

THE RING OF REGULAR FUNCTIONS OF AN ALGEBRAIC MONOID

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ABSTRACT. Let M be an irreducible normal algebraic monoid with unit group G . It is known that G admits a Rosenlicht decomposition, $G = G_{\text{ant}}G_{\text{aff}} \cong (G_{\text{ant}} \times G_{\text{aff}})/G_{\text{aff}} \cap G_{\text{ant}}$, where G_{ant} is the maximal anti-affine subgroup of G , and G_{aff} the maximal normal connected affine subgroup of G . In this paper we show that this decomposition extends to a decomposition $M = G_{\text{ant}}M_{\text{aff}} \cong G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} M_{\text{aff}}$, where M_{aff} is the affine submonoid $M_{\text{aff}} = \overline{G_{\text{aff}}}$. We then use this decomposition to calculate $\mathcal{O}(M)$ in terms of $\mathcal{O}(M_{\text{aff}})$ and $G_{\text{aff}}, G_{\text{ant}} \subset G$. In particular, we determine when M is an anti-affine monoid, that is $\mathcal{O}(M) = \mathbb{k}$.

1. INTRODUCTION

The theory of *affine* algebraic monoids has been investigated extensively over the last thirty years. See [11, 12, 18] for different accounts of these developments. More recently there has been some important progress on the structure of non-affine algebraic monoids. By generalizing a classical theorem of Chevalley, the authors of [6] prove that any normal algebraic monoid is an extension of an affine algebraic monoid by an abelian variety. This allows one to analyze the structure of such monoids in terms of more basic objects: affine monoids, abelian varieties and anti-affine algebraic groups.

To state our results we first introduce some notation. Let \mathbb{k} be an algebraically closed field. We work with algebraic varieties X over \mathbb{k} , that is, integral, separated schemes over \mathbb{k} . An algebraic group is assumed to be a smooth group scheme of finite type over \mathbb{k} . If X is an algebraic variety we denote by $\mathcal{O}(X)$ the ring of regular functions on X . If X is an affine variety and $I \subset \mathcal{O}(X)$ is an ideal, we denote by $\mathcal{V}(I) = \{x \in X : f(x) = 0 \forall f \in I\}$; if $Y \subset X$ is a subset, we denote by $\mathcal{I}(Y) = \{f \in \mathcal{O}(X) : f(y) = 0 \forall y \in Y\}$. If X is irreducible we denote by $\mathbb{k}(X)$ the field of rational functions on X . If A is any integral domain we denote by $[A]$ its quotient field. Hence, if X is an irreducible affine variety, then $\mathbb{k}(X) = [\mathcal{O}(X)]$.

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Let M be a connected, normal, *algebraic monoid* with unit group G (see Definition 2.1 below). The original motivation for this paper was to investigate the following basic question, first posed by M. Brion:

“How does one describe $\mathcal{O}(M)$, and when is it finitely generated?”

Although we do not answer this question completely, we obtain many remarkable results about $\mathcal{O}(M)$. Surprisingly, we find that $\mathcal{O}(M)$ is finitely generated if the characteristic of \mathbb{k} is positive, but this remains an open question in characteristic zero.

Let M and G be as above. By the results of [6], if $\alpha_G : G \rightarrow A$ is the Albanese morphism of G such that $\alpha_G(1_G) = 0_A$ (note the additive notation for A), then there exists a unique morphism $\alpha_M : M \rightarrow A$ such that $\alpha_M|_G = \alpha_G$. Furthermore, α_M is an affine morphism, and the scheme-theoretic fibers of α_M are normal varieties. The fiber at $1 \in A$ is M_{aff} , the unique irreducible, affine submonoid of M with unit group G_{aff} , the kernel of α_M . See Theorem 2.2 below.

The purpose of this paper is three-fold. First we identify $\mathcal{O}(M)$ in terms of the structure of M and M_{aff} ; see Theorem 3.1. We then identify conditions under which $[\mathcal{O}(M)] = [\mathcal{O}(G)]$; see Theorem 3.16. Finally, we determine the conditions under which $\mathcal{O}(M) = \mathbb{k}$ (that is when M is an *anti-affine* algebraic monoid, Theorem 3.19). In order to establish our results we define the notion of a *stable* algebraic monoid (see Definition 3.10).

To obtain our main results we make use of the generalized Chevalley decomposition presented in [6], which states that, if M is an irreducible monoid, then

$$M \cong G *_{G_{\text{aff}}} M_{\text{aff}},$$

where G_{aff} is the smallest affine algebraic group such that G/G_{aff} is an abelian variety (see Theorem 2.2 below). This structural result allows us to present a Rosenlicht decomposition $M = G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} M_{\text{aff}}$ that generalizes the corresponding decomposition $G = (G_{\text{ant}} \times G_{\text{aff}})/(G_{\text{aff}} \cap G_{\text{ant}})$ of G , where G_{ant} is the largest *anti-affine* subgroup of G . See [5] and Proposition 2.10 below. We then use this decomposition (of M) in Theorem 3.1 to calculate $\mathcal{O}(M)$ in terms of $\mathcal{O}(M_{\text{aff}})$ and $G_{\text{aff}} \cap G_{\text{ant}}$.

Next we identify a key central idempotent e of M such that $\mathcal{O}(M) = \mathcal{O}(eM)$ and such that eM is a *stable* monoid. Consequently, we reduce ourselves to the study of stable monoids, thereby obtaining a characterization of the algebraic monoids M such that $\mathcal{O}(M) = \mathbb{k}$, the *anti-affine* algebraic monoids. See Theorem 3.19.

Let M be an anti-affine algebraic monoid and let $e \in E(M)$ be the minimum idempotent of M . In Theorem 3.21 we show that the retraction $\ell_e : M \rightarrow eM$, $\ell_e(m) = em$, is Serre’s universal morphism from M to a commutative algebraic group (see [17, Thm. 8]).

We conclude the paper with Theorem 3.22. Here we show that there is an analogue of the Rosenlicht decomposition for a large class of normal, algebraic monoids. In particular the fiber $\varphi^{-1}(1)$, of the canonical map $\varphi : M \rightarrow \text{Spec}(\mathcal{O}(M))$, is an anti-affine monoid which we identify explicitly in terms of the internal structure of M .

2. PRELIMINARIES

In this section we assemble some of what is known about algebraic monoids with nonlinear unit groups. These results are due to M. Brion and the second-named author [3, 6, 4, 15].

Definition 2.1. An *algebraic monoid* is an algebraic variety M together with a morphism $m : M \times M \rightarrow M$ such that m is an associative product and there exists a neutral element $1 \in M$. The *unit group* of M is the group of invertible elements

$$G(M) = \{g \in M : \exists g^{-1}, gg^{-1} = g^{-1}g = 1\}.$$

We denote the *set of idempotent elements* by $E(M) = \{e \in M : e^2 = e\}$.

It has been proved that $G(M)$ is an algebraic group, open in M (see for example [14]). The structure of M is significantly influenced by the structure of $G(M)$. For instance, the $G(M) \times G(M)$ action by left and right multiplication, $(a, b) \cdot m = amb^{-1}$, has $G(M)$ as open orbit and, if $G(M)$ is dense in M , a unique closed orbit, which is the *Kernel* of M , i.e. the minimum, closed, nonempty subset I such that $MIM \subset I$.

Recall that if G is an algebraic group, then the Albanese morphism $p : G \rightarrow \mathcal{A}(G)$ fits into an exact sequence

$$1 \longrightarrow G_{\text{aff}} \longrightarrow G \xrightarrow{p} \mathcal{A}(G) \longrightarrow 0,$$

where G_{aff} is a normal connected affine algebraic group (since the group $\mathcal{A}(G)$ is commutative, its law will be denoted additively). Moreover, G_{aff} is the smallest affine algebraic subgroup such that G/G_{aff} is an abelian variety. This structure theorem is originally due to Chevalley, but now there is a modern proof in [7]. Recently it has been generalized from groups to monoids. The following theorem is a summary of this development.

Theorem 2.2 (Brion, Rittatore [3, 6, 4, 15]). *Let M be a normal irreducible algebraic monoid with unit group G . Then M admits a Chevalley decomposition:*

$$\begin{array}{ccccccc} 1 & \longrightarrow & M_{\text{aff}} = \overline{G_{\text{aff}}} & \longrightarrow & M = \overline{G} & \xrightarrow{p} & \mathcal{A}(G) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & G_{\text{aff}} & \longrightarrow & G & \xrightarrow{p} & \mathcal{A}(G) \longrightarrow 0, \end{array}$$

where $p : M \rightarrow \mathcal{A}(G) = G/G_{\text{aff}}$ and $p|_G : G \rightarrow \mathcal{A}(G)$ are the Albanese morphisms of M and G , respectively, and M_{aff} is an affine algebraic monoid. \square

If M is an irreducible, normal algebraic monoid with unit group G let Z^0 be the connected center of G . Let $Z^0 \cap G_{\text{aff}} \subseteq G$ be the scheme-theoretic intersection. $Z^0 \cap G_{\text{aff}}$ acts on $Z^0 \times M_{\text{aff}}$ by inverse followed by right multiplication on the first factor, and left translation on the second factor. Denote by $Z^0 *_{Z^0 \cap G_{\text{aff}}} M_{\text{aff}}$ the geometric quotient of this action. $G *_{G_{\text{aff}}} M_{\text{aff}}$ is defined similarly.

The following result is a consequence of Lemmas 3.1 and 3.5 of [6].

Theorem 2.3. *Let M be an irreducible, normal algebraic monoid and let Z^0 be the connected center of G . Then $\mathcal{A}(G) \cong Z^0 / (Z^0 \cap G_{\text{aff}})$ and*

$$M = GM_{\text{aff}} = Z^0 M_{\text{aff}} \cong G *_{G_{\text{aff}}} M_{\text{aff}} \cong Z^0 *_{Z^0 \cap G_{\text{aff}}} M_{\text{aff}}.$$

Definition 2.4. If M is an algebraic monoid with unit group, we define the *center of M* to be

$$\mathcal{Z}(M) = \{z \in M : zm = mz \ \forall m \in M\},$$

the set of central elements.

It is clear that $\mathcal{Z}(M)$ is a closed submonoid of M , with unit group $G(\mathcal{Z}(M)) = \mathcal{Z}(G)$, the center of G . However, one should be aware that this monoid is not necessarily connected. Moreover, the following example shows that the $\mathcal{Z}(G)$ is not necessarily dense in $\mathcal{Z}(M)$.

Example 2.5. Let $r, s \in \mathbb{N}$, $r \neq s$, and consider the affine algebraic monoid

$$N = \left\{ \begin{pmatrix} t^r & a \\ 0 & t^s \end{pmatrix} : t, a \in \mathbb{k} \right\}.$$

Then $G(N) = \left\{ \begin{pmatrix} t^r & a \\ 0 & t^s \end{pmatrix} : t \in \mathbb{k}^*, a \in \mathbb{k} \right\}$. The center of $G(N)$ is the finite subgroup $\mathcal{Z}(G) = \left\{ \begin{pmatrix} t^r & 0 \\ 0 & t^s \end{pmatrix} : t^r = t^s \right\}$. The center of N is $\mathcal{Z}(N) = \mathcal{Z}(G) \cup \{0\}$. However $\overline{\mathcal{Z}(G)} \neq \mathcal{Z}(N)$.

In particular the zero matrix is a central idempotent which does not belong to $\overline{\mathcal{Z}(G)}$.

In what follows we collect some results about the *Rosenlicht decomposition* of an algebraic group G . This decomposition depicts G as the product of two subgroups, one affine and the other anti-affine. We refer the reader to [16], [8, Sec. III.3.8] and [5] for proofs and further results about this decomposition.

Definition 2.6. A connected algebraic group G is *anti-affine* if $\mathcal{O}(G) = \mathbb{k}$.

Remark 2.7. It is easy to see that any anti-affine group is commutative. See for example [5, Lem. 1.1].

Theorem 2.8 (Rosenlicht decomposition). *Let G be a connected algebraic group. Then $\mathcal{O}(G)$ is a finitely generated algebra, in such a way that $\text{Spec}(\mathcal{O}(G))$ is an affine algebraic group.*

Let $\varphi_G : G \rightarrow \text{Spec}(\mathcal{O}(G))$ be the canonical morphism (the affinization). Then $G_{\text{ant}} = \text{Ker}(\varphi_G)$ is a connected subgroup, contained in the center of G . The subgroup G_{ant} is the largest anti-affine subgroup scheme of G . Equivalently, G_{ant} is the smallest normal subgroup scheme of G such that G/G_{ant} is affine.

Moreover, $G = G_{\text{aff}}G_{\text{ant}} \cong (G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{aff}} \cap G_{\text{ant}})$, and $G_{\text{aff}} \cap G_{\text{ant}}$ contains $(G_{\text{ant}})_{\text{aff}}$ as an algebraic group of finite index. \square

In loose terms, an anti-affine algebraic group G is a nonsplit extension of an abelian variety by an affine, commutative algebraic group. See [5, Thm. 2.7]. But in the case $\text{char}(\mathbb{k}) = p > 0$ the situation is considerably less complicated. Indeed we have the following simplifying result. See [5, Prop. 2.2].

Proposition 2.9. *Let G be an anti-affine, connected algebraic group over an algebraically closed field \mathbb{k} . If $\text{char}(\mathbb{k}) = p > 0$, then G is a semi-abelian variety; i.e. G is the extension of an abelian variety by an affine torus group*

$$1 \longrightarrow T \longrightarrow G \xrightarrow{p} \mathcal{A}(G) \longrightarrow 0.$$

The following result generalizes Rosenlicht's decomposition to the case of algebraic monoids. It is essential for determining the ring of regular functions on an algebraic monoid.

Proposition 2.10 (Rosenlicht decomposition for M). *Let M be a normal irreducible algebraic monoid with unit group G . Then*

$$M = G_{\text{ant}}M_{\text{aff}} \cong G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} M_{\text{aff}}.$$

Proof. Since $G = G_{\text{ant}}G_{\text{aff}}$, it follows that $M = GM_{\text{aff}} = G_{\text{ant}}M_{\text{aff}}$. The morphism $G_{\text{ant}} \times M_{\text{aff}} \rightarrow M$ ($g, m \mapsto gm$) induces a surjective morphism of algebraic monoids $\varphi : G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} M_{\text{aff}} \rightarrow M$, $\varphi([g, m]) = gm$. If $a, b \in G_{\text{ant}}$ and $m, n \in M_{\text{aff}}$ are such that $am = bn$, then $b^{-1}am = n$. It follows from Theorem 2.3 that $b^{-1}a \in G_{\text{aff}} \cap G_{\text{ant}}$, and thus $[a, m] = [b, n]$; that is, φ is injective. Since the restriction $\varphi : G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} G_{\text{aff}} \rightarrow G$ is an isomorphism, and noting that M is normal, it follows from Zariski's Main Theorem that φ is an isomorphism. \square

3. THE ALGEBRA OF REGULAR FUNCTIONS OF M

The following theorem is the key to understanding the ring of regular functions on an algebraic monoid.

Theorem 3.1. *Let M be a normal algebraic monoid. Then*

$$\mathcal{O}(M) \cong \mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}.$$

Proof. By Proposition 2.10, it follows that $M \cong G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} M_{\text{aff}}$, and hence $G_{\text{ant}} \times M_{\text{aff}} \rightarrow M$ is a geometric quotient. Thus

$$\begin{aligned} \mathcal{O}(M) &= \mathcal{O}(G_{\text{ant}} \times M_{\text{aff}})^{G_{\text{ant}} \cap G_{\text{aff}}} \\ &= (\mathcal{O}(G_{\text{ant}}) \otimes \mathcal{O}(M_{\text{aff}}))^{G_{\text{ant}} \cap G_{\text{aff}}} \\ &= \mathcal{O}(M_{\text{aff}})^{G_{\text{ant}} \cap G_{\text{aff}}}, \end{aligned}$$

where for the last equality we used that $\mathcal{O}(G_{\text{ant}}) = \mathbb{k}$. \square

Corollary 3.2. *Assume that $\text{char}(\mathbb{k}) = p > 0$. Then $\mathcal{O}(M)$ is a finitely generated algebra.*

Proof. Indeed, by Proposition 2.9, it follows that $(G_{\text{aff}} \cap G_{\text{ant}})^0$, the connected component of the identity, is a torus, and hence $\mathcal{O}(M) = \mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}$ is a finitely generated algebra. \square

Lemma 3.3. *Let $\phi : G \rightarrow H$ be a morphism of algebraic groups. Then $\phi(G_{\text{aff}}) \subseteq H_{\text{aff}}$ and $\phi(G_{\text{ant}}) \subseteq H_{\text{ant}}$. Furthermore these restriction maps are surjective provided ϕ is surjective.*

Proof. It follows from Lemma 1.5 of [5] that $\phi(G_{\text{ant}}) \subseteq H_{\text{ant}}$. On the other hand, if $\pi : H \rightarrow \mathcal{A}(H)$ is Chevalley's Albanese map, then the composite morphism, $\pi \circ \phi : G_{\text{aff}} \rightarrow \mathcal{A}(H)$, is trivial so that $\phi(G_{\text{aff}}) \subseteq H_{\text{aff}}$.

Now $H/\phi(G_{\text{ant}})$ is affine, as it is the image of G_{aff} . Thus $H_{\text{ant}} \subseteq \phi(G_{\text{ant}})$ since H_{ant} is the smallest normal subgroup with the property that H/H_{ant} is affine.

Finally let $K = \phi^{-1}(H_{\text{aff}})^0$. Then $K_{\text{aff}} = G_{\text{aff}}$, while $\phi(K_{\text{ant}})$ is the trivial subgroup of H_{aff} . Thus $\phi(G_{\text{aff}}) = \phi(K) = H_{\text{aff}}$. \square

Theorem 3.4. *Let M be a normal algebraic monoid and let $e \in E(M)$ be a central idempotent. Let $M_e = \{m \in M : me = e\}$ and let $G_e = \{g \in G(M) : ge = e\}$. Then*

(1) $M_e = \overline{G_e}$, and M_e is an irreducible algebraic monoid, with unit group $G_e \subset G_{\text{aff}}$. In particular, G_e and M_e are affine and $e \in M_{\text{aff}}$.

(2) The subset $eM \subset M$ is an algebraic monoid, closed in M , with identity element e . The morphism $\ell_e : M \rightarrow eM$, $m \mapsto em$, is a morphism of algebraic monoids, with $M_e = \ell_e^{-1}(1)$.

(3) The unit group of eM equals $G(eM) = eG$. Moreover, $(eG)_{\text{aff}} = e(G_{\text{aff}})$ and $(eG)_{\text{ant}} = e(G_{\text{ant}})$. In particular, the Chevalley decompositions of eM and $eG = G(eM)$ fit into the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & eM_{\text{aff}} & \longrightarrow & eM & \xrightarrow{p} & \mathcal{A}(G) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & eG_{\text{aff}} & \longrightarrow & eG & \xrightarrow{p} & \mathcal{A}(G) \longrightarrow 0. \end{array}$$

Furthermore $eM_{\text{aff}} = \overline{eG_{\text{aff}}} = \overline{eG_{\text{aff}}}$, and $eM = \overline{eG}$.

$$(4) eM = G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} eM_{\text{aff}} \cong G_{\text{ant}} *_{eG_{\text{aff}} \cap eG_{\text{ant}}} eM_{\text{aff}} \cong eG_{\text{ant}} *_{eG_{\text{aff}} \cap eG_{\text{ant}}} eM_{\text{aff}}.$$

Proof. (1) Since M is normal at e , it follows from [4, Corollary 2.2.5] (taking into account [4, Remark 3.1.3]) that M_e is an irreducible algebraic monoid, with unit group G_e . Since $M \cong G *_{G_{\text{aff}}} M_{\text{aff}}$, it follows that if $ge = e$, then $[g, e] = [1, e]$, and hence $g \in G_{\text{aff}}$, i.e. $G_e \subseteq G_{\text{aff}}$. That $e \in M_{\text{aff}}$ follows from [6, Corollary 2.4].

(2) Since $eM = \{x \in M : xe = x\}$, it is clear that eM is a closed subset. Hence, eM is an algebraic monoid and $\ell_e : M \rightarrow eM$, $\ell_e(m) = em$, is a morphism of algebraic monoids.

(3) Since $\ell_e : M \rightarrow eM$ is a surjective morphism of algebraic monoids, it follows that $G(eM) = eG$. In particular, $G \rightarrow eG$ is a surjective morphism of algebraic groups. Thus, by Lemma 3.3, $(eG)_{\text{aff}} = eG_{\text{aff}}$ and $(eG)_{\text{ant}} = eG_{\text{ant}}$. In particular eG/eG_{aff} is an abelian variety.

Now $eG \cong G/G_e$ while $G_e \subseteq G_{\text{aff}}$. Thus $\mathcal{A}(eG) = eG/(eG_{\text{aff}}) \cong G/G_{\text{aff}} = \mathcal{A}(G)$, with the Albanese morphism fitting into the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\ell_e} & eG \\ \alpha_G \downarrow & & \downarrow \alpha_{eG} \\ \mathcal{A}(G) = G/G_{\text{aff}} & \xleftarrow{\varphi} & \mathcal{A}(eG) = eG/(eG_{\text{aff}}), \end{array}$$

where $\varphi : eG/(eG_{\text{aff}}) \cong (G/G_e)/(G/G_e)_{\text{aff}} \rightarrow G/G_{\text{aff}}$ is the canonical isomorphism obtained by observing that $(G/G_e)_{\text{aff}} = G_{\text{aff}}/G_e$.

(4) follows from the description above and Proposition 2.10. \square

Remark 3.5. The reader should notice that

$$E(\overline{G_{\text{ant}} \cap G_{\text{aff}}}) = E(\overline{G_{\text{ant}}}).$$

Indeed, it follows from [5, Prop. 3.1] that $(G_{\text{ant}})_{\text{aff}} \subset G_{\text{aff}} \cap G_{\text{ant}}$. Since $E(N) \subset N_{\text{aff}}$ for any algebraic monoid ([6, Cor. 2.4]), it follows that $E(\overline{G_{\text{ant}}}) \subset E(\overline{G_{\text{ant}} \cap G_{\text{aff}}})$; the other inclusion being obvious.

Remark 3.6. Let M be a normal affine algebraic monoid and let $e \in E(M)$ be a central idempotent. Then $\ell_e : M \rightarrow M$, $\ell_e(m) = em$, is such that $\ell_e \circ \ell_e = \ell_e$. In other words, $\ell_e^* : \mathcal{O}(eM) \rightarrow \mathcal{O}(M)$ is a section for the canonical surjection $\mathcal{O}(M) \rightarrow \mathcal{O}(eM)$.

Corollary 3.7. *Let M be a normal algebraic monoid and let $e \in E(M)$ be a central idempotent. Then $\mathcal{O}(M) = \mathcal{O}(eM) \oplus \mathcal{I}(eM)$. In particular, if $e \in E(\overline{G_{\text{ant}}})$, then $\mathcal{O}(M) = \mathcal{O}(eM)$.*

Proof. The direct sum decomposition $\mathcal{O}(M) = \mathcal{O}(eM) \oplus \mathcal{I}(eM)$ is a direct consequence of Remark 3.6.

Now assume that $e \in E(\overline{G_{\text{ant}}})$ and let $f \in \mathcal{O}(M)$. If $x \in M$, then, by Proposition 2.10 and Theorem 3.1, $f(x) = f(g \cdot x)$ for all $g \in G_{\text{ant}}$. It follows from Remark 3.5 that $f(x) = f(ex)$, since $e \in \overline{G_{\text{ant}} \cap G_{\text{aff}}}$. In particular, if $f \in \mathcal{I}(eM)$, then $f(x) = f(ex) = 0$ for all $x \in M$. Thus $f \equiv 0$. \square

Definition 3.8. An algebraic monoid M is *anti-affine* if $\mathcal{O}(M) = \mathbb{k}$.

Let M be an algebraic monoid such that $G(M)$ is an anti-affine algebraic group. Then M is anti-affine. The converse is not true, as the following example shows.

Example 3.9. Let $T = \mathbb{k}^* \times \mathbb{k}^*$ be an algebraic torus of dimension 2, and consider the affine toric variety $T \subset \mathbb{A}^2$. Then \mathbb{A}^2 is an affine algebraic monoid with unit group T . Let A be a nontrivial connected abelian variety and consider an extension

$$0 \longrightarrow \mathbb{k}^* = T_1 \longrightarrow H \longrightarrow A \longrightarrow 0$$

of algebraic groups such that H is anti-affine. Let $G = (T \times H)/T_1$, where $T_1 \hookrightarrow T \times H$, $t \mapsto ((t, t), t)$. Then $G_{\text{aff}} = T$, $G_{\text{ant}} = H$, and $G_{\text{aff}} \cap G_{\text{ant}} = T_1 = \{(t, t) : t \in \mathbb{k}^*\} \subset T$.

The quotient $M = (\mathbb{A}^2 \times H)/T_1$ is an algebraic monoid with unit group G and with $M_{\text{aff}} \cong \mathbb{A}^2$. Thus

$$\mathcal{O}(M) = \mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}} = \mathbb{k}[x, y]^{T_1} = \mathbb{k}.$$

Hence, M is an anti-affine algebraic monoid while $G(M)$ is not an anti-affine algebraic group.

Definition 3.10. Let G be an algebraic group and let X be a G -variety. We say that the action is *generically stable* (equivalently that X is a *generically stable G -variety*) if there exists an open subset consisting of closed orbits. We say that an algebraic monoid M is *stable* if it is generically stable as a G_{ant} -variety. This is equivalent to M_{aff} being generically stable as a $(G_{\text{aff}} \cap G_{\text{ant}})$ -variety.

Remark 3.11. (1) Recall that any regular action is such that there exists an open subset of orbits of maximal dimension. Hence, an action $G \times X \rightarrow X$ is generically stable if and only if the set of orbits of maximal dimension contains an open subset consisting of closed orbits.

(2) If M is an algebraic monoid, then M is stable if and only if G_{ant} is closed in M . Indeed, the coset gG_{ant} , $g \in G$, is closed in M if and only if G_{ant} is closed in M .

Definition 3.12. Let G be an affine algebraic group acting on an affine variety X . We say that the action is *observable* if for every nonzero G -stable ideal $I \subset \mathcal{O}(X)$, $I^G \neq (0)$. Here we consider the induced action $G \times \mathcal{O}(X) \rightarrow \mathcal{O}(X)$, $(a \cdot f)(x) = f(a^{-1}x)$, for all $a \in G$, $x \in X$, $f \in \mathcal{O}(X)$.

The concept of an observable action is a generalization of the notion of an observable subgroup. Observable subgroups were introduced by Bialynicki-Birula, Hochschild and Mostow in [2] and have been researched extensively since then, notably by F. Grosshans (see [10] for a survey on this topic). Given an affine algebraic group G , a closed subgroup $H \subset G$ is said to be *observable* if G/H is a quasi-affine

algebraic variety. The equivalent definition of H being observable if every nonzero H -stable ideal $I \subset \mathcal{O}(G)$ has the property that $I^H \neq (0)$, was first recorded in [9], and then further generalized in [13]. We present here some of the basic results that we need in what follows. We include some of the proofs here for convenience.

Theorem 3.13. *Let G be an affine group acting on an affine variety X . Then the action is observable if and only if (1) $[\mathcal{O}(X)]^G = [\mathcal{O}(X)^G]$ and (2) the action is generically stable.*

Proof. We will only prove that if the action is observable, then conditions (1) and (2) hold. We refer the reader to [13, Thm. 3.10] for a complete proof.

Clearly $[\mathcal{O}(X)^G] \subset [\mathcal{O}(X)]^G$. Let $g \in [\mathcal{O}(X)]^G$, and consider the ideal $I = \{f \in \mathcal{O}(X) : fg \in \mathcal{O}(X)\}$. Clearly I is G -invariant, and hence there exists $f \in \mathcal{O}(X)^G \setminus \{0\}$ such that $fg \in \mathcal{O}(X)^G$.

Now let X_{\max} be the (open) subset of orbits of maximal dimension and let $Y = X \setminus X_{\max}$. Let $0 \neq f \in \mathcal{I}(Y)^G$. Then the affine open subset X_f is a G -stable subset contained in X_{\max} . Let $\mathcal{O} \subset X_f$ be an orbit. Since every orbit is open in its closure, it follows that if $\overline{\mathcal{O}} \neq \mathcal{O}$, then $\overline{\mathcal{O}} \cap Y \neq \emptyset$. Since f is constant on the orbits, it follows that $f|_{\mathcal{O}} = 0$ and hence $\mathcal{O} \subset Y$. That is a contradiction. \square

In [13] the reader can find examples showing that both conditions (1) and (2) of Theorem 3.13 are necessary. However, for some families of algebraic groups, generic stability of the action implies observability. This is the case when G is a reductive group (see [13, Thm. 4.7]), or when $G = U \times L$, where L is reductive and U is unipotent.

Proposition 3.14. *Let G be an affine algebraic group with Levi decomposition $G = L \times U$, as above. Assume that X is a generically stable affine G -variety. Then the action is observable.*

Proof. Let $I \subsetneq \mathcal{O}(X)$ be a nonzero G -stable ideal. Then $\mathcal{V}(I) \neq \emptyset$ is a proper closed subset. Since there exists an open subset of closed orbits, then there exists a closed orbit Z such that $Z \cap \mathcal{V}(I) = \emptyset$. Since L is reductive and Z is L -stable, it follows that there exists $f \in \mathcal{O}(X)^L$ such that $f \in I$, $f|_Z = 1$. Hence, $I^L \neq (0)$. Since U normalizes L , it follows that $I \cap \mathcal{O}(X)^L \neq \{0\}$ is a U -submodule, and hence $I^G = (I^L)^U \neq (0)$. \square

Proposition 3.15. *Let M be a stable algebraic monoid, with $G(M)$ an anti-affine algebraic group. Then $M = G(M)$.*

Proof. This is obvious from the definition. \square

Theorem 3.16. *Let M be a stable normal algebraic monoid. Then $[\mathcal{O}(M)] = [\mathcal{O}(G)]$.*

Proof. Theorem 3.1 guarantees that $[\mathcal{O}(M)] = [\mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}]$. Since M is stable, then, by Proposition 3.14 and Theorem 3.13, it follows that $[\mathcal{O}(M)] = [\mathcal{O}(M_{\text{aff}})]^{G_{\text{aff}} \cap G_{\text{ant}}}$. But M_{aff} and G_{aff} being affine varieties, it follows that $[\mathcal{O}(M)] = [\mathcal{O}(G_{\text{aff}})]^{G_{\text{aff}} \cap G_{\text{ant}}}$. Now, applying the same reasoning to the algebraic group G , we obtain that $[\mathcal{O}(M)] = [\mathcal{O}(G_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}] = [\mathcal{O}(G)]$. \square

Corollary 3.17. *Let M be a stable monoid. If M is anti-affine, then M is an (anti-affine) algebraic group.*

Proof. By Theorem 3.16, it follows that $G = G(M)$ is an anti-affine algebraic group. Hence, by Proposition 3.15, it follows that $M = G$. \square

Proposition 3.18. *Let M be a normal algebraic monoid and let $e \in E(\overline{G_{\text{ant}}})$ be the minimum idempotent. Then eM is a stable monoid.*

Proof. It follows from Lemma 3.3 that $eG_{\text{ant}} = (eG)_{\text{ant}}$. But $eG_{\text{ant}} \subseteq \overline{G_{\text{ant}}}$ since $e \in \overline{G_{\text{ant}}}$. Thus, by our choice of e , $E(\overline{eG_{\text{ant}}}) = \{e\}$. Thus $E((\overline{eG_{\text{ant}}})_{\text{aff}}) = \{e\}$ as well, and consequently $\overline{eG_{\text{ant}}}$ is a group. We conclude that $eG_{\text{ant}} = eG_{\text{ant}}$. Thus, by part (2) of Remark 3.11, eM is stable. \square

We now characterize normal, anti-affine, algebraic monoids.

Theorem 3.19. *Let M be a normal algebraic monoid. Then M is anti-affine if and only if eM is an anti-affine algebraic group, where e is the minimum idempotent of $\overline{G_{\text{ant}}}$. In particular, e is the minimum idempotent of M .*

Proof. First of all, we recall that $E(\overline{G_{\text{aff}} \cap G_{\text{ant}}}) = E(\overline{G_{\text{ant}}})$ (see Remark 3.5) and that $\mathcal{O}(M) = \mathcal{O}(eM)$ (see Corollary 3.7). Since, by Proposition 3.18, eM is a stable algebraic monoid, it follows from Corollary 3.17 that $\mathcal{O}(eM) = \mathbb{k}$ if and only if eM is an anti-affine algebraic group.

Finally, since eM is a group, it follows that $E(eM) = \{e\}$, and hence $ef = e$ for $f \in E(M)$. \square

The following examples indicate why, in Theorem 3.19, one must focus on the idempotents of $E(\overline{G_{\text{ant}}})$, rather than just any central idempotent.

Example 3.20. (1) Let N be an affine monoid of dimension $\dim N \geq 1$ with zero element 0_N . Then 0_N is a central idempotent of N . Let H be any anti-affine algebraic group. Then $M = N \times H$ is such that $\mathcal{O}(M) = \mathcal{O}(N)$, while $0_N M = H$ is an anti-affine algebraic group. Observe that 0_N is the minimum idempotent of M .

Here, $M_{\text{aff}} = N \times H_{\text{aff}}$, $G(M)_{\text{aff}} = G(N) \times H_{\text{aff}}$ and $G(M)_{\text{ant}} = \{1\} \times H$. Hence $G(M)_{\text{aff}} \cap G(M)_{\text{ant}} = \{1\} \times H_{\text{aff}}$, and thus $0 \notin E(\overline{G(M)_{\text{aff}} \cap G(M)_{\text{ant}}})$.

(2) Let N be an irreducible affine monoid, with zero element 0_N , such that $0_N \notin \overline{\mathcal{Z}(G)}$ (take for example N as in example 2.5). Let M be an algebraic monoid such that $M_{\text{aff}} = N$. Then, $0_N M = 0_N G(M)_{\text{ant}} \cong G(M)_{\text{ant}} / (G(M)_{\text{ant}})_{0_N}$ is an anti-affine algebraic group (see for example [5, Lemma 1.3]), whereas $\mathcal{O}(M) = \mathcal{O}(N)^{G(N) \cap G(M)_{\text{ant}}}$ is not necessarily equal to the field \mathbb{k} .

We now show that if M is anti-affine, then $\ell_e : M \rightarrow eM$ is Serre's universal morphism from the pointed variety (M, e) into a commutative algebraic group. See [17, Thm. 8] and [5, §2.4] for some basic properties of this morphism.

Theorem 3.21. *Let M be a normal anti-affine algebraic monoid, and let $e \in E(M)$ be its minimum idempotent. Then $\ell_e : M \rightarrow eM$ is Serre's universal morphism from*

the pointed variety (M, e) into a commutative algebraic group. In particular, Serre's morphism fits into the following short exact sequence of algebraic monoids:

$$1 \longrightarrow M_e \longrightarrow M \longrightarrow eM \longrightarrow 1.$$

Proof. Let $\sigma : M \rightarrow S$ be Serre's universal morphism from the pointed variety (M, e) into a commutative algebraic group. Since M is anti-affine, it follows that S is necessarily an anti-affine algebraic group (see for example [5, §2.4]). Moreover, it follows from Theorem 3.19 that eM is an anti-affine algebraic group. Thus we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ell_e} & eM \\ \sigma \downarrow & \nearrow \varphi & \\ S. & & \end{array}$$

Since $\varphi(0_S) = e$, it follows that φ is a morphism of algebraic groups (see for example [5, Lem.1.5]). Consider the associated short exact sequence

$$(3.1) \quad 0 \longrightarrow N \longrightarrow S \xrightarrow{\varphi} eM \longrightarrow 0.$$

Since eM is anti-affine with $\sigma(e) = 0_S$, it follows that $\sigma|_{eM} : eM \rightarrow S$ is a morphism of algebraic groups. Moreover, we have the following commutative diagram:

$$\begin{array}{ccccc} eM & \longrightarrow & M & \xrightarrow{\sigma} & S \\ \alpha_{eM} \downarrow & & \downarrow \alpha_M & & \downarrow \alpha_S \\ \mathcal{A}(eM) = \mathcal{A}(G) & \longrightarrow & \mathcal{A}(M) = \mathcal{A}(G) & \xrightarrow{\alpha_\sigma} & \mathcal{A}(S). \end{array}$$

Let $\gamma : \mathcal{A}(eM) \rightarrow \mathcal{A}(S)$ be the composition of the horizontal arrows. Then γ is a surjective morphism of algebraic groups. In particular, $\dim \mathcal{A}(S) \leq \dim \mathcal{A}(eM)$.

On the other hand, $\varphi \circ \sigma = \ell_e : M \rightarrow eM$, and thus $\sigma|_{eM}$ is a splitting for the exact sequence (3.1). It follows that $S \cong eM \times N$, and we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & eM & \longrightarrow & S = eM \times N & \longrightarrow & N \longrightarrow 0 \\ & & \alpha_{eM} \downarrow & & \downarrow \alpha_S & & \downarrow \alpha_N \\ & & \mathcal{A}(eM) & \xrightarrow{\gamma} & \mathcal{A}(S) & \xrightarrow{\beta} & \mathcal{A}(N) \longrightarrow 0. \end{array}$$

Notice that γ is surjective, while $\beta\gamma = 0$. Thus $\mathcal{A}(N) = 0$. It follows that N is an irreducible affine algebraic group. Since $\mathbb{k} = \mathcal{O}(S) \cong \mathcal{O}(eM) \otimes \mathcal{O}(N) \cong \mathcal{O}(N)$, it follows that N is a point, and thus $S \cong eM$. \square

We conclude this paper by extending the Rosenlicht decomposition of algebraic groups ([5, Prop. 3.1]) to the setting of algebraic monoids.

Theorem 3.22. *Let M be a normal algebraic monoid, with unit group G , and let $e \in E(\overline{G_{\text{ant}}})$ be the minimum idempotent of $\overline{G_{\text{ant}}}$. Assume that $\mathcal{O}(M)$ is finitely*

generated. Then $M_{\text{ant}} = M_e G_{\text{ant}}$ is an anti-affine algebraic monoid, and the sequence

$$1 \longrightarrow M_{\text{ant}} \longrightarrow M \xrightarrow{\varphi} \text{Spec}(\mathcal{O}(M))$$

is an exact sequence of algebraic monoids, with φ a dominant morphism.

Proof. Since $\mathcal{O}(M)$ is finitely generated, it follows that $N = \text{Spec}(\mathcal{O}(M))$ is an algebraic monoid. Moreover, the canonical morphism $\varphi : M \rightarrow N$ is a morphism of algebraic monoids. If we let $S = \varphi^{-1}(1)$, then we have the following commutative diagram. The top row is exact and the bottom row is left exact.

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_{\text{ant}} & \longrightarrow & G & \longrightarrow & \text{Spec}(\mathcal{O}(G)) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & S & \longrightarrow & M & \xrightarrow{\varphi} & N = \text{Spec}(\mathcal{O}(M)). \end{array}$$

Since $M = G_{\text{ant}} M_{\text{aff}} \cong G_{\text{ant}} *_{G_{\text{aff}} \cap G_{\text{ant}}} M_{\text{aff}}$ (Proposition 2.10) and $\mathcal{O}(M) \cong \mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}$ (Theorem 3.1), it follows that the restriction morphism $\varphi|_{M_{\text{aff}}}$ is induced by the inclusion $\mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}} \subset \mathcal{O}(M_{\text{aff}})$. In particular, $\varphi|_{M_{\text{aff}}}$ is dominant and the following diagram is commutative:

$$\begin{array}{ccccccc} 1 & \longrightarrow & S & \longrightarrow & M & \xrightarrow{\varphi} & N = \text{Spec}(\mathcal{O}(M)) \\ & & \uparrow & & \uparrow & & \parallel \\ & & G_{\text{aff}} \cap G_{\text{ant}} & \hookrightarrow & M_{\text{aff}} & \longrightarrow & N = \text{Spec}(\mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}). \end{array}$$

Assume that M is a stable algebraic monoid. Then $G_{\text{aff}} \cap G_{\text{ant}}$ is a commutative closed normal subgroup of M_{aff} , and hence it follows from Proposition 3.14 that $G_{\text{aff}} \cap G_{\text{ant}}$ is observable in M_{aff} . It follows from [13, Thm.3.18] that the *affinized quotient* $\varphi|_{M_{\text{aff}}} : M_{\text{aff}} \rightarrow \text{Spec}(\mathcal{O}(M_{\text{aff}})^{G_{\text{aff}} \cap G_{\text{ant}}}) = N$ is such that there exists an open subset $U \subset N$ so that $(\varphi|_{M_{\text{aff}}})^{-1}(u)$ is a closed $G_{\text{aff}} \cap G_{\text{ant}}$ -orbit for all $u \in U$. We claim that this implies that $(\varphi|_{M_{\text{aff}}})^{-1}(1_N) = G_{\text{aff}} \cap G_{\text{ant}}$. Indeed, observe that since $G_{\text{aff}} \cap G_{\text{ant}}$ is normal in M , it follows that $\varphi|_{M_{\text{aff}}}$ is $(G \times G)$ -equivariant. It suffices now to recall that $M = G_{\text{ant}} M_{\text{aff}}$, and thus

$$\varphi^{-1}(1_N) = G_{\text{ant}} (\varphi|_{M_{\text{aff}}})^{-1}(1_N) = G_{\text{ant}}.$$

Since M is stable, $e = 1$, and thus $M_e G_{\text{ant}} = G_{\text{ant}}$.

If M is not a stable monoid, let $e \in E(\overline{G_{\text{ant}}})$ be the minimum idempotent. Then by Corollary 3.7 it follows that $\mathcal{O}(M) = \mathcal{O}(eM)$. One then concludes from (the proof of) Corollary 3.7, that $\varphi|_{eM} : eM \rightarrow N \cong \text{Spec}(\mathcal{O}(eM))$ is the affinization morphism of eM . Recalling that $G(eM)_{\text{ant}} = eG_{\text{ant}}$, it follows that eM is stable, and we have an exact sequence

$$1 \longrightarrow eG_{\text{ant}} \longrightarrow eM \xrightarrow{\varphi} \text{Spec}(\mathcal{O}(eM))$$

that fits into the following commutative diagram of algebraic monoids, where the vertical sequence in the center is exact:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & & \downarrow & & & \\
 & & & M_e & & & \\
 & & \searrow & \downarrow & & & \\
 1 & \longrightarrow & S & \longrightarrow & M & \xrightarrow{\varphi} & \text{Spec}(\mathcal{O}(M)) \\
 & & \downarrow \ell_e & \downarrow \ell_e & & & \parallel \\
 1 & \longrightarrow & eG_{\text{ant}} & \longrightarrow & eM & \xrightarrow{\varphi} & \text{Spec}(\mathcal{O}(eM)) \\
 & & & & \downarrow & & \\
 & & & & 0. & &
 \end{array}$$

It follows that $S = \ell_e^{-1}(eG_{\text{ant}}) = M_eG_{\text{ant}}$.

To complete the proof we observe that $G(S) = G_eG_{\text{ant}}$, with $G_e \subset G_{\text{aff}}$. Hence $G(S)_{\text{ant}} = G_{\text{ant}}$. Since e is the minimum idempotent of the closure $\overline{G_{\text{ant}}} \subset M$, it follows that e is also the minimum idempotent of the closure $\overline{G_{\text{ant}}} \subset M_eG_{\text{ant}}$. It suffices now to observe that $e(M_eG_{\text{ant}}) = eG_{\text{ant}}$, and then apply Theorem 3.19. \square

Remark 3.23. One can check directly that the submonoid $M_{\text{ant}} = M_eG_{\text{ant}}$ of M is functorial. Indeed, let $\varphi : M \rightarrow N$ be a morphism of the algebraic monoids M and N , with unit groups G and H , respectively. It follows from Lemma 3.3 above that $\varphi(G_{\text{ant}}) \subseteq H_{\text{ant}}$. Thus if $e \in \overline{G_{\text{ant}}}$ is the minimal idempotent, then, by continuity, $\varphi(e) \in \overline{H_{\text{ant}}}$. Hence if $f \in \overline{H_{\text{ant}}}$ is the minimal idempotent, then $f\varphi(e) = \varphi(e)f = f$. Thus $\varphi(M_e) \subseteq N_{\varphi(e)} \subseteq N_f$. Hence,

$$\varphi(M_{\text{ant}}) = \varphi(M_e)\varphi(G_{\text{ant}}) \subseteq N_f H_{\text{ant}} = N_{\text{ant}}.$$

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REFERENCES

- [1] A. Bialynicki-Birula, *On induced actions of algebraic groups*, Ann. de l'Inst. Fourier, 43 no. 2 (1993), 365–368. MR1220274 (94c:14041)
- [2] A. Bialynicki-Birula, G. Hochschild, G.D. Mostow, *Extensions of representations of algebraic linear groups*, Amer. J. Math. **85** (1963), 131–144. MR0155938 (27:5871)
- [3] M. Brion, *Log homogeneous varieties*, Actas del XVI Coloquio Latinoamericano de Álgebra, 1–39, Revista Matemática Iberoamericana, Madrid, 2007. MR2500349
- [4] M. Brion, *Local structure of algebraic monoids*, Mosc. Math. J. **8** (2008), no. 4, 647–666. MR2499350 (2010b:14089)
- [5] M. Brion, *Anti-affine algebraic groups*, J. Algebra **321** (2009), no. 3, 934–952. MR2488561 (2009j:14059)
- [6] M. Brion, A. Rittatore *The structure of normal algebraic monoids*, Semigroup Forum **74** (2007), no. 3, 410–422. MR2321575 (2008c:20086)

- [7] B. Conrad, *A modern proof of Chevalley's theorem on algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), 1–18. MR1906417 (2003f:20078)
- [8] M. Demazure and P. Gabriel, *Groupes algébriques*, North Holland, Amsterdam, 1970. MR0302656 (46:1800)
- [9] W. Ferrer-Santos, A. Rittatore, *Actions and Invariants of Algebraic Groups*. Series: Pure and Applied Math., 268, Dekker-CRC Press, Florida, (2005). MR2138858 (2006c:14067)
- [10] F.D. Grosshans, *Observable groups and Hilbert's fourteenth problem*, Amer. J. Math. **95** (1973), 229–253. MR0325628 (48:3975)
- [11] M. S. Putcha, *Linear Algebraic Monoids*, London Math. Soc. Lecture Notes Series **133**, Cambridge University Press, Cambridge, 1988. MR964690 (90a:20003)
- [12] L. E. Renner, *Linear Algebraic Monoids*, Encyclopædia of Mathematical Sciences **134**, Invariant Theory and Algebraic Transformation Groups, V, Springer-Verlag, Berlin, 2005. MR2134980 (2006a:20002)
- [13] L. E. Renner, A. Rittatore, *Observable actions of algebraic groups*, Transform. Groups **14** (2009), no. 4, 985–999. MR2577204 (2011b:14102)
- [14] A. Rittatore, *Algebraic monoids and group embeddings*, Transformation Groups **3** (1998), no. 4, 375–396. MR1657536 (2000a:14056)
- [15] A. Rittatore, *Algebraic monoids with affine unit group are affine*, Transform. Groups **12** (2007), 601–605. MR2356324 (2008h:20097)
- [16] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443. MR0082183 (18:514a)
- [17] J.-P. Serre, *Morphismes universels et variété d'Albanese*, Séminaire Chevalley (1958–1959), Exposé No. 10, Documents Mathématiques **1**, Soc. Math. France, Paris, 2001.
- [18] L. Solomon, *An introduction to reductive monoids*, in Semigroups, formal languages and groups, York, 1993, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 466, 1995, 295–352. MR1630625 (99h:20099)
- [19] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, to appear in the Encyclopædia of Mathematical Sciences, subseries Invariant Theory and Algebraic Transformation Groups; arXiv: math.AG/0602228.

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