

THE IMAGE OF THE MAP FROM GROUP COHOMOLOGY TO GALOIS COHOMOLOGY

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ABSTRACT. We study the image of the natural map from group cohomology to Galois cohomology by using motivic cohomology of classifying spaces.

1. INTRODUCTION

Let k be a field of $ch(k) = 0$, which contains a p -th root of unity. Let G be a split affine algebraic group over k , V a faithful representation of G , and $k(V)^G$ the invariant field. When G is a finite group, we have the natural quotient map of groups $q : Gal(\bar{k}(\bar{V})/k(W)^G) \rightarrow G$, which induces the map from the group cohomology of G to the Galois cohomology of $k(V)^G$.

This map is also defined for connected groups (and finite groups) by using the classifying space BG defined by Totaro and Voevodsky ([To], [Vo4]). Given degree $* > 1$, let W_* be a faithful representation of G such that G acts freely on $W_* - S_*$ for some closed subset S_* with $codim_{W_*} S_* > *$, e.g., $W_* = V^{\oplus(*+1)}$. Then we can define the map (see §3 below)

$$q^* : H^*(BG; \mathbb{Z}/p) \rightarrow H^*(k(W_*)^G; \mathbb{Z}/p).$$

The purpose of this paper is to study the image $Im(q^*)$ by using the motivic cohomology defined by Suslin and Voevodsky [Vo1], [Vo2], [Vo3]. The image $Im(q^*)$ is called the *stable* cohomology in [Bo], [Bo-Pe-Ts]. The kernel $Ker(q^*) = Ng(G)$ is called the negligible ideal [Pe], [Sa].

Let $H^{*,*'}(X; \mathbb{Z}/p)$ be the *mod*(p) motivic cohomology. Let

$$0 \neq \tau \in H^{0,1}(Spec(k); \mathbb{Z}/p) \cong \mathbb{Z}/p.$$

Using an affirmative solution of the Bloch-Kato conjecture by Voevodsky (and hence the Beilinson-Lichtenbaum conjecture), the map q^* is decomposed as

$$q^* : H^{*,*}(BG; \mathbb{Z}/p) \rightarrow H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \rightarrow H^{*,*}(Spec(k(W_*)^G; \mathbb{Z}/p),$$

where $H^{*,*}(BG; \mathbb{Z}/p)/(\tau) = H^{*,*}(BG; \mathbb{Z}/p)/(\tau H^{*,*-1}(BG; \mathbb{Z}/p))$.

By the Beilinson-Lichtenbaum conjecture and the work of Bloch-Ogus [Bl-Og], we know that

$$H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \subset H_{Zar}^0(BG; H_{\mathbb{Z}/p}^*) \subset H^*(k(W_*)^G; \mathbb{Z}/p).$$

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Here $H_{\mathbb{Z}/p}^*$ is the Zariski sheaf induced from the presheaf $H_{et}^*(V; \mathbb{Z}/p)$ for an open subset V of BG . Therefore we see [Or-Vi-Vo]

$$Im(q^*) = H^*(BG; \mathbb{Z}/p)/(Ng) \cong H^{*,*}(BG; \mathbb{Z}/p)/(\tau).$$

Note that the ring on the right-hand side does not depend on the choice of W_* . We also note that the ideal $(Ng) = Ng(G)$ coincides with the coniveau filtration $N^1 H^*(BG; \mathbb{Z}/p)$ defined by Grothendieck.

In this paper we compute $Im(q^*)$ (when k is an algebraically closed field in \mathbb{C}) for abelian p -groups, O_n , SO_n , $Spin_n$, PGL_p , and exceptional groups. Extraspecial p -groups are also studied. For example, we see

$$H^*(BSpin_n; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2 \quad \text{for } n \geq 3.$$

Recall that the cohomological invariant $Inv^*(G; \mathbb{Z}/p)$ is (roughly speaking) a ring of natural maps $H^1(F; G) \rightarrow H^*(F; \mathbb{Z}/p)$ for finitely generated fields F over k . This ring is very well studied; for example, see [Ga-Me-Se]. In particular, it is very useful to compute the essential dimension $ed(G)$ of G ([Re], [Br-Re-Vi]). Moreover, Totaro proved that

$$Inv^*(G; \mathbb{Z}/p) \cong H_{Zar}^0(BG; H_{\mathbb{Z}/p}^*)$$

in a letter to Serre [Ga-Me-Se]. Hence $Im(q^*) \subset Inv^*(G; \mathbb{Z}/p)$.

We use these results for some parts of this paper. However, we also give new explanations of $Inv^*(G; \mathbb{Z}/p)$ for the case that k is an algebraically closed field in \mathbb{C} . For example, the image of a (topological) Stiefel-Whitney class w_i of the map

$$H^*(BO_n; \mathbb{Z}/2) \rightarrow H^*(BO_n; \mathbb{Z}/2)/(Ng) \subset Inv^*(O_n; \mathbb{Z}/2)$$

is indeed the Stiefel-Whitney class w_i defined by Milnor and Serre as the natural function from quadratic forms to Milnor K -theories.

All examples stated above are detected by abelian p -subgroups A of G , i.e., the restriction map

$$Res : H^*(BG; \mathbb{Z}/p)/(Ng) \rightarrow \Pi_A H^*(BA; \mathbb{Z}/p)/(Ng)$$

is injective. (Indeed, most of the above cases are detected by only one elementary abelian p -subgroup.)

Of course, this detected property does not hold for general G . However, to give examples is not so easy. Indeed, for a p -group G of exponent p , if $H^2(BG; \mathbb{Z}/p)/(Ng)$ is not detected by any $A \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, then G is a counterexample of Noether's problem, namely, $k(W_2)^G$ is not purely transcendental over k . The examples of Saltman and Bogomolov are essentially of these types [Sa].

For each $n > 1$, we give an example G_n of a p -group $p \geq 3$, such that $H^{2n}(G_n; \mathbb{Z}/p)$ is not detected by abelian p -subgroups, while it does not imply a counterexample of Noether's problem. Here the composition $Q_{2n-2} \dots Q_0(x)$ of Milnor operations is used to see $x \notin Ng(G)$ given $x \in H^{2n}(BG; \mathbb{Z}/p)$.

An outline of this paper is the following. In §2, we recall the properties of the motivic cohomology and coniveau filtrations. In §3, we study cohomology of groups $H^*(BG; \mathbb{Z}/p)$ and the definitions of stable cohomology and negligible ideals $Ng(G)$. In particular, we give conditions for $x \in Ng(G)$ by using cohomology operations. In §4, we recall the definition of the cohomological invariants. In §§5–8, we study cohomological invariants for concrete groups, e.g., abelian p -groups, O_n , SO_n , $Spin_n$, exceptional groups, and PGL_p . Extraspecial p -groups are studied in §9 and are also used in §§11–12. We recall Noether's problem in §10. Saltman's

example (which is a special p -group) is studied in §11. In the last section, we give an example for the above non-detected problem.

2. MOTIVIC COHOMOLOGY

Let X be a smooth (quasi-projective) variety. Let $H^{*,*'}(X; \mathbb{Z}/p)$ be the $mod(p)$ motivic cohomology defined by Voevodsky and Suslin.

Recall that the $(mod\ p)$ $BL(n, p)$ condition holds if

$$H^{m,n}(X; \mathbb{Z}/p) \cong H_{et}^m(X; \mu_p^{\otimes n}) \quad \text{for all } m \leq n.$$

It is known that the $BL(n, p)$ condition holds for $p = 2$ or $n = 2$ by Voevodsky [Vo1], [Vo2], and Merkurjev-Suslin, respectively. Recently M. Rost and V. Voevodsky ([Vo5], [Su-Jo], [Ro]) proved that the $BL(n, p)$ condition holds for each p and n . Hence the Bloch-Kato conjecture also holds. Therefore, in this paper, we *always assume* the $BL(n, p)$ -condition and also the Bloch-Kato conjecture for all n, p .

Moreover, we always assume that k contains a primitive p -th root of unity. For these cases, we see the isomorphism $H_{et}^m(X; \mu_p^{\otimes n}) \cong H_{et}^m(X; \mathbb{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k); \mathbb{Z}/p) \cong \mathbb{Z}/p$, so that

$$colim_i \tau^i H^{*,*'}(X; \mathbb{Z}/p) \cong H_{et}^*(X; \mathbb{Z}/p).$$

Recall that $\mathbb{Z}/p(n)$ [Vo1], [Vo2], [Vo3] is the complex of sheaves in Zariski topology such that $H^{m,n}(X; \mathbb{Z}/p) \cong H_{Zar}^m(X; \mathbb{Z}/p(n))$. Let α be the obvious map of sites from etale topology to Zariski topology so that $\alpha^*\mathbb{Z}/p(1) \cong \mu_p$ of etale sheaves and hence

$$H_{et}^m(X; \mathbb{Z}/p) \cong H_{et}^m(X; \mu_p^{\otimes n}) \cong H_{Zar}^m(X, R\alpha_*\alpha^*\mathbb{Z}/p(n)).$$

For $k \leq n$, let $\tau_{\leq k}R\alpha_*\alpha^*\mathbb{Z}/p(n)$ be the canonical truncation of $R\alpha_*\alpha^*\mathbb{Z}/p(n)$ of level k . Then we have the exact triangle in the derived category of Zariski sheaves

$$\rightarrow \tau_{\leq n-1}R\alpha_*\alpha^*\mathbb{Z}/p(n) \rightarrow \tau_{\leq n}R\alpha_*\alpha^*\mathbb{Z}/p(n) \rightarrow H_{\mathbb{Z}/p}^n[-n] \rightarrow,$$

where $H_{\mathbb{Z}/p}^n$ is the Zariski sheaf induced from the presheaf $H_{et}^n(V; \mathbb{Z}/p)$ for an open subset V of X . The Beilinson and Lichtenbaum conjecture (hence the $BL(n, p)$ -condition) (see [Vo2], [Vo5]) implies

$$\mathbb{Z}/p(k) \cong \tau_{\leq k}R\alpha_*\alpha^*\mathbb{Z}/p(n).$$

Hence we have

Lemma 2.1 ([Or-Vi-Vo], [Vo5]). *There is a long exact sequence*

$$\begin{aligned} \rightarrow H^{m,n-1}(X; \mathbb{Z}/p) \xrightarrow{\simeq} H^{m,n}(X; \mathbb{Z}/p) \\ \rightarrow H_{Zar}^{m-n}(X; H_{\mathbb{Z}/p}^n) \rightarrow H^{m+1,n-1}(X; \mathbb{Z}/p) \xrightarrow{\simeq} . \end{aligned}$$

In particular, we have

Corollary 2.2. *There is a short exact sequence*

$$0 \rightarrow H^{m,n}(X; \mathbb{Z}/p)/(\tau) \rightarrow H_{Zar}^{m-n}(X; H_{\mathbb{Z}/p}^n) \rightarrow {}_{\tau}H^{m+1,n-1}(X; \mathbb{Z}/p) \rightarrow 0,$$

where $H^{m,n}(X; \mathbb{Z}/p)/(\tau) = H^{m,n}(X; \mathbb{Z}/p)/(\tau H^{m,n-1}(X; \mathbb{Z}/p))$ and

$${}_{\tau}H^{m+1,n-1}(X; \mathbb{Z}/p) = Ker(\tau)|H^{m+1,n-1}(X; \mathbb{Z}/p).$$

Note that the above long exact sequence induces the τ -Bockstein spectral sequence

$$E(\tau)_1^{m,n} = H_{Zar}^{m-n}(X; H_{\mathbb{Z}/p}^n) \implies \text{colim}_i \tau^i H^{m,n}(X; \mathbb{Z}/p) \cong H_{et}^m(X; \mathbb{Z}/p).$$

On the other hand, the filtration *coniveau* is given by

$$N^c H_{et}^m(X; \mathbb{Z}/p) = \bigcup_Z \text{Ker}\{H_{et}^m(X; \mathbb{Z}/p) \rightarrow H_{et}^m(X - Z; \mathbb{Z}/p)\},$$

where Z runs in the set of closed subschemes of X of $\text{codim} = c$. Grothendieck wrote the E_1 -term of the spectral sequence induced from the above coniveau filtration,

$$E(c)_1^{c,m-c} \cong \Pi_{x \in X^{(c)}} H_{et}^{m-2c}(k(x); \mathbb{Z}/p) \implies H_{et}^m(X; \mathbb{Z}/p),$$

where $X^{(c)}$ is the set of primes of codimension c and $k(x)$ is the residue field of x . We can regard $i_{x*} H_{et}^{m-2c}(k(x); \mathbb{Z}/p)$ as a constant sheaf $H_{et}^{m-2c}(k(x); \mathbb{Z}/p)$ on $\{\bar{x}\}$ and extend it by zero to X . Then the differentials of the spectral sequence give us a complex on sheaves on X :

$$(2.9) \quad 0 \rightarrow H_{\mathbb{Z}/p}^q \rightarrow \Pi_{x \in X^{(0)}} i_{x*} H_{et}^q(k(x); \mathbb{Z}/p) \rightarrow \Pi_{x \in X^{(1)}} i_{x*} H_{et}^{q-1}(k(x); \mathbb{Z}/p) \rightarrow \dots \rightarrow \Pi_{x \in X^{(q)}} i_{x*} H_{et}^0(k(x); \mathbb{Z}/p) \rightarrow 0.$$

Bloch-Ogus [Bl-Og] proved that the above sequence of sheaves is exact and the E_2 -term is given by

$$E(c)_2^{c,m-c} \cong H_{Zar}^c(X, H_{\mathbb{Z}/p}^{m-c}).$$

In particular, we have

Corollary 2.3.

$$H_{Zar}^0(X; H_{\mathbb{Z}/p}^m) \cong \text{Ker}\{H_{et}^m(k(X); \mathbb{Z}/p) \rightarrow \Pi_{x \in X^{(1)}} H_{et}^{m-1}(k(x); \mathbb{Z}/p)\}.$$

By Deligne (footnote (1) in Remark 6.4 in [Bl-Og]) and Paranjape (Corollary 4.4 in [Pa]), it is proven that there is an isomorphism of the coniveau spectral sequence with the Leray spectral sequence for the map α . Hence we have

Theorem 2.4 (Deligne, Parajape). *There is the isomorphism*

$$E(c)_r^{c,m-c} \cong E(\tau)_{r-1}^{m,m-c}$$

for $r \geq 2$ of spectral sequences. Hence the filtrations are the same for

$$N^c H_{et}^m(X; \mathbb{Z}/p) = F_\tau^{m,m-c},$$

where

$$F_\tau^{m,m-c} = \text{Im}(\times \tau^c : H^{m,m-c}(X; \mathbb{Z}/p) \rightarrow H^{m,m}(X; \mathbb{Z}/p)).$$

3. COHOMOLOGY OF GROUPS

Let G be a reductive algebraic group over k acting on an affine variety W . A point $x \in W$ is called stable if the orbit Gx is closed and the stabilizer group $\text{Stab}(x)$ is a finite group. Let us write by X^s the set of stable points in X . Then X^s is an open subset of X and there is the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi} & W//G \\ \uparrow \text{open} & & \uparrow \text{open} \\ W^s & \xrightarrow{\phi^s} & W^s/G, \end{array}$$

where $W//G = \text{Spec}(k[W]^G)$, the geometric quotient of X . We also note that the invariant field $k(W)^G$ is a quotient field of $k[W]^G$ when $W^s \neq \emptyset$.

Suppose $W^s \neq \emptyset$. Then $H^{*,*'}(W//G; \mathbb{Z}/p) = H^{*,*'}(\text{Spec}(k[W]^G); \mathbb{Z}/p)$ and $k(W^s/G) = k(W//G) = k(W)^G$. Hence we have the diagram

$$\begin{CD} H^{*,*'}(W//G; \mathbb{Z}/p) @>>> H^{*,*'}(\text{Spec}(k(W)^G); \mathbb{Z}/p) \\ @VVV @=VV \\ H^{*,*'}(W^s/G; \mathbb{Z}/p) @>\psi^{*,*'}>> H^{*,*'}(\text{Spec}(k(W^s/G)); \mathbb{Z}/p). \end{CD}$$

We recall the definition of BG according to Voevodsky and Totaro (§6 in [Vo4], [To]). For a smooth embedding $X \subset Y$ of codimension c , we have the Gysin exact sequence [Vo1], [Vo2], [Vo3]

$$\rightarrow H^{*-2c,*'-c}(X; \mathbb{Z}/p) \xrightarrow{g_*} H^{*,*'}(Y; \mathbb{Z}/p) \rightarrow H^{*,*'}(Y - X; \mathbb{Z}/p) \rightarrow .$$

Since $H^{*,<0}(X; \mathbb{Z}/p) = 0$, we have the isomorphism

$$H^{*,*'}(Y; \mathbb{Z}/p) \cong H^{*,*'}(Y - X; \mathbb{Z}/p) \quad \text{if } *' < c.$$

The above isomorphism also holds for non-smooth X (e.g., (3.5) in [Vo4]).

For $n > 1$, let W_n be a faithful representation of G such that G acts freely on $W_n - S_n$ for some closed subset S_n with $\text{codim}_{W_n} S_n > n$.

Let us write $U_n = (W_n - S_n)$ simply. Let $U'_n = W'_n - S'_n$ be another such set. Then we can consider the maps

$$U_n \times_G W'_n \leftarrow U_n \times_G U'_n \rightarrow W_n \times_G U'_n$$

which imply the isomorphisms of motivic cohomology for $*' \leq n$ from the above result by the Gysin sequence

$$H^{*,*'}(U_n/G; \mathbb{Z}/p) \cong H^{*,*'}(U_n \times_G U'_n; \mathbb{Z}/p) \cong H^{*,*'}(U'_n/G; \mathbb{Z}/p).$$

Hence the motivic cohomology $H^{*,*'}(U_n/G; \mathbb{Z}/p)$ for $*' \leq n$ does not depend on the choice of such (W_n, S_n) (Proposition 6.1 in [Vo4]).

Let V be a faithful representation. Then we can take $W_n = V^{\oplus(n+1)}$, for example. The classifying space BG of G is defined as $\text{colim}_{n \rightarrow \infty} (U_n/G)$. The motivic cohomology of BG is given by (Corollary 6.2 in [Vo4])

$$H^{*,*'}(BG; \mathbb{Z}/p) \cong \text{lim}_{n \rightarrow \infty} H^{*,*'}(U_n/G; \mathbb{Z}/p).$$

Thus from the above diagram (taking $W = W_{*'}$ and $U_{*'} \subset W^s$), we have the map

$$\begin{aligned} \psi^{*,*'} : H^{*,*'}(BG; \mathbb{Z}/p) &\cong H^{*,*'}(U_{*'} / G; \mathbb{Z}/p) \\ &\rightarrow H^{*,*'}(\text{Spec}(k(U_{*'} / G)); \mathbb{Z}/p) \cong H^{*,*'}(\text{Spec}(k(W_{*'})^G); \mathbb{Z}/p). \end{aligned}$$

In particular, by the $BL(p, *)$ condition, letting $* = *'$ we have the map

$$\psi^*_{et} : H^*_{et}(BG; \mathbb{Z}/p) \rightarrow H^*(k(W_*)^G; \mathbb{Z}/p).$$

When G is a finite group, $H^*_{et}(BG; \mathbb{Z}/p)$ is isomorphic to the group cohomology $H^*(G; \mathbb{Z}/p)$. Let $\Gamma = \text{Gal}(k(\bar{V})/k(V)^G)$ be the absolute Galois group of the field $k(V)^G$. Then the group $G = \text{Gal}(k(V)/k(V)^G)$ is a quotient group of this absolute

Galois group Γ . Then the map ψ_{et}^* induces the map q^* from the quotient $q : \Gamma \rightarrow G$, i.e.,

$$q^* : H^*(G; \mathbb{Z}/p) = H_{et}^*(U_*/G; \mathbb{Z}/p)$$

$$\xrightarrow{\psi_{et}^*} H^*(k(U_*/G); \mathbb{Z}/p) \cong H^*(k(W_*)^G; \mathbb{Z}/p) \rightarrow H^*(k(V)^G; \mathbb{Z}/p) = H^*(\Gamma; \mathbb{Z}/p)$$

for $V \subset W_*$, e.g., $W_* = V^{\oplus(*+1)}$. Of course, we can identify $q^* = \psi_{et}^*$ if we take V by W_* (i.e., there is such an $S \subset V$ with $\text{codim}_V(S) > *$).

Lemma 3.1. *We have the isomorphism*

$$\text{Im}(\psi_{et}^*) \cong H^{*,*}(BG; \mathbb{Z}/p)/(\tau).$$

Proof. The composition

$$H^{*,*}(X; \mathbb{Z}/p) \rightarrow H_{Zar}^0(X; H_{\mathbb{Z}/p}^*) \rightarrow H^*(k(X); \mathbb{Z}/p)$$

is the restriction map for $\text{Spec}(k(X)) \rightarrow X$ from the following commutative diagram:

$$\begin{array}{ccc} H^{*,*}(X; \mathbb{Z}/p) & \longrightarrow & H^{*,*}(\text{Spec}(k(X)); \mathbb{Z}/p) \\ \downarrow & & \downarrow \cong \\ H_{Zar}^0(X; H_{\mathbb{Z}/p}^*) & \longrightarrow & H_{Zar}^0(\text{Spec}(k(X)); H_{\mathbb{Z}/p}^*) \cong H_{et}^*(k(X); \mathbb{Z}/p). \end{array}$$

Hence the above restriction map ψ_{et}^* is written as

$$\begin{aligned} \psi^{*,*} : H^{*,*}(BG; \mathbb{Z}/p) &\cong H^{*,*}(U_*/G; \mathbb{Z}/p) \\ &\rightarrow H_{Zar}^0(U_*/G; H_{\mathbb{Z}/p}^*) \rightarrow H^*(k(W_*)^G; \mathbb{Z}/p). \end{aligned}$$

For each field F , by the Bloch-Kato conjecture $H^*(F; \mathbb{Z}/p)$ is multiplicatively generated by elements in $H_{et}^1(F; \mathbb{Z}/p) \cong H^{1,1}(\text{Spec}(F); \mathbb{Z}/p)$. So

$$\psi^{*,*-1} : H^{*,*-1}(BG; \mathbb{Z}/p) \rightarrow H^{*,*-1}(\text{Spec}(k(W_*)^G; \mathbb{Z}/p) = 0.$$

Hence the map $\psi^{*,*}$ is expressed as a composition

$$H^{*,*}(BG; \mathbb{Z}/p) \rightarrow H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \rightarrow H^*(k(W_*)^G; \mathbb{Z}/p).$$

The first map is of course surjective, and we only need the injectivity of the second map. Indeed, from Corollaries 2.2 and 2.3, we see that

$$\begin{aligned} H^{*,*}(BG; \mathbb{Z}/p)/(\tau) &\cong H^{*,*}(U_*/G; \mathbb{Z}/p)/(\tau) \\ &\subset H_{Zar}^0(U_*/G; H_{\mathbb{Z}/p}^*) \subset H^*(k(W_*)^G; \mathbb{Z}/p). \end{aligned}$$

□

Remark. Let us write $\lim_{n \rightarrow \infty} k(W_n)^G = k(BG)$ and

$$H^{*,*'}(k(BG); \mathbb{Z}/p) = \lim_{n \rightarrow \infty} H^{*,*'}(\text{Spec}(k(W_n)^G); \mathbb{Z}/p).$$

Moreover, let

$$H^*(BG; H_{\mathbb{Z}/p}^*) = \lim_{n \rightarrow \infty} H^*(U_n/G, H_{\mathbb{Z}/p}^*).$$

Then we have the injections (by taking limits) of graded rings

$$\psi_{BG}^{*,*} : H^{*,*}(BG; \mathbb{Z}/p)/(\tau) \subset H_{Zar}^0(BG; H_{\mathbb{Z}/p}^*) \subset H^*(k(BG); \mathbb{Z}/p).$$

Almost all arguments about $H^*(k(W_*)^G; \mathbb{Z}/p)$ also hold if we take $H^*(k(BG); \mathbb{Z}/p)$ instead.

Recall the coniveau filtration given in §2:

$$N^c H_{et}^m(X; \mathbb{Z}/p) = \bigcup_Z \text{Ker}\{H_{et}^m(X; \mathbb{Z}/p) \rightarrow H_{et}^m(X - Z; \mathbb{Z}/p)\},$$

where Z runs in the set of closed subschemes of X of $\text{codim} = c$. From Theorem 2.4, we see

Corollary 3.2. $\text{Im}(\psi_{et}^*) \cong H_{et}^*(BG; \mathbb{Z}/p)/(N^1 H_{et}^*(BG; \mathbb{Z}/p))$.

We say an element $x \in H_{et}^*(BG; \mathbb{Z}/p)$ is (universal) negligible if $\psi_{et}^*(x) = 0$. Let us write the ideal generated by negligible elements by

$$\text{Ker}(\psi^*) = (Ng) = Ng(G).$$

Remark. Originally, Serre defines the negligible class $Ng(G)_S$ as follows. Let K be a field over k and $\Gamma_K = \text{Gal}(\bar{K}/K)$. The class $Ng(G)_S$ is defined as an ideal generated by elements $x \in H^*(BG; \mathbb{Z}/p)$ such that $\psi_K^*(x) = 0$ for all $\psi_K : \Gamma_K \rightarrow G$. In general, $Ng(G) \neq Ng(G)_S$, but we use only $Ng(G)$ as a negligible class in this paper.

From the above lemma, it is immediate that

$$\begin{aligned} Ng(G) &= N^1 H_{et}^*(BG; \mathbb{Z}/p) \\ &\cong \text{Im}(\times \tau | H^{*,* - 1}(BG; \mathbb{Z}/p) \rightarrow H^{*,*}(BG; \mathbb{Z}/p)), \end{aligned}$$

and we have

$$\text{Im}(\psi_{et}^*) \cong H^*(BG; \mathbb{Z}/p)/(Ng) \cong H^*(BG; \mathbb{Z}/p)/(N^1) \cong H^{*,*}(BG; \mathbb{Z}/p)/(\tau).$$

The Chow ring $CH^*(X)/p$ is identified with the motivic cohomology $H^{2*,*}(X; \mathbb{Z}/p)$ [Vo1], [Vo2], [Vo3]. The cycle map is written as

$$cl : CH^*(X)/p = H^{2*,*}(X; \mathbb{Z}/p) \xrightarrow{\times \tau^*} H^{2*,2*}(X; \mathbb{Z}/p) \cong H_{et}^{2*}(X; \mathbb{Z}/p).$$

Hence we see

Lemma 3.3. $\text{Im}(cl(CH^+(BG)/p)) \subset Ng(G)$.

Next consider the relation between cohomology operation and the ideal $Ng(G)$. Let $t_{\mathbb{C}} : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^*(X(\mathbb{C}); \mathbb{Z}/p)$ be the realization map [Vo1] for the inclusion $k \subset \mathbb{C}$. The motivic cohomology has (Bockstein, reduced powered) cohomology operations [Vo2], [Vo4]

$$\begin{aligned} \beta : H^{*,*'}(X; \mathbb{Z}/p) &\rightarrow H^{*+1,*'}(X; \mathbb{Z}/p), \\ P^i : H^{*,*'}(X; \mathbb{Z}/p) &\rightarrow H^{*+2i(p-1),*'+i(p-1)}(X; \mathbb{Z}/p), \end{aligned}$$

which are compatible with the usual (topological) cohomology operations by the realization map $t_{\mathbb{C}}$. Voevodsky defines the Milnor operation Q_i also in the mod p motivic cohomology

$$Q_i : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*+2p^i-1,*'+p^i-1}(X; \mathbb{Z}/p).$$

Here we define the weight degree by

$$w(x) = 2n - m \text{ (resp. } = n' - m')$$

for $0 \neq x \in H^{m,n}(X; \mathbb{Z}/p)$ (resp. $H^{m'}(X; H_{\mathbb{Z}/p}^{n'})$). Note that $w(x) \geq 0$ for any smooth X . (In fact, $H^{*,*'}(X; \mathbb{Z}/p) = 0$ for $* > 2*'$ and smooth X .) Similarly, we

also define the weight degree for cohomology operations, e.g., $w(\tau) = 2$, $w(P^i) = 0$, $w(Q_j) = -1$.

Let $\rho_p = (\xi_p) \in H^{1,1}(Spec(k); \mathbb{Z}/p) = k^*/(k^*)^p$, where ξ_p is the primitive p -th root of unity. The Q_i operation has the same property as the topological case with $mod(\rho_2)$. For example, Q_i is a derivation only with $mod(\rho_2)$.

Let A_p be the $mod\ p$ Steenrod algebra generated by all cohomology operations on $H^{*,*'}(X; \mathbb{Z}/p)$. (Voevodsky proved [Vo6] that A_p is multiplicatively generated by elements in $H^{*,*'}(Spec(k); \mathbb{Z}/p)$, P^j and Q_i .)

Lemma 3.4. *Let $\rho_p = 0 \in K_*^M(k)/p \cong H^*(Spec(k); \mathbb{Z}/p)$. Then the Steenrod algebra A_p acts on the $mod\ p$ etale cohomology $H_{et}^*(X; \mathbb{Z}/p)$ compatible with that on the motivic cohomology.*

Proof. In $H^{*,*'}(Spec(k); \mathbb{Z}/p)$, from Lemma 9.9 in [Vo4], we know

$$P^i(\tau) = 0 \text{ for } i > 0 \quad \text{and} \quad \beta(\tau) = \rho_p = 0.$$

(When $p = 2$, this means $Sq^{2i}(\tau) = 0$, $Sq^1(\tau) = 0$.)

When $p \geq 3$, the Cartan formula holds in the motivic cohomology (Proposition 9.7 in [Vo4]), and we have

$$P^i(\tau x) = \tau P^i(x) \text{ for } i > 0 \quad \text{and} \quad \beta(\tau x) = \tau \beta(x).$$

From the $BL(n, p)$ condition, $H_{et}^*(X; \mathbb{Z}/p) = colim_i \tau^i H^{*,*'}(X; \mathbb{Z}/p)$, which implies the lemma.

For $p = 2$, we also know from Proposition 9.7 in [Vo4] that

$$\begin{aligned} Sq^{2*}(xy) &= \sum_i Sq^{2i}(x)Sq^{2*-2i}(y) + \tau \sum_i Sq^{2i+1}(x)Sq^{2*-2i-1}(y), \\ Sq^{2*+1}(xy) &= \sum_j Sq^j(x)Sq^{2*+1-j}(y) + \rho_2 \sum_i Sq^{2i+1}(x)Sq^{2*-2i-1}(y). \end{aligned}$$

Since $\rho_2 = 0$, we see $Sq^{2i+1}(\tau) = 0$, and so $Sq^*(\tau x) = \tau Sq^*(x)$. This also induces the lemma. □

Lemma 3.5. *Let $\rho_p = 0$. Then for each element x in $H_{et}^*(BG; \mathbb{Z}/p)$, the images of cohomology (Bockstein, reduced) operations $\beta(x)$, $P^i(x)$ for $i > 0$ are in $Ng(G)$ (hence $x^p \in Ng(G)$). The image of the Gysin map $g_*(x)$ are also in $Ng(G)$.*

Proof. Let $x \in H^{*,*'}(BG; \mathbb{Z}/p)$. Then

$$P^i(x) \in H^{*+2(p-1)i, *(p-1)i}(BG; \mathbb{Z}/p).$$

On the other hand, the corresponding etale cohomology operation $P_{et}^i(x) \in H_{et}^*(X; \mathbb{Z}/p)$ is identified as a cohomology operation on $H^{*,*'}(BG; \mathbb{Z}/p)$. So we have

$$P_{et}^i(x) = \tau^{(p-1)i} P^i(x).$$

Hence these elements are in $Im(\tau) = Ng(G)$. Similar facts hold for the Bockstein operation since $\beta_{et}(x) = \tau \beta(x)$.

For a smooth embedding $X \subset Y$ of codimension c , the Gysin map is defined on

$$g_* : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*+2c, *+c}(Y; \mathbb{Z}/p).$$

By the same reason as the cases of cohomology operations, we obtain the lemma. □

Here we give a sufficient condition for $x \notin Ng(G)$. Recall the weight $w(x) = 2s' - s$ for the element (or operation) $x \in H^{s',s}(X; \mathbb{Z}/p)$, e.g., $w(\tau) = 2$, $w(Q_i) = -1$ and $w(P^i) = 0$.

Lemma 3.6. *Let $\rho_p = 0$. For $x \in H_{et}^n(BG; \mathbb{Z}/p)$, if $Q_{n-2} \dots Q_0(x) \neq 0$, then $x \notin Ng(G)$.*

Proof. At first we consider the motivic cohomology. Identify x as the element in $H^{n,n}(BG; \mathbb{Z}/p)$. Suppose that $x = \tau \tilde{x}$. So $w(\tilde{x}) = n - 2$ since $w(\tau) = 2$. The operation Q_i decreases the weight by one. Let $\psi = Q_{n-2} \dots Q_0(\tilde{x})$. Then $w(\psi) = -1$, so $\psi = 0$.

On the other hand, in etale cohomology,

$$cl(\psi) = Q_{n-2} \dots Q_0(x) \neq 0 \in H_{et}^*(BG; \mathbb{Z}/p)$$

for $cl = (\times \tau^{s-s'} : H^{s',s'}(BG; \mathbb{Z}/p) \rightarrow H_{et}^*(BG; \mathbb{Z}/p))$. This is a contradiction. \square

4. COHOMOLOGICAL INVARIANT

Recall that $H^1(k; G)$ is the first non-abelian Galois cohomology set of G , which represents the set of G -torsors over k . It is very important to study $H^1(k; G)$; for example, $H^1(k; O_n)$ is isomorphic to the set of isomorphism classes of non-degenerate quadratic forms over k of rank n . (For details, see the excellent book [Ga-Me-Se].)

The cohomological invariant is defined by

$$Inv^i(G, \mathbb{Z}/p) = Func(H^1(F; G) \rightarrow H^i(F; \mathbb{Z}/p)),$$

where $Func$ means the additive group of the morphisms of functors $H^1(F; G) \rightarrow H^i(F; \mathbb{Z}/p)$ on the category of field extensions F/k . (For the details, see [Ga-Me-Se].) The cohomological invariant is studied by many authors. The cohomological invariants $Inv^*(G; \mathbb{Z}/p)$ are computed (for example in [Ga-Me-Se]) for elementary abelian 2-groups, O_n, SO_n, G_2, \dots . It is also stated in [Ga-Me-Se] that for many G (but not all) $Inv^*(G; \mathbb{Z}/p)$ are detected by $Inv^*(H; \mathbb{Z}/p)$ for elementary abelian p -subgroups H .

Recall $H^*(BG; H_{\mathbb{Z}/p}^{s'}) = \lim_{n \rightarrow \infty} H^*(U_n/G; H_{\mathbb{Z}/p}^{s'})$ in the notation in §3. Let $x \in H^0(BG; H_{\mathbb{Z}/p}^i)$. Given a G -torsor E over F , we can construct $x(E) \in H_{et}^i(F; \mathbb{Z}/p)$ as follows. Roughly speaking, we can identify E as the pullback of the universal G -bundle by some map $f_E : Spec(F) \rightarrow BG$. So we can take $x(E) = f_E^*(x) \in H^0(Spec(F); H_{\mathbb{Z}/p}^i) = H_{et}^i(F; \mathbb{Z}/p)$. Indeed, Totaro proved [Ga-Me-Se] the following theorem in a letter to Serre.

Theorem 4.1 (Totaro). $Inv^*(G; \mathbb{Z}/p) \cong H^0(BG; H_{\mathbb{Z}/p}^*)$.

Therefore we see

Corollary 4.2. $Im(\psi_{et}^*) \cong H^*(BG; \mathbb{Z}/p)/(Ng) \subset Inv^*(G; \mathbb{Z}/p)$.

5. ABELIAN p -GROUPS

In §5 – §8, we will study $H^*(BG; \mathbb{Z}/p)/(Ng)$ for all concrete groups G , of which $Inv^*(G; \mathbb{Z}/p)$ are treated (and computed) in the book [Ga-Me-Se]. In this section we will first consider the case where G is an abelian p -group.

Let us write $H^{s',s'} = H^{s',s'}(Spec(k); \mathbb{Z}/p)$ and $H^* = H^{*,*} = K_*^M(k)/p$ so that $H^{s',s'} \cong H^*[s]$. Moreover, let us simply write $H^*(BG; \mathbb{Z}/p) = H_{et}^*(BG; \mathbb{Z}/p)$.

First consider the case $G = \mathbb{Z}/p^r$. We assume that k has a primitive p^r -th root of unity. The motivic cohomology of $B\mathbb{G}_m = \mathbb{P}^\infty$ is given as

$$H^{*,*'}(\mathbb{P}^\infty; \mathbb{Z}/p) \cong H^{*,*'}[y] \quad \text{with } \text{deg}(y) = (2, 1).$$

In the stable \mathbb{A}^1 -homotopy category, we have the cofibering (see (6.2) in [Vo4] for details)

$$B\mathbb{Z}/p^r \rightarrow E \xrightarrow{q_r} Th(E),$$

where $E = O(-p^r)$ is the $-p^r$ -th twisted line bundle (of the canonical one) of \mathbb{P}^∞ , $Th(E)$ is its Thom space and q_r is the composition map with p^r -times twist on \mathbb{P}^∞ and quotient map. Hence we have the long exact sequence

$$\rightarrow H^{*-2, *'-1}(\mathbb{P}^\infty; \mathbb{Z}/p) \xrightarrow{q_r^*} H^{*,*'}(\mathbb{P}^\infty; \mathbb{Z}/p) \rightarrow H^{*,*'}(B\mathbb{Z}/p^r; \mathbb{Z}/p) \rightarrow .$$

Here we see $q_r^*(x) = p^r y x = 0$ for all $x \in H^{*,*'}(\mathbb{P}^\infty; \mathbb{Z}/p)$. Hence we can compute (as Voevodsky computed in [Vo1], [Vo4] for $B\mathbb{Z}/p$)

$$H^{*,*'}(B\mathbb{Z}/p^r; \mathbb{Z}/p) \cong H^{*,*'}[y(r)] \otimes \{1, x(r)\},$$

where $\text{deg}(y(r)) = (2, 1)$, $\text{deg}(x(r)) = (1, 1)$ for all primes p .

Remark. If we take μ_{p^r} instead of \mathbb{Z}/p^r , then the above fact holds without the existence of the p^r -th root of the unity. However, we only consider \mathbb{Z}/p^r in this paper for the ease of arguments.

When $p = 2$ and $r = 1$, we know by Voevodsky [Vo2], [Vo4] that

$$x(1)^2 = \tau y(1) + \rho_2 x(1) \quad \text{with } \rho_2 = (-1) \in H^1 = k^*/(k^*)^2.$$

For the inclusion $i : \mathbb{Z}/p^s \subset \mathbb{Z}/p^r$ and quotient map $q : \mathbb{Z}/p^r \rightarrow \mathbb{Z}/p^s$, for $r > s$, we have

$$i^*(y(r)) = y(s), \quad i^*(x(r)) = 0, \quad q^*(y(s)) = 0, \quad q^*(x(s)) = x(r).$$

Considering the map q^* for $s = 1$, we see $x(r)^2 = 0$ when $r > 1$ and $p = 2$ (note that $\rho_2 = 0$ by the assumption of the existence of the 2^r -th root of unity). Of course $x(r)^2 = 0$ for $p > 2$ and all $r \geq 1$.

Moreover, we see $\text{Ker}(\tau)|_{H^{*,*'}(B\mathbb{Z}/p^r; \mathbb{Z}/p)} = 0$. Hence

$$\text{Inv}^*(\mathbb{Z}/p^r; \mathbb{Z}/p) \cong H^{*,*'}(B\mathbb{Z}/p^r; \mathbb{Z}/p)/(\tau) = H^*\{1, x(r)\}.$$

For each space X , considering the cofibering

$$B\mathbb{Z}/p^r \times X \rightarrow E \times X \xrightarrow{q_r \times id} Th(E) \times X$$

([Vo4]) and using $(q_r \times id)^* = 0$, we have

$$H^{*,*'}(B\mathbb{Z}/p^r \times X; \mathbb{Z}/p) \cong H^{*,*'}(B\mathbb{Z}/p^r; \mathbb{Z}/p) \otimes_{H^{*,*'}} H^{*,*'}(X; \mathbb{Z}/p);$$

namely, the Kunneth formula holds.

Let $G = \mathbb{Z}/p^{r_1} \oplus \dots \oplus \mathbb{Z}/p^{r_s}$ and assume the existence of the p^{r_i} -th root of unity. The motivic cohomology is given by the Kunneth formula

$$H^{*,*'}(BG; \mathbb{Z}/p) \cong H^{*,*'}[y_1(r_1), \dots, y_s(r_s)] \otimes \Lambda(x_1(r_1), \dots, x_s(r_s))$$

with $\deg(y_i(r_i)) = (2, 1)$ and $\deg(x_i(r_i)) = (1, 1)$ (when $p = 2$ and $r_i = 1, x_i(1)^2 = \tau y_i(1) + \rho x_i(1)$).

Theorem 5.1. *Let G be an abelian p -group, i.e., $G = \bigoplus_i \mathbb{Z}/(p^{r_i})$. Suppose that there are primitive p^{r_i} -th roots of unity. Then*

$$\text{Inv}^*(G; \mathbb{Z}/p) \cong H^*(G; \mathbb{Z}/p)/(Ng) \cong H^* \otimes \Lambda(x_1(r_1), \dots, x_s(r_s)),$$

where if $p = 2$ and $r_i = 1$, then $x_i(1)^2 = \rho x_i(1)$.

The elementary 2-group cases are stated in Theorem 16.4 in [Ga-Me-Se]. For ease of notation, let us write $x_i = x_i(1)$ and $y_i = y_i(1)$. The Q_i -operation acts on $H^{*,*'}(B\mathbb{Z}/p; \mathbb{Z}/p)$ by $Q_i(x) = y^{p^i}$ (while $Q_i(x(j)) = 0$ for all $j > 1$). We consider the Q_i action on $H^{*,*'}(B(\mathbb{Z}/p)^s; \mathbb{Z}/p)$. Each Q_i is a derivation with $\text{mod}(\rho_2)$ (when $p > 2$, Q_i is always a derivation), and hence

$$Q_0 \dots Q_{s-1}(x_1 \dots x_s) = \sum \text{sgn}(j_1, \dots, j_s) y_1^{p^{j_1}} y_2^{p^{j_2}} \dots y_s^{p^{j_s}} \neq 0 \text{ mod}(\rho_2),$$

where (j_1, \dots, j_s) are permutations of $(0, \dots, s - 1)$. Thus we see that this case satisfies the sufficient condition of Lemma 3.6; in fact, $x_1 \dots x_s \notin Ng(G)$.

Let us say that an element $\bar{x} \in H^*(BG; \mathbb{Z}/p)/(Ng)$ is *detected* by an elementary abelian p -subgroup A if $\text{Res}(\bar{x}) \neq 0$ for

$$\text{Res} : H^*(BG; \mathbb{Z}/p)/(Ng) \rightarrow H^*(BA; \mathbb{Z}/p)/(Ng).$$

Lemma 5.2. *Let k be an algebraically closed field and $x \in H^n(BG; \mathbb{Z}/p)$. If the corresponding element $\bar{x} \in H^n(BG; \mathbb{Z}/p)/(Ng)$ is detected by an elementary abelian p -subgroup, then $Q_{n-1} \dots Q_0(x) \neq 0$ in $H^*(BG; \mathbb{Z}/p)$.*

Proof. Let A be an elementary p -subgroup of G such that

$$\text{Res}(\bar{x}) \neq 0 \in H^*(BA; \mathbb{Z}/p)/(Ng) \cong \Lambda(x_1, \dots, x_N).$$

Take a subgroup A' of A of $\text{rank}_p(A') = n$ such that

$$\text{Res}(\bar{x}) = x_1 \dots x_n \in H^*(BA'; \mathbb{Z}/p)/(Ng) \cong \Lambda(x_1, \dots, x_n).$$

Then we can take the restriction for motivic cohomology,

$$\text{Res}_{MO}(x) = x_1 \dots x_n + y \in H^{*,*'}(BA'; \mathbb{Z}/p),$$

where $y = \sum y_I x_{i_1} \dots x_{i_s}$ for $s < n$ and $y_I \in \mathbb{Z}/p[\tau] \otimes \mathbb{Z}/p[y_1, \dots, y_n]$. Here note that $Q_0 \dots Q_{n-1}(y) = 0$, since $Q_0 \dots Q_{n-1}(x_{i_1} \dots x_{i_s}) = 0$. Thus we see in etale cohomology that

$$\text{Res}_{et}(Q_0 \dots Q_{n-1}x) = Q_0 \dots Q_{n-1}(x_1 \dots x_n) \neq 0 \in H^*(BA'; \mathbb{Z}/p).$$

Hence $Q_{n-1} \dots Q_0(x) \neq 0$ in $H^*(BG; \mathbb{Z}/p)$. □

6. CASES $G = O_n$ AND SO_n

Hereafter, in this paper we always assume that k is an *algebraically closed* field in \mathbb{C} unless otherwise stated.

It is well known that

$$\begin{aligned} H^*(BO_n; \mathbb{Z}/2) &\cong H^*((B(\mathbb{Z}/2)^n; \mathbb{Z}/p)^{S_n}) \\ &\cong \mathbb{Z}/2[x_1, \dots, x_n]^{S_n} \cong \mathbb{Z}/2[w_1, \dots, w_n], \end{aligned}$$

where S_n is the n -th symmetric group and w_i is the Stiefel-Whitney class representing the i -th elementary symmetric function of the x_i 's. Let us write this by $y_i = x_i^2$. Then

$$Q_{i-1} \dots Q_0(w_i) = y_1^{p^{i-1}} y_2^{p^{i-2}} \dots y_i + \dots \neq 0 \in H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2),$$

and hence $w_i \notin Ng(G)$.

Recall the Wu formula

$$Sq^i w_k = \sum_{j=0}^i \binom{k-j-1}{i-j} w_{k+i-j} w_j \quad (0 \leq i \leq k).$$

Many cases of products of $w_i w_j$ are in $Ng(G)$, e.g., $w_i^2 \in Ng(G)$. More precisely, the motivic cohomology of BO_n is computed for $k = \mathbb{C}$ (Theorem 8.1 in [Ya2]):

$$H^{*,*'}(BO_n; \mathbb{Z}/2) \cong H^{*,*'}(B(\mathbb{Z}/2)^n; \mathbb{Z}/2)^{S_n}.$$

In particular, we see $Ker(\tau) = 0$.

Given $x \in H^*(BG; \mathbb{Z}/p)$, let us define the weight $w(x)$ as the smallest weight $w(x')$ such that $t_{\mathbb{C}}(x') = x$ with $x' \in H^{*,*'}(BG; \mathbb{Z}/p)$. Note that monomials $\tau^{-j_k} w_{i_1} \dots w_{i_k}$ do not make the homogeneous base of $H^{*,*'}(B(\mathbb{Z}/2)^n; \mathbb{Z}/2)^{S_n}$. For example,

$$w(w_{2i+1} + w_{2i} w_1) = 2i - 1 \quad \text{but} \quad w(w_{2i+1}) = w(w_{2i} w_1) = 2i + 1.$$

This fact follows from $Q_0(w_{2i}) = Sq^1(w_{2i}) = w_{2i+1} + w_{2i} w_1$.

In fact, the weight of the symmetric polynomial

$$t = \sum x_1^{2i_1+1} \dots x_k^{2i_k+1} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q} \quad \text{in } H^*((B\mathbb{Z}/2)^{\times n}; \mathbb{Z}/2)$$

(with $0 \leq i_1 \leq \dots \leq i_k, 0 \leq j_1 \leq \dots \leq j_q$) is given by $w(t) = k$.

Let $t \notin Ng(G)$. Then there is $t' \in H^{*,*'}(BG; \mathbb{Z}/p)$ (but not in $H^{*,*-1}(BG; \mathbb{Z}/p)$) with $t_{\mathbb{C}}(t') = t$. So we have

$$w(t) = w(t') = 2 * - * = deg(t).$$

This implies that $i_1 = \dots = j_q = 0$, and hence t is the symmetrization of $x_1 \dots x_i$, which is w_i .

Since $Ker(\tau) = 0$, we have the following theorem from Corollary 2.2.

Theorem 6.1.

$$Inv^*(O_n; \mathbb{Z}/2) \cong H^*(BO_n; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_1, \dots, w_n\}.$$

Indeed, $Inv(O_n; \mathbb{Z}/2)$ is well known (Theorem 17.3 in [Ga-Me-Se]) for general k :

$$Inv^*(O_n; \mathbb{Z}/2) \cong H^*\{1, w_1, \dots, w_n\}.$$

We consider the multiplicative structure of $Inv^*(O_n; \mathbb{Z}/2)$. We still know that $Sq^1(w_{2i}) = w_{2i+1} + w_{2i} w_1 \in Ng(O_n)$, and hence we have $w_{2i+1} = w_{2i} w_1$ in $Inv^*(O_n; \mathbb{Z}/2)$.

By Rost and Kahn [Ka], the divided power operation can be defined in $K_*^M(F)/p$ compatible with fields F over k (and hence $Inv^*(G; \mathbb{Z}/p)$) if $\sqrt{-1} \in k$ (i. e. , $\rho = 0$ in $K_1^M(k)/2$). Vial [Via] showed that the divided power operations are only compatible maps (natural maps) with field extensions over k . Moreover, Becher [Be] showed that $\gamma_n(w_2) = w_{2n}$. (See also Milnor, p. 133 in [Mi].)

Theorem 6.2 (Becher [Be], [Via]). *Let $\rho = 0 \in K_1^M(k)/2$. Then $Inv^*(O_n; \mathbb{Z}/2)$ is generated by w_1 and w_2 as an H^* -ring with divided powers by*

$$\gamma_i(w_2) = w_{2i}, \quad w_{2i+1} = w_{2i}w_1.$$

Next consider the case $G = SO_n$. It is well known that

$$H^*(BSO_n; \mathbb{Z}/2) \cong H^*(BO_n; \mathbb{Z}/2)/(w_1) \cong \mathbb{Z}/2[w_2, \dots, w_n].$$

Let $n = 2m + 1 = \text{odd}$. For this case, there is the isomorphism

$$O_{2m+1} \cong SO_{2m+1} \times \mathbb{Z}/2.$$

Let $p : O_n \rightarrow SO_n$ be the projection and $i : SO_n \rightarrow O_n$ the inclusion. We consider the induced map p^* and i^* on the mod 2 motivic cohomology of their classifying spaces. Since $i^*(w_1) = 0 \in H^{*,*'}(BSO_n; \mathbb{Z}/2)$, we see that $w_{2i+1} \in Ng(G)$ from the above theorem (in fact, $Sq^1(w_{2i}) = w_{2i+1}$ in $H^*(BSO_n; \mathbb{Z}/2)$). Moreover, $p^*w_{2i} = w_{2i} \neq 0 \text{ mod}(Ng(G))$ in $H^*(BO_n; \mathbb{Z}/2)$. We also note that $Ker(\tau) = 0$ for this case. Thus we have

Theorem 6.3. *For $G = SO_{2m+1}$, we have*

$$Inv^*(G; \mathbb{Z}/2) \cong H^*(BG; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_2, \dots, w_{2m}\}.$$

Moreover, $Inv^*(SO_{2m+1}; \mathbb{Z}/2)$ is still computed (Theorem 19.1 in [Ga-Me-Se]) for general k ; $Inv^*(SO_{2m+1}; \mathbb{Z}/2) \cong H^*\{1, w_2, \dots, w_{2m}\}$.

Now consider the case $n = 2m = \text{even}$. In this case the mod 2 motivic cohomology is not yet computed even $k = \mathbb{C}$ for $n > 4$. However, we compute $H^{*,*}(BSO_n; \mathbb{Z}/2)/(\tau)$ easily. Consider the inclusion

$$SO_{2m-1} \xrightarrow{i_1} SO_{2m} \xrightarrow{i_2} SO_{2m+1}.$$

Since the restriction map for $i < m$,

$$i_1^*(w_{2i}) \notin Ng(SO_{2m-1}) \quad \text{and} \quad i_2^*(w_{2i+1}) \in Ng(SO_{2m}),$$

we see that $w_{2i} \notin Ng(SO_{2m})$ but $w_{2i+1} \in Ng(SO_{2m})$. Moreover, we know that $w(w_{2m}) = 2m - 2$ from Lemma 9.2 in [Ya2]. In fact,

$$w(w_{2i}) = 2i \text{ for } i < m \quad \text{but} \quad w(w_{2m}) = 2m - 2.$$

On the other hand, the monomial w_I not of the form w_{2i} are all in $Ng(SO_{2m+1})$, and so $i_2^*(w_I) \in Ng(SO_{2m})$.

Thus we have

Theorem 6.4. $H^*(BSO_{2m}; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2\{1, w_2, \dots, w_{2m-2}\}$.

Next we study $Inv^*(SO_{2m}; \mathbb{Z}/2)$. (Let $n = 2m$ as well.) There is an element (Lemma 9.3 in [Ya2]) in the motivic cohomology

$$u_{m-1} \in H^{n, n-2}(BSO_n; \mathbb{Z}/2) \quad \text{with} \quad \tau u_{m-1} = 0.$$

So from Corollary 2.2, there is a non-zero element

$$u \in H^0(BSO_n; H_{\mathbb{Z}/2}^{n-1})/(H^{n-1, n-1}(BSO_n; \mathbb{Z}/2)/(\tau)).$$

Hence we can take $0 \neq u \in Inv^{n-1}(SO_n; \mathbb{Z}/2)$, but it is only determined with $\text{mod}(H^*(BSO_n; \mathbb{Z}/2)/(Ng))$.

On the other hand, in [Ga-Me-Se] it is proved (Theorem 20.6) for general k :

$$Inv^*(SO_{2m}; \mathbb{Z}/2) \cong H^*\{1, w_2, \dots, w_{n-2}\} \oplus (Im(I_\delta)).$$

Here, when $k = \mathbb{C}$, $Im(I_\delta) \cong \mathbb{Z}/2\{u\}$ with $deg(u) = n - 1$ from Proposition 20.1 in [Ga-Me-Se]. (Here I_δ can be taken canonically, but u cannot.) Thus we see

Theorem 6.5. *Let $G = SO_{2m}$ and $m \geq 2$. Then for $deg(u) = 2m - 1$,*

$$Inv^*(G; \mathbb{Z}/2) \cong H^*(BG; \mathbb{Z}/2)/(Ng) \oplus \mathbb{Z}/2\{u\}.$$

From 22.10 in [Ga-Me-Se], it is known that

$$Res : Inv^*(G; \mathbb{Z}/p) \rightarrow Inv^*(H; \mathbb{Z}/p)$$

is injective for $p = 2$, $G = SO_n$ and $H = (\mathbb{Z}/2)^{n-1}$. Hence for

$$Res_H : H^0(BSO_n; H_{\mathbb{Z}/2}^*) \rightarrow H^0(B(\mathbb{Z}/2)^{n-1}; H_{\mathbb{Z}/2}^*)$$

we have $Res_H(u) = x_1 \dots x_{n-1}$. Of course $u \notin H^{*,*}(BSO_n; \mathbb{Z}/2)/(\tau)$, but $x_1 \dots x_{n-1} \in H^{*,*}(B(\mathbb{Z}/2)^{n-1}; \mathbb{Z}/2)/(\tau)$. So note that for a map $X \rightarrow Y$, we see that

$$f_K : Ker(\tau)|H^{*+1,*'-1}(Y; \mathbb{Z}/p) \rightarrow Ker(\tau)|H^{*+1,*'-1}(X; \mathbb{Z}/p)$$

is (in general) not injective, even if $f_H : H^{*-}*'(Y; H_{\mathbb{Z}/p}^*) \rightarrow H^{*-}*'(X; H_{\mathbb{Z}/p}^*)$ is injective.

7. Spin_n AND EXCEPTIONAL GROUPS

By the definition, we have the short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow Spin_n \xrightarrow{j} SO_n \rightarrow 1.$$

Recall that w_i is the Stiefel-Whitney class in $H^*(BSO_n; \mathbb{Z}/2)$. Let Δ be a spin representation (which is any (orthogonal) representation which restricts to non-zero on $\mathbb{Z}/2$). Let $w_i(\Delta) = \Delta^*(w_i)$ be the induced Stiefel-Whitney class. We also write by the same letter w_i the Stiefel-Whitney class $j^*(w_i)$ of the usual representation $j : Spin_n \rightarrow SO_n$.

Let 2^h be the the Radon-Hurwitz number (for details see [Qu], p. 210). Then the $mod(2)$ cohomology of $BSpin_n$ was computed by Quillen:

$$H^*(BSpin_n; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_{2^h}(\Delta)] \otimes \mathbb{Z}/2[w_2, \dots, w_n]/(w_2, Q_i w_2 | 0 \leq i \leq h).$$

By the result of Becher (Theorem 6.2), we have

Theorem 7.1. $H^*(BSpin_n; \mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$ for $n > 2$.

Proof. Recall representations $j : Spin_n \rightarrow SO_n$ and $\Delta : Spin_n \rightarrow SO_N$. We consider the induced map in Galois cohomology,

$$j^* : H_{et}^*(k(W_*)^{SO_n}; \mathbb{Z}/2) \rightarrow H_{et}^*(k(W_*)^{Spin_n}; \mathbb{Z}/2).$$

Moreover, by Rost and Kahn [Ka], the divided powers naturally act on $K_*^M(F)/p$ for a field F over k .

Our case is $k = \mathbb{C}$. By Quillen's result, we see $j^*(w_2) = 0$. From Becher's theorem (Theorem 6.2), we get

$$j^*(w_{2i}) = j^*(\gamma_i(w_2)) = \gamma_i(j^*(w_2)) = 0.$$

It is known (Table (6.2) in [Qu]) that $h > 1$ for $n > 2$ and hence $H^2(BSpin_n; \mathbb{Z}/2) = 0$. The fact that $w_2(\Delta) = 0$ implies $w_{2^h}(\Delta) = 0$ if $n > 2$. □

Corollary 7.2. *For $n > 2$ and $* > 0$, there is the isomorphism*

$$Inv^*(Spin_n; \mathbb{Z}/2) \cong Ker(\tau)|H^{*+1,*-1}(BSpin_n; \mathbb{Z}/2).$$

The $mod(2)$ motivic cohomology of $BSpin_7$ is computed in Theorem 9.6 in [Ya2], while it is determined only with the assumption of the existence of the Gottlieb transfer of motivic theories. We can easily see the above theorem also from concrete computation. Moreover, there are τ -torsion elements

$$y_2 \in H^{4,2}(Spin_7; \mathbb{Z}/2), \quad y'_2 \in H^{5,3}(BSpin_7; \mathbb{Z}/2).$$

Therefore we can take $u \in H^0(BSpin_7, H^3_{\mathbb{Z}/2}), v \in H^0(BSpin_7, H^4_{\mathbb{Z}/2})$.

Theorem 7.3 ([Ga-Me-Se]).

$$Inv^*(BSpin_7; \mathbb{Z}/2) \cong \mathbb{Z}/2\{1, u, v\}, \quad |u| = 3, |v| = 4.$$

We consider exceptional Lie group types G_2, F_4 and split E_6 . In [Ga-Me-Se], it is proved (Theorem 18.1, Theorem 22.5) that

$$Inv^*(G_2; \mathbb{Z}/2) \cong Inv^*(E_6; \mathbb{Z}/2) \cong H^*\{1, u\}, \\ Inv^*(F_4; \mathbb{Z}/2) \cong H^*\{1, u, f_5\}.$$

(Unfortunately, we cannot re-explain f_5 by using $H^{*,*'}(BF_4; \mathbb{Z}/2)$, since it has not yet been computed.)

Moreover, the restriction map of cohomological invariant for an elementary abelian 2-subgroup of rank 3 (resp. rank 5) is injective for G_2, E_6 (resp. for F_4); see 22.10 in [Ga-Me-Se].

Theorem 7.4. *Let $G = G_2, F_4, E_6$. Then $H^*(BG; \mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$.*

Proof. It is known by Borel ([Bor], (6.8) in [Ko-Ya]), that the inclusion $i : G_2 \rightarrow Spin_7$ induces the epimorphism $i^* : H^*(BSpin_7; \mathbb{Z}/2) \rightarrow H^*(BG_2; \mathbb{Z}/2)$. Hence the result for $G = G_2$ follows from $H^*(BSpin_7; \mathbb{Z}/2)/(Ng) = \mathbb{Z}/2$.

It is also known by Borel ([Bor], page 786 in [Ko-Ya]) that the inclusion $i' : Spin_9 \rightarrow F_4$ induces the injection $i'^* : H^*(BF_4; \mathbb{Z}/2) \rightarrow H^*(BSpin_9; \mathbb{Z}/2)$. The groups $Spin_9$ and F_4 have the same maximal abelian 2-group H (of rank 5). We consider the restriction to this H . The fact that $Res(H^*(BSpin_9; \mathbb{Z}/2)/(Ng)) = \mathbb{Z}/2$ implies the results for F_4 , because the restriction is injective from [Ga-Me-Se].

It is known that the inclusion $G_2 \subset E_6$ induces the isomorphism

$$Inv^*(E_6; \mathbb{Z}/2) \cong Inv^*(G_2; \mathbb{Z}/2)$$

(page 52 in [Ga-Me-Se]). Hence the result for E_6 follows from that for G_2 . □

Here we consider an example for the above theorem. The $mod(2)$ cohomology of BG_2 is well known:

$$H^*(BG_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7].$$

Let us write by c_4, c_6, c_7 the Chern classes with $c_i = w_i^2$.

Proposition 7.5 ([Ya2]). *The cohomology $H^*(BG_2; \mathbb{Z}/2)$ is multiplicatively generated by c_4, c_6, c_7 , and $Image(Q_i)$ for $0 \leq i \leq 2$, except for elements w_4, w_6, w_4w_6 and $w_4w_6w_7$.*

The motivic cohomology G_2 is studied in [Ya2]. There we consider a virtual element a with $deg(a) = (3, 3)$ so that $\tau^7c_7a = w_6w_7w_4, \tau Q_0a = w_4, \tau^2Q_1a = w_6$ and $\tau^4Q_2a = w_6w_4$. This virtual element a exists in

$$H^0(BG_2; H^3_{\mathbb{Z}/2}) \cong Inv^3(G_2; \mathbb{Z}/2).$$

Indeed, this element restricts $x_1x_2x_3$ in $H^0(B(\mathbb{Z}/2)^3; H_{\mathbb{Z}/2}^3)$ (as in the arguments before Theorem 7.4), and it represents the Rost cohomological invariant. So we see that $H^*(BG_2; \mathbb{Z}/2)/(Ng) \cong \mathbb{Z}/2$ as well. Moreover, we see there is

$$y_2 \in \text{Ker}(\tau)|H^{4,2}(BG_2; \mathbb{Z}/2)$$

as the case SO_4 .

Next we consider the odd prime cases where $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3)$ or $(E_8, 5)$. In these cases, each exceptional Lie group has two conjugacy classes of maximal elementary abelian p -groups. One is the subgroup of a maximal torus and the other is a non-toral A . Let us write $i_A : A \rightarrow G$ and $i_T : T \rightarrow G$ as the inclusions. Tezuka and Kameko proved ([Ka], [Ka-Ya]) that the following map is injective:

$$i_A^* \times i_T^* : H^*(BG; \mathbb{Z}/p) \rightarrow H^*(BA; \mathbb{Z}/p) \times H^*(BT; \mathbb{Z}/p);$$

namely, $H^*(BG; \mathbb{Z}/p)$ is detected by A and T . Note that $H^*(BT; \mathbb{Z}/p)$ is a polynomial algebra generated by 2-dimensional elements of the weight degree zero, and so $H^*(BT; \mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p$.

On the other hand, in 22.10 in [Ga-Me-Se], it is stated that the restriction map of $\text{Inv}^*(G; \mathbb{Z}/p)$ to some (maximal) elementary abelian p -group H is an injection for these groups. Hence this is $H = A$ and

$$H^*(BG; \mathbb{Z}/p)/(Ng) \subset H^*(BA; \mathbb{Z}/p)/(Ng) \cong \Lambda(x_1, \dots, x_s).$$

Theorem 7.6. *Let $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3)$ or $(E_8, 5)$. Then*

$$H^*(BG; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p.$$

Proof. The restriction image of $H^*(BG; \mathbb{Z}/p)$ to A is studied in [Ka-Ya]. Images are generated as a ring by Chern classes and

$$Q_I(x_1 \dots x_s) \in H^*(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_s] \otimes \Lambda(x_1, \dots, x_s),$$

where $Q_I = Q_{i_1} \dots Q_{i_r}$ for some $I \neq \emptyset$. (Note that $x_1 \dots x_s \notin \text{Im}(i_A^*)$.) Hence $i_A^*(H^+(BG; \mathbb{Z}/p)) \in Ng(A)$; that is, $\text{Res}(H^+(BG; \mathbb{Z}/p)/(Ng)) = 0$.

By the above injectivity results from [Ga-Me-Se], we see the desired result $H^+(BG; \mathbb{Z}/p)/(Ng) = 0$. □

Here we give an example of the above theorem. The *mod* 3 cohomology of BF_4 is completely determined by Toda.

Theorem 7.7 ([Tod]). *The cohomology $H^*(BF_4; \mathbb{Z}/3)$ is given by*

$$\mathbb{Z}/3[z_{36}, z_{48}] \otimes (\mathbb{Z}/3[z_4, z_8] \otimes \{1, z_{20}, z_{20}^2\} + \mathbb{Z}/3[z_{26}] \otimes \Lambda(z_9) \otimes \{1, z_{20}, z_{21}, z_{25}\})$$

(here the above two terms have the intersection $\{1, z_{20}\}$), where subindices mean their degree, i. e. , $|z_i| = i$.

Indeed, we can compute ([Tod], [Ka-Ya]) $z_{26}|A = Q_0Q_1Q_2(u_3)$, $z_{36}|A = c_{3,1}$, $z_{48}|A = c_{3,2}$, $z_4|A = Q_0(u_3)$, $z_8|A = Q_1(u_3)$, $z_{20}|A = Q_2(u_3)$, $z_9|A = Q_0Q_1(u_3)$, $z_{21}|A = Q_0Q_2(u_3)$, $z_{25}|A = Q_1Q_2(u_3)$. Here $u_3 = x_1x_2x_3$ and z_{36} and z_{48} are represented by Chern classes of a minimal faithful complex representation. This shows that $H^*(BF_4; \mathbb{Z}/3)/(Ng) = \mathbb{Z}/3$ as well.

We have the Kunneth formula for the etale cohomology with coefficients \mathbb{Z}/p (since k is algebraically closed). Since $Ng(G)$ is an ideal, we have the surjection

$$H^*(BG_1; \mathbb{Z}/p)/(Ng) \otimes H^*(BG_2; \mathbb{Z}/p)/(Ng) \rightarrow H^*(B(G_1 \times G_2); \mathbb{Z}/p)/(Ng).$$

The above map is not an isomorphism, in general, because there is the possibility that $x_1 \otimes x_2 \in Ng(G_1 \times G_2)$ but $x_1 \notin Ng(G_1)$, $x_2 \notin Ng(G_2)$.

However, when $H^*(BG_i; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$, the above map is of course an isomorphism. If we can prove $H^*(BG; \mathbb{Z}/p)/(Ng) = \mathbb{Z}/p$ for $(E_7, p = 2)$ and $(E_8, p = 2, 3)$, then the same fact holds for all 2-connected linear algebraic groups G .

8. PGL_p

Let p be an odd prime and denote by PGL_p the projective group which is the quotient of the general linear group GL_p by the center \mathbb{G}_m . Its ordinary $mod(p)$ cohomology and the Chow ring are known by Vistoli [Vi] and Kameko-Yagita [Ka-Ya].

To state the cohomology $H^*(BPGL_p; \mathbb{Z}/p)$, we recall the Dickson algebra. Let $A \cong (\mathbb{Z}/p)^n$ be an elementary abelian p -group of rank n , and $H^*(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n)$. The Dickson algebra is

$$D_n = \mathbb{Z}/p[y_1, \dots, y_n]^{GL_n(\mathbb{F}_p)} \cong \mathbb{Z}/p[c_{n,0}, \dots, c_{n,n-1}]$$

with $|c_{n,i}| = 2(p^n - p^i)$. The invariant ring under $SL_n(\mathbb{F}_p)$ is also given by

$$SD_n = \mathbb{Z}/p[y_1, \dots, y_n]^{SL_n(\mathbb{F}_p)} \cong D'_n \otimes \mathbb{Z}/p[e_n],$$

where $D_n = \mathbb{Z}/p[c_{n,0}] \otimes D'_n$, namely $D'_n = \mathbb{Z}/p[c_{n,1}, \dots, c_{n,n-1}]$. Moreover, $e_n^{p-1} = c_{n,0}$. We also recall Mui's result by using Q_i according to Kameko and Mimura [Ka-Mi]:

$$H^*(BA; \mathbb{Z}/p)^{SL_n(\mathbb{F}_p)} \cong D'_n \oplus SD_n \otimes \Lambda(Q_0, \dots, Q_{n-1})\{u_n\},$$

where $u_n = x_1 \dots x_n$ and $e_n = Q_0 \dots Q_{n-1} u_n$.

Theorem 8.1 ([Vi], [Ka-Ya]). *There is an additive isomorphism*

$$H^*(BPGL_p; \mathbb{Z}/p) \cong M \oplus N,$$

where $M \cong \mathbb{Z}/p[z_4, z_6, \dots, z_{2p}]$ with $deg(z_i) = i$ as modules (but not rings) and

$$N \cong SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \cong \mathbb{Z}/p[e_2, c_{2,1}] \otimes \Lambda(Q_0, Q_1)\{u_2\}$$

with $|e_2| = 2(p+1)$ and $|c_{2,1}| = 2(p^2 - p)$.

Here note that $e_2 = Q_0 Q_1(u_2)$, and $c_{2,1}$ is represented by some Chern class and $c_{2,1} \in M$ in the above theorem (but $e_2, c_{2,1}e_2 \in N$).

Theorem 8.2 ([Vi], [Ka-Ya]). *The cycle map*

$$cl : CH^*(BPGL_p)/p \rightarrow H^{2*}(BPGL_n; \mathbb{Z}/p)$$

is injective and

$$CH^*(BPGL_p)/p \cong Im(cl) = M \oplus SD_2\{Q_0 Q_1 u_2\}.$$

Hence we have from Lemmas 3.3 and 3.5

Theorem 8.3. $H^*(BPGL_p; \mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p\{1, u_2\}$ with $|u_2| = 2$.

More strongly, it is shown in [Ga-Me-Se] that

$$Inv^*(PGL_p; \mathbb{Z}/p) \cong H^*(BPGL_p; \mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p\{1, u_2\}.$$

Throughout §5–§8, we have studied all groups G which are studied in [Ga-Me-Se] except for the symmetric group S_n . The permutations induce the natural representation $S_n \rightarrow O_n$. Let A be the subgroup of S_n generated by transpositions $(2i - 1, 2i)$ for $1 \leq i \leq [n/2]$ so that $A \cong (\mathbb{Z}/2)^{[n/2]}$. Thus we have the map

$$\text{Inv}^*(O_n; \mathbb{Z}/2) \rightarrow \text{Inv}^*(S_n; \mathbb{Z}/2) \rightarrow \text{Inv}^*(A; \mathbb{Z}/2)^{S_{[n/2]}}.$$

It is shown in [Ga-Me-Se] that the last map is isomorphic.

Theorem 8.4 ([Ga-Me-Se]). *The cohomology $H^*(BS_n; \mathbb{Z}/2)/(Ng)$ is isomorphic to*

$$\text{Inv}^*(S_n; \mathbb{Z}/2) \cong \text{Inv}^*(O_{[n/2]}; \mathbb{Z}/2) \cong \mathbb{Z}/2\{w_1, \dots, w_{[n/2]}\}.$$

Moreover, it is shown in [Ga-Me-Se] that $\text{Inv}^*(S_n; \mathbb{Z}/p) = \mathbb{Z}/p$ for $p \geq 3$. Unfortunately, we cannot find a good way to re-explain these facts from the cohomology $H^*(BS_n; \mathbb{Z}/p)$, although they are quite well studied, e.g., [Ad-Mi], [Ma-Mi].

9. EXTRASPECIAL p -GROUPS

In §11-§12, we give examples of groups for undetected problems. The groups (special p -groups) are deeply related with extraspecial p groups p_+^{1+2n} . In this section we recall and explain the cohomology of extraspecial p -groups. Similar computations are used in §11-§12. In addition, we give some partial results for the cohomological invariant of p_+^{1+2n} .

At first, we assume that p is an odd prime. The extraspecial p -group $E_n = p_+^{1+2n}$ (see §4 in Chapter 4 of [Suz]) is the group such that its exponent is p , its center $Z(G)$ is $C \cong \mathbb{Z}/p$ and there is an extension

$$0 \longrightarrow C \xrightarrow{i} E_n \xrightarrow{\pi} V \longrightarrow 0$$

with $V = \bigoplus^{2n} \mathbb{Z}/p$.

Remark. The definition of 2_+^{1+2n} is quite different, e.g., *exponent* $\neq 2$; see the last parts of this section below. One of their reasons is the following. For each finite group $a, b, c \in G$, if $[a, b] = c \in Z(G)$, then we see (e.g., (3.7) in Chapter 4 of [Suz])

$$a^n b^n = (ab)^n c^{n(n-1)/2},$$

e.g., if a, b and c are order p , then so is ab for each odd p , but not for $p = 2$.

We can take generators $a_1, \dots, a_{2n}, c \in E_n$ such that $\pi(a_1), \dots, \pi(a_{2n})$ (resp. c) make a base of V (resp. C) such that

$$[a_{2i-1}, a_{2i}] = c \quad \text{for } 1 \leq i \leq n$$

and other pairs commute. We note that E_n is also the central product of the n -copies of E_1 (Theorem 4.18 (i) in Chapter 4 of [Suz]):

$$E_n \cong E_1 \cdots E_1 = E_1 \times_{\langle c \rangle} E_1 \cdots \times_{\langle c \rangle} E_1.$$

Let us write the cohomologies (for details, see [Te-Ya])

$$H^*(BC; \mathbb{Z}/p) \cong \mathbb{Z}/p[u] \otimes \Lambda(z), \quad \beta z = u,$$

$$H^*(BV; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_{2n}] \otimes \Lambda(x_1, \dots, x_{2n}), \quad \beta x_i = y_i,$$

identifying the dual of a_i (resp. c) with x_i (resp. z). These elements $x_i = \pi^*(x_i)$ also represent elements in $H^*(BE_n; \mathbb{Z}/p)$ so that

$$x_i : E_n \xrightarrow{pr} \langle a_i \rangle \cong \mathbb{Z}/p \quad \text{in } \text{Hom}(E_n; \mathbb{Z}/p) \cong H^1(BE_n; \mathbb{Z}/p).$$

The central extension is expressed by

$$f = \sum_{i=1}^n x_{2i-1}x_{2i} \in H^2(BV; \mathbb{Z}/p).$$

Hence $\pi^*f = 0$ in $H^2(BE_n; \mathbb{Z}/p)$. We consider the Hochschild-Serre spectral sequence

$$E_2^{*,*'} \cong H^*(BV; \mathbb{Z}/p) \otimes H^{*'}(BC; \mathbb{Z}/p) \implies H^*(BE_n; \mathbb{Z}/p).$$

Since $\pi^*(f) = 0$, we see that $f = 0$ in $E_\infty^{2,0}$. So there is an element $v \in E_r^{*,+}$ such that $d_r(v) = f$. By dimensional reason, it is the first non-zero differential $d_2(z) = f$.

The Cartan-Serre transgression theorem says (e.g., Theorem 4.8.1 in [Ben]) that if $x \in E_n^{0,n-1}$ with $d_n(x) = y$, and ϕ is a Steenrod (reduced and Bockstein) operation of degree r , then $\phi(x)$ survives to $E_{n+r}^{0,n+r-1}$ and $d_{n+r}(\phi(x)) = \phi(y)$.

Hence $u = Q_0(z)$ survives to $E_3^{0,2}$, which means that $d_2(u) = 0$, and

$$\begin{aligned} d_3(u) &= d_3(Q_0(z)) = Q_0(d_2(z)) = Q_0(f) = Q_0\left(\sum x_{2i-1}x_{2i}\right) \\ &= \sum Q_0(x_{2i-1})x_{2i} - x_{2i-1}Q_0(x_{2i}) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1}. \end{aligned}$$

Clearly, the above element is not in the ideal (f) . In particular,

$$E_4^{*,0} \cong \mathbb{Z}/p[y_1, \dots, y_{2n}] \otimes \Lambda(x_1, \dots, x_{2n}) / (f, Q_0(f)) \supset \Lambda(x_1, \dots, x_n) / (f).$$

Lemma 9.1. *The map $E_3^{*,0} \rightarrow E_\infty^{*,0} \subset H^*(BE_n; \mathbb{Z}/p)$ induces the injection*

$$\Lambda(x_1, \dots, x_{2n}) / (f) \subset H^*(BE_n; \mathbb{Z}/p).$$

Proof. We consider a similar group E'_n such that its center is $C \cong \mathbb{Z}/p$, and there is an extension

$$0 \longrightarrow C \xrightarrow{i} E'_n \xrightarrow{\pi} V' \longrightarrow 0,$$

but $V' = \bigoplus^{2n} \mathbb{Z}_p$ such that there is the quotient map $q : E'_n \rightarrow E_n$. We also consider the spectral sequence

$$\bar{E}_2^{*,*'} \cong H^*(BV'; \mathbb{Z}/p) \otimes H^{*'}(BC; \mathbb{Z}/p) \implies H^*(BE'_n; \mathbb{Z}/p).$$

Here $H^*(BV'; \mathbb{Z}/p) \cong \Lambda(x_1, \dots, x_{2n})$ since $y_i = 0$ in $H^*(BV'; \mathbb{Z}/p)$. The first non-zero differential is $d_2(z) = f$. (Hence $d_2(u) = d_2(Q_0(z)) = 0$ from the transgression theorem.) So we have

$$\bar{E}_3^{*,*'} \cong \mathbb{Z}/p[u] \otimes (\Lambda(x_1, \dots, x_n) / (f) \oplus K),$$

where $K = Ker(d_2)|_{\bar{E}_2^{*,*'}} \subset \bar{E}_2^{*,*'}.$

Here note that $\bar{E}_3^{*,*'}$ is multiplicatively generated by elements u and $\bar{E}_3^{*,*'}$ with $*' \leq 1$. However, we see that

$$d_3(u) = \sum y_{2i-1}x_{2i} - y_{2i}x_{2i-1} = 0.$$

Therefore all $d_r = 0$ for all $r \geq 3$ by dimensional reason; that is, $\bar{E}_3^{*,*'} \cong \bar{E}_\infty^{*,*'}.$ In particular, we have the injection

$$\Lambda(x_1, \dots, x_n) / (f) \subset \bar{E}_3^{*,0} \cong \bar{E}_\infty^{*,0} \subset H^*(BE'_n; \mathbb{Z}/p).$$

Hence from the map

$$E_3^{*,0} \rightarrow H^*(BE_n; \mathbb{Z}/p) \rightarrow H^*(BE'_n; \mathbb{Z}/p),$$

we obtain the result. □

Unfortunately, we cannot yet prove

$$\Lambda(x_1, \dots, x_{2n})/(f) \subset H^*(BE_n; \mathbb{Z}/p)/(Ng).$$

However, when $n = 1$, from Theorem 3.3 in [Ya2] we see

Proposition 9.2. *For an odd prime p , we have the isomorphism*

$$H^*(Bp_+^{1+2}; \mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p\{1, x_1, x_2, a'_1, a'_2\} \quad \text{deg}(a'_i) = 2.$$

When $p = 2$, the situation becomes a bit better. The extraspecial 2-group $D(n) = 2_+^{1+2n}$ is defined (Theorem 4.18 in Chapter 4 of [Suz]) as the n -th central extension of the dihedral group D_8 of order 8. (Hence the exponent is 4.) It has the central extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow D(n) \rightarrow V \rightarrow 0$$

with $V = \bigoplus^{2n} \mathbb{Z}/2$. Hence $H^*(BV; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_{2n}]$.

Then using the Hochschild-Serre spectral sequence, Quillen proved [Qu]

$$H^*(BD(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_{2n}]/(f, Q_0(f), \dots, Q_{n-2}(f)) \otimes \mathbb{Z}[w_{2^n}(\Delta)].$$

(In fact, when n is the real case (i.e., $n = -1, 0, 1 \pmod{8}$), the cohomology $H^*(BSpin_n; \mathbb{Z}/2)$ injects into $H^*(BD(n); \mathbb{Z}/2)$.) Here $w_{2^n}(\Delta)$ is the Stiefel-Whitney class of any 2^n -dimensional representation Δ which restricts non-zero on the center.

Moreover, Quillen proves the following two theorems (Theorem 5.10-11 in [Qu]).

Theorem 9.3 ([Qu]). *$H^*(BD(n); \mathbb{Z}/2)$ is detected by the product of cohomology of maximal elementary abelian subgroups.*

Theorem 9.4 ([Qu]). *The non-zero Stiefel-Whitney $w_i(\Delta)$ are those of degrees 2^n and $2^n - 2^i$ for $0 \leq i < n$.*

In fact, $w_i(\Delta)$ generates the Dickson algebra in the cohomology of a maximal elementary abelian 2-group modulo the center of $D(n)$.

Proposition 9.5. *When $n > 2$, we have the surjection*

$$\Lambda(x_1, \dots, x_{2n})/(f) \rightarrow H^*(BD(n); \mathbb{Z}/2)/(Ng).$$

Proof. The fact that $w_2(\Delta) = 0$ follows from Quillen's second theorem above, since $n > 2$. Hence we have $w_{2^n}(\Delta) \in Ng(D(n))$ from Becher's theorem (Theorem 6.2). Of course $x_i^2 = Q_0(x_i) \in Ng(D(n))$. Thus we prove the theorem. □

Considering the restriction to the subgroup $A = \langle x_1, x_3, \dots, x_{2n-1} \rangle$, we see that

$$\Lambda(x_1, x_3, \dots, x_{2n-1}) \subset H^*(BD(n); \mathbb{Z}/2)/(Ng).$$

From Theorem 9.3, $H^*(BD(n); \mathbb{Z}/2)$ is detected by elementary abelian 2-groups. However, we cannot yet see that this is so for $H^*(BD(n); \mathbb{Z}/2)/(Ng)$; namely, whether or not the above epimorphism is indeed an isomorphism.

10. NOETHER’S PROBLEM

In this section, we recall Noether’s problem and consider the relation to the detected problem for $H^*(BG; \mathbb{Z}/p)/(Ng)$. In particular, we will show that for a p -group G of exponent p , if $H^2(BG; \mathbb{Z}/p)/(Ng)$ is not detected by any $A \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, then G is a counterexample of Noether’s problem; namely, $k(W)^G$ is not purely transcendental over k .

Let K be a finitely generated extension over k . We recall the definition of unramified cohomology of $H^*(K; \mathbb{Z}/p)$ according to Saltman, Peyre and Colliot-Thelene [Sa], [Pe], [Co].

We denote by $P(K/k)$ the set of discrete valuation rings A of rank one such that $k \subset A \subset K$ and that the fraction field $Fr(A)$ of A is K . If A belongs to $P(K/k)$, then for the residue field κ_A , we can define the residue map $\partial_A : H^*(K; \mathbb{Z}/p) \rightarrow H^{*-1}(\kappa_A; \mathbb{Z}/p)$ as follows.

Let \hat{K}_A be the completion, \hat{K}_A^{nr} the maximal unramified extension of \hat{K}_A and \bar{K}_A an algebraic closure of \hat{K}_A . Put $I_A = Gal(\bar{K}_A/\hat{K}_A^{nr})$ and $G_A = Gal(\bar{K}_A/\hat{K}_A)$,

$$\bar{K}_A \xrightarrow{I_A} \hat{K}_A^{nr} \xrightarrow{G_A/I_A} \hat{K}_A \xrightarrow{\quad} K.$$

Then ∂_A is defined as the composition of maps

$$\begin{aligned} \partial_A : H^*(K; \mathbb{Z}/p) &\rightarrow H^*(\hat{K}_A; \mathbb{Z}/p) \\ &\xrightarrow{proj.} H^{*-1}(G_A/I_A; \mathbb{Z}/p) \otimes H^1(I_A; \mathbb{Z}/p) \cong H^{*-1}(\kappa_A; \mathbb{Z}/p). \end{aligned}$$

Here we used the fact that $G \cong I_A \oplus (G_A/I_A)$ ([Sa]) and $I_A \cong \hat{\mathbb{Z}}$. Moreover, $H^*(\hat{\mathbb{Z}}; \mathbb{Z}/p) \cong \mathbb{Z}/p$ if $*$ = 1 and $\cong 0$ for $*$ > 1.

Then we can define the unramified cohomology

$$H_{nr}^*(K; \mathbb{Z}/p) = \bigcap_{A \in P(K/k)} Ker(H^*(K; \mathbb{Z}/p) \xrightarrow{\partial_A} H^{*-1}(\kappa_A; \mathbb{Z}/p)).$$

Hereafter (to the end of this paper) we assume that G is a *finite* group. According to Peyre [Pe], we define a subring $H_{nr}^*(G; \mathbb{Z}/p)$ of $H^*(BG; \mathbb{Z}/p)$ as follows. Let $P(G)$ be the set of elements $g \in G$ such that $I = \langle g \rangle \cong \mathbb{Z}/p^s$ for some $s \geq 1$ but $g \neq h^p$ for any $h \in G$. Then the centralizer is written as $Z_G(I) \cong I \oplus H$ with $H = Z_G(I)/I$. Let us write by ∂_g the composition map

$$\begin{aligned} \partial_g : H^*(BG; \mathbb{Z}/p) &\rightarrow H^*(BZ_G(I); \mathbb{Z}/p) \\ &\xrightarrow{proj.} H^{*-1}(BH; \mathbb{Z}/p) \otimes H^1(BI; \mathbb{Z}/p) \cong H^{*-1}(BH; \mathbb{Z}/p) \end{aligned}$$

using $H^1(BI; \mathbb{Z}/p) \cong \mathbb{Z}/p$. Then define the unramified cohomology by

$$H_{nr}^*(G; \mathbb{Z}/p) = \bigcap_{g \in P(G)} Ker(H^*(BG; \mathbb{Z}/p) \xrightarrow{\partial_g} H^{*-1}(BH; \mathbb{Z}/p)).$$

From the above two definitions of unramified cohomologies, we have the following theorem.

Theorem 10.1 (Peyre, [Pe]). *Let V be a faithful representation of G . Recall that q is the quotient map $q : Gal(\bar{k}(\bar{V})/k(V)^G) \rightarrow G$. Then we have $q^*(H_{nr}^*(G; \mathbb{Z}/p)) \subset H_{nr}^*(k(V)^G; \mathbb{Z}/p)$.*

Remark. We can define the similar map

$$\bar{\partial}_g : H^*(BG; \mathbb{Z}/p)/(Ng) \rightarrow H^{*-1}(BH; \mathbb{Z}/p)/(Ng)$$

using $H^1(BI; \mathbb{Z}/p) \cong H^1(BI; \mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p$. We also define the unramified cohomology by

$$\bar{H}_{nr}^*(G; \mathbb{Z}/p) = \bigcap_{g \in P(G)} Ker(\bar{\partial}_g) \subset H^*(BG; \mathbb{Z}/p)/(Ng).$$

Then we can prove $q^*(\bar{H}_{nr}^*(G; \mathbb{Z}/p)) \subset H_{nr}^*(k(W_*)^G; \mathbb{Z}/p)$.

It is well known (Theorem 4.1.5 in [Co]) that if K is purely transcendental over k (i.e., $K \cong k(x_1, \dots, x_n)$ for indeterminate x_i), then

$$H^*(k; \mathbb{Z}/p) \cong H_{nr}^*(K; \mathbb{Z}/p).$$

(We assume k is an algebraically closed field, and so $H^+(k; \mathbb{Z}/p) = 0$.) Hence if K is purely transcendental over k , then $H_{nr}^+(K; \mathbb{Z}/p) = 0$.

Corollary 10.2. *Let $x \in H^+(BG; \mathbb{Z}/p)$ such that $x \in H_{nr}^*(G; \mathbb{Z}/p)$ but $x \notin Ng(G)$. Then $q^*(x) \neq 0$ in $H_{nr}^*(k(W_*)^G; \mathbb{Z}/p)$. Hence $k(W_*)^G/k$ is not purely transcendental.*

Proof. From the preceding theorem, we have

$$q^*(x) \in H_{nr}^*(k(W_*)^G; \mathbb{Z}/p) \subset H^*(k(W_*)^G; \mathbb{Z}/p).$$

From the explanation just before Lemma 3.1, we see

$$\psi_{et}^* = q^* : H^*(BG; \mathbb{Z}/p)/(Ng) \subset H_{Zar}^0(BG; H_{\mathbb{Z}/p}^*) \subset H^*(k(W_*)^G; \mathbb{Z}/p).$$

Hence we see $q^*(x) \neq 0$. □

Corollary 10.3. *Let G be a p -group of exponent p . If $H^2(BG; \mathbb{Z}/p)/(Ng) \neq 0$ and it is not detected by $\mathbb{Z}/p \times \mathbb{Z}/p$, then $k(W_2)^G$ is not purely transcendental.*

Proof. Suppose that $x \notin H_{nr}^2(G; \mathbb{Z}/p)$. Then there is g such that $\partial_g(x) \neq 0$ for

$$\partial_g : H^2(BG; \mathbb{Z}/p) \xrightarrow{pr_1} H^1(BI; \mathbb{Z}/p) \otimes H^1(BH; \mathbb{Z}/p) \xrightarrow{pr_2} H^1(BH; \mathbb{Z}/p).$$

So $pr_1(x) = x_1 \otimes x_2 \neq 0$, where $x_1 \in H^1(BI; \mathbb{Z}/p) \cong \mathbb{Z}/p$ and $x_2 \in H^1(BH; \mathbb{Z}/p)$. Identify $x_2 \in Hom(H; \mathbb{Z}/p) \cong H^1(BH; \mathbb{Z}/p)$. Note the exponent of H is p . Hence we can take the subgroup $\mathbb{Z}/p \cong J \subset H$ such that $x_2|_J \neq 0$.

Then the corresponding element $\bar{x} \in H^2(BG; \mathbb{Z}/p)/(Ng)$ is detected by $I \oplus J$, because $Res(\bar{x}) = \bar{x}_1 \otimes \bar{x}_2$ for

$$H^1(BI; \mathbb{Z}/p) \otimes H^1(BJ; \mathbb{Z}/p) \cong H^2(B(I \oplus J); \mathbb{Z}/p)/(Ng) \cong \mathbb{Z}/p\{\bar{x}_1 \otimes \bar{x}_2\}.$$

Hence if $\bar{x} \neq 0$ is not detected by each $\mathbb{Z}/p \oplus \mathbb{Z}/p$, then $x \notin Ng(G)$ and $x \in H_{nr}^2(G; \mathbb{Z}/p)$. Thus we have the desired result from Corollary 10.2. □

11. SALTMAN'S EXAMPLE

In §11 and §12, we assume $p \geq 3$. Saltman [Sa] showed that $H_{nr}^2(k(W_2)^{Gs}; \mathbb{Q}/\mathbb{Z}) \neq 0$ for some special p -group Gs . We will give a short proof of the fact that

$$H_{nr}^2(k(W_2)^{Gs}; \mathbb{Z}/p) \neq 0.$$

A p -group G is called a special p -group if $Z(G) = [G, G] \cong (\mathbb{Z}/p)^s$ for $s \geq 1$. (When $s = 1$, we call it an extraspecial p -group.) Let G_s be the special p -group of exponent p defined by

$$0 \rightarrow \langle c, c_3, c_4 \rangle \rightarrow G_s \rightarrow \langle a_1, a_2, a_3, a_4 \rangle \rightarrow 0$$

with $[a_1, a_2] = [a_3, a_4] = c, [a_1, a_3] = c_3, [a_1, a_4] = c_4,$

and the other pairs commute. Note that

$$G_s/\langle c_3, c_4 \rangle \cong p_+^{1+4}, \quad \langle c_3, a_1, a_3 \rangle \cong p_+^{1+2}, \quad \langle c_4, a_1, a_4 \rangle \cong p_+^{1+2}.$$

Of course such a group G_s exists because the set

$$\{c^{i_1} c_3^{i_2} \dots a_4^{i_7} \mid 0 \leq i_s \leq p-1, 1 \leq s \leq 7\}$$

forms the group by the multiplication given by the commutator and $c^p = c_3^p = \dots = a_4^p = 1$. (For $p = 2$, there does not exist a G_s with exponent 2. See the Remark in §9 and see [Suz] for properties of special groups.)

Let us write

$$a^\Lambda = a_1^{\lambda_1} \dots a_4^{\lambda_4} \quad \text{for } \Lambda = (\lambda_1, \dots, \lambda_4) \in (\mathbb{Z}/p)^4,$$

$$a^M = a_1^{\mu_1} \dots a_4^{\mu_4} \quad \text{for } M = (\mu_1, \dots, \mu_4) \in (\mathbb{Z}/p)^4.$$

The commutator is given by the definition

$$[a^\Lambda, a^M] = c^d c_3^{d_3} c_4^{d_4}, \quad \text{where } d = \lambda_1 \mu_2 - \lambda_2 \mu_1 + \lambda_3 \mu_4 - \lambda_4 \mu_3,$$

$$d_3 = \lambda_1 \mu_3 - \lambda_3 \mu_1, \quad d_4 = \lambda_1 \mu_4 - \lambda_4 \mu_1.$$

Lemma 11.1. *Let $\langle \Lambda, M \rangle$ and $\langle \Lambda', M' \rangle$ generate the same 2-dimensional vector space in $(\mathbb{Z}/p)^4$. Then $[a^\Lambda, a^M] = 0$ if and only if $[a^{\Lambda'}, a^{M'}] = 0$.*

Proof. First note by the definition of G_s that

$$Z(G_s) = [G_s, G_s] = \langle c, c_3, c_4 \rangle.$$

For each Λ, M , we see $a^\Lambda a^M = a^{\Lambda+M} c(\Lambda, M)$ with $c(\Lambda, M) \in Z(G_s)$. We also note that $[a^\Lambda c(0), a^M] = [a^\Lambda, a^M]$ for each $c(0) \in Z(G_s)$.

Suppose $[a^\Lambda, a^M] = 0$. Let $\Theta_i = \beta_i \Lambda + \gamma_i M$ with $\beta_i, \gamma_i \in \mathbb{Z}/p$ for $i = 1, 2$. Note that

$$[a^{\beta_1 \Lambda} a^{\gamma_1 M}, a^\Lambda] = [(a^\Lambda)^{\beta_1} c(1) (a^M)^{\gamma_1} c(2), a^\Lambda] = [(a^\Lambda)^{\beta_1} (a^M)^{\gamma_1}, a^\Lambda] = 0$$

for some $c(1), c(2) \in Z(G_s)$, since $[a^\Lambda, a^M] = 0$. Similarly, we have

$$0 = [a^{\beta_1 \Lambda} a^{\gamma_1 M}, a^{\lambda_2 \Lambda} a^{\gamma_2 M}] = [a^{\Theta_1} c(3), a^{\Theta_2} c(4)] = [a^{\Theta_1}, a^{\Theta_2}]$$

for some $c(3), c(4) \in Z(G_s)$. □

Recall that $x_i \in H^1(BG_s; \mathbb{Z}/p)$ is the dual of a_i (and $\beta(x_i) = y_i$); namely,

$$x_i : G_s \xrightarrow{pr} \langle a_i \rangle \cong \mathbb{Z}/p \in Hom(G_s; \mathbb{Z}/p).$$

Hence $x_i | \langle a^\Lambda \rangle = \lambda_i$. Moreover, the above central extension (representing G_s) is written as an element in $H^2(B(\mathbb{Z}/p)^4; (\mathbb{Z}/p)^3)$,

$$(x_1 x_2 + x_3 x_4) \{c\} + x_1 x_3 \{c_3\} + x_1 x_4 \{c_4\},$$

identifying $H^*(B(\mathbb{Z}/p)^4; (\mathbb{Z}/p)^3) \cong H^*(B(\mathbb{Z}/p)^4; \mathbb{Z}/p) \{c, c_3, c_4\}$ (see also [Te-Ya] for details).

Lemma 11.2. *The 3-dimensional cohomology $H^3(BGs; \mathbb{Z}/p)$ contains the \mathbb{Z}/p -module*

$$A = \mathbb{Z}/p\{y_i x_j | 1 \leq i, j \leq 4\} / (Q_0(f), Q_0(x_1 x_3), Q_0(x_1 x_4)).$$

Proof. Consider the central extension

$$0 \rightarrow \langle c, c_3, c_4 \rangle \rightarrow G \rightarrow V = \langle a_1, \dots, a_4 \rangle \rightarrow 0$$

and the induced spectral sequence

$$E_2^{*,*'} \cong \mathbb{Z}/p[y_1, \dots, y_4, u, u_3, u_4] \otimes \Lambda(x_1, \dots, x_4, z, z_3, z_4)$$

converging to $H^*(BGs; \mathbb{Z}/p)$.

The first non-zero differential is computed as the cases of extraspecial p -groups

$$d_2(z) = f, \quad d_2(z_3) = x_1 x_3, \quad d_2(z_4) = x_1 x_4.$$

The second non-zero differential is

$$d_3(u) = Q_0(f), \quad d_3(u_3) = Q_0(x_1 x_3), \quad d_3(u_4) = Q_0(x_1 x_4).$$

Thus we see $A \cong E_4^{3,0} \cong E_\infty^{3,0} \subset H^3(BG; \mathbb{Z}/p)$ by dimensional reasons of differentials. □

Theorem 11.3 (Saltman, [Sa]). *We have*

$$x_1 x_2 \in H_{nr}^2(Gs; \mathbb{Z}/p) \quad \text{but} \quad x_1 x_2 \notin Ng(Gs).$$

Hence $k(W_2)^{Gs}$ is not purely transcendental.

Proof. From the preceding lemma, we see that $Q_0(x_1 x_2) \neq 0$ in $H^3(BGs; \mathbb{Z}/p)$. Hence we have $x_1 x_2 \notin Ng(Gs)$ from Lemma 3.6.

Next we will show $x_1 x_2 \in H_{nr}(Gs; \mathbb{Z}/p)$. Suppose this is not the case. By the definition (or Corollary 10.3), this means that there are elements $g \in Gs$ and $h \in Z_{Gs}(\langle g \rangle)$ such that

$$x_i | \langle g \rangle \neq 0, \quad x_j | \langle h \rangle \neq 0 \quad \text{for } \{i, j\} = \{1, 2\}.$$

Let $g = a^\Lambda c(g)$ with $c(g) \in Z(Gs)$. From Lemma 11.1, we can take

$$\lambda_1 = 1, \lambda_2 = 0, \quad \Lambda = (1, 0, \lambda_3, \lambda_4).$$

(Hence $x_1 | \langle g \rangle = 1$ and $x_2 | \langle g \rangle = 0$.) Then we can take $h = a^M \in Z(\langle g \rangle)$ so that $x_2 | \langle h \rangle = 1$ and $h \notin \langle g \rangle$, which means that

$$\mu_1 = 0, \mu_2 = 1, \quad M = (0, 1, \mu_3, \mu_4).$$

Since $[h, g] = 0$, we see that $d_3 = 0$ and $d_4 = 0$. Hence we have

$$\mu_3 = \lambda_3 \mu_1 = 0 \quad \text{and} \quad \mu_4 = \lambda_4 \mu_1 = 0$$

(and hence $M = (0, 1, 0, 0)$). Therefore $d = 1 \times 1 - 0 + 0 - 0 = 1$. This is a contradiction. Hence we have proved $x_1 x_2 \in H_{nr}(Gs; \mathbb{Z}/p)$. □

12. NON-DETECTED EXAMPLES

In this section, we study the cohomology of some special p -group G_m for $m \geq 1$ and an element $\bar{\xi}_m \in H^{2m}(BG'_m; \mathbb{Z}/p)/(Ng)$ which is not detected by elementary abelian p -groups. (However, we see $\xi_m \notin H_{nr}(G'_m; \mathbb{Z}/p)$ for $m \geq 2$.)

Let G' be the special p -group of exponent p defined by

$$0 \rightarrow \langle c, c_3, c_4 \rangle \rightarrow G' \rightarrow \langle a_1, a_2, a_3, a_4 \rangle \rightarrow 0$$

with $[a_1, a_2] = [a_3, a_4] = c, \quad [a_1, a_3] = c_3, \quad \text{and} \quad [a_2, a_4] = c_4,$

and the other pairs commute. (Such a group exists by the reason similar to that for G_s stated in §11.) The difference between the definitions of G_s and G' is just $[a_i, a_4] = c_4$ for $i = 1$ or 2 . However, note that G' has the rank 2 elementary abelian p -subgroup

$$\langle a_1 a_3^{-1}, a_2 a_4^{-1} \rangle \cong \mathbb{Z}/p \oplus \mathbb{Z}/p,$$

while G_s does not.

Let G'_m be the central product of $(m - 1)$ -th copies of G' and one G_s :

$$G'_m = G_s \cdot G' \cdot \dots \cdot G' = G_s \times_{\langle c \rangle} G' \times_{\langle c \rangle} \dots \times_{\langle c \rangle} G',$$

which is generated by $a_1, \dots, a_{4m}, c, c_3, c_4, c_7, \dots, c_{4m-1}, c_{4m}$. The comutators are given for $1 \leq i \leq m$:

$$[a_{4i-3}, a_{4i-2}] = [a_{4i-1}, a_{4i}] = c,$$

$$[a_{4i-3}, a_{4i-1}] = c_{4i-1}, \quad \text{and} \quad [a_{4i-2}, a_{4i}] = c_{4i} \text{ for } i \neq 1,$$

but $[a_1, a_4] = c_4$.

Let us write $a^\Lambda = a_1^{\lambda_1} \dots a_{4m}^{\lambda_{4m}}$ and $\Lambda = (\lambda_1, \dots, \lambda_{4m})$. Note also that $x_i | \langle a^\Lambda \rangle = \lambda_i$. The commutator is given by the definition

$$[a^\Lambda, a^M] = c^d c_3^{d_3} c_4^{d_4} \dots c_{4m}^{d_{4m}},$$

where $d = \sum_i (\lambda_{4i-3} \mu_{4i-2} - \lambda_{4i-2} \mu_{4i-3} + \lambda_{4i-1} \mu_{4i} - \lambda_{4i} \mu_{4i-1}),$

$$d_{4i-1} = \lambda_{4i-3} \mu_{4i-1} - \lambda_{4i-1} \mu_{4i-3}, \quad d_{4i} = \lambda_{4i-2} \mu_{4i} - \lambda_{4i} \mu_{4i-2} \quad (i \neq 1),$$

and $d_4 = \lambda_1 \mu_4 - \lambda_4 \mu_1$.

By the arguments similar to the proof of Lemma 11.1, we have

Lemma 12.1. *Let $\langle \Lambda_1, \dots, \Lambda_{2m} \rangle$ and $\langle \Lambda'_1, \dots, \Lambda'_{2m} \rangle$ generate the same $2m$ -dimensional vector space in $(\mathbb{Z}/p)^{4m}$. Then $\langle a^{\Lambda_1}, \dots, a^{\Lambda_{2m}} \rangle$ is an abelian group if and only if $\langle a^{\Lambda'_1}, \dots, a^{\Lambda'_{2m}} \rangle$ is as well.*

Lemma 12.2. *Let $\xi_m = x_1 x_2 x_5 x_6 \dots x_{4m-3} x_{4m-2} \in H^*(BG'_m; \mathbb{Z}/p)$. Then the corresponding element $\bar{\xi}_m \in H^*(BG'_m; \mathbb{Z}/p)/(Ng)$ is not detected by elementary abelian p -groups.*

Proof. Suppose that $\bar{\xi}_m$ is detected by

$$\langle g_1, g_2, \dots, g_{2m} \rangle \cong \mathbb{Z}/p \oplus \dots \oplus \mathbb{Z}/p$$

such that $\bar{\xi}_m | \langle g_1, \dots, g_{2m} \rangle \neq 0$, and $\bar{x}_i | \langle g_{j_i} \rangle = x_i | \langle g_{j_i} \rangle \neq 0$ for some j_i . By using the preceding lemma, we can take generators adequately so that $g_i = a^{\Lambda_i}$ with

$$\begin{aligned} \Lambda_1 &= (1, 0, *, * | 0, 0, *, * | \dots | 0, 0, *, *), \\ \Lambda_2 &= (0, 1, *, * | 0, 0, *, * | \dots | 0, 0, *, *), \\ \Lambda_5 &= (0, 0, *, * | 1, 0, *, * | \dots | 0, 0, *, *), \\ \Lambda_6 &= (0, 0, *, * | 0, 1, *, * | \dots | 0, 0, *, *), \\ &\dots \\ \Lambda_{4m-3} &= (0, 0, *, * | 0, 0, *, * | \dots | 1, 0, *, *), \\ \Lambda_{4m-2} &= (0, 0, *, * | 0, 0, *, * | \dots | 0, 1, *, *). \end{aligned}$$

Let $i > 1$. Consider the commutativity of $L = \Lambda_{4i-3}$ (e.g., $\lambda_{4i-3} = 1$ and $\lambda_{4i-2} = 0$) and $M = \Lambda_1$ ($\mu_{4i-3} = \mu_{4i-2} = 0$). Since $d_{4i-1} = \lambda_{4i-3}\mu_{4i-1} - \lambda_{4i-1}\mu_{4i-3} = 0$, we see that

$$\mu_{4i-1} = 0 \quad \text{from } \mu_{4i-3} = 0 \text{ and } \lambda_{4i-3} = 1.$$

Similarly, we have $\mu_{4i} = 0$ from the commutativity of Λ_{4i-2} and Λ_1 . Thus we see that

$$\Lambda_1 = (1, 0, *, * | 0, 0, 0, 0 | \dots | 0, 0, 0, 0).$$

We also have $\Lambda_2 = (0, 1, *, * | 0, 0, 0, 0 | \dots | 0, 0, 0, 0)$. Let $\Lambda = \Lambda_1$ and $M = \Lambda_2$. By the commutativity and the facts $d_3 = d_4 = 0$, we see that $\mu_3 = \mu_4 = 0$; that is,

$$\Lambda_2 = (0, 1, 0, 0 | 0, 0, 0, 0 | \dots | 0, 0, 0, 0).$$

However, this is a contradiction. Indeed,

$$d = \sum_i (\lambda_{4i-3}\mu_{4i-2} - \lambda_{4i-2}\mu_{4i-3} + \lambda_{4i-1}\mu_{4i} - \lambda_{4i}\mu_{4i-1}) = 1 \neq 0.$$

□

Lemma 12.3. *We have $\xi_m \notin Ng(G'_m)$. Indeed, $Q_0 \dots Q_{2m-2}(\xi_m) \neq 0$ in $H^*(BG'_m; \mathbb{Z}/p)$.*

Proof. If $Q_0 \dots Q_{2m-2}(\xi_m) \neq 0$ in $H^*(BG'_m; \mathbb{Z}/p)$, then we have $\xi_m \notin Ng(G'_m)$ from Lemma 3.6. So we will prove $Q_0 \dots Q_{2m-2}(\xi_m) \neq 0$.

We recall that G'_m is the central product of G s and copies of G' , which means that

$$0 \rightarrow \bigoplus_{i=1}^{m-1} \mathbb{Z}/p \rightarrow Gs \times G' \times \dots \times G' \xrightarrow{pr.} G'_m \rightarrow 0.$$

We consider the composition map

$$\begin{aligned} f^* : H^*(BG'_m; \mathbb{Z}/p) &\xrightarrow{pr.} H^*(BGs; \mathbb{Z}/p) \otimes H^*(BG'; \mathbb{Z}/p)^{\otimes(m-1)} \\ &\rightarrow H = H^*(BGs; \mathbb{Z}/p) \otimes H^*(B(\mathbb{Z}/p)^2; \mathbb{Z}/p)^{\otimes(m-1)}. \end{aligned}$$

Here the last map is the restriction map for $(\mathbb{Z}/p)^2 \subset G'$. Indeed, we have the injection $f : Gs \oplus (\mathbb{Z}/p)^{2m-2} \subset G'_m$.

The above restriction map is explicitly written as

$$\begin{aligned} H^*(BG'; \mathbb{Z}/p) &\rightarrow H^*(B\langle a_{4i-3}a_{4i-1}^{-1}, a_{4i-2}a_{4i}^{-1} \rangle; \mathbb{Z}/p) \\ &\cong \mathbb{Z}/p[y_{4i-3}, y_{4i-2}] \otimes \Lambda(x_{4i-3}, x_{4i-2}), \end{aligned}$$

where $y_{4i-3}, -y_{4i-1} \mapsto y_{4i-3}$ and $y_{4i-2}, -y_{4i} \mapsto y_{4i-2}$.

We want to show that $f^*(Q_0 \dots Q_{2m-2}(\xi_m)) \neq 0$. For $i > 1$, we note that

$$Q_{2i-3}(x_{4i-3}) = y_{4i-3}^{p^{2i-3}}, \quad Q_{2i-2}(x_{4i-2}) = y_{4i-2}^{p^{2i-2}}.$$

From Lemma 11.2, we also note that

$$Q_0(x_1x_2) = y_1x_2 - y_2x_1 \neq 0 \quad \text{in } H^*(BGs; \mathbb{Z}/p).$$

Here we give an order for elements in the cohomology

$$H = H^*(B(Gs \oplus (\mathbb{Z}/p)^{2m-2}); \mathbb{Z}/p).$$

We define the lexicographic order $<$ by

$$y_5^{i_5} y_6^{i_6} \dots y_{4m-3}^{i_{4m-3}} y_{4m-2}^{i_{4m-2}} x < y_5^{j_5} y_6^{j_6} \dots y_{4m-3}^{j_{4m-3}} y_{4m-2}^{j_{4m-2}} x'$$

if there is an s such that $i_s < j_s$ and $i_k = j_k$ for all $s < k$, where

$$x, x' \in H^*(BGs; \mathbb{Z}/p) \otimes \Lambda(x_5, x_6, \dots, x_{4m-3}, x_{4m-2}).$$

The operation Q_i is a derivation, and we have

$$\begin{aligned} & f^*(Q_0 \dots Q_{2m-2}(\xi_m)) \\ &= Q_0(x_1x_2)Q_1(x_5)Q_2(x_6) \dots Q_{2m-3}(x_{4m-2})Q_{2m-2}(x_{4m-2}) \text{ mod}(F) \\ &= (y_1x_2 - y_2x_1)y_5^p y_6^{p^2} \dots y_{4m-3}^{p^{2m-3}} y_{4m-2}^{p^{2m-2}} \text{ mod}(F), \end{aligned}$$

where (F) is the filtration generated by elements of the smaller order rather than the element on the right hand side. In particular, it is non-zero. \square

Remark. To see $x \in H_{nr}^*(G; \mathbb{Z}/p)$, from the definition of the unramified cohomology by Peyre, we need to check that

$$\partial_g(x) = 0 \in H^{*-1}(BH; \mathbb{Z}/p)$$

for all $H \oplus \mathbb{Z}/p \cong H \oplus \langle g \rangle \subset G$. When $* > 2$ these H need not be abelian. This seems that $x \in H_{nr}^*(G; \mathbb{Z}/p)$ is something of a rare case. Indeed, from the (proof of) the preceding lemma, we see that $Gs \oplus (\mathbb{Z}/p)^{2m-2} \subset G'_m$ and

$$\begin{aligned} \partial_{a_{4m-2}}(\xi_m) &= f^*(\xi_m)/(x_{4m-2}) = x_1x_2x_5x_6 \dots x_{4m-3} \\ &\neq 0 \in H^{2m-1}(B(Gs \oplus (\mathbb{Z}/p)^{2m-3}); \mathbb{Z}/p). \end{aligned}$$

So $\xi_m \notin H_{nr}^{2m}(G'_m; \mathbb{Z}/p)$ for $m \geq 2$. (However, we do not yet see $q^*(\xi_m) \notin H_{nr}^*(k(W_*)^{G'_m}; \mathbb{Z}/p)$.)

Corollary 12.4. *There is an element in $H^{2m}(BG'_m; \mathbb{Z}/p)/(Ng)$ which is not detected by abelian subgroup. Namely, the restriction map*

$$Res : H^{2m}(BG'_m; \mathbb{Z}/p)/(Ng) \rightarrow \Pi_{A, \text{abelian}} H^{2m}(BA; \mathbb{Z}/p)/(Ng)$$

is not injective.

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