)/[\text{mod}_1 B_Y/G_Y]$, $Y \in \mathcal{A}_0$, established above, induce isomorphisms of translation quivers $\Gamma_{K[T, T^{-1}]} \cong (\Gamma_{B_Y/G_Y})_2$, where $(\Gamma_{B_Y/G_Y})_2$ is the translation subquiver of Γ_{B_Y/G_Y} formed by all modules from $\text{ind}_2 B_Y/G_Y$. Therefore, the translation subquiver $(\Gamma_A)_2$ of Γ_A formed by all modules from $\text{ind}_2 A$ is isomorphic to the disjoint union of $|\mathcal{A}_0|$ copies of the translation quiver $\Gamma_{K[T, T^{-1}]}$. Since the translation subquiver $(\Gamma_A)_1$ of Γ_A formed by all modules from $\text{ind}_1 A$ is the orbit translation quiver $\Gamma_{R/G}$, the Auslander-Reiten quiver Γ_A of A has the required decomposition

$$\Gamma_A = (\Gamma_{R/G}) \sqcup \left(\coprod_{\mathcal{A}_0} \Gamma_{K[T, T^{-1}]} \right).$$

This shows statement (iii).

For statement (i), observe that, since R is cycle-finite, every finite convex subcategory of R is of polynomial growth, by Theorem 2.2. Then applying [DS1, Lemma 3] we conclude that there exists a positive integer m such that, for any dimension d , there exists a finite number of $K[T]$ - A -bimodules M_i , $1 \leq i \leq n_d \leq d^m$, which are finitely generated and free as left $K[T]$ -modules such that all but a finite number of isoclasses of modules in $\text{ind}_1 A$ of dimension d are of the form $K[T]/(T - \lambda) \otimes_{K[T]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \dots, n_d\}$. This shows that the category $\text{mod}_1 A$ of finite dimensional A -modules of the first kind is of polynomial growth, hence tame. Further, it follows from (ii) that every module X in $\text{ind}_2 A$ is isomorphic to $K[T, T^{-1}]/(T - \mu)^r \otimes_{K[T, T^{-1}]} F_\lambda(Y)$ for some $Y \in \mathcal{A}_0 = \mathcal{A}_0(R, G)$, $\mu \in K \setminus \{0\}$ and $r \geq 1$. For each $Y \in \mathcal{A}_0$, consider the $K[S]$ - A -bimodules

$$Q_{Y,r} = K[S, T, T^{-1}]/(T - S)^r \otimes_{K[T, T^{-1}]} F_\lambda(Y), r \geq 1,$$

which are clearly finitely generated and free as left $K[S]$ -modules. Moreover, for every simple $K[S]$ -module $K[S]/(S - \mu)$, $\mu \in K \setminus \{0\}$, the (right) A -modules

$$K[S]/(S - \mu) \otimes_{K[S]} Q_{Y,r} \quad \text{and} \quad K[T, T^{-1}]/(T - \mu)^r \otimes_{K[T, T^{-1}]} F_\lambda(Y)$$

are isomorphic. Therefore, the category $\text{mod}_2 A$ of finite dimensional A -modules of the second kind is tame. Summing up, we conclude that A is tame, so (i) also holds. \square

We now exhibit properties of indecomposable locally finite dimensional modules over cycle-finite locally bounded categories established along the proof of Theorem 3.1.

Corollary 3.2. *Let R be a connected cycle-finite locally bounded K -category, Y a module in $\text{Ind } R$, and G a torsion-free admissible group of K -linear automorphisms of R . Then Y is a weakly G -periodic R -module if and only if the stabilizer G_Y of Y is nontrivial.*

Proposition 3.3. *Let R be a connected cycle-finite locally bounded K -category, G a torsion-free admissible group of K -linear automorphisms of R , and Y a weakly G -periodic R -module. Then the following statements hold:*

- (i) $B_Y = \text{supp } Y$ is an infinite convex subcategory of R whose every finite convex subcategory is representation-finite.
- (ii) $G_{B_Y} = G_Y$ is an infinite cyclic group.
- (iii) There is a Galois covering $F^{B_Y} : \tilde{B}_Y \rightarrow \tilde{B}_Y/H_Y = B_Y$, where \tilde{B}_Y is a simply connected locally bounded K -category, whose every finite convex subcategory is representation-finite, and $B_Y = F^{B_Y}(L_Y)$ for an H_Y -periodic line L_Y in \tilde{B}_Y .
- (iv) $Y \cong F_\lambda^{B_Y}(M_{L_Y})$, where $F_\lambda^{B_Y} : \text{Mod } \tilde{B}_Y \rightarrow \text{Mod } B_Y$ is the push-down functor associated to F^{B_Y} and M_{L_Y} is the canonical linear indecomposable \tilde{B}_Y -module associated to the H_Y -periodic line L_Y .
- (v) For a weakly G -periodic R -module Z with $\text{supp } Z = \text{supp } Y$, we have $Y \cong Z$.

As a direct consequence of Corollary 3.2 and [DS3, Proposition 2.3], described in Section 1, we obtain also the following result.

Corollary 3.4. *Let R be a connected cycle-finite locally bounded K -category and G a torsion-free admissible group of K -linear automorphisms of R . Then R is G -exhaustive if and only if G acts freely on the isoclasses in $\text{Ind } R$.*

It would be interesting to know if the G -exhaustivity of R , in the above corollary, forces R to be locally support-finite, and consequently the equality $\text{Ind } R = \text{ind } R$ (see [DS3, Proposition 2.5]).

The next result confirms Brenner's conjecture for the tame algebras having cycle-finite Galois coverings.

Theorem 3.5. *Let R be a connected cycle-finite locally bounded K -category, G a torsion-free admissible group of K -linear automorphisms of R , and $A = R/G$. Let*

$$0 \longrightarrow \tau_A X \longrightarrow \bigoplus_{i=1}^{s(X)} Y_i \longrightarrow X \longrightarrow 0$$

be an almost split sequence in $\text{mod } A$, where the modules Y_i are indecomposable. Then $s(X) \leq 5$. Moreover, if $s(X) = 5$, then one of the modules Y_i is projective-injective.

Proof. This follows from Theorems 2.6 and 3.1, because the Auslander-Reiten quiver $\Gamma_{K[T, T^{-1}]}$ consists of a K^* -family of stable tubes of rank one. \square

We also mention that the assumption on the group G to be torsion-free, posed in the above results, is justified by examples presented in [GP].

4. CRITERION FOR POLYNOMIAL GROWTH

The aim of this section is to establish a criterion for an algebra with a cycle-finite Galois covering to be of polynomial growth (respectively, domestic type).

Theorem 4.1. *Let R be a connected cycle-finite locally bounded K -category, G a torsion-free admissible group of K -linear automorphisms of R , and $A = R/G$. Then*

- (i) *A is of polynomial growth if and only if the number of G -orbits of isoclasses of weakly G -periodic R -modules is finite.*
- (ii) *A is domestic if and only if R does not contain a convex subcategory which is tubular and the number of G -orbits of isoclasses of weakly G -periodic R -modules is finite.*

Proof. Since every finite convex subcategory of R is of polynomial growth, as a cycle-finite category, we know by [DS1, Lemma 3] that the category $\text{mod}_1 A$ of finite dimensional A -modules of the first kind is of polynomial growth. Moreover, by Theorem 2.4, $\text{mod}_1 A$ is domestic if and only if R does not contain a finite convex subcategory which is a tubular algebra. We will discuss now the representation type of the category $\text{mod}_2 A$ of finite dimensional A -modules of the second kind. We keep the notation from the proof of Theorem 3.1. It follows from Theorem 3.1 that every indecomposable module in $\text{mod}_2 A$ is isomorphic to a module $\Phi^Y(V) = V \otimes_{K[T, T^{-1}]} F_\lambda(Y)$ for some weakly G -periodic R -module $Y \in \mathcal{A}_0 = \mathcal{A}_0(R, G)$ and some indecomposable finite dimensional $K[T, T^{-1}]$ -module V . Hence, if \mathcal{A}_0 is finite, then the category $\text{mod}_2 A$ is of domestic type. Therefore, if \mathcal{A}_0 is finite, then A is of polynomial growth, and A is domestic if and only if R does not contain a finite convex subcategory which is a tubular algebra.

Assume that \mathcal{A}_0 is infinite. We will show that then the category $\text{mod}_2 A$ is not of polynomial growth, and consequently A is not of polynomial growth, and then not domestic. We will apply arguments similar to those used in the proof of [Sk9, Proposition 1.2]. We know from Proposition 3.3 that, for each weakly G -periodic R -module $Y \in \mathcal{A}(R, G)$, its support $B_Y = \text{supp } Y$ admits a simply connected Galois covering $F^{B_Y} : \tilde{B}_Y \rightarrow \tilde{B}_Y/H_Y = B_Y$ such that $B_Y = F^{B_Y}(L_Y)$ for a weakly H_Y -periodic line L_Y in \tilde{B}_Y , and $Y \cong F_\lambda^{B_Y}(M_Y)$ for the canonical linear \tilde{B}_Y -module $M_Y = M_{L_Y}$ associated to L_Y . Moreover, every finite convex subcategory of B_Y is a representation-finite standard algebra.

For each dimension d , denote by $\mu_A^2(d)$ the least number of $Y \in \mathcal{A}_0 = \mathcal{A}_0(R, G)$ such that every module X in $\text{ind}_2 A$ of dimension d is isomorphic to $\Phi^Y(V)$ for some $Y \in \mathcal{A}_0$ and some indecomposable finite dimensional $K[T, T^{-1}]$ -module V .

Suppose that $\mu_A^2(d) \leq d^m$ for a fixed positive integer m and all $d \geq 1$. Let $Y^{(n)}$, $n \geq 1$, be pairwise different modules from \mathcal{A}_0 . Then it follows from Proposition 3.3 that the support categories $B^{(n)} = B_{Y^{(n)}}$, $n \geq 1$, are also pairwise different. Since $A = R/G$ is a finite category, there exists an object x in R whose G -orbit Gx intersects infinitely many categories $B^{(n)}$, $n \geq 1$. Without loss of generality, we may assume that x belongs to all categories $B^{(n)}$, $n \geq 1$.

Observe that, for each $n \geq 1$, there are only finitely many categories B_Y , $Y \in \mathcal{A} = \mathcal{A}(R, G)$, containing x and with B_Y/G_Y having at most r objects. Further, if $R = KQ_R/I_R$, for an admissible ideal I_R in KQ_R , then there is a common bound on the length of paths in Q_R which do not belong to I_R , because $A = R/G$ is finite. This implies that there are two categories $B' = B^{(i)}$ and $B'' = B^{(j)}$ such that the intersection $B' \cap B''$ contains the common category $F'(L') = F''(L'')$, where $F' = F^{B^{(i)}}$, $F'' = F^{B^{(j)}}$, L' is a finite subline of the line $L_{Y^{(i)}}$ of $\tilde{B}^{(i)}$ of the form

$$a' \longrightarrow y'_1 \longleftarrow \dots \longleftarrow y'_p \longleftarrow t' \longrightarrow z'_s \longrightarrow \dots \longrightarrow z'_1 \longleftarrow b',$$

L'' is a finite subline of the line $L_{Y^{(j)}}$ of $\tilde{B}^{(j)}$ of the form

$$a'' \longrightarrow y''_1 \longleftarrow \dots \longleftarrow y''_p \longleftarrow t'' \longrightarrow z''_s \longrightarrow \dots \longrightarrow z''_1 \longleftarrow b'',$$

with $F'(a') = a = F''(a'')$, $F'(y'_l) = y_l = F''(y''_l)$, for $l \in \{1, \dots, p\}$, $F'(t') = t = F''(t'')$, $F'(z'_k) = z_k = F''(z''_k)$, for $k \in \{1, \dots, s\}$, and $F'(b') = b = F''(b'')$. Let $g \in G_{Y'}$ be such that $gy_1 \notin B''$ and B' contains a convex subcategory $C = F'(v)$, for a subline v of L' of the form

$$y'_1 \longleftarrow \dots \longleftarrow y'_p \longleftarrow t' \longrightarrow z'_s \longrightarrow \dots \longrightarrow z'_1 \longleftarrow \dots \longrightarrow c_1,$$

with $F'(c_1) = gy_1$. Similarly, let $h \in G_{Y''}$ be such that $hy_1 \notin B'$ and B'' contains a convex subcategory $D = F''(w)$ for a subline w of L'' of the form

$$y''_1 \longleftarrow \dots \longleftarrow y''_p \longleftarrow t'' \longrightarrow z''_s \longrightarrow \dots \longrightarrow z''_1 \longleftarrow \dots \longrightarrow d_1,$$

with $F''(d_1) = hy_1$. For each positive integer n , denote by C^n the full subcategory of B' given by the objects of $C, gC, \dots, g^{n-1}C$. Observe that $C^n = F'(v^n)$, where v^n is a convex connected subline of L' with ends y'_1 and c_n such that $F'(c_n) = g^n y_1$. Similarly, for each positive integer r , denote by D^r the full subcategory of B'' given by the objects of $D, hD, \dots, h^{r-1}D$. Again, $D^r = F''(w^r)$, where w^r is a convex connected subline of L'' with ends y''_1 and d_r such that $F''(d_r) = h^r y_1$.

Let M be the larger of the numbers of objects in C and in D . Take a prime number q such that

$$2^q - 2 > 2^{q-1} > q^{m+3} \quad \text{and} \quad q > M^{m+1}.$$

For any sequence $(\mathbf{n}, \mathbf{r}) = (n_1, r_1, n_2, r_2, \dots, n_s, r_s)$ of integers $n_i, r_i \in \mathbb{N}$ with $s \geq 1$, $\sum_{i=1}^t (n_i + r_i) = q$, $\sum_{i=1}^s n_i > 0$, $\sum_{i=1}^s r_i > 0$, we denote by $E(\mathbf{n}, \mathbf{r})$ the full subcategory of R given by the objects of

$$C^{n_1}, g^{n_1} D^{r_1}, h^{r_1} g^{n_1} C^{n_2}, g^{n_2} h^{r_1} g^{n_1} D^{r_2}, \dots, g^{n_s} h^{r_{s-1}} \dots h^{r_1} g^{n_1} D^{r_s},$$

and by B_f the full subcategory of R consisting of the objects $f^i x$, $i \in \mathbb{Z}$, $x \in E(\mathbf{n}, \mathbf{r})$, where $f = h^{r_s} g^{n_s} \dots h^{r_1} g^{n_1}$. Observe that B_f is a convex subcategory of R whose every finite convex subcategory is a standard representation-finite algebra, admits

a simply connected Galois covering $F^{(f)} : \tilde{B}_f \rightarrow \tilde{B}_f/H_f = B_f$, and \tilde{B}_f contains an H_f -periodic line L_f such that $F^{(f)}(L_f) = B_f$. Further, the image $Y_f = F_\lambda^{(f)}(M_f)$ of the canonical linear indecomposable module $M_f = M_{L_f}$ associated to L_f by the push-down functor $F_\lambda^{(f)} : \text{MOD } \tilde{B}_f \rightarrow \text{MOD } B_f$ is a weakly G -periodic R -module with $B_f = \text{supp } Y_f$. Moreover, the stabilizer G_{Y_f} of Y_f is the infinite cyclic group generated by f .

Let $(\mathbf{n}, \mathbf{r}) = (n_1, r_1, \dots, n_s, r_s)$ and $(\mathbf{n}', \mathbf{r}') = (n'_1, r'_1, \dots, n'_t, r'_t)$ be two sequences of nonnegative integers satisfying the above condition, and $f = h^{r_s} g^{n_s} \dots h^{r_1} g^{n_1}$, $f' = h^{r'_t} g^{n'_t} \dots h^{r'_1} g^{n'_1}$. Then the isoclasses of Y_f and $Y_{f'}$ belong to the same G -orbit if and only if one of the following two conditions holds:

$$(1) \quad (n'_1, r'_1, \dots, n'_t, r'_t) = (b_i, r_i, n_{i+1}, r_{i+1}, \dots, n_s, r_s, n_1, r_1, \dots, n_{i-1}, r_{i-1}, c_i, 0)$$

for some $i \geq 1$ and $b_i, c_i \in \mathbb{N}$ with $b_i + c_i = n_i$,

or

$$(2) \quad (n'_1, r'_1, \dots, n'_t, r'_t) = (0, d_i, n_{i+1}, r_{i+1}, \dots, n_s, r_s, n_1, r_1, \dots, n_{i-1}, r_{i-1}, n_i, e_i)$$

for some $i \geq 1$ and $d_i, e_i \in \mathbb{N}$ with $d_i + e_i = r_i$.

Therefore, there exist $(2^q - 2)/q$ pairwise different G -orbits of isoclasses of weakly G -periodic R -modules of the form Y_f . For any element $f = h^{r_s} g^{n_s} \dots h^{r_1} g^{n_1}$ associated to a sequence $(\mathbf{n}, \mathbf{r}) = (n_1, r_1, \dots, n_s, r_s)$, satisfying the imposed conditions, and an element $\lambda \in K^* = K \setminus \{0\}$, consider the A -module

$$\mathbb{X}(\lambda, f) = K[T, T^{-1}]/(T - \lambda) \otimes_{K[T, T^{-1}]} F_\lambda(Y_f).$$

Then $\mathbb{X}(\lambda, f)$ is a module in $\text{ind}_2 A$ with $\dim_K \mathbb{X}(\lambda, f) \leq Mq$. Moreover, $\mathbb{X}(\lambda, f) \cong \mathbb{X}(\lambda', f')$ if and only if $\lambda = \lambda'$ and the isoclasses of Y_f and $Y_{f'}$ belong to the same G -orbit. Therefore, we infer that

$$\sum_{d \leq Mq} \mu_A^2(d) \geq (2^q - 2)/q.$$

Hence there exists $p \leq Mq$ such that $\mu_A^2(p) \geq (2^q - 2)/(Mq^2)$. On the other hand, we have $\mu_A^2(p) \leq p^m \leq (Mq)^m$ and so $2^q - 2 \leq M^{m+1} q^{m+2}$. But, by our choice of q , we have $2^q - 2 > 2^{q-1} > q^{m+3} > M^{m+1} q^{m+2}$, a contradiction. Therefore, if $\mathcal{A}_0 = \mathcal{A}_0(R, G)$ is infinite, then $\text{mod}_2 A$ is not of polynomial growth. \square

5. EXAMPLES

Here we exhibit some examples illustrating the main results of the paper.

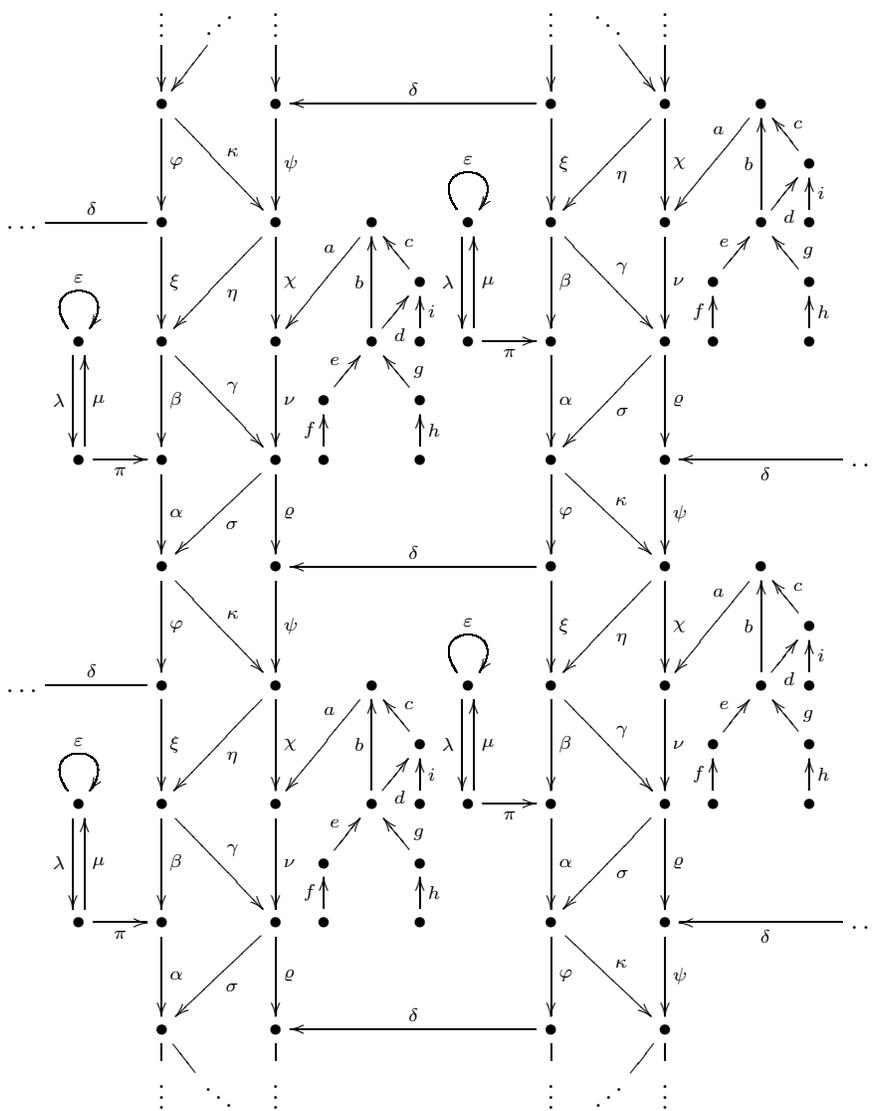
5.1. Let R be a locally bounded interval-finite K -category such that every finite convex subcategory of R is of finite representation type, G a torsion-free admissible group of K -linear automorphisms of R , and $A = R/G$. Then R is a cycle-finite category and, applying Theorem 3.1, we conclude that A is tame. We note that the special biserial algebras, playing the fundamental role in the representation theory, admit such Galois coverings (see [DS3], [ES], [PoS]).

5.2. A triangular locally bounded interval-finite K -category R is called *strongly simply connected* if, for every finite convex subcategory C of A , the first Hochschild

convex hereditary subcategory of type \tilde{A}_3 given by the objects 9, 10, 11. We then conclude that A does not admit a strongly simply connected Galois covering.

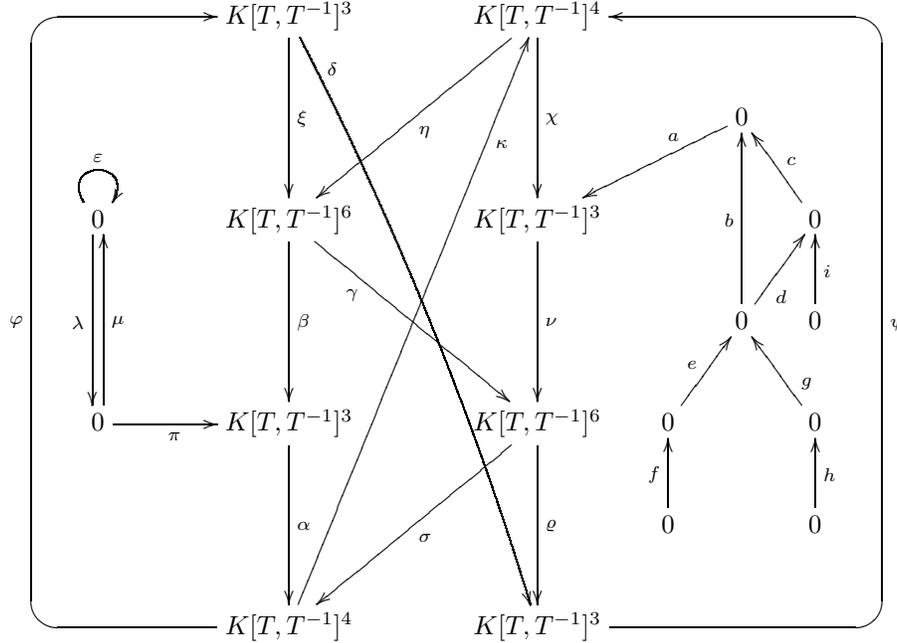
For K of characteristic 2, the convex subcategory P of A given by the objects 17 and 18 is a penny-farthing, and hence is a nonstandard representation-finite algebra. Hence, for K of characteristic 2, the algebra A does not admit a simply connected (even triangular) Galois covering.

The algebra A admits a Galois covering $R \rightarrow R/G = A$, where R is the bound quiver category $K\Delta/J$ with Δ the quiver of the form



J the ideal of the path category $K\Delta$ of Δ generated by all elements of the form $\alpha\varphi, \alpha\kappa, \sigma\varphi, \sigma\kappa, \gamma\sigma, \gamma\rho, \nu\rho, \nu\sigma, \eta\gamma, \eta\beta, \xi\beta, \xi\gamma, \varphi\delta, \delta\psi, \psi\eta, \psi\chi, \kappa\chi, \kappa\eta, a\nu, ba, dca, ic, eb, gb - gdc, \pi\alpha, \lambda\pi, \mu\lambda\mu, \varepsilon^2 - \lambda\mu, \mu\lambda - \mu\varepsilon\lambda$, and G the free abelian group of rank 2 generated by the canonical horizontal and vertical shifts of $R = K\Delta/J$. Clearly, R is a locally bounded interval-finite K -category. Applying the standard representation theory techniques one shows that the convex hull of the support of an indecomposable finite dimensional R -module X is either a representation-finite subcategory containing one of the penny-farthings given by the arrows $\varepsilon, \lambda, \mu$, is contained in one of the tubular subcategories of tubular type $(3, 3, 3)$ given by the arrows b, c, d, e, f, g, h, i , or is a representation-finite subcategory of the special biserial subcategory of R given by the subquiver of Δ consisting of all arrows except the arrows $\varepsilon, \lambda, \mu, b, d, e, f, g, h, i$. Therefore, R is a cycle-finite locally bounded category, and hence, applying Theorem 3.1, we conclude that A is a tame algebra. Further, applying Proposition 3.3, we conclude that the number of G -orbits of isoclasses of weakly G -periodic R -modules is infinite, and hence, by Theorem 4.1, A is not of polynomial growth. A typical weakly G -periodic R -module Y is of the

$K \xleftarrow{\delta=1} K$ at the bottom. Then the $K[T, T^{-1}]$ - A -bimodule $F_\lambda(Y)$ is of the form



where $\xi, \eta, \gamma, \nu, \alpha, \sigma, \kappa, \psi, \beta, \varphi, \delta, \chi, \varrho$ are given by the matrices

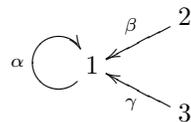
$$\xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \nu, \eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \kappa = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \psi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

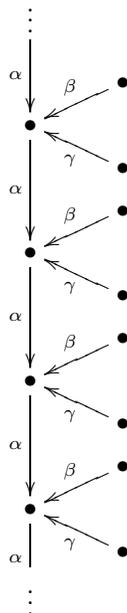
$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \varrho, \varphi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \delta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ T & 0 & 0 \end{bmatrix}, \chi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5.4. We now present an example of a tame algebra A which admits a Galois covering $R \rightarrow R/G = A$ with infinite cyclic group G and R an interval-finite tame locally bounded K -category having infinitely many pairwise nonisomorphic weakly G -periodic modules with the support equal to R . This will show that the assumption on R to be cycle-finite is necessary for the validity of Proposition 3.3.

Let A be the bound quiver algebra KQ/I given by the quiver Q of the form

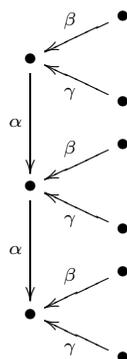


and the ideal I of the path algebra KQ generated by α^2 . The algebra A admits a Galois covering $R \rightarrow R/G = A$, where R is the bound quiver category $K\Delta/J$ with Δ the quiver of the form



J the ideal of the path category $K\Delta$ of Δ generated by all elements of the form α^2 , and G is the infinite cyclic group generated by the canonical vertical shift of $R = K\Delta/J$.

Then R is an interval-finite strongly simply connected locally bounded K -category which does not contain a hypercritical finite convex subcategory (see [U] for the list of hypercritical algebras), and consequently R is tame, by [BPS, Corollary 1]. On the other hand, R contains the finite convex subcategory $R' = K\Delta'/J'$ given by a subquiver Δ' of Δ of the form



and the ideal J' of $K\Delta'$ generated by α^2 , which is a pg -critical algebra in the sense of [NS]. In particular, by [Sk8, Theorem 4.1], R' is not of polynomial growth, and hence is not cycle-finite. Therefore, R is a tame strongly simply connected locally bounded K -category which is not cycle-finite. Moreover, it is not hard to show that there are infinitely many pairwise nonisomorphic weakly G -periodic R -modules with

support equal to R . We also mention that A is a tame algebra. For example, by [Bru, Lemma 5.3], A degenerates (in the variety of K -algebras of dimension 8) to the special biserial bound quiver algebra $\bar{A} = K\bar{Q}/\bar{I}$ given by the quiver \bar{Q} of the form

$$\alpha \circlearrowleft 1 \xleftarrow{\sigma} 2 \circlearrowright \varepsilon$$

and the ideal \bar{I} of the path algebra $K\bar{Q}$ generated by α^2 and ε^2 . Since \bar{A} is tame (see [DS3, (5.2)]), A is also tame, by [Ge] or [CB3].

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