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ESTIMATES FOR INVARIANT METRICS ON $\mathbb{C}$-CONVEX DOMAINS

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Abstract. Geometric lower and upper estimates are obtained for invariant metrics on $\mathbb{C}$-convex domains containing no complex lines.

1. Introduction and results

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. For a domain $D \subset \mathbb{C}^n$ the Carathéodory and Kobayashi (pseudo)metrics are defined in the following way (cf. [16], p. 16 and p. 90):

$$\gamma_D(z;X) = \sup\{|f'(z)X| : f \in \mathcal{O}(D,\mathbb{D}), f(z) = 0\},$$
$$\kappa_D(z;X) = \inf\{\alpha \geq 0 : \exists \varphi \in \mathcal{O}(\mathbb{D},D) : \varphi(0) = z, \alpha\varphi'(0) = X\},$$

where $z \in D$, $X \in \mathbb{C}^n$ and $f'(z)$ is the Fréchet derivative. It is clear that $\gamma_D \leq \kappa_D$.

Recall that a domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if any non-empty intersection with a complex line is a simply connected domain (cf. [1, 14]). By a fundamental theorem due to L. Lempert (see [18]), the equality $\gamma_D = \kappa_D$ takes place for any so-called strongly linearly convex domain D (see also [16], Miscellanea C). This remains true for any convex domain, since it can be exhausted by smooth bounded strictly convex domains. A similar result holds for $\mathbb{C}$-convex domains with C^2 boundary (see [15]).

A domain $D \subset \mathbb{C}^n$ is said to be *linearly convex* (respectively, *weakly linearly convex*) if for any $a \in \mathbb{C}^n \setminus D$ (for any $a \in \partial D$) there exists a complex hyperplane through a which does not intersect D.

Recall that the following implications hold (cf. [1], Theorem 2.3.9 and [14], Theorem 4.6.8):

$$\mathbb{C}\text{-convexity} \Rightarrow \text{linear convexity} \Rightarrow \text{weak linear convexity}.$$

Moreover, these three notions coincide in the case of C^1-smooth domains in dimension greater than 1 (cf. [1], Corollary 2.5.6 and [14], Corollary 4.6.9). On the other hand, any planar domain is linearly convex but any annulus is not $\mathbb{C}$-convex.

(A) For $\mathbb{C}$-convex domains we shall prove the following results for the boundary behavior of the Carathéodory and Kobayashi metrics.

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Proposition 1. *Let D be a $\mathbb{C}$-convex domain containing no complex line through $z \in D$ in the direction of X. Then*

$$\frac{1}{4} \le \gamma_D(z;X)d_D(z,X) \le \kappa_D(z;X)d_D(z,X) \le 1,$$

where

$$d_D(z,X) = \sup\{r > 0 : z + \lambda X \in D \text{ if } |\lambda| < r\}$$

is the distance from z to ∂D in direction X.

The constant $\frac{1}{4}$ can be replaced by $\frac{1}{2}$ in the case of convex domains (see [2], Theorem 4.1). On the other hand, the constant $\frac{1}{4}$ is the best one in the planar case as the image $D = \mathbb{C} \setminus [1/4, \infty)$ of $\mathbb{D}$ under the Koebe function $\frac{z}{(1+z)^2}$ shows. It is clear that the upper constant 1 is attained if, for example, $D = \mathbb{D}$.

Corollary 2. *For any $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ one has that $\kappa_D \le 4\gamma_D$.*

Recall that if a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ contains a complex line, then it is linearly equivalent to the Cartesian product of $\mathbb{C}$ and a $\mathbb{C}$-convex domain in $\mathbb{C}^{n-1}$ (see [28]).

For a boundary point a of a domain $D \subset \mathbb{C}^n$ denote by L_a the set of all vectors $X \in \mathbb{C}^n$ for which there exists $\varepsilon > 0$ such that $\partial D \supset \Delta_X(a,\varepsilon) = \{a + \lambda X : |\lambda| < \varepsilon\}$.

The following result is a consequence of Proposition 1.

Proposition 3. *Let a be a boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$.*

(i) Then

$$\lim_{z \to a} \gamma_D(z;X) = \infty$$

locally uniformly in $X \notin L_a$.[1]

(ii) If ∂D is C^1-smooth at a, then L_a is a linear space. Moreover, for any non-tangential cone Λ with vertex at a there is a constant $c > 0$ such that

$$\limsup_{\Lambda \ni z \to a} \kappa_D(z;X) \le c$$

locally uniformly in the unit vectors $X \in L_a$.

(B) Next we shall discuss types related to a (C^∞-)smooth boundary point a of a domain $D \subset \mathbb{C}^n$ and a vector $X \in (\mathbb{C}^n)_*$. Denote by m_a the (usual) type of a, i.e. the maximal order of contacts of non-trivial analytic discs through a and ∂D at the point a. Replacing analytic discs by complex lines, we define the linear type l_a of a. We may also define $l_{a,X}$ as the order of contact of the line through a in direction of X and ∂D at a. Then $m_a \ge l_a = \sup_X l_{a,X}$. Note that if $l_{a,X} < \infty$, then $X \notin L_a$.

Proposition 4. *Let a be a smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ and let $X \in (\mathbb{C}^n)_*$ with $l_{a,X} < \infty$. Denote by n_a the inner normal to ∂D at a. Then there exist a neighborhood U of a and a constant $c > 1$ such that*

$$c^{-1} d_D(z) \le d_D(z,X)^{l_{a,X}} \le c\, d_D(z), \quad z \in D \cap U \cap n_a,$$

where d_D is the distance to ∂D.

Combining Propositions 1 and 4 we immediately obtain an extension of the main result in [17] from the convex to the $\mathbb{C}$-convex case.

[1] This means that for any $M > 0$ there are neighborhoods U of a and V of X such that $\gamma_D(z;Y) > M$ for any $z \in D \cap U$ and $Y \in V$.

Corollary 5. *Under the notation of Proposition* 4*, there is a constant* $c > 0$ *such that*

$$c^{-1}(d_D(z))^{-1/l_{a,X}} \le \gamma_D(z;X) \le \kappa_D(z;X) \le c(d_D(z))^{-1/l_{a,X}}.$$

The main result in [20] (see also [4]) states that $m_a = l_a$ for convex domains. The same remains true for a $\mathbb{C}$-convex domain.

Proposition 6. *If a is a smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then $m_a = l_a$.*

Remark. We like to mention that the proof in [4] immediately implies the above proposition in dimension 2. But we do not know if the criterion in [4] (for the equality $m_a = l_a$) holds for any $\mathbb{C}$-convex domain.

Moreover, in the case of infinite type we have the following result.

Proposition 7. *If a is a C^1-smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then ∂D contains no non-trivial analytic disc through a if only if $L_a = \{0\}$.*

Remark. Some of the above propositions in **(A)** and **(B)** have local versions. In this connection recall that there is a localization principle for the Kobayashi metric of any hyperbolic domain (cf. [16]).

(C) Now we are going to discuss multitypes of boundary points. Recall that a smooth finite type pseudoconvex boundary point a of a domain $D \subset \mathbb{C}^n$ is said to be semiregular [8] (or h-extendible [32]) if its Catlin multitype $\mathcal{M}(a)$ coincides with its D'Angelo type $\Delta(a)$. Based on the fact that the usual type is equal to the line type in the case of convex domains, it is shown in [31] that if a is a smooth convex point (not necessarily of finite type), then $\mathcal{L}(a) = \mathcal{M}(a) = \Delta(a)$, where $\mathcal{L}(a)$ denotes the linear multitype of a.

We shall say that a is a $\mathbb{C}$-*convex boundary point* of a domain $D \subset \mathbb{C}^n$ if there is a neighborhood U of D such that $D \cap U$ is a $\mathbb{C}$-convex domain.

Proposition 8. [2] *If a is a smooth $\mathbb{C}$-convex boundary point of a domain $D \subset \mathbb{C}^n$, then $\mathcal{L}(a) = \mathcal{M}(a) = \Delta(a)$.*

Then the main result in [32] implies the following.

Corollary 9. *Any smooth finite type $\mathbb{C}$-convex boundary point a of a domain $D \subset \mathbb{C}^n$ is a local (holomorphic) peak point. Moreover, there is a neighborhood U of a and a domain $\mathbb{C}^n \supset G \supset \overline{D \cap U} \setminus \{a\}$ such that $a \in \partial G$ is a peak point w.r.t. the algebra $A(G)$.*

This corollary is also a direct consequence of the main result in [9], where local holomorphic support functions which depend smoothly on the boundary points are constructed.

We point out that the assumption of smoothness is essential as the domain $D = \mathbb{D} \setminus [0,1)$ may show. It is easy to see that the points from the deleted interval are not peak points for $A(D)$.

On the other hand, in [30], the following result is claimed.

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[2] The same result may be found in [7]; the proof there is related on good local coordinates and on the proof in [31], whereas our proof is based on the simple geometric Lemma 15 and on the proof in [31].

Proposition 10. *Let $D \subset \mathbb{C}^n$ be a bounded convex domain. Then $a \in \partial D$ is a peak point w.r.t. $A(D)$ if and only if $L_a = \{0\}$.*

For the convenience of the reader, we shall prove this result.

Note that there is a smooth convex bounded domain $D \subset \mathbb{C}^2$ containing no non-trivial analytic discs in the boundary but some of the boundary points (not of finite type) are not peak points w.r.t. $A^\alpha(D)$ for any $\alpha > 0$ (see [29]).

Also note that the main result in [23] (see also [33] and [5]) and Proposition 8 give the following fact about the boundary behavior of invariant metrics (see also [3, 19]).

Corollary 11. *Let a be a finite type $\mathbb{C}$-convex boundary point of a smooth bounded pseudoconvex domain $D \subset \mathbb{C}^n$. Let $\mathcal{M}(a) = (m_1, \dots, m_n)$ be the Catlin multitype of a ($m_1 = 1$ and $m_2 \le \dots \le m_n$ are even numbers). Denote by n_a the inner normal to ∂D at a. There is a basis $\{e_1, \dots, e_n\}$ (e_1 is the complex normal vector and $\{e_2, \dots, e_n\} \subset T_a^{\mathbb{C}}(\partial D)$) and a constant $c > 1$ such that for any $X = \sum_{j=1}^n X_j e_j$ we have*

$$c^{-1} \le \liminf_{n_a \ni z \to a} F_D(z; X) \left(\sum_j^n \frac{|X_j|}{(d_D(z))^{1/m_j}} \right)^{-1}$$
$$\le \limsup_{n_a \ni z \to a} F_D(z; X) \left(\sum_j^n \frac{|X_j|}{(d_D(z))^{1/m_j}} \right)^{-1} \le c.$$

Here F_D is any of the Carathéodory, Kobayashi or Bergman metrics (see below for the definition of the last metric).

We point out that this corollary implies Proposition 4 in the finite type case, showing in addition that for any $X \in (\mathbb{C}^n)_*$ there is $j = 1, \dots, n$ with $l_{a,X} = m_j$.

(D) Finally, we turn to the main part in this paper, namely, the boundary behavior of the Bergman metric of $\mathbb{C}$-convex domains. Denote by $L_h^2(D)$ the Hilbert space of all holomorphic functions f on a domain $D \subset \mathbb{C}^n$ that are square-integrable and by $||f||_D$ the L_2-norm of f. Let K_D be the restriction to the diagonal of the Bergman kernel function of D. It is well known that (cf. [16], p. 171)

$$K_D(a) = \sup\{|f(a)|^2 : f \in L_h^2(D),\ \|f\|_D \le 1\}.$$

If $K_D(z) > 0$ for some point $z \in D$, then the Bergman metric $B_D(z; X),\ X \in \mathbb{C}^n$, is well-defined and can be given by the equality (cf. [16], Theorem 6.2.5)

$$B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}},$$

where $M_D(z; X) = \sup\{|f'(z)X| : f \in L_h^2(D), \|f\|_D = 1,\ f(z) = 0\}$.

Recall that (cf. [16])

$$\gamma_D \le B_D.$$

On the other hand, there exists a constant $c_n > 0$, depending only on n such that for any convex domain $D \subset \mathbb{C}^n$, containing no complex line, the following inequality holds (see [26], Theorem 2):

$$B_D \le c_n \gamma_D.$$

This fact extends to any $\mathbb{C}$-convex domain as the following theorem shows.

Theorem 12. *There exists a constant $c_n > 0$, depending only on n, such that for any $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, containing no complex lines,*[3] *one has that*

$$B_D(z;X)d_D(z,X) \le c_n, \quad z \in D, X \in (\mathbb{C}^n)_*.$$

In particular, by Proposition 1,

$$\frac{\kappa_D}{4} \le B_D \le 4c_n\gamma_D.$$

To prove Theorem 12, we shall need a lower geometric estimate for the Bergman kernel. For this, similarly to the convex case (see [26]; see also [11, 12, 7] and compare with [6, 20, 21, 22]), we introduce the following geometric objects related to an arbitrary domain $D \subset \mathbb{C}^n$, containing no complex lines.

For $z^0 \in D =: D_0 \subset \mathbb{C}^n =: H_0$ define $d_{1,D}(z^0) := \operatorname{dist}(z^0, \partial D) = d_D(z^0)$. Fix an $a^1 \in \partial D$ such that $||a^1 - z^0|| = d_{1,D}(z^0)$. Let $l_1 = z^0 + V_1$ be the complex line passing through z^0 and a^1. Let $H_1 := V_1^\perp$ be the $(n-1)$-dimensional complex space orthogonal to V_1. Set $D_1 := D_0 \cap (z^0 + H_1)$ and $d_{2,D}(z^0) := \operatorname{dist}_{z^0+H_1}(z^0, \partial_{z^0+H_1} D_1)$. Then fix a point $a^2 \in \partial_{z^0+H_1}(D_1)$ with $\|a^2 - z^0\| = d_{2,D}(z^0)$. Denote by $l_2 = z^0 + V_2$ the complex line through z^0 and a^2. Note that $V_2 \subset V_1^\perp$. Put $H_2 := V_2^\perp \cap H_1$ and define $D_2 := D_1 \cap (z^0 + H_2)$. Continuing the previous procedure we are led to an orthonormal basis (arising from the complex lines $l_1, \dots, l_n$), positive numbers $d_{1,D}(z^0), \dots, d_{n,D}(z^0)$ and points $a^1, \dots, a^n$ with $a^j \in \partial_{z^0+H_{j-1}} D_{j-1}$ and $\|a^j - z^0\| = d_{j,D}(z^0)$.

Set

$$p_D(z^0) := d_{1,D}(z^0) \cdots d_{n,D}(z^0).$$

Using these numbers we get the following estimates for the Bergman kernel.

Theorem 13. *Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain containing no complex lines. Then*

$$\frac{1}{(16\pi)^n} \le K_D(z)p_D^2(z) \le \frac{(2n)!}{(2\pi)^n}.$$

Recall that the constant 16 can be replaced by 4 in the case of convex domains (cf. [26]).

The previous and next results extend earlier ones treating convex domains of finite type (cf. [6, 21, 22]) and the proofs here are easier and purely geometric. Take a vector $X \in \mathbb{C}^n$. For any point $z \in D$, decompose X w.r.t. to the orthogonal basis mentioned above, i.e. $X = (X_1(z), \dots, X_n(z))$.

Then the following result is a consequence of Proposition 1 and Theorem 12 .

Proposition 14. *There exists a constant $c_n > 1$, depending only on n, such that for any $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, containing no complex lines, one has that*

$$c_n^{-1} \le F_D(z;X)\left(\sum_j^n \frac{|X_j(z)|}{d_{j,D}(z)}\right)^{-1} \le c_n,$$

where F_D denotes any of the Carathéodory, Kobayashi or Bergman metrics.

This result is in the spirit of Corollary 11.

[3] Under the given assumptions D is biholomorphic to a bounded domain (cf. [28]) and hence B_D is well defined.

Remark. Proposition 3 and Corollary 5 hold for the Bergman metric, if the domain contains no complex lines. (In fact, then Proposition 3 transports the main result in [13] and a result in [25] from the convex to the $\mathbb{C}$-convex case.) Moreover, these and the other results for the Bergman kernel and metric have local versions on bounded pseudoconvex domains due to the localization principle for the Bergman invariants (cf. [16]).

2. Proofs

Proof of Proposition 1. The upper bound is trivial and holds for any domain D, since it contains the disc with center z and radius $d_D(z,X)$ in direction X.

To prove the lower bound, we may assume that $||X|| = 1$. Denote by l the complex line trough z in direction X and choose $a \in l \cap \partial D$ such that $||z-a|| = d_D(z,X)$. Consider a complex hyperplane H through a such that $D \cap H = \emptyset$ and denote by G the projection of D onto l in direction H. Note that G is a simply connected domain (cf. [1], Theorem 2.3.6 and [14], Proposition 4.6.7), $a \in \partial G$ and $d_G(z) = ||z-a||$. It remains to apply the Koebe theorem to get that

$$\gamma_D(z;X) \geq \gamma_G(z;1) \geq \frac{1}{4d_G(z)}. \qquad \square$$

Many of the next proofs will be based on the following geometric property of weakly linearly convex domains (see also [35] and (for the finite type case) [7]).

Lemma 15. *Assume that a weakly linearly convex domain $G \subset \mathbb{C}^n$ contains the unit disc $\mathbb{D}_j$ in the j-th complex coordinate line for any $j = 1,\dots,n$. Then G contains the convex hull of $\bigcup_{j=1}^n \mathbb{D}_j$, i.e.*

$$E := \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j| < 1\} \subset G.$$

Proof. For any $\varepsilon \in (0,1)$ there is $\delta > 0$ such that

$$X_\varepsilon := \bigcup_{j=1}^n \Big(\delta\mathbb{D} \times \dots \times \delta\mathbb{D} \times \underbrace{\varepsilon\mathbb{D}}_{j-th \text{ place}} \times \delta\mathbb{D} \times \dots \times \delta\mathbb{D}\Big) \subset G.$$

Note that

$$\widehat{X}_\varepsilon \subset G,$$

where $\widehat{X}_\varepsilon$ is the smallest linearly convex set containing X_ε. Moreover,

$$\widehat{X}_\varepsilon = \{z \in \mathbb{C}^n : \forall b \in \mathbb{C}^n : < z,b >= 1 \ \exists a \in X_\varepsilon : < a,b >= 1\},$$

(cf. [1], p. 17, and [14], Proposition 4.6.2). Then $\widehat{X}_\varepsilon$ is a balanced domain and, therefore, convex (see [28], Proposition 2). Hence,

$$E_\varepsilon := \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j| < \varepsilon\} \subset \widehat{X}_\varepsilon \subset G, \quad \varepsilon \in (0,1),$$

which proves Lemma 15. $\square$

Remark. The same argument implies that G contains the convex hull of any balanced domain lying in G. In particular, the maximal balanced domain lying in G is convex (see also [35]).

Proof of Proposition 3. (i) Assuming the contrary, we may find an $r > 0$ and sequences $D \supset (z_j)_j,\ z_j \to a,\ \mathbb{C}^n \supset (X_j)_j,\ X_j \to X \notin L_a$ such that $\gamma_D(z_j; X_j) \le \dfrac{1}{4r}$. Note that, by Proposition 1, $d_D(z_j; X_j) \ge r$ (this is trivial if D contains the complex line through z_j in direction X_j). Then $\Delta_{X_j}(z_j, r) \subset D_r = D \cap \mathbb{B}_n(a, 2r)$ for any large j. Note that D_r is a (weakly) linearly convex open set. It is easy to see that D_r is taut, i.e. the family $\mathcal{O}(\mathbb{D}, D_r)$ is normal (cf. [27]). Hence $\Delta_X(a, r) \subset \partial D$; a contradiction.

(ii) Recall that ∂D is C^1-smooth. Therefore, for any two linearly independent vectors $X, Y \in L_a$, we may find a neighborhood U of a and a number $\varepsilon > 0$ such that $\Delta_X(z, \varepsilon) \subset D$ and $\Delta_Y(z, \varepsilon) \subset D$ for $z \in D \cap U \cap \Lambda$. It follows by Lemma 15 that $\Delta_{X+Y}(z, \varepsilon') \subset D$ for some $\varepsilon' > 0$. We get as in (i) that $\Delta_{X+Y}(a, \varepsilon') \subset \partial D$. Therefore, L_a is a linear space.

Then, choosing a basis in L_a and applying Lemma 15, we see that there are a neighborhood U of a and a number $c > 0$ such that $\Delta_X(z, c) \subset D$ for any $z \in D \cap U \cap \Lambda$ and any unit vector $X \in L_a$. Now the desired estimates follow by Proposition 1. $\square$

Proof of Proposition 4. We may assume that $\mathrm{Re}(z_1) < 0$ is the inner normal direction to ∂D at $a = 0$. Let $r(z) = \mathrm{Re}(z_1) + o(|z_1|) + \rho('z)$ be a smooth defining function of D near 0.

For any small $\delta > 0$ we have that $\delta = d_D(\delta_n)$, where $\delta_n = (-\delta,' 0)$. Set $L_\delta(\zeta) = -\delta_n + \zeta X$, $\zeta \in \mathbb{C}^n$.

We shall consider two cases.

1. $l_{a,X} = 1$. This means that $X_1 \ne 0$. Then $r(L_\delta(\zeta)) = -\delta + \mathrm{Re}(\zeta X_1) + o(|\zeta|)$. It follows that $L_\delta(\zeta) \in D$ if $|\zeta| < \frac{\delta}{2|X_1|}$ and δ is small enough. This proves the left-hand side inequality.

The opposite inequality follows by the inequality $r(L_\delta(2\delta/X_1)) > 0$ which holds for any small $\delta > 0$.

2. $l_{a,X} \ge 2$. This means that $X_1 = 0$. Then $r(L_\delta(\zeta)) = -\delta + \rho(\zeta' X)$. Since $\rho(\zeta' X) \le c|\zeta|^l$ for some $c > 0$, we conclude that $L_\delta(\zeta) \in D$ if $c|\zeta|^l < \delta$. This implies the left-hand side inequality.

To prove the opposite inequality, we have to find $c_1 > 0$ such that for any small $\delta > 0$ there is a ζ with $|\zeta|^l = c_1^{-1}\delta$ and $\rho(\zeta' X) \ge \delta$. Since D is (weakly) linearly convex, it follows that $\rho(\zeta' X) = h(\zeta) + o(|\zeta|^l) \ge 0$, where

$$h(\zeta) = \sum_{j+k=l} a_{jk} \zeta^j \overline{\zeta}^k \not\equiv 0.$$

Then the homogeneity of h implies that $h \ge 0$. Moreover, since $h \not\equiv 0$ we may find a ζ with $|\zeta| = 1$ and $h(\zeta) > c_1$ for some $c_1 > 0$. Then the constant c_1 does the job for any small $\delta > 0$. $\square$

Proof of Proposition 6. The inequality $l_a \le m_a$ is trivial. To prove the opposite one, we may assume that $l_a < \infty$. It follows from Propositions 1 and 4 that

$$\liminf_{D \cap n_a \ni z \to a} \gamma_D(z; X) d^{1/l_a} \ge c_X > 0.$$

Hence, $m_a \le l_a$ by Corollary 2 in [34] (in fact, lim sup instead of lim inf above is sufficient). $\square$

Proof of Proposition 7. We shall use the same notation as in the proof of Proposition 4. It is enough to show that if $\varphi : \mathbb{D} \to \partial D$ is a non-trivial analytic disc with $\varphi(0) = 0$, then $L_a \neq \{0\}$. Since ∂D is smooth near a, it follows that there is a $c > 0$ such that $\varphi_\delta(\zeta) = -\delta_n + \varphi(\zeta) \in D$ if $\delta < c$ and $|\zeta| < c$. Let $m = \operatorname{ord}_0 \varphi$ and $X = \frac{\varphi^{(m)}(0)}{m!}$. Denoting by $\kappa_D^{(m)}$ the Kobayashi metric of order m (cf. [34] for this notion), it follows that $\kappa_D^{(m)}(\delta_n; X) \leq 1/c$. Since $\gamma_D \leq \kappa_D^{(m)}$, we get as in the proof of Proposition 3 (i) that $\Delta_X(a, c/4) \subset \partial D$. $\square$

Proof of Proposition 8. The proof can be done following line by line the proofs in [31]. We only point out how to replace the arguments there that use convexity. We may assume that D is a $\mathbb{C}$-convex domain and $a = 0$. Following the notation from Proposition 4, let $r(z) = Re(z_1) + o(|z_1|) + \rho('z)$ be a defining function of D which is smooth near 0.

page 841: Let $X, Y \subset \mathbb{C}^{n-1}$ be such that $\rho(\zeta X) \leq C|\zeta|^m$ and $\rho(\zeta Y) \leq C|\zeta|^m$. We have to show that $\rho(\zeta(X+Y)/2) \leq C|\zeta|^m$. For this, fix $\zeta \neq 0$ and take $\delta = C|\zeta^m|$. Then $\Delta_X(\delta_n, |\zeta|) \subset D$, $\Delta_Y(\delta_n, |\zeta|) \subset D$ and hence $\Delta_{(X+Y)/2}(\delta_n, |\zeta|) \subset D$ by Lemma 15. This implies the desired inequality.

We may do the same to get the formula (2.13) on page 845.

Our Proposition 6 is an extension of Theorem C which is invoked on page 845.

It remains to show Proposition 2 on page 843. Let $k_2, \dots, k_n$ be even integers such that $\rho(\zeta e_j) \leq C|\zeta|^{k_j}$ for any $j = 2, \dots, n$. It is enough to prove that $D^\alpha \overline{D}^\beta \rho(0) = 0$ for any n-tuples $\alpha = (\alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_2, \dots, \beta_n)$ of non-negative integers with $w_{\alpha,\beta} = \sum_{j=2}^n \frac{\alpha_j + \beta_j}{k_j} < 1$. Since $\Delta_{e_j}(C\delta_n, \delta^{1/k_j}) \subset D$ for any $\delta > 0$, it follows by Lemma 11 that $\rho('z/n) < C\delta$ for any z with $|z_j|^{k_j} < \delta$. In particular, if $\rho_t('z) = \rho(t^{1/k_2} z_2, \dots, t^{1/k_n} z_n)$, $t > 0$, then

$$0 \leq \rho_t('z/n) < Ct, \quad 'z \in \mathbb{D}^{n-1}. \tag{1}$$

Now let $s = \min\{w(\alpha, \beta) : D^\alpha \overline{D}^\beta \rho(0) \neq 0\}$. Then

$$\lim_{t \to 0} t^{-s} \rho_t('z) = \sum_{w(\alpha,\beta)=s} D^\alpha \overline{D}^\beta \rho(0) 'z^\alpha \overline{'z}^\beta$$

locally uniformly in $'z$. Assuming $s < 1$, the inequality 1 implies that the last polynomial vanishes, a contradiction. $\square$

Proof of Proposition 10. First let $L_a \neq \{0\}$. This means that $\Delta_X(a, r) \subset \partial D$ for some $r > 0$ and $X \in (\mathbb{C}^n)_*$. By convexity, $\Delta_X(c, r/2) \subset D$ for any $c = ta + (1-t)b$ if $b \in D$ and $t \in (0, 1/2]$. Now the maximum principle implies that a is not a peak point.

Now let $L_a = \{0\}$. We may assume that $a = 0$ and $D \subset \{z \in \mathbb{C}^n : \operatorname{Re}(z_1) < 0\}$. Then e^{z_1} is an entire weak peak function for $\overline{D}$ at 0. Setting $H = \{z \in \mathbb{C}^n : \operatorname{Re}(z_1) < 0\}$. It follows that this implies $\operatorname{supp}\mu \subset D_1 = \partial D \cap H$ for any representing measure μ for 0 w.r.t. $A(D)$. Since $L_0 = \{0\}$, it follows that 0 is a boundary point of the convex set D_1. Then there exists an entire function which is a weak peak function for D_1 at 0 (we need such a function to be in $A(D)$). We get as above that supp μ is contained in some $(n-2)$-dimensional space. Repeating this procedure, it follows that $\operatorname{supp}\mu \subset \partial D \cap l$, where l is a complex line. Since 0 is a boundary point of the last convex set, then there is an entire function which is a peak function for $\partial D \cap l$ at 0. So $\operatorname{supp}\mu = \{0\}$, i.e. 0 is a peak point w.r.t. $A(D)$ (cf. [10]). $\square$

Proof of Theorem 13. We first prove the lower bound. Fix $z^0 \in D$. Using a translation and then successive rotations we may assume (see the description of the numbers $d_{j,D}$) that $z^0 = 0$, $H_j = \{0\} \times \mathbb{C}^{n-j}$, $j = 1, \dots, n-1$, and $a^j = (0, a^j_j, 0) \in \mathbb{C}^{j-1} \times \mathbb{C} \times \mathbb{C}^{n-j}$ with $d_{j,D}(z^0) = |a^j_j|$.

Recall that D is $\mathbb{C}$-convex. Therefore, there exist affine hyperplanes $a^j + W_{j-1}$ through a^j which do not intersect D. Note that $W_1 \cap H_1$ is orthogonal to a^2, i.e. $W_1 \cap H_1 \subset \{0\} \times \mathbb{C}^{n-2}$. Hence W_1 is given by the equation $\alpha_{1,1} z_1 + z_2 = 0$. Moreover, using a similar argument, the equations for W_j, $j = 0, \dots, n-1$, are the following ones:

$$\alpha_{j,1} z_1 + \cdots + \alpha_{j,j} z_j + z_{j+1} = 0.$$

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be the linear mapping given by the matrix A whose rows are given by the vectors $(\alpha_{j,1}, \dots, \alpha_{j,j}, 1, 0, \dots, 0)$, $j = 0, \dots, n-1$. Define $G = F(D)$ and observe that G is again $\mathbb{C}$-convex. Note that $K_D(0) = K_G(0)$ since $\det A = 1$. Finally, put $G_j := \pi_j(G)$, where π_j is the projection onto the j-th coordinate axis. Then (see [1]) G_j is a simply connected domain, $j = 1, \dots, n$, and $G \subset G_1 \times \cdots \times G_n$. Hence

$$K_D(0) \geq K_{G_1 \times \cdots \times G_n}(0) = K_{G_1}(0) \cdots K_{G_n}(0).$$

Since G_j is simply connected, using the Koebe theorem we get

$$\sqrt{\pi K_{G_j}(0)} = \gamma_{G_j}(0; 1) \geq \frac{1}{4 d_{G_j}(0)}.$$

Note that $F(a^j) \in \partial G$, its j-th coordinate is a^j_j, and the affine hyperplane $\{z \in \mathbb{C}^n : z_j = a^j_j\}$ does not intersect G. Hence $a^j_j \in \partial G_j$; in particular, $d_{j,D}(z^0) = |a^j_j| \geq d_{G_j}(0)$, which finally gives the lower bound.

To show the upper bound, consider the dilatation of coordinates

$$\Phi(z) = (z_1 / d_{1,D}(z^0), \dots, z_n / d_{n,D}(z^0))$$

and set $\tilde{G} = \Phi(D)$. Hence

$$K_D(z^0) = \frac{K_{\tilde{G}}(0)}{p_D^2(z^0)}.$$

Then the upper bound follows from Lemma 15 and the following formula (cf. [16, 25]):

$$K_E(0) = \frac{(2n)!}{(2\pi)^n}. \qquad \square$$

Proof of Theorem 12. The proof can be done following line by line the proof of Theorem 2 in [26] and using Theorem 13 and Lemma 15. For convenience of the reader, we include a complete proof.

We shall use the geometric constellation in the proof of Theorem 13. Let $X \in (\mathbb{C}^n)_*$ and fix a $k \in J := \{j : X_j \neq 0\}$. Then

$$\Psi_k(z) := (z_1 - \frac{X_1}{X_k} z_k, \dots, z_{k-1} - \frac{X_{k-1}}{X_k} z_k, z_k, z_{k+1} - \frac{X_{k+1}}{X_k} z_k, \dots, z_n - \frac{X_n}{X_k} z_k)$$

is a linear mapping with Jacobian equal to 1 and $Y^k := \Psi_k(X) = (0, \dots, 0, X_k, 0, \dots, 0)$. Let Δ_j be the disc in the j-th coordinate plane with center at 0 and radius

$d_{j,D}(0)$ if $j \neq k$, and $d'_k := |X_k| d_D(0,X)$ if $j = k$. Then $\Delta_j \subset D_k := \Psi_k(D)$ and, by Lemma 15,

$$D_k \supset E_k = \{z \in \mathbb{C}^n : \frac{|z_k|}{d'_k} + \sum_{j=1, j\neq k}^{n} \frac{|z_j|}{d_j} < 1\}.$$

Hence

$$M_D(0;X) = M_{D_k}(0;Y^k) \leq M_{E_k}(0;Y^k) = C \frac{d_{k,D}(0)}{|X_k| p_D(0) d_D^2(0,X)},$$

where $C_n := M_E(0;e_1) = \sqrt{\frac{(2(n+1))!}{6(2\pi)^n}}$ (cf. [26]) and e_1 is the first basis vector. Applying the lower bound in Theorem 13, we obtain that

$$B_D(0;X) = \frac{M_D(0;X)}{\sqrt{K_D(0)}} \leq \frac{c'_n d_{k,D}(0)}{|X_k| d_D^2(0,X)}, \quad 1 \leq k \leq n, \tag{2}$$

where $c'_n = (4\sqrt{\pi})^n C_n = 2^n \sqrt{\frac{2^{n-1}(2(n+1))!}{3}}$. It remains to apply Lemma 15 to get that

$$\frac{1}{d_D(0,X)} \leq \sum_j^n \frac{|X_j(z)|}{d_{j,D}(z)} \tag{3}$$

and then to choose $c_n = nc'_n$. □

Proof of Proposition 14. It follows by (2) and the inequality

$$B_D(z;X) \geq \frac{1}{4 d_D(z,X)}$$

that

$$\frac{|X_j(z)|}{d_{j,D}(z)} \leq \frac{4c'_n}{d_D(z)}.$$

Hence,

$$\frac{1}{d_D(z,X)} \leq \sum_j^n \frac{|X_j(z)|}{d_{j,D}(z)} \leq \frac{4c_n}{d_D(z,X)}, \tag{4}$$

where $c_n = nc'_n$. Then (2) and (3) imply that

$$(16c_n)^{-1} \leq F_D(z;X) \left(\sum_j^n \frac{|X_j(z)|}{d_{j,D}(z)} \right)^{-1} \leq c_n. \quad \square$$

Remark. Assume that a domain $D \subset \mathbb{C}^n$ is smooth and weakly linearly convex near a boundary point a of finite type m. Then the quotient $\frac{(d_{j,D})^m}{d_D}$ is bounded from above near a for $j = 1, \ldots, n$ (see [7]). Since a is a local peak point for D at a, it follows that there is a neighborhood U of a and a constant $c > 0$ such that

$$\kappa_D(z;X) \geq \frac{c||X||}{(d_D(z))^{1/m}}, \quad z \in D \cap U;$$

the same estimate holds for B_D if D is pseudoconvex (not necessarily bounded - use e.g. localization results in [24]).

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