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A LIMITING FREE BOUNDARY PROBLEM RULED BY ARONSSON'S EQUATION

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ABSTRACT. We study the behavior of a p-Dirichlet optimal design problem with volume constraint for p large. As the limit of p goes to infinity, we find a limiting free boundary problem governed by the infinity-Laplacian operator. We find a necessary and sufficient condition for uniqueness of the limiting problem and, under such a condition, we determine precisely the optimal configuration for the limiting problem. Finally, we establish convergence results for the free boundaries.

1. INTRODUCTION

Let Ω be a smooth bounded domain in the Euclidean space $\mathbb{R}^n$ and α a fixed positive number less than the Lebesgue measure of Ω. An optimal design problem with volume constraint can be generally written as

$$\text{Min } \left\{ \mathfrak{J}(\mathcal{O}) \;\middle|\; \mathcal{O} \subset \Omega \quad \text{and} \quad \mathcal{L}^n(\mathcal{O}) \leq \alpha \right\}. \tag{1.1}$$

For most applications, $\mathfrak{J}(\mathcal{O})$ has an integral representation involving functions which are linked to the competing configuration $\mathcal{O}$ by a prescribed partial differential equation (PDE).

The modern history of this line of research probably starts at the pioneering work of N. Aguilera, W. Alt and L. Caffarelli [2]. In that paper, the authors address the question of minimizing the Dirichlet integral when prescribed the volume of the zero set. C. Lederman in [17] establishes similar results for the nonhomogeneous minimization problem, $\int |Du|^2 - gu$. N. Aguilera, L. Caffarelli and J. Spruck [4] considered the minimization problem (1.1) for $\mathfrak{J}(\mathcal{O}) = \int_\Omega \Delta u dX$, where u is the harmonic function in $\mathcal{O}$, taking a prescribed boundary data φ on $\partial\Omega$ and zero on $\partial\mathcal{O}$. This is a model for an optimal shape problem in heat conduction theory with nonconstant temperature distribution. Nonlinear optimal design problems with nonconstant temperature distribution were treated in [22]. The common feature of the aforementioned works is that all of them are governed by the Laplacian operator. Their fine analyses rely on the revolutionary work of W. Alt and L. Caffarelli [3].

Just recently the study of optimal design problems ruled by degenerate quasilinear operators was successfully developed. This theory is the starting point for the main goal of the present work, which we now describe. Let us consider the problem of minimizing the p-Dirichlet integral with a given positive boundary data

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f and with the maximum volume of the support prescribed. More precisely, let us consider the following free boundary optimization problem:

$$(\mathfrak{P}_p) \quad \min\left\{\int_\Omega |\nabla u(X)|^p dX \;\Big|\; u \in W^{1,p}(\Omega),\ u = f \text{ on } \partial\Omega,\ \mathcal{L}^n\left(\{u > 0\}\right) \le \alpha\right\}.$$

Existence of a minimizer as well as smoothness properties of its free boundary have been established in [10] and [21]. Further generalizations are addressed in [25] and [26]. In the present paper, we are interested in the asymptotic behavior, as p goes to infinity, of optimal shapes to problem $(\mathfrak{P}_p)$. Analytical and geometric features of a limiting free boundary reveals asymptotic information upon the optimal design problem $(\mathfrak{P}_p)$. Driven by classical considerations, we are led to consider the following *limiting problem*:

$$(\mathfrak{P}_\infty) \qquad \min\left\{\mathrm{Lip}(u) \;\big|\; u \in W^{1,\infty}(\Omega),\ u = f, \text{ on } \partial\Omega,\ \mathcal{L}^n\left(\{u > 0\}\right) \le \alpha\right\},$$

where $\mathrm{Lip}(u)$ is the Lipschitz constant of u:

$$\mathrm{Lip}(u) = \sup_{x,y} \frac{|u(x) - u(y)|}{|x - y|}. \tag{1.2}$$

Our first concern is to prove that any sequence of minimizers u_p to problem $(\mathfrak{P}_p)$ converges (up to a subsequence) to a solution, u_∞, of the limiting problem $(\mathfrak{P}_\infty)$. In addition, we are interested in finding the partial differential equation (PDE) u_∞ satisfies in its set of positivity. In this direction and enforcing the fact that u_∞ is an extremal for the Lipschitz minimization problem, we show that u_∞ is indeed an absolute minimizer for the Lipschitz constant within its set of positivity, $\Omega_\infty := \{u_\infty > 0\}$. That is, it minimizes the Lipschitz constant in every subdomain of Ω_∞ when testing against functions with the same boundary data; see [5]. Hence, it is an ∞-harmonic function in its positivity set. This information is the content of the first theorem in this paper that we state now.

Theorem 1. *Let u_p be a minimizer of $(\mathfrak{P}_p)$. Then, up to a subsequence,*

$$u_p \to u_\infty, \qquad \text{as } p \to \infty,$$

uniformly in $\overline{\Omega}$ and weakly in every $W^{1,q}(\Omega)$ for $1 < q < \infty$, where v_∞ is a minimizer of $(\mathfrak{P}_\infty)$. The limiting function u_∞ satisfies the PDE, $\Delta_\infty u_\infty = 0$, in $\{u_\infty > 0\}$ in the viscosity sense. Here $\Delta_\infty u := DuD^2u(Du)^t$ is the infinity-Laplacian.

It is known that under the assumptions Ω convex and $f \equiv$ const., one can prove uniqueness for problem $(\mathfrak{P}_p)$, [23] (see also [1, 12, 13, 14, 16] for related Bernoulli-type problems). However, uniqueness is not expected in general for problem $(\mathfrak{P}_p)$. Surprisingly enough, under a mild compatibility condition upon $\mathrm{Lip}(f)$, Ω, and α which does not involve any convexity assumption on Ω, we prove uniqueness for the limiting problem $(\mathfrak{P}_\infty)$. In particular, in this case any sequence of solutions to problem $(\mathfrak{P}_p)$ converges to the same optimal limiting configuration. Such a result can be read as an *asymptotic uniqueness phenomenon* for problem $(\mathfrak{P}_p)$. In addition, we have precisely found the optimal shape for the limiting problem $(\mathfrak{P}_\infty)$; that is, it reveals where and how optimal configurations $\Omega_p := \{u_p > 0\}$ stabilize (see also Remark 2).

More precisely, for our next theorem we shall work under the following geometric compatibility condition:

$$\text{(H)} \qquad \mathcal{L}^n\left(\bigcup_{y\in\partial\Omega} B_{\frac{f(y)}{\mathrm{Lip}(f)}}(y)\cap\Omega\right) \ge \alpha.$$

It is understood that if f is constant, then (H) is automatically satisfied.

Theorem 2. *Assume* (H)*, and let $\lambda^\star$ be the unique positive real number such that the domain*

$$\Omega^\star := \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\lambda^\star}}(x)\cap\Omega$$

has Lebesgue measure precisely α. Then the function u_∞, defined as

$$\begin{cases} \Delta_\infty u_\infty &= 0 \qquad \text{in } \Omega^\star,\\ u_\infty &= f \qquad \text{on } \partial\Omega,\\ u_\infty &= 0 \qquad \text{on } \partial\Omega^\star\cap\Omega \end{cases}$$

is the unique minimizer for problem $(\mathfrak{P}_\infty)$. Hence, if u_p is a minimizer of $(\mathfrak{P}_p)$, then the whole sequence u_p converges, $u_p \to u_\infty$, uniformly in $\overline{\Omega}$ and weakly in every $W^{1,q}(\Omega)$ for $1<q<\infty$. In addition, u_∞ is given by the formula,

$$u_\infty = \max_{y\in\partial\Omega}\left(f(y) - \lambda^\star|x-y|\right)_+ .$$

Remark 1. As mentioned above, Theorem 2 applies in particular to an important physical situation, namely heat conduction problems with evenly heated domains, i.e., $f\equiv T$ (constant).

Remark 2. From the applied point of view, Theorem 2 provides a rigorous mathematical proof for the empirical, and widely employed, intuition that says that the configuration $\Omega^\star$ should be approximately an optimal way of insulating a given body Ω with temperature distribution f.

Remark 3. The fact that the equation that rules the limit configuration is Aronsson's equation $-\Delta_\infty u = 0$ is not surprising. Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) appear naturally as limits of p-harmonic functions (solutions to $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$), [6], and have applications to optimal transport problems, [9], [11], image processing, etc.; see the survey [5].

In view of Theorem 2, it becomes natural to inquire what happens if condition (H) is violated. In this direction, we show that (H) is a necessary and sufficient condition for uniqueness to problem $(\mathfrak{P}_\infty)$. Indeed, if (H) does not hold, we manage to find multiple solutions for problem $(\mathfrak{P}_\infty)$. Nevertheless, we could prove the existence of a minimal one.

Theorem 3. *Assume that* (H) *does not hold. Then there exists infinitely many minimizers for the limit problem $(\mathfrak{P}_\infty)$. The function*

$$u_\infty(x) = \max_{y\in\partial\Omega}\left(f(y) - \mathrm{Lip}(f)|x-y|\right)_+$$

is a minimizer with the measure of its positivity set

$$\{u_\infty > 0\} = \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x)\cap\Omega$$

strictly less than α. Moreover, u_∞ is the minimal solution, in the sense that any minimizer v_∞ verifies $v_\infty(x)\ge u_\infty(x)$.

Remark 4. Note that the support of the minimal minimizer for problem $(\mathfrak{P}_\infty)$ is given by the set $\Omega^\star := \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x) \cap \Omega$.

Finally, we study geometric properties of the limiting free boundary, $\partial\{u_\infty > 0\}$, as well as convergence issues of the free boundaries $\partial\{u_p > 0\}$. The next theorem we state shows that the the limiting free boundary enjoys the appropriate geometric features suitable for the study of its geometric measure properties.

Theorem 4. *Let u_p be extremals to problem $(\mathfrak{P}_p)$, and assume that a subsequence (still labelled as u_p) is such that $u_p \to u_\infty$ uniformly in $\overline{\Omega}$ and weakly in every $W^{1,q}(\Omega)$ for $1 < q < \infty$. Then u_∞ is uniformly Lipschitz continuous in Ω, grows linearly away from the free boundary, and is strongly nondegenerate. That is, for a constant $\gamma > 0$,*

$$u_\infty(x) \geq \gamma \operatorname{dist}(x, \partial\{u_x > 0\}), \quad \forall x \in \Omega_\infty := \{u_\infty > 0\},$$

and for any fixed free boundary point $x_0 \in \partial\{u_\infty > 0\}$ there holds

$$\sup_{B_r(x_0)} u_\infty \geq \gamma r.$$

The strategy for showing Theorem 4 is to revisit the p-Dirichlet optimization problem $(\mathfrak{P}_p)$ and verify that these properties hold uniformly in p. As a by-product of this analysis, we obtain convergence of the free boundaries $\partial\{u_p > 0\}$ in the Hausdorff metric.

Theorem 5. *Let u_p be a sequence of minimizers for problem $(\mathfrak{P}_p)$, and assume that a subsequence of u_p is such that $u_p \to u_\infty$ uniformly in $\overline{\Omega}$ and weakly in every $W^{1,q}(\Omega)$ for $1 < q < \infty$, u_∞ being a solution to $(\mathfrak{P}_\infty)$. Then*

$$\partial\{u_p > 0\} \longrightarrow \partial\{u_\infty > 0\}, \qquad \text{as } p \to \infty,$$

in the Hausdorff distance.

The variational optimization problem $(\mathfrak{P}_p)$ relates, to some extent, to Bernoulli-type problems governed by the p-Laplacian operator. This is done through a constant free boundary condition proven to hold for minimizers of problem $(\mathfrak{P}_p)$. Indeed, it has been shown (see [10], [21]) that $|\nabla u_p| = \lambda_{u_p}$ for a positive constant λ_{u_p} along its free boundary $\partial\{u_p > 0\}$. This is the so-called free boundary condition for the optimization problem $(\mathfrak{P}_p)$—a key piece of information when studying geometric measure as well as smoothness properties of the free boundary. In this direction we have proven the following convergence of free boundary conditions.

Theorem 6. *Let u_p be a sequence of minimizers for problem $(\mathfrak{P}_p)$ and $|\nabla u_p| = \lambda_p$ along $\partial\{u_p > 0\}$. Denote $\Omega_\infty := \{u_\infty > 0\}$. Then, up to a subsequence, $(u_p, \lambda_p) \to (u_\infty, \lambda_\infty)$, with $0 < \lambda_\infty < \infty$ and*

$$\lim_{\substack{x \to \partial\Omega_\infty \\ x \in \Omega_\infty}} \frac{u_\infty(x)}{\operatorname{dist}(x, \partial\Omega_\infty)} = \lambda_\infty.$$

When Ω is convex and f is constant, Theorem 6 can be seen in connection to the results of J. J. Manfredi, A. Petrosyan and H. Shahgholian [18], who study convergence issues, as $p \to \infty$, for Bernoulli-type problems.

The rest of the paper is organized as follows: In the next section we prove Theorem 1. In Section 3 we study the limit problem under condition (H) and in

Section 4 we deal with the complementary case. Finally, in Section 5 we include some uniform bounds for the sequence u_p (showing uniform nondegeneracy of the free boundary), and we study the convergence of the free boundaries.

2. Proof of Theorem 1

In this section we prove Theorem 1. The key to the proof is to find bounds for the energy $\left(\int_\Omega |\nabla u_p|^p\right)^{1/p}$ of a minimizer that are independent of p.

Proof of Theorem 1. Let us fix hereafter a Lipschitz extension of f, which we will denote by v, among functions in the set

$$K_\infty = \left\{\varphi \in W^{1,\infty}(\Omega) \;\middle|\; \varphi = f, \text{ on } \partial\Omega, \; |\{\varphi > 0\}| = \alpha\right\}. \tag{2.1}$$

Clearly, since Ω is bounded, v competes in the minimization problem $(\mathfrak{P}_p)$. Thus, using v as a test function in problem $(\mathfrak{P}_p)$, we obtain

$$\left(\int_\Omega |\nabla u_p|^p\right)^{1/p} \le \left(\int_\Omega |\nabla v|^p\right)^{1/p} \le \operatorname{Lip}(v)|\Omega|^{1/p} \le C,$$

where C is a constant independent of p. With an exponent $q < \infty$ fixed, we obtain

$$\left(\int_\Omega |\nabla u_p|^q\right)^{1/q} \le \left(\int_\Omega |\nabla u_p|^p\right)^{1/p} |\Omega|^{p/(q(p-q))} \le \operatorname{Lip}(v)|\Omega|^{1/p+p/(q(p-q))} \le C.$$

Therefore, the sequence u_p is uniformly bounded in $W^{1,q}(\Omega)$, and its weak limit as $p \to \infty$ (let us call it u_∞) verifies

$$\left(\int_\Omega |\nabla u_\infty|^q\right)^{1/q} \le \operatorname{Lip}(v)|\Omega|^{1/q} \le C.$$

Taking $q \to \infty$ and performing a diagonal argument, we obtain a subsequence (that will still be named as u_p) that converges weakly in every $W^{1,q}(\Omega)$, $1 < q < \infty$ to a limit $u_\infty \in W^{1,\infty}(\Omega)$ such that

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le \operatorname{Lip}(v).$$

Let us now turn our attention towards estimating the Lebesgue measure of $\{u_\infty > 0\}$. Fix $\epsilon > 0$. Thanks to uniform convergence, for p large enough, there holds

$$\{u_\infty > \epsilon\} \subset \{u_p > 0\}.$$

Hence we conclude that

$$\mathcal{L}^n\left(\{u_\infty > 0\}\right) = \lim_{\epsilon \to 0} \mathcal{L}^n\left(\{u_\infty > \epsilon\}\right) \le \alpha.$$

Therefore, we have proved that u_∞ is an extremal for the limit problem $(\mathfrak{P}_\infty)$.

It remains to prove that u_∞ is indeed ∞-harmonic in its set of positivity. Following [7], let us recall the definition of viscosity solution.

Definition 2.1. Consider the boundary value problem

$$F(x, Du, D^2u) = 0 \qquad \text{in } \Omega. \tag{2.2}$$

(1) A lower semicontinuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$ we have

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \ge 0.$$

(2) An upper semicontinuous function u is a subsolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$ we have

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(3) Finally, u is a viscosity solution if it is a super- and a subsolution.

If we have a weak p-harmonic function (in the sense of distribution) that is continuous, then it is a viscosity solution. This is the content of our next result.

Lemma 2.1. *Let u be a continuous weak solution of $\Delta_p u = 0$ in some domain Ω for $p > 2$. Then u is a viscosity solution of*

$$-(p-2)|Du|^{p-4}\Delta_\infty u - |Du|^{p-2}\Delta u = 0 \qquad \text{in } \Omega. \tag{2.3}$$

Proof. Let $x_0 \in \Omega$ and a test function ϕ such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$-(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x_0) - |D\phi|^{p-2}\Delta\phi(x_0) \geq 0.$$

Assume that this is not the case. Then there exists a radius $r > 0$ such that

$$-(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x) - |D\phi|^{p-2}\Delta\phi(x) < 0,$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u-\phi)(x)$, and let $\psi(x) = \phi(x) + m/2$. This function ψ verifies $\psi(x_0) > u(x_0)$ and

$$-\mathrm{div}(|D\psi|^{p-2}D\psi) < 0.$$

Multiplying by $(\psi - u)^+$ extended by zero outside $B(x_0, r)$, we get

$$\int_{\{\psi > u\}} |D\psi|^{p-2} D\psi D(\psi - u) < 0.$$

Taking $(\psi - u)^+$ as test function in the weak form of the equation, we get

$$\int_{\{\psi > u\}} |Du|^{p-2} Du D(\psi - u) = 0.$$

Hence,

$$C(N,p)\int_{\{\psi > u\}} |D\psi - Du|^p \leq \int_{\{\psi > u\}} \langle |D\psi|^{p-2}D\psi - |Du|^{p-2}Du, D(\psi - u)\rangle < 0,$$

a contradiction. This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution runs as above, and we omit the details. $\square$

We are now ready to prove that the limit $\lim_{p_i\to\infty} u_{p_i} = u_\infty$ satisfies the desired PDE in its set of positivity. In fact, let us check that $-\Delta_\infty u_\infty = 0$ in the viscosity sense in the set $\{u_\infty > 0\}$. Let us recall the standard proof. Let ϕ be a smooth test function such that $u_\infty - \phi$ has a strict maximum at $x_0 \in \{u_\infty > 0\}$. Since u_{p_i} converges uniformly to u_∞ we get that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \Omega$ with $x_i \to x_0$ and moreover we have that $u_{p_i} > 0$ in a whole fixed neighborhood of x_0 (and therefore $u_{p_i}(x_i) > 0$ and every u_{p_i} is p-harmonic there). Next, we use the fact that u_{p_i} is a viscosity solution of $-\Delta_p u_p = 0$ in the set $\{u_{p_i} > 0\}$, and we obtain

$$-(p_i-2)|D\phi|^{p_i-4}\Delta_\infty\phi(x_i) - |D\phi|^{p_i-2}\Delta\phi(x_i) \leq 0. \tag{2.4}$$

If $D\phi(x_0) = 0$, we get $-\Delta_\infty \phi(x_0) \leq 0$. If this is not the case, we have that $D\phi(x_i) \neq 0$ for large i and then

$$-\Delta_\infty \phi(x_i) \leq \frac{1}{p_i - 2} |D\phi|^2 \Delta\phi(x_i) \to 0, \quad \text{as } i \to \infty.$$

We conclude that

$$-\Delta_\infty \phi(x_0) \leq 0.$$

That is, u_∞ is a viscosity subsolution of $-\Delta_\infty u_\infty = 0$.

A similar argument shows that u_∞ is also a supersolution and therefore a solution of $-\Delta_\infty u_\infty = 0$ in Ω. The proof of Theorem 1 is completed. □

3. Proof of Theorem 2

In this section we deal with the situation in which we have uniqueness for the limit problem. We will assume that condition (H) holds, that is,

$$\mathcal{L}^n \left(\bigcup_{y \in \partial\Omega} B_{\frac{f(y)}{\mathrm{Lip}(f)}}(y) \cap \Omega \right) \geq \alpha.$$

Note that with the notation of the statement of Theorem 2 this implies that

$$\lambda^\star \geq \mathrm{Lip}(f).$$

This fact is crucial in the course of the next proof.

Proof of Theorem 2. Let v_∞ be a minimizer for problem $(\mathfrak{P}_\infty)$. Existence of such a minimizer is obtained by Theorem 1. Let us denote

$$\Omega_\infty := \{v_\infty > 0\} \subset \Omega.$$

For each free boundary point $y \in \partial\Omega_\infty$, let $x \in \partial\Omega$ be a point satisfying

$$|x - y| = \mathrm{dist}(y, \partial\Omega).$$

Using the Lipschitz continuity of v_∞, we obtain the following estimate:

$$f(x) \leq \mathrm{Lip}(v_\infty)|x - y|. \tag{3.1}$$

From (3.1), we immediately conclude that

$$\bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(v_\infty)}}(x) \cap \Omega \subset \Omega_\infty, \tag{3.2}$$

which, in particular, implies

$$\mathcal{L}^n \left(\bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(v_\infty)}}(x) \cap \Omega \right) \leq \alpha = \mathcal{L}^n \left(\bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^\star}}(x) \cap \Omega \right). \tag{3.3}$$

From above we obtain

$$\lambda^\star \leq \mathrm{Lip}(v_\infty). \tag{3.4}$$

On the other hand, let

$$\Omega^\star := \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^\star}}(x) \cap \Omega.$$

Then, u_∞, defined as the solution to

$$\begin{cases} \Delta_\infty u_\infty &= 0 \quad \text{in } \Omega^\star, \\ u_\infty &= f \quad \text{on } \partial\Omega, \\ u_\infty &= 0 \quad \text{on } \partial\Omega^\star \cap \Omega \end{cases}$$

competes in the minimization problem ($\mathfrak{P}_\infty$), thus

$$\mathrm{Lip}(u_\infty) \geq \mathrm{Lip}(v_\infty). \tag{3.5}$$

In the sequel, we will use the fact that u_∞ is the best Lipschitz extension of the boundary data f on $\partial\Omega$ and 0 on $\partial\Omega^\star \cap \Omega$ together with the geometric compatibility condition (H) to bridge these inequalities. For that we consider the auxiliary barrier function

$$\psi(x) := \max_{y \in \partial\Omega} \left(f(y) - \lambda^\star |x - y| \right)_+ .$$

We initially verify that

$$\mathrm{Lip}(\psi) = \lambda^\star. \tag{3.6}$$

To see this fact, let us first show that $\mathrm{Lip}(\psi) \leq \lambda^\star$. Let x_1 and x_2 be two points in Ω. We assume $0 < \psi(x_1) < \psi(x_2)$. Let y_1 and y_2 be such that

$$\psi(x_i) = f(y_i) - \lambda^\star |x_i - y_i|, \quad \text{i=1,2}.$$

From the definition of ψ, we know

$$\psi(x_1) \geq f(y_2) - \lambda^\star |x_1 - y_2|.$$

We now estimate

$$\begin{aligned} 0 < \psi(x_2) - \psi(x_1) &\leq f(y_2) - \lambda^\star |x_2 - y_2| - \left(f(y_2) - \lambda^\star |x_1 - y_2| \right) \\ &\leq \lambda^\star \left(|x_1 - y_2| - |x_2 - y_2| \right) \\ &\leq \lambda^\star |x_1 - x_2|. \end{aligned}$$

This shows that $\mathrm{Lip}(\psi) \leq \lambda^\star$. To see the reverse inequality, we argue as follows. Given $x_1 \in \Omega$, let y_1 such that

$$\psi(x_1) = f(y_1) - \lambda^\star |x_1 - y_1|.$$

Now, let $x_2 = x_1 + \gamma(x_1 - y_1)$ and choose γ small such that $x_2 \in \Omega$. For this x_2 we also have that

$$\psi(x_2) = f(y_1) - \lambda^\star |x_2 - y_1|,$$

and this implies

$$|\psi(x_1) - \psi(x_2)| = \lambda^\star |x_1 - x_2|,$$

showing that $\mathrm{Lip}(\psi) \geq \lambda^\star$.

Our next step is to check that ψ matches the desired boundary conditions. It is clear from its definition that

$$\psi\big|_{\partial\Omega^\star} = 0.$$

Proving ψ agrees with f on $\partial\Omega$ is equivalent to showing that

$$f(x) = \max_{y \in \partial\Omega} \left\{ \left(f(y) - \lambda^\star |x - y| \right)_+ \right\}, \quad \forall x \in \partial\Omega. \tag{3.7}$$

Let us assume, for the sake of contradiction, that (3.7) does not hold. This will imply that there exist two points x, y on $\partial\Omega$ with

$$\lambda^\star |x - y| < f(y) - f(x).$$

That is,

$$\lambda^\star < \mathrm{Lip}(f) = \sup_{x,y\in\partial\Omega} \left\{ \frac{|f(x)-f(y)|}{|x-y|} \right\},$$

which contradicts (H).

As a remark, note that when we take two points $x, y \in \partial\Omega$, we get

$$|\psi(x)-\psi(y)| = |f(x)-f(y)| \le \mathrm{Lip}(f)|x-y|.$$

Thus, $\mathrm{Lip}(\psi) = \max(\lambda^\star, \mathrm{Lip}(f)) = \lambda^\star$.

Once it is verified that ψ has the same boundary condition as u_∞, we know

$$\mathrm{Lip}(u_\infty) \le \mathrm{Lip}(\psi) = \lambda^\star. \tag{3.8}$$

Now let us show that u_∞ coincides with the barrier

$$\psi(x) = \max_{y\in\partial\Omega} \left(f(y) - \lambda^\star|x-y|\right)_+ .$$

We have that u_∞ is a minimizer for the limit problem, hence we must have

$$u_\infty(x) \ge \max_{y\in\partial\Omega} \left(f(y) - \lambda^\star|x-y|\right)_+ .$$

In fact, if we assume that this is not the case, then there exists x_0 such that $u_\infty(x_0) < \psi(x_0)$. Now, considering quotients that involve x_0 and points on $\partial\Omega$, we can easily conclude that $\mathrm{Lip}(u_\infty) > \lambda^\star = \mathrm{Lip}(\psi)$, a contradiction since ψ is a competitor in the limit problem.

Therefore, we obtain that both functions have the same positivity set (both sets have the same measure and one is included in the other).

Now, arguing as before, assume that there exists x_0 such that $u_\infty(x_0) > \psi(x_0)$. In this case, comparing quotients defining the Lipschitz constant with x_0 and points on the boundary of the positivity set, we get $\mathrm{Lip}(u_\infty) > \lambda^\star = \mathrm{Lip}(\psi)$. This contradicts again the fact that u_∞ is optimal for the limit problem.

Combining (3.2), (3.3), (3.4), and (3.8), together with the fact that u_∞ and ψ are ∞-harmonic in Ω^* with the same value on the boundary of this set, we end the proof of Theorem 2. □

4. Proof of Theorem 3

Now let us show that when (H) does not hold there is no uniqueness for minimizers of the limit problem.

Proof of Theorem 3. As before, let $\lambda^\star$ be such that

$$\Omega^\star := \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\lambda^\star}}(x) \cap \Omega$$

has Lebesgue measure precisely α, and assume that (H) does not hold. That is,

$$\mathrm{Lip}(f) > \lambda^\star.$$

Let

$$D := \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x) \cap \Omega.$$

We have

$$\mathcal{L}^n(D) < \alpha.$$

By our previous result we have that

$$\psi(x) := \max_{y\in\partial\Omega} \left(f(y) - \mathrm{Lip}(f)|x-y|\right)_+$$

is an extremal for the limit problem with measure $\mathcal{L}^n(D)$.

Now, let v_∞ be an extremal for the limit problem with measure α. Then, as $v_\infty = f$ on $\partial\Omega$, we have

$$\mathrm{Lip}(v_\infty) \geq \mathrm{Lip}(f) = \mathrm{Lip}(\psi).$$

On the other hand ψ is a competitor in the limit problem with measure α and hence

$$\mathrm{Lip}(\psi) \geq \mathrm{Lip}(v_\infty).$$

We conclude that

$$\mathrm{Lip}(\psi) = \mathrm{Lip}(v_\infty) = \mathrm{Lip}(f),$$

and then ψ is also a maximizer for the limit problem.

Moreover, we have that

$$\psi(x) \leq v_\infty(x), \qquad x \in D,$$

if not the Lipschitz constant of v_∞ is greater than $\mathrm{Lip}(\psi)$. Indeed, let us assume that there exists $x_0 \in D$ such that

$$\psi(x_0) > v_\infty(x_0).$$

That is,

$$\max_{y\in\partial\Omega} \left(f(y) - \mathrm{Lip}(f)|x_0-y|\right)_+ > v_\infty(x_0).$$

From that we get that there exists $y \in \partial\Omega$ such that

$$f(y) - \mathrm{Lip}(f)|x_0-y| > v_\infty(x_0),$$

which is to say that (using that $v_\infty = f$ on $\partial\Omega$)

$$v_\infty(y) - v_\infty(x_0) > \mathrm{Lip}(f)|x_0-y|,$$

which clearly implies

$$\mathrm{Lip}(f) < \mathrm{Lip}(v_\infty).$$

Therefore, we have that ψ is the *minimal* extremal for the limit problem, and hence we obtain the following estimate for the support of any extremal v_∞:

$$D := \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x) \cap \Omega \subset \{v_\infty > 0\}.$$

Now, let

$$D_\delta := \bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x) \cap \Omega + B(0,\delta),$$

with δ small such that

$$\mathcal{L}^n(D_\delta) < \alpha.$$

In this set D_δ, let us consider v_∞ the solution to

$$\begin{cases} \Delta_\infty v_\infty = 0 & \text{in } D_\delta, \\ v_\infty = f & \text{on } \partial\Omega, \\ v_\infty = 0 & \text{on } \partial D_\delta. \end{cases}$$

Since $D \subset D_\delta$, we have

$$\mathrm{Lip}(v_\infty) = \mathrm{Lip}(f).$$

To prove this fact, let us consider in the set D_δ the boundary value

$$F(x) = \begin{cases} f(x) & x \in \partial\Omega, \\ 0 & x \in \partial D_\delta \cap \Omega. \end{cases}$$

This boundary datum F is a Lipschitz function with Lipschitz constant given by

$$\mathrm{Lip}(F) = \sup_{x,y \in \partial D_\delta} \frac{|F(x) - F(y)|}{|x-y|}.$$

Let us estimate this Lipschitz constant $\mathrm{Lip}(F)$. If $x, y \in \partial D_\delta \cap \Omega$, then

$$\frac{|F(x) - F(y)|}{|x-y|} = 0 < \mathrm{Lip}(f).$$

When $x, y \in \partial\Omega$, clearly

$$\frac{|F(x) - F(y)|}{|x-y|} \le \mathrm{Lip}(f).$$

And finally when $x \in \partial\Omega$ and $y \in \partial D_\delta \cap \Omega$, we have

$$\frac{|F(x) - F(y)|}{|x-y|} = \frac{|f(x)}{|x-y|} < \mathrm{Lip}(f).$$

We are using the fact that $D \subset \Omega_\delta$ and hence the distance $|x-y|$ is bigger than $f(x)/\mathrm{Lip}(f)$. To see this fact, just take $y \in \partial D$, then for any $x \in \partial\Omega$, we have

$$f(x) - \mathrm{Lip}(f)|x-y| \le 0,$$

which is to say

$$|x-y| \le \frac{f(x)}{\mathrm{Lip}(f)}.$$

Therefore, we conclude that

$$\mathrm{Lip}(F) = \mathrm{Lip}(f),$$

and since v_∞ has the same Lipschitz constant as F (it is its best possible Lipschitz extension), we get that

$$\mathrm{Lip}(v_\infty) = \mathrm{Lip}(f).$$

Hence v_∞ is also an extremal for the limit problem that is positive on $\partial D \subset (D_\delta)^o$ (the strong maximum principle holds for ∞-harmonic functions), and hence we conclude that $v_\infty \neq \psi$.

With these estimates we can conclude that there is no strict monotonicity with respect to the measure in the limit problem. $\square$

Now, we can state further consequences of our previous results.

Theorem 7. *Assume that*

$$\beta := \mathcal{L}^n \left(\bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x) \cap \Omega \right) < \alpha.$$

Then we have

$$\lim_{p\to\infty} \mathfrak{P}_p(\alpha) = \lim_{p\to\infty} \mathfrak{P}_p(\beta)$$

in the sense that if u_p is an extremal for $\mathfrak{P}_p(\alpha)$ and v_p is an extremal for $\mathfrak{P}_p(\beta)$, then

$$\lim_{p\to\infty} \left(\int_\Omega |\nabla u_p(X)|^p dX \right)^{1/p} = \lim_{p\to\infty} \left(\int_\Omega |\nabla v_p(X)|^p dX \right)^{1/p}.$$

Moreover,

$$v_p \to \psi \qquad \text{and} \qquad u_p \to u_\infty$$

uniformly in $\overline{\Omega}$ *with*

$$\mathrm{Lip}(u_\infty) = \mathrm{Lip}(\psi) = \mathrm{Lip}(f) \qquad \text{and} \qquad \psi(x) \le u_\infty(x).$$

One possible conclusion of this fact is that the boundary datum f is so that the limit problem has many solutions and hence we are "wasting measure" when considering the problem with α instead of β. In fact, the value of the minimum for $\mathfrak{P}_p(\alpha)$ and for $\mathfrak{P}_p(\beta)$ are almost the same for p large and the minimal solution of the limit problem is ψ (which is the unique minimizer for $\mathfrak{P}_\infty(\beta)$).

5. Uniform estimates and free boundary convergence issues

This section is devoted to establishing Theorems 4, 5, and 6. To this end we shall revisit the study of the p-Dirichlet energy minimization problem with volume constraint, $(\mathfrak{P}_p)$ carried out in [21] and in [10]. Our strategy is to seize uniform-in-p properties and afterwards explore their impact on the limiting problem $(\mathfrak{P}_\infty)$.

It is well established in the literature that ordinary techniques from the calculus of variations are not suitable to approach directly optimal design problems with volume constraints. Indeed, to establish existence of a minimizer for problem $(\mathfrak{P}_p)$ requires a careful analysis, involving penalty methods, and geometric measure perturbation techniques.

A penalized version of problem $(\mathfrak{P}_p)$ can be easily set up. Indeed, for each $L > 0$, let

$$\varrho_L(t) := L(t-\alpha)^+. \tag{5.1}$$

We then define the L-penalized problem for the p-Dirichlet integral as

$$(\mathfrak{P}_p^L) \qquad \min\left\{ \int_\Omega |\nabla u(X)|^p dX + \varrho_L\left(\{u>0\}\right) \;\Big|\; u \in W^{1,p}(\Omega),\ u = f \text{ on } \partial\Omega \right\}.$$

Notice that problem $(\mathfrak{P}_p^L)$ does not involve a volume constraint anymore, thus the proof of existence of a minimizer, u_p^L, for problem $(\mathfrak{P}_p^L)$ follows a standard scheme from the calculus of variations. It is also simple to check that $u_p^L \ge 0$ and $\Delta_p u_p^L$ is a nonnegative Radon measure supported on $\partial\{u_p^L > 0\}$. In particular, u_p^L is p-harmonic in its positivity set, that is, u_p^L satisfies the following PDE:

$$\Delta_p u_p^L = 0, \qquad \text{in } \{u_p^L > 0\}.$$

Although locally $C^{1,\alpha}$ within $\{u_p^L > 0\}$, notice that Lipschitz is the optimal regularity for u_p^L in Ω. This is due to the fact that ∇u_p^L jumps from positive slope to zero along the free boundary $\partial\{u_p^L > 0\}$. Indeed it has been proven in [21, 10] that for each L fixed u_p^L is locally Lipschitz continuous in Ω. Our next lemma gives the precise dependence of the Lipschitz norm of u_p^L with respect to p and the penalty charge L. This lemma is essentially taken from [25]. We present a proof here as a courtesy to the readers.

Lemma 1. *Let* u_p^L *be a minimizer for* $(\mathfrak{P}_p^L)$. *Then,*

$$\|\nabla u_p^L\|_{L^\infty(\Omega)} \le C L^{1/p},$$

where C *is a constant that depends only on dimension,* f *and* α.

Proof. Since we are interested in the limiting problem, we will only deal with the case $p \gg 1$. We will follow the approach suggested in [3], keeping track of the precise constants that appear on the estimates. From the minimality of u_p^L, we deduce, for any ball $B = B_d(x_0) \subset \Omega$ centered at a free boundary point, i.e., $x_0 \in \partial\{u_p^L > 0\}$, that there holds

$$L \cdot \mathcal{L}^n \left(\{x \in B_d(x_0) \mid u_p^L(x) = 0\}\right) \geq c_0 \left(\int_\Omega \left|\nabla \left(u_p^L - \mathfrak{h}_p\right)(x)\right|^p dx\right), \tag{5.2}$$

where $\mathfrak{h}_p$ is the p-harmonic function in $B_d(x_0)$ which agrees with u_p^L on $\partial B_d(x_0)$ and c_0 is a constant that depends only upon dimension. For any direction ν, we define

$$r_\nu := \min\left\{r \mid \frac{1}{4} \leq r \leq 1 \text{ and } u_p^L(x_0 + dr\nu) = 0\right\}$$

if such a set is nonempty; otherwise, we put $r_\nu = 1$. Taking into account that

$$u_p^L(x_0 + dr_\nu\nu) = 0$$

whenever $r_\nu < 1$, we can compute,

$$\begin{aligned} \mathfrak{h}_p(x_0 + dr_\nu\nu) &= \int_{r_\nu}^1 \frac{d}{dr}(u_p^L - \mathfrak{h}_p)(x_0 + dr\nu)dr \\ &\leq d \cdot (1 - r_\nu)^{1/p'} \times \left[\int_{r_\nu}^1 |\nabla(\mathfrak{h}_p - u_p^L)(x_0 + r\nu)|^p dr\right]^{1/p}. \end{aligned} \tag{5.3}$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$. Now, by the Harnack inequality, we know that

$$\inf_{B_{\frac{2}{3}d}(x_0)} \mathfrak{h}_p \geq c_1 \mathfrak{h}_p(x_0) \tag{5.4}$$

for a constant $c_1 > 0$ that depends only on dimension (see, for instance, [15]). Let us consider the following barrier function, b, given by

$$\left\{ \begin{array}{rcll} \Delta_p b &=& 0 & \text{in } B_1(0) \setminus B_{\frac{2}{3}}(0), \\ b &=& 0 & \text{on } \partial B_1(0), \\ b &=& c_1 & \text{in } \overline{B_{\frac{2}{3}}(0)}, \end{array} \right. \tag{5.5}$$

where c_1 is the universal constant in (5.4). By the Hopf maximum principle, there exists a universal constant $c_2 > 0$, depending only on dimension, such that

$$b(x) \geq c_2 \left(1 - |x|\right). \tag{5.6}$$

By the maximum principle and (5.6) we can write

$$\mathfrak{h}_p(x_0 + dx) \geq \mathfrak{h}_p(x_0) \cdot b(x) \geq c_2 \mathfrak{h}_p(x_0) \cdot (1 - |x|). \tag{5.7}$$

Combining (5.2) and (5.7), we end up with

$$d^p \cdot \left[\int_{r_\nu}^1 |\nabla(\mathfrak{h}_p - u_p^L)(x_0 + r\nu)|^p dr\right] \geq c_3 \mathfrak{h}_p^p(x_0) \cdot (1 - r_\nu). \tag{5.8}$$

Integrating (5.8) with respect to ν over $\mathbb{S}^{n-1}$ and taking into account the definition of r_ν, we find

$$\left(\frac{\mathfrak{h}_p(x)}{d}\right)^p \cdot \int_{B_d(x) \setminus B_{d/4}(x)} \chi_{\{u_p^L = 0\}} dx \leq C_4 \int_{B_d(x)} \left|\nabla \left(\mathfrak{h}_p - u_p^L\right)(x)\right|^p dx. \tag{5.9}$$

If we replace, in all of our arguments so far, $B_{d/4}(x)$ by $B_{d/4}(\overline{x})$, for any $\overline{x} \in \partial B_{d/2}(x)$, we obtain

$$\left(\frac{\mathfrak{h}_p(x)}{d}\right)^p \cdot \int_{B_d(x)\setminus B_{d/4}(\overline{x})} \chi_{\{u_p^L=0\}} dx \leq \tilde{C}_4 \int_{B_d(x)} \left|\nabla\left(\mathfrak{h}_p - u_p^L\right)(x)\right|^p dx \tag{5.10}$$

for every $\overline{x} \in \partial B_{d/2}(x)$.

Integrating (5.10) with respect to $\overline{x}$, yields

$$\left(\frac{\mathfrak{h}_p(x)}{d}\right)^p \cdot \left|\left\{x \in B_d(x) \mid u_p^L(x) = 0\right\}\right| \leq C_5 \int_{B_d(x)} \left|\nabla\left(\mathfrak{h}_p - u_p^L\right)(x)\right|^p dx. \tag{5.11}$$

Now we argue as follows: let $\rho := \mathrm{dist}(x, \partial\{u_p^L > 0\})$ and for each $0 < \delta \ll 1$, denote $\mathfrak{h}_p^\delta$ the p-harmonic function in $B_{\rho+\delta}(x)$ that agrees with u_p^L on $\partial B_{\rho+\delta}(x)$. Combining (5.2) and (5.11) together with standard elliptic estimate, we deduce

$$u_p^L(x) = \mathfrak{h}_\delta(x) + \mathrm{o}(1) \leq C_6 L^{1/p}(\rho + \delta) + \mathrm{o}(1), \quad \text{as } \delta \searrow 0, \tag{5.12}$$

for a constant C_6 that depends on dimension, f and α. Letting $\delta \searrow 0$ in (5.12) we finally conclude

$$u_p^L(x) \leq C_6 L^{1/p} \mathrm{dist}\left(x, \partial\Omega_\lambda^\star\right),$$

which clearly implies that u_p^L is Lipschitz continuous up to the free boundary $\partial\{u_p^L > 0\}$ and $\|\nabla u_p^L\|_\infty \lesssim L^{1/p}$. Lemma 1 is proved. □

Another important piece of information concerns uniform nondegeneracy.

Lemma 2. *Let $x \in \{u_p^L > 0\}$ be a free boundary point. Then*

$$L^{-1/p}\underline{c} \cdot \mathrm{dist}\left(x, \partial\{u_p^L > 0\}\right) \leq u_p^L(x) \tag{5.13}$$

for a constant $\underline{c}$ that depends only upon dimension, f and α. Moreover, the following strong nondegeneracy holds

$$\sup_{B_r(x_0)} u_p^L \geq L^{-1/p}\underline{c}_1 r \tag{5.14}$$

for any free boundary point $x_0 \in \partial\{u_p^L > 0\}$. The constant $\underline{c}_1$ depends only on dimension, f and α and is independent of p.

The proof of Lemma 2 is, by now, classical in variational free boundary theory. It relies on "cutting" a small hole around the free boundary point and comparing the result with the original optimal design. For further details we refer the readers to [25], Theorem 6.2. As observed in the proof of Lemma 1, the fact that $\underline{c}$ and $\underline{c}_1$ are universal is a consequence of a uniform-in-p Harnack inequality and a uniform-in-p Hopf boundary maximum principle. We skip the details here.

The penalty method strategy is based on the idea that if L is large enough (but still finite), one expects that minimizers for $(\mathfrak{P}_p^L)$ would rather prefer to obey the volume constraint, $\mathcal{L}^n\left(\{u_p^L > 0\}\right) \leq \alpha$. Therefore, they will be a solution for the original problem, $(\mathfrak{P}_p)$. Such a strategy does work, [21], [10], and [25], however it relies on a fine geometric measure perturbation approach. The following theorem is a consequence of the analysis carried out in [25, section 7].

Lemma 3. *There exists a universal constant C, depending only on dimension, f and α, but independent of p, such that if*

$$L \geq Cp,$$

then

$$\mathcal{L}^n\left(\{u_p^L > 0\}\right) \le \alpha.$$

Therefore, u_p^{Cp} is a solution to problem $(\mathfrak{P}_p)$.

It is important to notice that any minimizer, u_p, of problem $(\mathfrak{P}_p)$ is also a minimizing function for problem $(\mathfrak{P}_p^{Cp})$. As a consequence, combining Lemmas 1, 2, and 3, we obtain the following theorem with estimates that are uniform in p.

Theorem 8. *There exists a constant $K > 0$, depending on dimension, f and α, but independent of p, such that for any solution u_p of $(\mathfrak{P}_p)$, there holds*

$$\|\nabla u_p\|_{L^\infty(\Omega)} \le K. \tag{5.15}$$

Moreover, u_p grows linearly uniform-in-p away from the free boundary; that is, for a constant $\gamma > 0$ independent of p,

$$u_p(x) \ge \gamma \operatorname{dist}(x, \partial\{u_p > 0\}), \quad \forall x \in \{u_p > 0\}. \tag{5.16}$$

In addition, u_p is uniformly strong nondegenerate; that is, for any fixed free boundary point $x_0 \in \partial\{u_p > 0\}$,

$$\sup_{B_r(x_0)} u_p \ge \gamma r, \tag{5.17}$$

where $\gamma > 0$ is independent of p.

Proof of Theorem 4. Notice that $\lim_{p\to\infty} p^{1/p} = 1$. Passing the limit as p goes to infinity in (5.15), (5.16), and (5.17), we prove Theorem 4. □

Theorem 8 actually gives more qualitative information than Theorem 4 itself. Indeed, with Theorem 8 we can address free boundary convergence issues. In what follows we prove convergence of the free boundaries in the Hausdorff metric, Theorem 5.

Proof of Theorem 5. For any set $A \subset \mathbb{R}^n$, and $\varepsilon > 0$ fixed, let $\Gamma_\varepsilon(A)$ denote the ε-neighborhood of A; that is,

$$\Gamma_\varepsilon(A) := \left\{x \in \mathbb{R}^n \;\middle|\; \operatorname{dist}(x, A) < \varepsilon\right\}.$$

We have to show that given $\varepsilon > 0$, for $p \gg 1$, depending on $\varepsilon > 0$, there hold

$$\partial\{u_p > 0\} \subset \Gamma_\varepsilon\left(\partial\{u_\infty > 0\}\right)$$

and

$$\partial\{u_\infty > 0\} \subset \Gamma_\varepsilon\left(\partial\{u_p > 0\}\right).$$

Let ξ be an arbitrary point on $\partial\{u_p > 0\}$, and let us assume, for sake of contradiction, that $\xi \notin \Gamma_\varepsilon(\partial\{u_\infty > 0\})$; that is,

$$\operatorname{dist}(\xi, \partial\{u_\infty > 0\}) \ge \varepsilon.$$

If $u_\infty(\xi) > 0$, then by linear growth we would have

$$u_\infty(\xi) \ge \gamma \operatorname{dist}(\xi, \partial\{u_\infty > 0\}) \ge \gamma\varepsilon.$$

Thus, from uniform convergence, if $p \gg 1$, $u_p(\xi) \ge \frac{2}{3}\gamma\varepsilon$, driving us to a contradiction. If we assume $u_\infty(\xi) = 0$, then $u_\infty\big|_{B_\varepsilon(\xi)} \equiv 0$. However, by strong nondegeneracy, we know that

$$\sup_{B_{\frac{\varepsilon}{2}}} u_p \ge \gamma_0 \frac{\varepsilon}{2},$$

and again it would drive us to a contradiction on the uniform convergence of u_p to u_∞. We have proven

$$\partial\{u_p > 0\} \subset \Gamma_\varepsilon \left(\partial\{u_\infty > 0\}\right).$$

The other inclusion is proven similarly. □

Proof of Theorem 6. Initially, let us recall some further facts from the p-Dirichlet minimization problem $(\mathfrak{P}_p)$. Recall that the free boundary $\partial\{u_p > 0\}$ is a $C^{1,\alpha}$ smooth surface up to an $\mathcal{H}^{n-1}$ closed and negligible set (see [10], [21], [8]). From the free boundary condition $|\nabla u_p| = \lambda_p$, we deduce that

$$\lim_{\substack{x \to \partial\Omega_p \\ x \in \Omega_p}} \frac{u_p(x)}{\mathrm{dist}(x, \partial\Omega_p)} = \lambda_p. \tag{5.18}$$

Hereafter, Ω_p denotes the set of positivity of u_p. From uniform convergence, $u_p \rightrightarrows u_\infty$, given a point $x \in \Omega_\infty$, we may assume $x \in \Omega_p$ for p sufficiently large. Now, from the free boundary convergence result, Theorem 5, there holds

$$\mathrm{dist}(x, \partial\Omega_\infty) = \mathrm{dist}(x, \partial\Omega_p) + o(1), \qquad \text{as } p \nearrow \infty. \tag{5.19}$$

Here, $o(1)$ is an error that goes to zero as p goes to infinity. Thus, using once more the Hausdorff metric convergence of the free boundary and uniform convergence of u_p to u_∞ together with (5.18) and (5.19), we reach the chain

$$\begin{aligned} \frac{u_\infty(x)}{\mathrm{dist}(x, \partial\Omega_\infty)} &= \frac{u_p(x)}{\mathrm{dist}(x, \partial\Omega_p)} + o(1) \\ &= \lambda_p + o(\mathrm{dist}(x, \partial\Omega_p)) + o(1) \\ &= \lambda_\infty + o(\mathrm{dist}(x, \partial\Omega_\infty)) + o(1). \end{aligned}$$

Letting $p \to \infty$, the proof of Theorem 6 is complete. □

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